Research Article

Modeling of an aircraft fire extinguishing process with a porous medium equation

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Abstract

The aim of this work is to provide a formulation of a non-linear diffusion model in the form of a Porous Medium Equation (PME) with application to a fire extinguishing process in an aircraft engine nacelle. The work starts by describing some relevant publications currently related to fire suppression modelling methods with emphasis in the aerospace sector. The PME is then introduced highlighting some key relevant features (particularly the finite speed of propagation) compared to the classical Heat Equation (HE). We will refer as u to the extinguisher or suppressor concentration in the media, which is postulated to be governed by a PME equation of the form:

$$u_t = \Delta u^m + |x|^\sigma u^p, \tag{1}$$

$$u(x,0) = u_0(x) \in \mathbb{L}^\infty_{loc}(\mathbb{R}^\mathbb{N}), \tag{2}$$

where

$$m > 1, \sigma > 0, p < 1, \tag{3}$$

$$(x,t) \in Q_T = \mathbb{R}^\mathbb{N} \times (0,T) \tag{4}$$

Without losing generality and in virtue of the mass transfer application, we will consider that any solution is $u \geq 0$. The set of equation and conditions expressed from (1) to (4) will be referred as problem P. From a mathematical perspective, the main areas of analysis are related to the existence of solutions, the obtaining of particular solutions as asymptotic approach and the application or particularization to a representative aircraft engine nacelle domain where a fire may happen.

Keywords Non-linear diffusion · Reaction · Mass transfer · Fire extinguishing · Aircrafts · Engines

1 Problem description and objectives

The fire extinguishing system design is subjected to the regulations provided by each territorial agency in which the aircraft is intended to operate. The European EASA and the American FAA (in addition to other worldwide regulatory agencies) require the aircraft to be equipped with fire detection and suppression systems. A short fire suppression is required to ensure the safety of any aircraft operation and to avoid serious damages to personnel and crew. Typically, the aircraft fire system designers conduct costly testing campaigns to ensure the regulatory requirements are met (see the remarkable examples in [1, 2]). In addition to the testing activities, a modelling exercise,

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involving computer codes in CFD, arises to support the construction of a process capable of predicting the fire suppression behaviour in any condition not limited to the tested scenario. Kim et al. [3] studied a computational numerical algorithm to understand the efficiency and flow properties of a halon 1301 (as typical fire suppressant in the aerospace sector) extinguishing process with the aim to optimize the design process emphasizing the effect of the diameter and potential damages in the suppressant piping. Another remarkable analysis, not directly related with the aerospace sector, is given by Harish and Venkatasubbaiah [4]. These authors investigated the fire hazards and flame propagation due to liquified natural gas (LNG) using a CFD model.

In some cases, the testing activities may have more relevance than the requirements in accuracy of the employed set of equations. In this sense, it is considered that a realistic operating campaign will permit to understand the behaviour of the fire suppression system for each operational condition. Penteado [5] (cited in [6]) proposed an experimental test to determine the behaviour of the fire suppressant substance in an aircraft un-pressurized zone (cargo compartment beneath the passenger area). In addition, Kurokawa et al. [6] proposed a method consisting on the employment of the lumped parameter method with pure mass continuity equations (no diffusion derived) to determine the fire suppressant agent concentration along the cargo hold area. In this case, the authors discussed about the need for a complete testing campaign to fully validate the modeling concept followed.

We highlight the fact that the methods typically used in the cited references consist on a balance between numerical fire suppression models (typically CFD codes) and testing activities. In the present work, we make use of analytical assessments in stead of numerical codes. This analytical approach has an inherent feature: for each step, we keep the attention in the model and its potential outcomes. We assess each calculated step and determine its accuracy and representativeness with the fire extinguishing process aimed to model. This possibility differs from the pure numerical approach where we do not have a clear understanding of the representativeness of solutions until a calibrating testing campaign is executed. To support the analytical approach presented and to determine the values of the different parameters involved in the proposed model P, a minimum set of testing activities are described.

In all cited previous analysis, fire modelling has been formulated in terms of a classical linear order two diffusion validated with extensive testing campaigns and numerical codes. The modelization in the classical diffusion is given by the fluid mechanics equations that have conditioned the basis for any modelling exercise in fire suppression. Galea and Markatos [7] established the basic models for fire suppression in aircraft design. These modelling philosophy has been used for the design and simulation of fire suppression means in pressurized and non-pressurized zones. The authors typified two different models depending whether the fire simulation is performed in discrete points or in a particular zone. In both cases, the driven diffusion was given by the classical linear term in any of the involved variables related with diffusion (fire propagation speed, temperature and smoke). Alternatively, we introduce a non-linear diffusion in the form of a Porous Medium Equation (PME). The PME is classified within the nonlinear parabolic partial differential equations scope (p. 85 in [8]). In a PME, the non-linearity is given in the diffusion term which introduces a set of properties differing from the classical order two diffusion. We purport to model the fire extinguisher evolution when the concentration pressure to avoid *crowding* makes the gas to travel. Let consider a sub-region of the domain where the extinguishing substance starts to increase. The pressure makes the gas to move towards other sub-region with a finite propagation speed. This phenomena can be modelled by the homogeneous PME [8]:

$$u_t = \nabla \cdot (u^{m-1} \nabla u), \quad m > 1, \tag{5}$$

where u^{m-1} is known as the pressure term.

Additionally, we propose a reaction term of the form $|x|^{\sigma} u^p, \sigma > 0$ and $p < 1$, that can be justified as follows:

When a extinguisher starts to populate any region, this medium has no substance of fire suppressor. Thus, it is considered that the time growing rate is high and positive at the beginning, but with less growing rate when the extinguisher increases due to the saturation of the agent in the media:

$$u_t = u^p, \quad p < 1. \tag{6}$$

Additionally, we consider that the agent is not homogeneous spatially distributed. This leads to the further increasing of extinguisher in certain locations. Mathematically speaking, we can think on:

$$u_t = F(x) u^p, \tag{7}$$

where $F(x)$ is a smooth function that permits to characterize the time growing rate depending on the location. We consider:

$$F(x) = |x|^{\sigma}, \quad \sigma > 0. \tag{8}$$

The selection of $F(x)$ responds to:

$$u_t \to \infty, \tag{9}$$

whenever:

$$|x| \to \infty, \tag{10}$$

to model a heterogeneous distributed concentration in which the location of the agent discharge is assumed to be qualitatively far.

One key question that will arise during our study is the finite time blow-up phenomena. We will proof if such property can be given in certain locations; Indeed, if the agent concentration growing rate goes to infinity with the space variable, it may induce the own concentration to go to infinity in a finite time.

To illustrate the mathematical topics introduced, we stress that a similar equation (but with a reaction term not depending on $|x|^\sigma$) was studied by De Pablo and Vázquez in [9]. In particular, the authors showed that solutions to the problem:

$$u_t = \Delta u^m + u^p, \qquad p < 1, \tag{11}$$

does not exhibit local blow-up.

Additionally, R. Ferreira et al. showed [10] the existence of blow up for a equation of the form:

$$u_t = \Delta u^m + u^{p(x)}, \tag{12}$$

where $p(x)$ is a smooth function with bounds (p_-, p_+). They showed that when the integration domain $\Omega = \mathbb{R}^{\mathbb{N}}$, there exists local (finite in time) blow-up provided that the following condition is met:

$$1 < p_- \le p_+ \le 1 + 2/N. \tag{13}$$

The coefficient $p^* = 1 + 2/N$ is denoted as Fujita exponent and is well known to be a boundary between values of p motivating finite time blow-up (as expressed under the condition (13)) and values of p providing global blow up, for which the following condition is shown:

$$p_- > 1 + 2/N. \tag{14}$$

Iagar and Sánchez studied a similar equation compared to the problem P. The authors classified the behaviour of blow-up profiles for the case of strong critical weight reaction ($\sigma > 2$) [11] and for the case of strong reaction ($\sigma > 2(1-p)/(m-1)$) [12]. In our case, we are not specifically concerned about the blow-up patterns as the real application intuition suggests that the fire suppressant agent will not *explode* at any finite time, therefore the parameter σ adopts a particular value (see the Eq. (220) in Sect. 3) shown to provide global solutions in Sect. 2.3.

Nonetheless and for completeness, we present some key features of the blow-up phenomena with the intention to determine a particular criterion to ensure that no blow-up is given, particularly, we show the existence of a critical exponent $p*$:

$$p^* = sign_+\left(1 - \frac{\sigma(m-1)}{2}\right), \tag{15}$$

such that for $p > p^*$, there exists blow up in finite time while for $p \le p^*$ there exist global solutions. In a physical intuition, this means:

- Finite time Blow-up: The solutions goes to infinity for a given finite time due to the cumulative effect of the reaction term. This phenomena is well known in the study of parabolic operators and has become a source of investigations [13]. Considering that the blow-up is given at $t = T$, we can express the finite time blow-up phenomena as:

  $$|u(x,T)| \to \infty. \tag{16}$$

  In a physical sense, a finite time blow-up corresponds to an extreme invasion from the fire suppressor that provokes the solutions to increase suddenly up to a theoretical infinity.
- Global solutions: The solutions evolve with no blow-up in finite time. This means, that solutions can go to infinity, nonetheless, this will happen in a infinite time as well. Then, we can read:

  $$u(x, T \to \infty) \to \infty. \tag{17}$$

  In this case, the physical intuition suggests that the solutions are not bounded unless we limit the exposure time (i.e. we make T finite).

Due to the degeneracy of the diffusion coefficient ($D(u)$), any solution cannot be classically defined, in case it is locally null for a certain time. This is the case of compact support initial conditions and solutions. As a consequence, the theory developed employs a generalization on the way solutions are defined, i.e. we focus our attention in weak solutions.

We will say

$$u \in Q_T = \mathbb{R}^{\mathbb{N}} \times (0,T), \tag{18}$$

is a weak solution to the problem P, if for every t, such that $0 \le t \le T$; and for every test function

$$\phi \in C^\infty(Q_T), \tag{19}$$

with compact support, the following identity holds:

$$\int_{\mathbb{R}^N} u(t)\,\phi(t) = \int_{\mathbb{R}^N} u(0)\,\phi(0) + \int_0^t \int_{\mathbb{R}^N} [u\,\phi_t + u^m\,\Delta\phi + |x|^\sigma u^p\,\phi] ds. \tag{20}$$

Note that when we refer to a subsolution (minimal) or a supersolution (maximal), the $'' ='' $ in the last equation is replaced by $'' \leq''$ and $'' \geq''$ respectively.

