Travelling waves and instability in a Fisher-KPP problem with a non-linear advection and a high order diffusion

July 7, 2021

José Luis Díaz Palencia
joseluis.diaz@ufv.es

Universidad Francisco de Vitoria

Escuela Politécnica Superior, Universidad Francisco de Vitoria, Ctra. Pozuelo-Majadahonda Km 1,800, 28223, Pozuelo de Alarcón, Madrid

Abstract: The instability of Travelling Waves (TW) in high order operators has been a source of investigation in the last years. One of the principal researching aspects is related with the characterization of such instability in the proximity of stationary solutions. Instability shall be understood as the existence of oscillatory exponential bundles of solutions. In this article, the stationary solutions are defined by a KPP-Fisher reaction term. The intention along this analysis is to study existence, uniqueness, TW instability and a characterization of a local inner region with positive monotone behaviour of solutions (also called positive inner region) in contrast with an outer region where the instabilities or oscillations of the TW are an inherent property. Furthermore, a sharp estimation of the TW propagation speed to ensure the existence of such positive inner region is obtained.

Keywords: *High order diffusion, Travelling Waves, Homotopy, Fisher-KPP problem, Evans Functions.*

Mathematics Subject Classification Codes: 35K92, 35K91, 35K55.

1 Problem description and objectives

The study of reaction-diffusion models requires a specific treatment on the diffusion. Diffusion based on the Random Walk model (see [1] and references listed) has been generalized to account for a wider complexity in spatial movement. Several approaches to diffusion have been followed based on the Landau-Ginzburg free energy concept [2], [3]. The free energy accounts for a generalization beyond the classical Fickian diffusion. Cohen and Murray [2] derived a particular expression for the free energy of a non-homogeneous spatial pattern. The authors showed that such free energy shall depend on the gradient of a concentration, i.e. $\frac{1}{2}k(\nabla u)^2$. Departing from this free energy and making use of the chemical potential, the associated diffusion is given by a fourth order spatial operator (see [2] for a detailed review). This is particularly relevant and leads to the loss of regularity in the operator. Typically, diffusion is considered to come from a Fick law that relates linearly the concentration flux with the associated concentration gradient. Such linear relation leads to the classical gaussian order two operator that exhibits positivity and regularity. On the contrary, a fourth order diffusion emerges from a non-homogeneous free energy quadratically related with the concentration gradient. The introduction of such fourth order operator induces a set of properties related with oscillatory patterns that are generically referred as instabilities.

In the 1930s, Fisher [5], proposed a reaction-diffusion model to understand the interaction process of genes. In parallel, Kolmogorov, Petrovskii and Piskunov [4] proposed the same equation in combustion theory. In both cases, the models considered a gaussian order two diffusion with a non-linear reaction of the form $f(u) = u(1-u)$. The authors introduced the concept of Travelling Wave (TW) solutions to understand the propagation features of the species involved.

The Fisher-KPP model has been subjected to a remarkable mathematical research to explore possible further applications in biology or ecology (See [6], [7], [8]) and non-Newtonian fluids [28]. In some cases, the fourth order operators emerge from known order two diffusion problems. As an example, the instabilities close degenerate points given by classical Fisher-Kolmogorov equation (see [12] and references listed there) led to propose the Extended Fisher-Kolmogorov equation to model the behaviour of bistable systems [14], [13] and [19]. A detail review of the cited studies suggests that the high order operator induces a set of instabilities in the proximity of critical points. Consequently, it is possible to think that such instabilities shall be applicable to the problem along this paper involving a Fisher-KPP reaction term with non-linear advection.

Focusing on the high order operators theory, some remarkable recent achievements are related with advances in the De Giorgi's conjecture for a fourth order Allen-Cahn equation [10]. In addition, the Fisher-KPP model has been analyzed as a p-Laplacian Porous Medium Equation in [11].

The problem discussed along this article is formulated with a non-linear advection. Note that Montaru has studied a degenerate parabolic problem with nonlinear advection [20] to show local well posedness in some appropriate functional spaces. In addition, it presents regularity results for global behaviour of solutions.

As it has been discussed, the analyzed problem (P) involves a high order operator, non-linear advection and a Fisher-KPP reaction term:

$$\begin{gathered} u_t = -\Delta^2 u + c \cdot \nabla u^q + u(a-u), \\ u_0(x), \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ u_{0,x} \, ; \in L^q(\mathbb{R}), \quad q>1, \quad a>0. \end{gathered} \tag{1}$$

The problem P main features emerge from the Travelling Waves (TW) theory applied to high order operators. The TW kind of solutions are of wide interest in the natural sciences [22], [23] and [24] and has been applied to high order operators in numerous analysis [15], [17], [21] and [26].

The present work provides a characterization of TW instabilities. In addition, a sharp estimation of a TW-speed in a positive inner region is presented together with a global characterization. Finally, in the appendix some ideas are introduced to explore existence and uniqueness of solutions as an abstract evolution of a bounded continuous single parametric (with t) operator.

Note that the mathematical analysis in the TW domain starts with a step kind initial data $u_0(x) = H(-x) \in L^1_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ where H refers to the Heaviside function. In the proximity of the stationary solutions, the L^1 space replaces the L^1_{loc} so that any formulated result can be understood as the non-linear evolution from L^1 into a Hilbert-Sobolev functional space H^4 and weighted H^4_ρ .

The methodology is based on analytical and numerical exercises for which the Matlab software has been used, in particular the function bvp4c. The numerical method used is based on an implicit Runge-Kutta with an interpolant extension [25]. The collocation method requires to specify the boundary conditions, in this case are given by the stationary solutions. Along this paper, the number of nodes in the integration domain has been adjusted at 10000 and the absolute error fixed at 10^{-6}.