One of the intentions of this work is to analyze the existence and to determine a characterization of maximal and minimal solutions for the problem *P*. Such study does not prevent us to analyze uniqueness of solutions.

We will see that the sign of the parameter

$$\gamma = m\sigma + 2(1-\sigma)p + \sigma, \tag{21}$$

plays and important role to understand the applicable solutions:

- When $\gamma < 2$, the non-Lipschitz reaction is relevant. Thus, the existence of solutions is guaranteed whenever the initial data $u_0(x) > 0$. While in the case of $u_0 \equiv 0$, the proof of existence is more subtle and, in general, we will show that there exist two particular solutions (the maximal and minimal solutions) that are key to demonstrate existence and to bound the family of possible solutions.
- When $\gamma \geq 2$, the existence of solutions is shown with the help of the so-called self-similar solution which is obtained as a minimal asymptotic behavior. Additionally, the degeneracy of the diffusion implies that uniqueness cannot be guaranteed in case $u = 0$ in a ball B_R. In this case, a minimal solution can be proved to exist with the property of finite speed propagation and a maximal solution positive for each time $0 \leq t \leq T$ and all $x \in \mathbb{R}^N$

In summary, the most general problem (*P*) is:

$$\begin{aligned} &u_t = \Delta u^m + |x|^\sigma u^p, \\ &u(x,0) = u_0(x) \in \mathbb{L}^\infty_{loc}(\mathbb{R}^N), \end{aligned} \tag{22}$$

where

$$\begin{aligned} &m > 1\,, \sigma > 0, p < 1, \\ &(x,t) \in Q_T = \mathbb{R}^N \times (0,T). \end{aligned} \tag{23}$$

Without losing generality and in virtue of the application to a fire suppressor process, it is considered that any solution $u \geq 0$.

2 Mathematical theory

2.1 Source-type solutions and comparison of the heat equation versus the porous medium equation

This section has the aim of presenting the fundamental (or source-type) solutions for the HE firstly and for the PME secondly. In both cases, the initial condition is given in the form of a finite pulsed mass (*M*):

$$u(x,0) = M\delta(x), \tag{24}$$

where $\delta(x)$ represents the Dirac function at the spatial coordinate origin.

The homogeneous equation to solve for the HE is of the form:

$$u_t = \Delta u, \tag{25}$$

and for the PME:

$$u_t = \Delta u^m. \tag{26}$$

The positivity property in the HE is the main feature for comparison with the PME source solution that does not exhibit positiviy everywhere in its domain. This property is used as the basis for modelling the fire extinguisher mass transfer whose behaviour is not positive in all the domain of interest.

2.1.1 Heat equation source-solution

The process of obtaining a fundamental solution is based on studying the class of solutions that are invariant under the scaling group in the variables (x, t, u) which give the so-called self-similar form [14]:

$$u(x,t) = t^{-\alpha} f(\eta), \tag{27}$$

where

$$\eta = xt^{-\beta}. \tag{28}$$

The exponents α and β are called self-similarity exponents and the function *f* is called the self-similar profile.

The solution adopts the form of:

$$f(x,t) = M\frac{1}{t^{N/2}} e^{-\frac{|x|^2}{4t}}. \tag{29}$$

This fundamental or source-type solution, normally named as Gaussian kernel, is represented in Fig. 1. It represents a visual representation of the property referred as infinite speed of propagation that naturally appears as a consequence of the HE resolution. Starting from a single and finite mass at an isolated spacial point $(u(x,0) = M\delta(x))$,

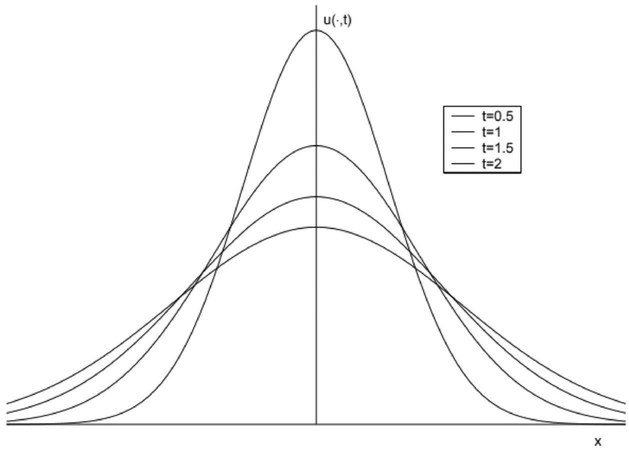

Fig. 1 The source-type evolution solution for the Heat Equation. It is to be highlighted the positivity condition everywhere. (Source reference [8])

the solution evolves towards positivity everywhere in the domain. This property is the basis for a certain comparison with the PME source-type solution.

2.1.2 Porous medium equation source-solution

The scaling of the variables, done for the HE, can be followed for the source solution of the PME:

$$u(x,t) = t^{-\alpha} f(\eta),\ \eta = xt^{-\beta}. \tag{30}$$

$$\underbrace{-\alpha t^{-(\alpha+1)} f(\eta) + t^{-\alpha}(-\beta) t^{-(\beta+1)} x \cdot f_\eta(\eta)}_{u_t} = \underbrace{t^{-(\alpha m+2\beta)} f^m_{\eta\eta}(\eta)}_{\Delta u^m}. \tag{31}$$

$$- \alpha t^{-(\alpha+1)} f(\eta) - t^{-(\alpha+1)} \beta\eta \cdot f_\eta(\eta) - t^{-(\alpha m+2\beta)} f^m_{\eta\eta}(\eta) = 0. \tag{32}$$

The elliptic differential equation for the self-similar profile is set after removing the time dependence in the Eq. (32). Hence, we have:

$$\alpha + 1 = \alpha m + 2\beta, \tag{33}$$

so that:

$$\alpha(m-1) + 2\beta = 1. \tag{34}$$

We arrive at one equation expressing a relation between the self-similar exponents α and β; Therefore, another relation is required to determine two particular values for each exponent. This second relation is given by the energy conservation during the evolution:

$$\int_{\mathbb{R}^N} t^{-\alpha} f(xt^{-\beta}) dx = M. \tag{35}$$

If we make the following change of variable:

$$\eta = xt^{-\beta}, \tag{36}$$

we shall take into account that the term dx is a volume magnitude to represent a differential in the whole space $\mathbb{R}^N$, therefore operating with volumes, we have:

$$\|x\|^N_{\mathbb{R}^N} = \|\eta\|^N_{\mathbb{R}^N} t^{N\beta}. \tag{37}$$

Then, we have that the differential in volumes are given by:

$$d\|x\|^N_{\mathbb{R}^N} = d\|\eta\|^N_{\mathbb{R}^N} t^{N\beta}. \tag{38}$$

Note that it is usual to simply write:

$$dx = d\eta t^{N\beta}, \tag{39}$$

to represent the volume integral, so that we have:

$$\int_{\mathbb{R}^N} t^{-\alpha} f(xt^{-\beta}) dx = t^{-\alpha+\beta N} \int_{\mathbb{R}^N} f(\eta) d\eta = M. \tag{40}$$

If we remove the time-dependency in the previous equation, we have $\alpha = \beta N$, so that we read the following set of algebraic equations:

$$\alpha(m-1) + 2\beta = 1, \tag{41}$$

$$\alpha = \beta N. \tag{42}$$

After resolution for the variables α and β, we have:

$$\alpha = \frac{N}{N(m-1)+2}, \tag{43}$$

$$\beta = \frac{1}{N(m-1)+2}. \tag{44}$$

It is still pending to solve the following elliptic differential equation for the self-similar profile f:

$$f^m_{\eta\eta}(\eta) + \beta\eta \cdot f_\eta(\eta) + \alpha f(\eta) = 0. \tag{45}$$

As we did in the previous section, we search for non-negative solutions with a radial symmetric profile. After the substitution of the Laplacian by its corresponding radial coordinates, we arrive at the following expression:

$$\frac{1}{r^{N-1}}[(r^{N-1}(f^m)')' + \beta r^N f' + r^{N-1} N\beta f] = 0. \tag{46}$$

Which can be re-written as:

$$\frac{1}{r^{N-1}}[(r^{N-1}(f^m)')' + (\beta r^N f)'] = 0 \\ \rightarrow (r^{N-1}(f^m)' + \beta r^N f)' = 0. \tag{47}$$

We can solve the first integral to have:

$$r^{d-1}(f^m)' + \beta r^d f = C. \tag{48}$$

As we did with the HE we require that $f \rightarrow 0$, whenever $r \rightarrow \infty$. Hence, we determine $C = 0$ and the Eq. (48) reads as:

$$(f^m)' + \beta r f = 0. \tag{49}$$

The Eq. (49) can be solved using ordinary differential equations techniques:

$$\begin{aligned} &\frac{df^m}{f} = -\beta r dr; \\ &\frac{mf^{m-1}df}{f} = -\beta r dr, \\ &mf^{m-2}df = -\beta r dr, \\ &\frac{m}{m-1}f^{m-1} = -\frac{\beta}{2}r^2 + C. \end{aligned} \tag{50}$$

The profile solution is:

$$f(r) = \left(A - \frac{\beta(m-1)}{2m}r^2\right)^{\frac{1}{m-1}}, \tag{51}$$

and in the variable η:

$$f(\eta) = \left(A - \frac{\beta(m-1)}{2m}|\eta|^2\right)^{\frac{1}{m-1}}. \tag{52}$$

Finally, the source-type solution adopts the following expression after substitution in the expression (30):

$$u(x,t) = t^\alpha \left(A - \frac{\beta(m-1)}{2m}|x|^2 t^{-2\beta}\right)^{\frac{1}{m-1}}, \tag{53}$$

where:

$$\alpha = \frac{N}{N(m-1)+2}, \tag{54}$$

and

$$\beta = \frac{1}{N(m-1)+2}. \tag{55}$$

We provide the graphical representation of the self-similar solution (Fig. 2) with the aim of comparing with the same graphic obtained in the HE case. The graphical representation for the PME manifests a relevant difference in the character of the fundamental profile. Namely, the fundamental profile of the PME is not positive everywhere in the domain as we had with the HE.

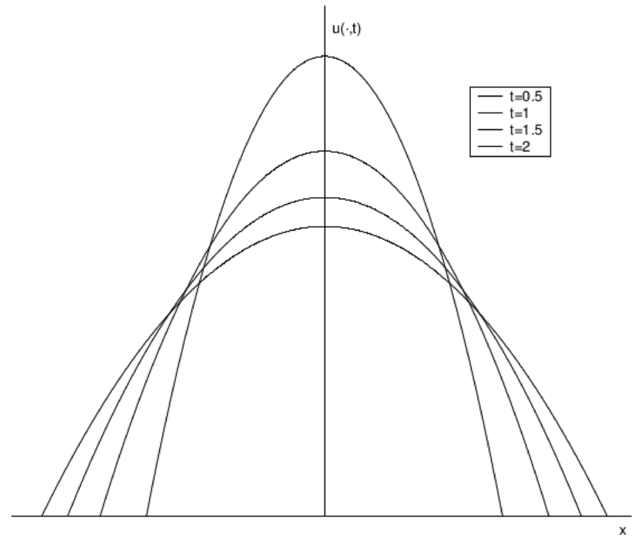

Fig. 2 The source-type evolution solution for the Homogenous PME. It is to be highlighted the non-negativity everywhere. (Source reference [8])

2.1.3 Comparison of the fundamentals solutions for the HE and PME

The contrast between both solutions of the HE and PME can be summarized as follows [8]:

- HE: A non-negative solution of the heat equation is automatically positive everywhere in its domain of definition.
- PME: Disturbances from the level $u = 0$ propagate in time with finite speed.

To make the comparison more intuitive, we can think how the initial and finite mass evolves in the HE and in the PME. For the HE the initial mass provides positivity everywhere as the gaussian kernel is positive for any $t > 0$. Nonetheless, the evolution of the PME is not positive everywhere; indeed, the support of the solution in the spatial domain propagates with a finite speed introducing a propagation front that turns the domain from zero to positivity. This propagating support evolves precisely in the (x, t) space following the expression:

$$t = \left(\frac{A2m}{\beta(m-1)}\right)^{\frac{1}{\beta}}. \tag{56}$$

The finite propagation feature of the PME is very important in the development of this work and shall be considered as a property that will appear when solving the PME with a forcing-reaction term. This property permits to model diffusion problems in which a propagating front appears as a result of the evolution. In mass transfer applications, a substance, invading or propagating in the domain, moves with finite propagation speed until it covers the whole domain. In our case, this substance is represented by the fire extinguishing agent. In the PME, the finite speed is given by the propagation of the function support that shifts the null state to positivity, which can be interpreted as the existence of substance.

2.2 Existence of solutions

To show existence of solutions, we operate with a truncation to bound the $|x|^\sigma$ term globally in $\mathbb{R}^N$:

$$|x|_\epsilon^\sigma = \begin{bmatrix} |x|^\sigma \ when \ 0 \le |x| < \epsilon \\ \epsilon^\sigma \ when \ |x| \ge \epsilon \end{bmatrix}. \quad (57)$$

Thus, we consider the following non-Lipschitz problem, named as P_ϵ:

$$\begin{aligned} &u_t = \Delta u^m + |x|_\epsilon^\sigma u^p \le \Delta u^m + \epsilon^\sigma u^p \ in \ Q_{T_\epsilon} = \mathbb{R}^N \times \left[0, T_\epsilon\right], \\ &u(x,0) = u_0(x) \ge 0, \\ &p < 1; \ m > 1 \ N \ge 1 \end{aligned} \quad (58)$$

The condition of a non-Lipschitz reaction term has implications on the study of existence of solutions. One of them is the impossibility to show uniqueness for any value of u, particularly when $u = 0$ or when u increases from zero to positivity. Our effort is, hence, focused on determining the existence and characterizing two particular solutions, named as the maximal and the minimal solutions, so that any other solution will exist between them.

Theorem 2.2.1 *There exists two particular solutions to the problem P_ϵ referred as maximal solution u^M and minimal solution u_m existing in $\left[0, T_\epsilon\right]$ with $T_\epsilon(\epsilon, \|u_0\|_*)$ such that any solution to problem P_ϵ satisfies*:

$$u_m \le u^\epsilon \le u^M.$$

Proof We firstly construct a Lipschitz function depending on a parameter δ:

$$f_\delta(s) = \begin{bmatrix} \epsilon^\sigma \delta^{(p-1)} s \ for \ 0 \le s < \delta \\ \epsilon^\sigma s^p \ for \ s \ge \delta \end{bmatrix}, \quad (59)$$

so that in the limit for $\delta \to 0$, we recover the original term u^p $(p < 1)$ (see Fig. 3 together with the Eq. (59)).

For building the maximal solution, we consider the following problem P_ϵ^M:

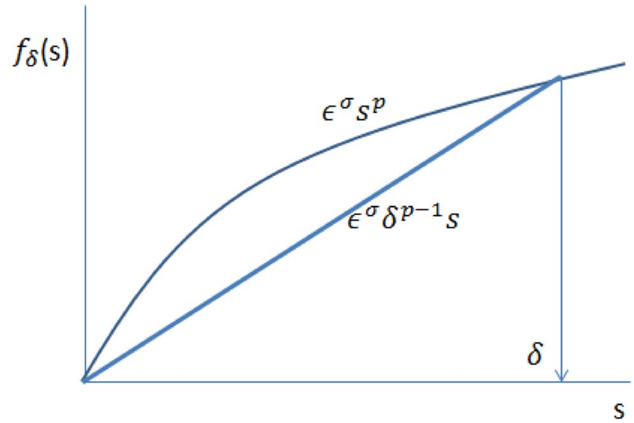

Fig. 3 The function $f_\delta(s)$ is used to approximate the non-Lipschitz problem by a Lipschitz one. Note that in the limit $\delta \to 0$, we recover the original non-Lipschitz term

$$\begin{aligned} &u_t = \Delta u^m + f_\delta(u) \ in \ Q_{T_{\epsilon,\delta}} = \mathbb{R}^N \times \left[0, T_{\epsilon,\delta}\right], \\ &u(x,0) = u_0(x) + \nu \ for \ x \in \mathbb{R}^N \ and \ \nu > 0. \end{aligned} \quad (60)$$

ν is selected such that:

$$f_\delta(u_0 + \nu) > f(u_0), \quad (61)$$

which gives:

$$\nu > |f_\delta^{-1} f(u_0) - u_0|. \quad (62)$$

The Lipschitz constant for the expression $f_\delta(s)$ can be obtained as follows:

$$\epsilon^\sigma (s_1^p - s_2^p) \le \epsilon^\sigma L(s_1 - s_2) \le \epsilon^\sigma \delta^{(p-1)} L(s_1 - s_2).$$

We remind that $p < 1$, therefore the last inequality make sense for $|\delta| < 1$. We shall consider the Lipschitz constant as:

$$\epsilon^\sigma L \delta^{(p-1)}. \quad (63)$$

The problem P_ϵ^M has a unique solution, existing for a time interval $T_{\epsilon,\delta}$ given by the following expression [15]:

$$T_{\epsilon,\delta} \ge \frac{1}{L\delta^{(p-1)}\epsilon^\sigma(m-1)} \log\left(1 + Lc\delta^{(p-1)}\epsilon^\sigma(m-1)\|u_0 + \nu\|_\infty^{1-m}\right). \quad (64)$$

The problem has, now, three different parameters: ϵ used to bound the forcing term, δ used to approximate the non-Lipschitz problem by a Lipschitz one and the paremeter ν that shall be chosen to ensure the maximality of u^M.

For a given ϵ, we can make $\delta \to 0$, to recover the non-Lipschitz problem. Then, it is possible to determine the following condition for the existence time:

$$T_{\epsilon,\delta\to 0} \ge 0. \quad (65)$$

Or explicitly with δ:

$$T_{\epsilon,\delta\to 0} \geq \frac{1}{L\epsilon^{\sigma}(m-1)}\delta^{1-p}. \tag{66}$$

This condition means that the existence time is given in the proximity of any δ.

To recover the forcing term $|x|^{\sigma}$, we shall impose $\epsilon \to \infty$ while to recover the non-Lipschitz problem we shall impose $\delta \to 0$. To account for both effects, we can take:

$$\epsilon = \frac{1}{\delta^{a}}, \ a > 0, \tag{67}$$

to joinly evaluate both parameters, ϵ and δ, in the limit with $\delta \to 0$ and $\epsilon \to \infty$.

Previously and first of all, we make $\delta \to \infty$ and $\epsilon \to 0$. We show that in this case, we recover the result obtained in the Lipschitz case (we remind that $\sigma > 0$).

$$T_{\epsilon,\delta} \geq \frac{\delta^{(1-p)}\delta^{a\sigma}}{L(m-1)}\log\left(1 + c\frac{1}{\delta^{(1-p)}\delta^{a\sigma}}L(m-1)\|u_0 + v\|_{\infty}^{1-m}\right), \tag{68}$$

$$T_{\epsilon\to 0,\delta\to\infty} \geq c\|u_0 + v\|_{\infty}^{1-m}, \tag{69}$$

for any $a > 0$.

Even when the value for $T_{\epsilon,\delta}$ has been obtained in the limit for $\epsilon \to 0$, it can be applied for any positive ϵ and by extension for any local single point in $x \in \mathbb{R}^{\mathbb{N}}$ as the function $|x|^{\sigma} \in \mathbb{L}^{\infty}_{loc}$.

Nonetheless, to recover the original problem we shall require $\delta \to 0$ and $\epsilon \to \infty$. In this case, we operate with the term $\delta \to 0$:

$$T_{\epsilon,\delta} \geq \frac{\delta^{(1-p)}\delta^{a\sigma}}{L(m-1)}\log\left(c\frac{1}{\delta^{(1-p)}\delta^{a\sigma}}L(m-1)\|u_0 + v\|_{\infty}^{1-m}\right), \tag{70}$$

$$T_{\epsilon,\delta} \geq 0, \tag{71}$$

Or explicitly with δ:

$$T_{\epsilon,\delta\to 0} \geq \frac{1}{L(m-1)}\delta^{1-p+a\sigma}. \tag{72}$$

for any $a > 0$.

This case corresponds to the existence of global blow-up as it will be shown afterwards in Sect. 2.3. This implies that, if we select two values, one arbitrary for δ sufficiently small and other one for ϵ sufficiently large; we are in a position to calculate a value for $T_{\epsilon,\delta}$ and, therefore, to ensure the existence of a local maximal solution, not hidden by the global blow-up that seems, previous to any formal proof, to be an inherent feature of our problem. The existence of a maximal solution is supported by the precise calculation of such solution in Sect. 2.2.1.

For building the minimal solution, we consider the following problem P_{ϵ}^{m}:

$$\begin{aligned} &u_t = \Delta u^m + f_{\delta}(u) \ Q_{T_{\epsilon,\delta}} = \mathbb{R}^{\mathbb{N}} \times \left[0, T_{\epsilon,\delta}\right], \\ &u(x,0) = u_0(x) \ for \ x \in \mathbb{R}^{\mathbb{N}} \ and \ \delta > 0. \end{aligned} \tag{73}$$

The problem P_{ϵ}^{m} has a unique solution [15] existing for a time interval $(0, T_{\epsilon,\delta})$. Any solution, u_{δ}^{m}, to the problem P_{ϵ}^{m} is a subsolution to the problem P_{ϵ} and to the original problem P. Indeed the approximation $f_{\delta}(u)$ of the non-Lipschitz function u^p satisfies:

$$\begin{aligned} f_{\delta}(u) \leq \epsilon^{\sigma}u^p \leq |x|^{\sigma}u^p, \\ u_{\delta}^{m} \leq u. \end{aligned} \tag{74}$$

Given $\delta_1 > \delta_2$, we have $f_{\delta_1}(u) < f_{\delta_2}(u)$, such that for an arbitrary decreasing sequence of $\delta\}s$, there exist a non-decreasing sequence of u_{δ}^{m} that satisfies $u_{\delta}^{m} \leq u$, such that in the limit with $\delta \to 0$ we can establish:

$$u^m = \lim_{\delta\to 0} u_{\delta}^{m}. \tag{75}$$

Note that u^m is a minimal solution to the problem P_{ϵ} and to the problem P (in virtue of the ordered property in (74)), indeed, u^m has been obtained under the change of the reaction original term u^p by a Lipschitz function from below $f_{\delta}(u)$. □

The provided proof of Theorem 2.2.1 is based on the approximation to a Non-Lipschitz problem from a Lipschitz one. The Non-Lipschitz condition of the reaction term implies that uniqueness cannot hold. It has been proved that two particular solutions, maximal and minimal, exist. The determination of both solutions, with a classification in accordance with the problem data, is done in the immediate following sections.

2.2.1 Discussion about types of solutions

We have shown the existence of a maximal and a minimal solution, when the non-Lipschitz reaction imposes non-uniqueness. It is, now, the intention to obtain such solution profiles together with the expected types of solutions depending on the problem P data.

We consider the initial condition of the form:

$$u_0 \equiv 0, \tag{76}$$

or,

$$u_0 = 0 \ in \ B_R = \{|x - x_0| < R\}. \tag{77}$$

In this case, and whenever $u_0 \to 0$, the reaction term predominates over the diffusive term (indeed the reaction term has $p < 1$ while the diffusion $m > 1$). Based on the fact that the non-Lipschitz reaction is predominant, we can think on two different solutions: The minimal solution to the problem P of the elementary form $u_m = 0$ and a maximal positive solution that can be shown to be:

$$u_\tau^M = |x|^{\sigma/(1-p)}(1-p)^{1/(1-p)}(t-\tau)^{1/(1-p)}, \tag{78}$$

for any $\tau > 0$.

To show the structure of such maximal solution, we start by a function of the form:

$$u_\tau^M = |x|^\theta k(t-\tau)^\alpha. \tag{79}$$

Introducing the expression (79) into the problem P, we have:

$$|x|^\theta k\alpha(t-\tau)^{\alpha-1} = m\theta(m\theta-1)|x|^{m\theta-2}k^m(t-\tau)^{m\alpha} + |x|^{p\theta+\sigma}k^p(t-\tau)^{p\alpha}. \tag{80}$$

The above expression gives the following values for θ and α provided the reaction term predominates over the diffusion:

$$\theta = \frac{\sigma}{1-p}, \quad \alpha = \frac{1}{1-p}, \quad k = (1-p)^{\frac{1}{1-p}}. \tag{81}$$

The postulated maximal solution adopts the following form:

$$u_{\tau\to 0}^M = |x|^{\sigma/(1-p)}(1-p)^{\frac{1}{1-p}}(t)^{1/(1-p)}. \tag{82}$$

In Theorem 2.4.1 , we will show that any maximal solution is positive.

Once the solution starts to be positive, the local time evolution provides a positive and growing solution if the spatial term in the reaction predominates over the diffusion:

$$p\theta + \sigma > m\theta - 2, \quad m\sigma + 2(1-\sigma)p + \sigma < 2, \tag{83}$$

which shall be met for a maximal solution of the form (82). In fact, this condition can be used to state the following results to understand the expected type of solutions depending on the data parameters for P:

- $m\sigma + 2(1-\sigma)p + \sigma \geq 2$.
  The diffusion predominates and finite speed of propagation shall be considered whenever the solution is null in a certain ball B_R. This kind of solutions, where finite speed is given, are characterized in Theorem 2.3.2.
- $m\sigma + 2(1-\sigma)p + \sigma < 2$.
  The reaction is relevant, and particularly, the non-Lipschtiz condition provides non-uniqueness. Two particular solutions, $u^m = 0$ and $u_{\tau\to 0}^M$ have been proved to exist (Theorem 2.2.1). In this case, the finite speed of propagation. We highlight that the two solutions, $u^m = 0$ and $u_{\tau\to 0}^M$, have been obtained based on the reaction term properties.

2.3 Precise minimum order of growth and solution

The intention, now, is to establish a minimum order of growth for the positive solutions to the problem P, but considering that the initial condition is a compactly supported function. This fact will permit to obtain the precise evolution of the support. The interest of a compactly supported function is focused on understanding the evolution of a smooth function whose support is null, and therefore, we can expect finite propagation speed due to the degeneracy of the diffusivity when $u \to 0$ in accordance with the parameters conditions derived in Sect. 2.2.1. Furthermore, the propagation of gas substances (for instance a fire extinguisher) can be modelled with a compactly supported function to understand the dynamic of the such propagation.

We firstly develop a self-similar solution to the problem P that provides two cases to distinguish: The global evolution problem and the blow-up in finite time case. We define a critical parameter p^* accordingly. This result is compiled in the following theorem:

Theorem 2.3.1 *There exist a critical exponent p^* defined as:*

$$p^* = sign_+\left(1 - \frac{\sigma(m-1)}{2}\right), \tag{84}$$

such that for:

$$p > p^*, \tag{85}$$

there exists blow up in finite time, while for:

$$p \leq p^*, \tag{86}$$

there exists a global solution.

Proof We look for self-similar profiles of the form:

$$E(x,t) = t^{-\alpha}f(|x|t^\beta), \quad \chi = |x|t^\beta. \tag{87}$$

We make N = 1 for simplification purposes. The involved components, in the problem P, adopt the following forms:

$$u_t = -\alpha t^{-\alpha-1} f + \beta \underbrace{|x| t^{\beta}}_{\chi} t^{-\alpha-1} f',$$
$$\Delta u^m = t^{-\alpha m} t^{2\beta} f^m_{xx},$$
$$|x|^{\sigma} u^p = \chi^{\sigma} t^{-\sigma\beta - \alpha p} f^p. \tag{88}$$

Upon substitution into P

$$-\alpha t^{-\alpha-1} f + \beta \underbrace{|x| t^{\beta}}_{\chi} t^{-\alpha-1} f' = t^{-\alpha m} t^{2\beta} f^m_{xx} + \chi^{\sigma} t^{-\sigma\beta-\alpha p} f^p. \tag{89}$$

And comparing the exponents of each variable t in the expression (89), we arrive at:

$$-\alpha - 1 = -\alpha m + 2\beta,$$
$$\alpha m - 2\beta = \alpha p + \beta\sigma. \tag{90}$$

The solutions for α and β are:

$$\alpha = \frac{\sigma+2}{\sigma(m-1)+2(p-1)},$$
$$\beta = \frac{m-p}{\sigma(m-1)+2(p-1)}. \tag{91}$$

Note that the term:

$$\sigma(m-1) + 2(p-1), \tag{92}$$

is common to α and β and in the blow-up in finite time case, it must be positive; while for the existence of a global solution, it must be negative (refer to the form of the self-similar profile in (87) where the time exponent is $-\alpha$). This two qualitative different behaviour of the solutions to the problem P can be clearly separated thanks to the definition of the critical exponent

$$0 < p^* < 1. \tag{93}$$

For the finite time blow up case, we have:

$$\sigma(m-1) + 2(p-1) > 0,$$
$$p > p^* = sign_+\left(1 - \frac{\sigma(m-1)}{2}\right). \tag{94}$$

Where the function $sign_+$ returns zero whenever:

$$\left(1 - \frac{\sigma(m-1)}{2}\right) < 0. \tag{95}$$

The complementary case provides the criteria for the existence of global in time solutions in Q_T: $p \le p^*$. □

The following theorem provides us with the evolution of a positive point $u(x_0, t_0) > 0$ and the evolution of the support, given a compactly supported initial data. It can be stated as:

Theorem 2.3.2 *Let u be a solution to problem P, such that $u(x_0, t_0) > 0$ for a given point in Q_T, then the following evolutions hold*:

- $u(x_0, t) \ge c_1(x_0)(t - t_0)^{-\alpha}$ *for any*
$$t > t_0, \tag{96}$$
and
$$\alpha = \frac{\sigma + 2}{\sigma(m-1) + 2(p-1)}. \tag{97}$$
- $u(x, t) > 0$ *for any*
$$t > t_0, \tag{98}$$
such that
$$|x - x_0| < c_2(x)(t - t_0)^{\beta}, \tag{99}$$
where
$$\beta = \frac{p - m}{\sigma(m-1) + 2(p-1)}. \tag{100}$$

And where:

$$c_1(x_0) = |x_0|^{\frac{\sigma}{1-p}} (-\alpha + \beta N)^{\frac{1}{p-1}}. \tag{101}$$

$$c_2(x) = c_{supp} |x|^{\frac{\sigma(m-1)}{2(1-p)}}, \tag{102}$$

being,

$$c_{supp} = \frac{(-\alpha + \beta N)^{\frac{m-1}{2(p-1)}}}{\left(\frac{(m-1)\beta}{2m}\right)^{1/2}}. \tag{103}$$

Proof The stated results are obtained from a lower estimation to the reaction term (see the coming term $h_{\epsilon,n}$ to be characterized), so that comparison can be applied with an explicit subsolution.

The proof of the theorem starts by considering the following problem P_s:

$$u_t = \Delta u^m + h_{\epsilon,n}, \tag{104}$$

where:

$$h_{\epsilon,n} = n^{\sigma} min\left[u^p, \epsilon^{p-1} u\right], \tag{105}$$

for:

$$n > 0 \;\; and \;\; \epsilon > 0, \tag{106}$$

understood as parameters such that in the limit with:

$$n \to \infty \;\; and \;\; \epsilon \to 0, \tag{107}$$

we recover the original term $|x|^\sigma u^p$.

The function $h_{\epsilon,n}$ satisfies the Lipschitz condition and, as a consequence, solutions exist [15].

The solution to the problem P_s can be obtained using a self-similar structure. The self-similar form is as per (87). The following equation holds for the determination of an exact solution profile for the term $f(|x|t^\beta)$, being $\chi = |x|t^\beta$ in the case of $x \in \mathbb{R}^{\mathbb{N}}$:

$$\begin{aligned} &-\alpha t^{-\alpha-1} f + \beta \chi t^{-\alpha-1} f' \\ &\quad = t^{-\alpha m}(f^m)'' + \frac{N-1}{\chi}(f^m)' + h_{\epsilon,n}, \end{aligned} \tag{108}$$

where:

$$h_{\epsilon,n}(f,t) = n^\sigma min\left[f^p, \epsilon^{p-1} t^{\alpha(p-1)} f\right]. \tag{109}$$

Note that we write

$$h_{\epsilon,n}(f,t) = n^\sigma min\left[t^{-\alpha p} f^p, \epsilon^{p-1} t^{-\alpha} f\right]. \tag{110}$$

This last expression and the expression in (109) have exactly the same intersection $f = t^\alpha \epsilon$. For simplification purposes, we make the calculations with the expression in (109) operating with the linear term:

$$\epsilon^{p-1} t^{\alpha(p-1)} f. \tag{111}$$

We can select a time t (to be determined), such that we have:

$$h_{\epsilon,n}(f,t) \geq n^\sigma \, c \, f \tag{112}$$

and c can be chosen as $c = n^{-\sigma}(-\alpha + \beta N)$ for simplification during the resolution of (108).

The profile $f(\chi)$ must satisfy the coming equation in (115) for each time (we assume $t = 1$). We consider that:

$$h_{\epsilon,n}(f,t) = n^\sigma \, c \, f, \tag{113}$$

such that, the equation reads:

$$-\alpha f + \beta \chi f' = (f^m)'' + \frac{N-1}{\chi}(f^m)' + (-\alpha + \beta N) f. \tag{114}$$

$$\beta \chi f' = (f^m)'' + \frac{N-1}{\chi}(f^m)' + \beta N f. \tag{115}$$

We have an elliptic equation with a known solution [16]:

$$f(\chi) = (A - B\chi^2)^{\frac{1}{m-1}}, \tag{116}$$

where:

$$\begin{aligned} &A > 0, \\ &B = \tfrac{(m-1)\beta}{2m}. \end{aligned} \tag{117}$$

This solution is valid for a sufficiently large time to hold the inequality (112) and to be determined as:

$$\begin{aligned} &n^\sigma min\left[f^p, \epsilon^{p-1} t^{\alpha(p-1)} f\right] \geq n^\sigma \, c \, f, \\ &min\left[f^p, \epsilon^{p-1} t^{\alpha(p-1)} f\right] \geq c \, f. \end{aligned} \tag{118}$$

And in the sublinear case:

$$\epsilon^{p-1} t^{\alpha(p-1)} \geq n^{-\sigma}(-\alpha + \beta N). \tag{119}$$

We perform, now, the change of variable:

$$n = \frac{1}{\epsilon}, \tag{120}$$

to jointly evaluate the effect of both ϵ and n^σ. Indeed, we recover the original problem P when we make $\epsilon \to 0$ and $n^\sigma \to \infty$.

We can obtain a explicit value of t_ϵ from the expression (119):

$$t_\epsilon = (-\alpha + \beta N)^{\frac{-1}{\alpha(1-p)}} \epsilon^{-1/\alpha} \left(\frac{1}{\epsilon}\right)^{\frac{\sigma}{\alpha(1-p)}}, \tag{121}$$

such that the solution in (116) is a subsolution provided that:

$$t \geq t_\epsilon. \tag{122}$$

It is particularly interesting to make the limit with $\epsilon \to 0$. In this case, we have two cases to distinguish:

- Blow up case $\alpha > 0$:

$$t_\epsilon = (-\alpha + \beta N)^{\frac{-1}{\alpha(1-p)}} \frac{1}{\epsilon^{\frac{\sigma}{\alpha(1-p)} + \frac{1}{\alpha}}} \to \infty. \tag{123}$$

- Global solution case $\alpha < 0$:

$$t_\epsilon = (-\alpha + \beta N)^{\frac{1}{|\alpha|(1-p)}} \epsilon^{\frac{\sigma}{|\alpha|(1-p)} + \frac{1}{|\alpha|}} \to 0. \tag{124}$$

The self-similar solution is a subsolution for $t \geq t_\epsilon$ as it has been obtained by approximating the reaction term by the function $h_{\epsilon,n}$. Note that, on one side, the blow up case represents a singularity as the self-similar estructure blows-up in a finite time. On the other side, the self-similar solution is a subsolution for any $t \geq t_\epsilon$ in case a global solution exists.

Any solution to the problem P_s is, indeed, a subsolution to the problem P as $h_{\epsilon,n} \leq n^\sigma u^p$. We can further assess this condition by letting y to be a solution to the problem P

and u a solution to the problem P_s, starting at $t = t_\epsilon$. For any $\tau > t_\epsilon$, we have:

$$y(x,\tau) \geq u(x,t_\epsilon)\ x \in \mathbb{R}^{\mathbb{N}}, \tag{125}$$

for any $t \geq 0$, we have:

$$y(x,\tau+t) \geq u(x,t_\epsilon+t)\ x \in \mathbb{R}^{\mathbb{N}}, \tag{126}$$

and in the limit with $\tau \to 0$:

$$y(x,t) \geq u(x,t)\ x \in \mathbb{R}^{\mathbb{N}}. \tag{127}$$

Showing that $u(x, t)$ solution of the problem P_s is indeed a subsolution to the problem P.

Coming back to the expression (116): The precise evolution of the global solutions is given by directly obtaining the evolution of the maximum value in the function (116) for $\chi = 0$. The intention is to have a growing evolution starting at the positive A.

$$u(x,t) = A^{\frac{1}{m-1}} t^{-\alpha}, \qquad \alpha = \frac{\sigma+2}{\sigma(m-1)+2(p-1)}. \tag{128}$$

A value for A can be determined from the expression (118):

$$min\left[f^{p-1}, \epsilon^{p-1} t^{\alpha(p-1)}\right] \geq n^{-\sigma}(-\alpha+\beta N). \tag{129}$$

To obtain A, we make $\chi = 0$:

$$min\left[A^{\frac{p-1}{m-1}}, \epsilon^{p-1} t^{\alpha(p-1)}\right] \geq n^{-\sigma}(-\alpha+\beta N). \tag{130}$$

Our solution departs from the point $f(\chi = 0)$, which is the minimum point as the time evolves due to the increasing behaviour of the global solution as per the expression (128) with $\alpha < 0$. Therefore we can determine A considering the following expression:

$$A^{\frac{p-1}{m-1}} = n^{-\sigma}(-\alpha+\beta N), \tag{131}$$

$$A = n^{\frac{\sigma(m-1)}{1-p}}(-\alpha+\beta N)^{\frac{m-1}{p-1}} = c(\alpha,\beta,N,m,p) n^{\frac{\sigma(m-1)}{1-p}}. \tag{132}$$

Upon recovering of the independent variable $|x|$:

$$A(x) = c(\alpha,\beta,N,m,p)|x|^{\frac{\sigma(m-1)}{1-p}}. \tag{133}$$

Eventually, the minimum growing evolution of the point with $f(\chi = 0)$ is:

$$y_m(x,t) = |x|^{\frac{\sigma}{1-p}}(-\alpha+\beta N)^{\frac{1}{p-1}} t^{-\alpha}. \tag{134}$$

This last expression provides the proof of the first part of the theorem considering that:

$$c_1(x) = |x|^{\frac{\sigma}{1-p}}(-\alpha+\beta N)^{\frac{1}{p-1}}. \tag{135}$$

Now, our intention is to determine the time evolution of the support of $f(\chi)$. For this purpose, we firstly assess the χ values determining such support of f:

$$f(\chi) = 0 \Rightarrow \chi = \left(\frac{A}{B}\right)^{(1/2)},$$

$$\chi_{supp} = \frac{1}{\left(\frac{(m-1)\beta}{2m}\right)^{1/2}} c^{(1/2)}(\alpha,\beta,N,m,p)|x|^{\frac{\sigma(m-1)}{2(1-p)}} = c_{supp}|x|^{\frac{\sigma(m-1)}{2(1-p)}}. \tag{136}$$

The evolution of the self-similar solution support in the (x, t) hiperspace:

$$|x|_{supp} = \frac{1}{c_{supp}} t^{\frac{2(m-p)(1-p)}{(\sigma(m-1)+2(p-1))^2}}. \tag{137}$$

Coming back to the second part of the theorem enunciation, we can calculate the value of $c_2(x)$ as:

$$c_2(x) = c_{supp}|x|^{\frac{\sigma(m-1)}{2(1-p)}}. \tag{138}$$

The theorem is, therefore, shown and the final results are as per the following expressions:

- $u(x_0,t) \geq c(\alpha,\beta,N,m,p)|x_0|^{\frac{\sigma(m-1)}{1-p}}(t-t_0)^{-\alpha}$ for any $t > t_0$ and

$$\alpha = \frac{\sigma+2}{\sigma(m-1)+2(p-1)}. \tag{139}$$

- $u(x,t) > 0$ for any $t > t_0$ and

$$|x-x_0| < c_{supp}|x|^{\frac{\sigma(m-1)}{2(1-p)}}(t-t_0)^{\beta}, \tag{140}$$

where

$$\beta = \frac{m-p}{\sigma(m-1)+2(p-1)}. \tag{141}$$

In addition, note that:

$$c_{supp} = \frac{(-\alpha+\beta N)^{\frac{m-1}{2(p-1)}}}{\left(\frac{(m-1)\beta}{2m}\right)^{1/2}}, \tag{142}$$

and,

$$c(\alpha,\beta,N,m,p) = (-\alpha+\beta N)^{\frac{m-1}{2(p-1)}} \tag{143}$$

□

Once we have shown the evolution of the solutions for the problem P, with compactly supported initial data, we proceed to enunciate the conditions required for a unique solution.

2.4 Uniqueness

Our objective is, now, to establish the required conditions, so that there exists only one solution to the problem P. Essentially, uniqueness of solutions leads to consider only positive initial data:

$$u_0 \geq \phi > 0, \tag{144}$$

so that the reaction term, $R(x,u) = |x|^\sigma u^p$, is Lipschitz in the interval $[\phi, \infty)$. The following theorem aims to show that the maximal solution u^M, as per (78), is the unique solution to P provided the initial data is positive.

Theorem 2.4.1 *Let u be a solution to problem P, such that*

$$u_0(x) > 0, \tag{145}$$

and $v \in \mathbb{R}^+$ *with*:

$$u \geq v, \tag{146}$$

for all $0 \leq t < T$. *Then u coincides with the maximal solution to problem P*.

Proof Firstly, we perform the usual truncation to the term $|x|^\sigma$ as per (57), so that we define the problem P_ϵ.

If we consider v as the maximal solution to the problem P_ϵ in $0 \leq t < T$ and $x \in \mathbb{R}^N$, we have that the following expression holds for every test function $\phi \in C^\infty(Q_T)$ with compact support in x:

$$\begin{aligned}0 &\leq \int_{\mathbb{R}^N} (v-u)(t)\,\phi(t) \\ &= \int_0^t \int_{\mathbb{R}^N} [(v-u)\,\phi_t + (v^m - u^m)\Delta\phi \\ &\quad + |x|_\epsilon^\sigma (v^p - u^p)\phi)]ds.\end{aligned} \tag{147}$$

So that:

$$\begin{aligned}0 &\leq \int_{\mathbb{R}^N} (v-u)(t)\,\phi(t) \\ &\leq \int_0^t \int_{\mathbb{R}^N} [(v-u)\,\phi_t + (v^m - u^m)\Delta\phi + \epsilon^\sigma (v^p - u^p)\phi)]ds.\end{aligned} \tag{148}$$

To ensure the convergence of the integrals in (148), we require

$$\phi \in C^\infty(Q_T) \cap \mathbb{L}^1(Q_T). \tag{149}$$

The following function will be of help during the integral assessment:

$$a(\epsilon,s) = \begin{bmatrix} \frac{v(\epsilon,s)^m - u(\epsilon,s)^m}{v(\epsilon,s) - u(\epsilon,s)} \;\; for \;\; v \not\equiv u \\ mv^{m-1} \;\; otherwise \end{bmatrix}. \tag{150}$$

Given two fixed values for ϵ and $s = T$, the last expression is bounded satisfying that:

$$0 \leq a(\epsilon,s) \leq c_0(m, \|u_0\|_\infty, T). \tag{151}$$

We try the following test function:

$$\phi(|x|,s) = e^{k(T-s)}(1+|x|^2)^{-\gamma}, \tag{152}$$

for some constant k and γ.

The determination for γ is given by the condition related to the compact support and integral convergence in $\mathbb{R}^N$. Indeed:

$$\int_{\mathbb{R}^N;|x|\to\infty} e^{k(T-s)}(1+|x|^2)^{-\gamma}|x|^\sigma dx \sim e^{k(T-s)} \int_{\mathbb{R}^N;|x|\to\infty} (|x|)^{-2\gamma+\sigma} dx \sim e^{k(T-s)}(|x|)^{-2\gamma+\sigma} x^N \to 0 \tag{153}$$

when $|x| \to \infty$.

The condition in (153) holds for

$$\gamma > \frac{\sigma + N}{2}. \tag{154}$$

In addition, the test function satisfies the following expressions:

$$\begin{aligned}&\phi_t = -k\phi, \\ &\Delta_{|x|}\phi \leq c_1(\gamma, N)\phi,\end{aligned} \tag{155}$$

we have:

$$\phi_t + a\Delta\phi \leq (-k + a(\epsilon,s)\,c_1(\gamma,N))\phi. \tag{156}$$

We are particularly interested in making:

$$\phi_t + a\Delta\phi \leq 0, \tag{157}$$

as it will be shown shortly.

For this purpose, we can consider a k sufficiently large satisfying:

$$k > a(\epsilon,s)\,c_1(\gamma,N). \tag{158}$$

In such a case, the inequality in (148) can be rewritten as:

$$0 \leq \int_{\mathbb{R}^{\mathbb{N}}} (v-u)(t)\,\phi(t) \leq \int_0^t \int_{\mathbb{R}^{\mathbb{N}}} [(v-u)\,\phi_t + a(v-u)\Delta\phi + \epsilon^{\sigma}(v^p - u^p)\phi)]ds, \tag{159}$$

and

$$0 \leq \int_{\mathbb{R}^{\mathbb{N}}} (v-u)(t)\phi(t) \leq \int_0^t \int_{\mathbb{R}^{\mathbb{N}}} [(v-u)[\phi_t + a\Delta\phi] + \epsilon^{\sigma}(v^p - u^p)\phi)]ds. \tag{160}$$

Considering that:

$$\phi_t + a\Delta\phi \leq 0, \tag{161}$$

we have:

$$0 \leq \int_{\mathbb{R}^{\mathbb{N}}} (v-u)(t)\,\phi(t) \leq \int_0^t \int_{\mathbb{R}^{\mathbb{N}}} \epsilon^{\sigma}(v(|x|_{\epsilon}, s)^p - u(|x|_{\epsilon}, s)^p)\phi(|x|_{\epsilon}, s)ds. \tag{162}$$

The proof of Theorem 2.4.1 will succeed if we demonstrate that the right hand side of the inequality in (162) is cero or tends to cero under certain suitable conditions that involve making $\epsilon \to \infty$.

Given a positive initial data, the minimum positive solution is [9]:

$$u(\epsilon, s) = \epsilon^{\sigma/(1-p)}(1-p)^{(1-p)}(t)^{1/(1-p)}, \tag{163}$$

So that:

$$0 \leq \epsilon^{\sigma}(v^p - u^p)(|x|_{\epsilon}, s) \leq \epsilon^{\sigma}\frac{p(v-u)}{u^{(1-p)}} \leq \frac{p}{1-p}\,\frac{v-u}{s}. \tag{164}$$

Therefore, we can write:

$$\int_{\mathbb{R}^{\mathbb{N}}} (v-u)(t)\,\phi(t) \leq \frac{p}{1-p}\int_0^t \int_{\mathbb{R}^{\mathbb{N}}} (v(|x|_{\epsilon}, s) - u(|x|_{\epsilon}, s))\phi(|x|_{\epsilon}, s)s^{-1}ds. \tag{165}$$

We define now the following function:

$$f(t) = \int_{\nu}^t \int_{\mathbb{R}^{\mathbb{N}}} (v-u)\phi s^{-1}ds, \tag{166}$$

where $\nu \to 0$.

This ν can be considered as the one mentioned in the postulations of the theorem, when we established $u \geq \nu > 0$, as it is a free parameter that we can make positive and tending to cero. The derivative is as follows:

$$\dot{f}(t) = t^{-1}\int_{\mathbb{R}^{\mathbb{N}}} (v-u)(t)\phi(t)dx;\ t\dot{f}(t) = \int_{\mathbb{R}^{\mathbb{N}}} (v-u)(t)\phi(t)dx. \tag{167}$$

The inequality (165) can be expressed as:

$$t\dot{f}(t) \leq \frac{p}{1-p}f(t) \tag{168}$$

The ordinary differential equation in (168) has the solution:

$$f(t) = ct^{p/(1-p)}, \tag{169}$$

for a constant c to be determined.

Given $\epsilon > 0$ such that $\nu < t < T$ we have:

$$f(\nu) = c\nu^{p/(1-p)} \to c = \frac{f(\nu)}{\nu^{p/(1-p)}}. \tag{170}$$

Finally, the solution to (168) is:

$$f(t) \leq f(\nu)\left(\frac{t}{\nu}\right)^{p/(1-p)}. \tag{171}$$

In the limit $t \to \nu$, we have $f(t) \to 0$ whenever $f(\nu) \to 0$. Therefore, our problem has resulted in the searching of a suitable function, such that

$$f(\nu) \to 0, \tag{172}$$

whenever $t \to \nu$.

In order to find the suitable $f(\nu)$, we can arrange the inequality (165) aiming to obtain another upper estimation that after comparison with the expression in (171), will support the finding of that suitable $f(\nu)$.

$$\int_{\mathbb{R}^{\mathbb{N}}} (v-u)(t)\,\phi(t) \leq \int_0^t \int_{\mathbb{R}^{\mathbb{N}}} (v^p - u^p)(\tau)\phi(\tau)\epsilon^{\sigma}dxd\tau \leq \int_0^t \left(\int_{\mathbb{R}^{\mathbb{N}}} \phi(\tau)\epsilon^{\sigma}dx\right)^{1-p}\left(\int_{\mathbb{R}^{\mathbb{N}}} (v-u)\phi(\tau)\epsilon^{\sigma}dx\right)^{p}d\tau. \tag{173}$$

Note that the integral:

$$\int_{\mathbb{R}^{\mathbb{N}}} \phi(\tau)\epsilon^{\sigma}dx, \tag{174}$$

is bounded in $\mathbb{R}^{\mathbb{N}}$:

$$\int_0^t \left(\int_{\mathbb{R}^{\mathbb{N}}} \phi(\tau)\epsilon^{\sigma}dx\right)^{1-p} \leq c(t \to T). \tag{175}$$

And the inequality in (173) is expressed as:

$$\int_{\mathbb{R}^N} (v-u)(t)\,\phi(t) \le \int_0^t \int_{\mathbb{R}^N} (v^p - u^p)(\tau)\phi(\tau)\epsilon^\sigma dx d\tau$$
$$\le c(T)\int_0^t \left(\int_{\mathbb{R}^N} (v-u)\phi(\tau)\epsilon^\sigma dx\right)^p d\tau. \tag{176}$$

We have obtained two upper estimates for every $0 \le s = \tau \le T$ and $x \in \mathbb{R}^N$ in (165) and (176). Making both of them coincide and obtaining the value for the integrand $v-u$ we arrive at:

$$\int_{\mathbb{R}^N} (v-u)(s)\phi(s)dx = \left(c(T)\frac{1-p}{p}\tau\right)^{1/(1-p)} \epsilon^{\frac{\sigma p}{1-p}}. \tag{177}$$

And now considering any of the two upper bounds (in this case (165)) we have:

$$\frac{p}{1-p}\int_0^t \left(c(T)\frac{1-p}{p}\tau\right)^{1/(1-p)} \epsilon^{\frac{\sigma p}{1-p}} \tau^{-1} d\tau. \tag{178}$$

The integral in (178) can be solved considering that after integration the $|x|$ variable is introduced within the truncation in (104) to obtain:

$$\frac{p}{1-p}\int_0^t \left(c(T)\frac{1-p}{p}\tau\right)^{1/(1-p)} \epsilon^{\frac{\sigma p}{1-p}} \tau^{-1} d\tau$$
$$= c(T,p,\sigma)\epsilon^{\frac{p\sigma}{1-p}} t^{\frac{1}{1-p}}. \tag{179}$$

Now we approximate $t \to v \to 0$ and $\epsilon \to \infty$. To simplify the balance between both conflicting parts of the integral, we consider

$$\epsilon = \frac{1}{v^a}, \tag{180}$$

for any $a > 0 \in \mathbb{R}$ to be chosen. Hence, the expression (179) can be reformulated in terms of v only:

$$c(T,p,\sigma)\epsilon^{\frac{p\sigma}{1-p}} t^{\frac{1}{1-p}} = c(T,p,\sigma)\left(\frac{1}{v^a}\right)^{\frac{p\sigma}{1-p}} v^{\frac{1}{1-p}}, \tag{181}$$

for

$$a < \frac{1}{p\sigma}. \tag{182}$$

and making $v \to 0$, we finally arrive at:

$$\int_{\mathbb{R}^N} (v-u)(t)\,\phi(t) \le c(T,p,\sigma)\left(\frac{1}{v^a}\right)^{\frac{p\sigma}{1-p}} v^{\frac{1}{1-p}} \to 0. \tag{183}$$

In virtue of the expression in (183), we ensure, hence, that $u \equiv v$ in Q_T for any $T > 0$. □

2.5 Finite propagation

The finite propagation is a well known property of the PME equation [8] (see the discussion in Sect. 1). Our intention, now, is to show that finite propagation holds [9] . This is remarkable for the case when the diffusion is relevant compared to reaction, nonetheless, it will appear, in a less extent, when reaction predominates over diffusion due to the introduction of the PME operator.

Theorem 2.5.1 *For the case when the diffusion is important, i.e.:*

$$m\sigma + 2(1-\sigma)p + \sigma \ge 2, \tag{184}$$

with

$$u_0(x) \in \mathbb{L}^1(\mathbb{R}^N) \cap \mathbb{L}^\infty(\mathbb{R}^N), \tag{185}$$

and

$$u_0 \equiv 0, \tag{186}$$

in some ball $B(x_0, R)$. Then, any minimal solution to the problem P satisfies:

$u(x,t) \equiv 0$ for some ball $B\left(x_0, \frac{R}{2n}\right)$ for any $n \in \mathbb{N}^+$ and t between $0 < t < \tau$ (τ sufficiently small)

Proof For simplicity, we make $x_0 = 0$ and $R = 1$. The proof of the theorem relies upon finding a local supersolution whose behaviour in the selected ball determines the local behaviour of the postulated minimal solution.

Firstly, we define the following change of variable to work with the pressure term:

$$v = \frac{m}{m-1}u^{m-1}. \tag{187}$$

So that:

$$u_t = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} v^{\frac{2-m}{m-1}} v_t, \tag{188}$$

$$\Delta u^m = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} \left(v^{\frac{2-m}{m-1}} |\nabla v|^2 + v^{\frac{1}{m-1}} \Delta v\right). \tag{189}$$

Upon substitution, the problem P is, then, transformed into:

$$v_t = (m-1)v\Delta v + |\nabla v|^2 + \mu |x|^\sigma v^\delta, \tag{190}$$

$$\delta = \frac{p+m-2}{m-1},$$
$$\mu = m\left(\frac{m-1}{m}\right)^\delta. \tag{191}$$

When $v \to 0$, the laplacian term in Eq. (190) vanishes leading to a first order spatial equation:

$$v_t \sim |\nabla v|^2 + \mu |x|^\sigma v^\delta. \tag{192}$$

This equation is of the first order type that propagates along characteristics. Therefore, in the search of potential solutions, we will search for linear distributions involving the time and spatial variables, i.e. solutions of the form:

$$v(x,t) = g(x + ct), \tag{193}$$

where g is a suitable function and c is the propagation speed along characteristics.

The intention now is to find a suitable maximal solution for the Eq. (190) in the assumption that diffusion is relevant, solutions will not blow-up and will preserve the bound condition given at the initial data. A formal proof of this statement is out of the scope of this section, nonetheless, the bound condition of the PME operator, when starting with bound initial data, has been shown in Lemma 3.3 of [8].

we consider the following function in the search of a maximal solution:

$$w(x,t) = a\left(ct + r - \tfrac{1}{n}\right)_+, \quad r = |x|;\ n \in \mathbb{N}. \tag{194}$$

Both a and $c > 0$ are constants to be determined. In particular, given $0 \le \tau \le 1$, we can impose:

$$c\tau = \frac{1}{2n}, \tag{195}$$

where c shall be determined.

Under this condition, we have:

$$w(x,t) \equiv 0 \quad for \quad r < \frac{1}{2n} \quad and \quad 0 \le t \le \tau. \tag{196}$$

It is clear that any solution to the Eq. (190) is bounded for

$$0 \le t \le \tau, \tag{197}$$

because u_0 is bounded according to the theorem condition

$$u_0(x) \in \mathbb{L}^1(\mathbb{R}^{\mathbb{N}}) \cap \mathbb{L}^\infty(\mathbb{R}^{\mathbb{N}}), \tag{198}$$

and the diffusion is relevant compared to reaction. Then, we have:

$$v(x,t) \le K \ for\ x \in \mathbb{R}^{\mathbb{N}}\ 0 \le t \le \tau\ and\ K(\sigma, p, \|u_0\|_\infty). \tag{199}$$

Our intention is to make $w(x, t)$ as a maximal solution:

$$w(x,t) \ge v(x,t), \tag{200}$$

$$a\left(ct + r - \frac{1}{n}\right)_+ \ge K. \tag{201}$$

We can select any $r > \frac{1}{n}$, for example we establish:

$$r = \frac{2}{n}. \tag{202}$$

Thus, for $t = 0$ we have:

$$a\left(\frac{2}{n} - \frac{1}{n}\right)_+ \ge K, \tag{203}$$

$$a \ge nK. \tag{204}$$

We have built a supersolution, such that:

$$w(x,t) \ge v(x,t), \tag{205}$$

in $r = \frac{2}{n}$ and $0 \le t \le \tau$. Once we have established a suitable condition for the constant a, the next intention is to precise another criteria for c. The value of c shall be chosen in such a way that $w(x, t)$ is a supersolution not only for:

$$r = \frac{2}{n}, \tag{206}$$

but for the range:

$$0 < r < \frac{2}{n}, \tag{207}$$

and in the time interval:

$$0 \le t \le \tau. \tag{208}$$

$w(x, t)$ is a supersolution if it satisfies:

$$w_t \ge (m-1) w \Delta w + |\nabla w|^2 + \mu |x|^\sigma w^\delta, \tag{209}$$

and considering that:

$$w_t = ac;\ w_r = a;\ w_{rr} = 0, \tag{210}$$

the following value for c is obtained:

$$c \ge a + \mu \left(\frac{2}{n}\right)^\sigma a^{\delta - 1} \left(c\tau + \frac{1}{n}\right)^\delta. \tag{211}$$

For the values of a and c derived in expressions (204) and (211) respectively, the function $w(x, t)$ is a supersolution locally:

$$w(x,t) \ge u(x,t), \tag{212}$$

for

$$0 < |x| < \frac{2}{n}, \tag{213}$$

and

$$0 \leq t \leq \tau. \tag{214}$$

This inequality (212) permits to conclude on the proof of the theorem, as any maximal local solution satisfies the null criteria in a region of the selected ball. By direct argument, any minimal solution $u(x, t)$ satisfies the theorem postulations. □

3 Application to a fire extinguishing process in an aircraft engine nacelle

The application exercise consists on modelling the propagation of the extinguishing agent in a domain given by a propeller engine nacelle. Our intention is to determine the concentration of fire suppressant making use of the obtained results, especially the non-existence of a positivity condition as expressed in Sect. 2.1.

The methodology used is based on calibrating the existing parameters in Problem P with real aircraft testing activities. In contrast with typical CFD models involving classical diffusion (see Sect. 1, the modelling exercise in accordance to Problem P does not require extensive testing campaigns.

In addition to the non-linear PME diffusion, the problem P presents a reaction term that aims to introduce the following aspects:

- Agent saturation: Once the extinguisher agent discharges into the domain, the process is fast initially as no agent exists in the media; nonetheless, during the discharge the agent concentration increases leading to reduce the rate of change in the concentration. This principle is introduced by the term u^p ($p < 1$).
- Agent heterogeneous distribution: The discharge nozzles are located in different places all over the domain. Therefore, we shall consider that the rate of time-change in the agent concentration varies with the position. This is the objective when introducing the term u^m ($m > 1$).

The modelling process is as follows: The problem P is applied to an aircraft propeller engine, in which a fire extinguisher agent has been discharged during flight at a certain given pressure altitude and ambient temperature. The engine nacelle has been divided into partitions that are of help for representing the positions of the agent concentration measuring sensors (Fig. 4). In each of the sections A, B, C and D, a sensor has been placed following a longitudinal fix represented by x. This means that the agent concentration is measured following the stream ventilation flow passing through the nacelle. In addition, we consider that any radial dimension is negligible compared to the axial variable x.

Table 1 provides the results of the measuring sensors located at each section. The measuring time has been selected at 3 s as it is enough for stabilized measurements while the process dynamic is still active, i.e. the discharge bottle is still full providing pressurized agent. It is highlighted that the the selection of the 3 s is particular to the discharge process we are aiming to model. In any other case, it is important to select the time frame for measuring capture.

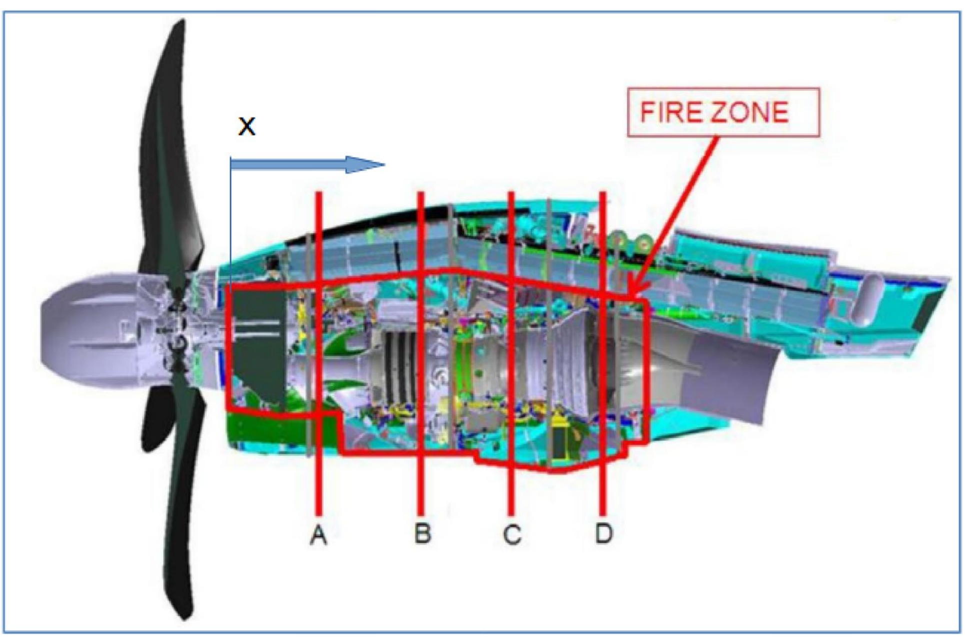

Fig. 4 Propeller engine nacelle area representation. The fire zone is divided into four partition for sensoring allocation purposes

Table 1 Sensor volumetric measurements at t = 3 s. The t = 3 s is selected to fix a common reference time for model parameter obtaining

Sensor section	Longitudinal offset (m)	% Volumetric concentration at t = 3 s
A	1,200	7,5
B	2,000	10,5
C	2,800	14,5
D	3,600	23,5

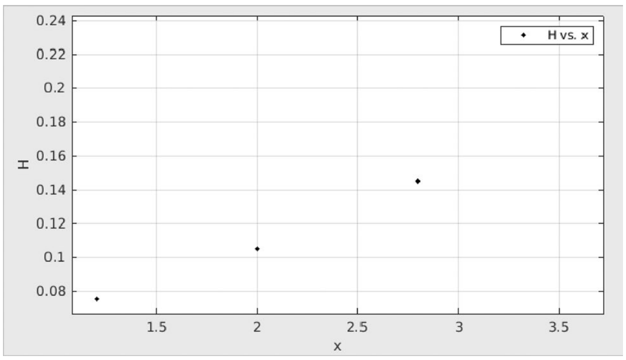

Fig. 5 Halon concentration (H) expressed per unit, as function of the offset (x in meters)

During the testing condition, it is possible to consider that the reaction term $|x|^{\sigma}u^{p}$ is more important than the non-linear diffusion Δu^{m}. This assumption is sensible as we can think that the discharging time is qualitatively fast. Under this assumption, we can make use of the expression (78) to determine values for σ and p. Previously, the data in Table 1 is represented in Fig. 5

Figure 5 data is adjusted to a potential law of the form:

$$u = 0,1461|x|^{0,3277}, \tag{215}$$

which can be compared to expression (78) that adopts the specific form:

$$u_{\tau}^{M} = |x|^{\sigma/(1-p)}(1-p)^{1/(1-p)}(3)^{1/(1-p)}, \tag{216}$$

then, we have:

$$(1-p)^{1/(1-p)}(3)^{1/(1-p)} = 0,1461, \tag{217}$$

which provides a value for p:

$$p = 0,78. \tag{218}$$

Note that $p < 1$as it was previously assumed. Additionally, we can obtain a value for σ:

$$\frac{\sigma}{1-p} = 0,3277, \tag{219}$$

which provides

$$\sigma = 0,072 \tag{220}$$

Under the assumption that the reaction predominates over the diffusion, the expression (181) provides the following condition for the PME diffusion order:

$$m < 6,67, \tag{221}$$

therefore we admit the value

$$m = 2. \tag{222}$$

The existence of global solutions requires a certain condition to be met for the involved parameters as resulting from Theorem 2.3.1

$$p \leq sign_{+}\left(1 - \frac{\sigma(m-1)}{2}\right), \tag{223}$$

which is met according to

$$0,78 < \left(1 - \frac{0,072(2-1)}{2}\right) = 0,964. \tag{224}$$

Thus, the solution describing the behavior of the halon concentration exits as per the model P along the line characterized by Table 1. This solution adopts the form:

$$u = |x|^{0,3277}\, 0,00102\, t^{4,5}, \tag{225}$$

where

$$0 < u < 1, \tag{226}$$

and x and t are expressed in meters and seconds respectively.

Our next intention is to obtain the propagation front that results when considering the non-linear diffusion. For this purpose we consider the results as per Theorem 2.3.2. Particularly, the positivity of the solution (i.e. the existence of extinguisher concentration) is provided in the frame:

$$|x| < c_2(x)\, t^{\beta}, \tag{227}$$

where c_2 is given by (102), and:

$$c_{supp} = \frac{(-\alpha + \beta N)^{\frac{m-1}{2(p-1)}}}{\left(\frac{(m-1)\beta}{2m}\right)^{1/2}} = 0,775. \tag{228}$$

The values of α and β can be obtained from the expression in (91):

$$\alpha = -5,63; \;\; \beta = 3,33. \tag{229}$$

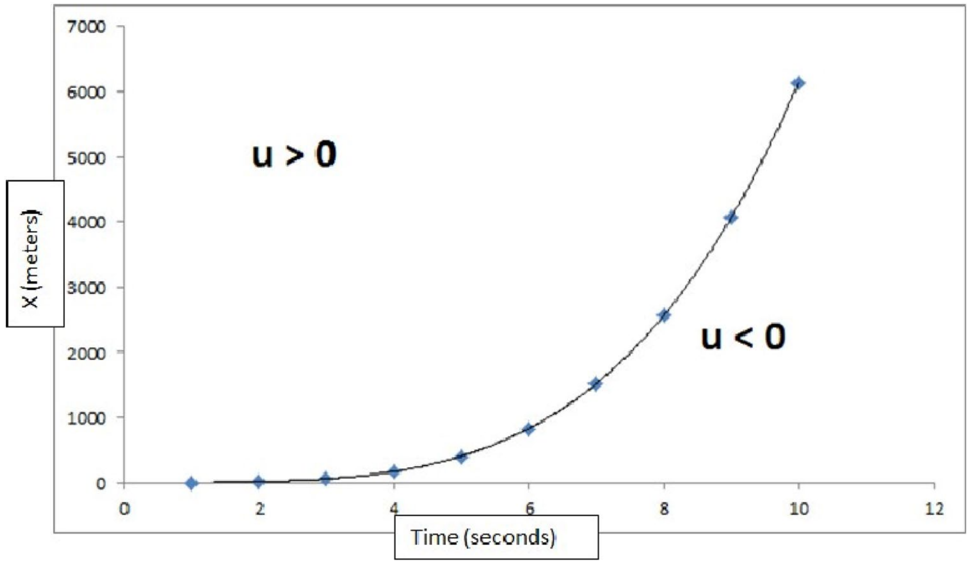

Fig. 6 The propagation of extinguisher is given by the line shifting from cero to positivity

Then, we can conclude to have:

$$u > 0, \tag{230}$$

whenever

$$|x| < 0,738\, t^{3,92}. \tag{231}$$

The shifting to positivity propagation is given by the expression

$$|x| = 0,738\, t^{3,92}, \tag{232}$$

and represented in Fig. 6.

Based on the results compiled in Fig. 6, it is possible to determine the requested time to ensure that the extinguisher concentration reaches the complete domain. For this purpose, we consider the an engine shaped cowling geometry as given in Fig. 7. When the diffusion and reaction have made the extinguisher concentration to propagate along the engine cowl, we have

$$|x| = 4,012m, \tag{233}$$

and the time required to ensure the propagation has reached the entire engine cowl is given by the graph in Fig. 6

$$t = 1,4s. \tag{234}$$

This obtained time represents the minimum required to ensure that the discharging agent has reached the whole domain of interest. Nonetheless, it does not permit to ensure that a fire can be extinguished. For this purpose, it is necessary to know, before hand, the required concentration for fire suppression. Let assume that the minimum extinguisher concentration is 6% in volume. It is possible to determine the minimum required time to ensure $u \geq 0,06$ with the expression (225):

$$t = \left(\frac{u}{|x|^{0,3277}\, 0,00102} \right)^{\frac{1}{4,5}} = 2,24s. \tag{235}$$

This value makes sense when compared with the data in Table 1. Note that all sensors are measuring more than 6% for a time beyond the assessed in (235).

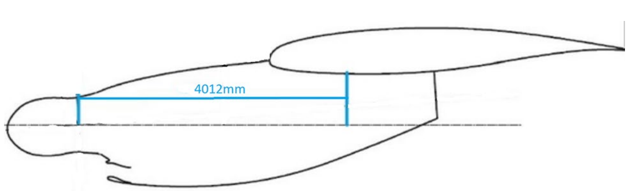

Fig. 7 Engine cowl geometry. The blue line represents the length in the geometry where the extinguisher discharges

4 Conclusions

The problem *P* proposed with a Porous Medium Equation (PME) to model a fire extinguisher process in an aircraft engine nacelle has been discussed with a mathematical approach stressing aspects related with existence, uniqueness and behaviour of finite speed solutions. The application exercise set the evidences in the use of a PME with a non-Lipschitz reaction to model fire extinguishing processes in aircraft geometries. The information provided has permitted to ask global questions, such us, What is the required time to ensure that the propagating extinguisher is capable of extinguish an engine fire? and, How is the extinguisher propagating front in areas of the engine nacelle?

In addition, finite values for the model parameters p, m and σ have been shown to exist, and, further, the combination of such values has been shown to provide existence of global solutions. This means that no blow-up is given at finite time as suggested by the natural process evolution intuition in which the agent concentration does not increase suddenly at a certain time.

Compliance with ethical standards

Conflict of interest The author states that there is no conflict of interest.

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