Note that previous to any analysis in the TW domain, the reader can consult the Appendix where some initials estimations, existence and uniqueness of solutions are provided.

2 General a priori definitions

Firstly, consider the following norm:

$$\|F\|_\rho = \int_{\mathbb{R}} \rho(y) \sum_{k=0}^{4} |D^k F(y)| dy, \tag{2}$$

where $D = \frac{d}{dy}$, $F \in H^4_\rho(\mathbb{R}) \subset L^1_\rho(\mathbb{R}) \subset L^1(\mathbb{R})$ and the weight ρ is defined as (see [26] together with [20]):

$$\rho(y) = e^{a_0|y|^{\frac{4}{3}} - \frac{1}{y^q}\frac{1}{t^\gamma}\int_0^t (\|F_x(s)\|^q+1)ds}, \tag{3}$$

$a_0 > 0$ is a small constant and $\gamma > q+1$.

Now, consider the homogeneous problem:

$$F_t = LF, \tag{4}$$

where $L = (-\Delta^2 + qF^{q-1}c\cdot\nabla)$ defines the spatial operator.

Aspects related with existence and uniqueness of solutions are analyzed in the appendix upon definition of a strongly continuous bounded operator formulated according to the semi-group theory.

3 Travelling Waves

The TW profiles are expressed as $u(x,t) = \varphi(\xi)$, $\xi = x \cdot n_d - \lambda t \in \mathbb{R}$, where n_d is a unitary vector in $\mathbb{R}^N$ that defines the TW-propagation direction. λ is the TW-speed and $\varphi : \mathbb{R} \to (0,\infty)$ belongs to $L^1_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and in the proximity of the stationary solutions given by the Fisher-KPP reaction $\varphi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap H^4_\rho$ or $\varphi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap H^4$.

Note that two TW are equivalent under translation $\xi \to \xi + \xi_0$ and symmetry $\xi \to -\xi$.

For the sake of simplicity, the vector n_d is $n_d = (1,0,...,0)$, then $u(x,t) = \varphi(\xi)$, $\xi = x - \lambda t \in \mathbb{R}$.

In addition, the advection coefficient c is considered in the same direction as n_d. The problem P (1) is then transformed into:

$$-\lambda\varphi' = -\varphi^{(4)} + c(\varphi^q)_x + \varphi(a-\varphi), \tag{5}$$

for which the following lemma to characterize the propagating TW holds:

Lemma 3.1. *The TW speed λ is positive (TW moves from $\xi \to -\infty$ to $\xi \to \infty$) if the following condition is met for the advection coefficient:*

$$c > \frac{-\frac{1}{2}a + \frac{1}{3}}{q\left(2 - \frac{1}{q-1}\right)}. \tag{6}$$

The wave speed λ is negative if the ‘ $>$ ‘ *is replaced by* ‘ $<$ ‘. *In addition, the wave speed stops for an advection coefficient making* ‘ $=$ ‘ *in the previous expression.*

Proof. Multiply (5) by φ':

$$-\lambda(\varphi')^2 = -\varphi^{(4)}\varphi' + c(\varphi^q)_x\varphi' + \varphi\varphi' a - \varphi^2\varphi', \tag{7}$$

such that:

$$\begin{aligned}\int \varphi^{(4)}\varphi' &= \varphi'\varphi^{(3)} - \int \varphi^{(3)}\varphi^{(2)} = \varphi'\varphi^{(3)} - \left(\varphi^{(2)}\varphi^{(2)} - \int \varphi^{(2)}\varphi^{(3)}\right) \\ &= \varphi'\varphi^{(3)} - \varphi^{(2)}\varphi^{(2)} + \int \varphi^{(2)}\varphi^{(3)} = \varphi'\varphi^{(3)} - \varphi^{(2)2} + \varphi'\varphi^{(3)} - \int \varphi^{(4)}\varphi'.\end{aligned} \tag{8}$$

Then

$$\int \varphi^{(4)}\varphi' = \frac{1}{2}(2\varphi'\varphi^{(3)} - \varphi^{(2)2}). \tag{9}$$

The integral is assessed between $-\infty$ and $+\infty$ where in the asymptotic approach:

$$\begin{aligned}\varphi'(-\infty) = \varphi^{(2)}(-\infty) = \varphi^{(3)}(-\infty) = 0, \\ \varphi'(\infty) = \varphi^{(2)}(\infty) = \varphi^{(3)}(\infty) = 0.\end{aligned} \tag{10}$$

Therefore:

$$\int_{-\infty}^{\infty} \varphi^{(4)}\varphi' = 0. \tag{11}$$

The next involved integral is:

$$\int \varphi\varphi' = \varphi\varphi - \int \varphi\varphi' \to \int_{-\infty}^{\infty} \varphi\varphi' = \frac{1}{2}\left[\varphi^2(\infty) - \varphi^2(-\infty)\right] = \frac{1}{2}\left[0-1\right] = -\frac{1}{2}. \tag{12}$$

The advection term is assessed as:

$$c\int(\varphi^q)_x\varphi' = c(\varphi^q)_x\varphi - c\int q(q-1)\varphi^{q-1}\varphi' - cq\int\varphi^q\varphi^{(2)}. \tag{13}$$

Note that:

$$\int\varphi^{q-1}\varphi' = \varphi^{q-1}\varphi - \int(\varphi^{q-1})_x\varphi, \tag{14}$$

$$\int\varphi^q\varphi^{(2)} = \varphi^q\varphi' - \int(\varphi^q)_x\varphi'. \tag{15}$$

After compilation:

$$(1-q)c\int(\varphi^q)_x\varphi' = -cq(q-1)\varphi^q + cq(q-1)\int\varphi(\varphi^{q-1})_x. \tag{16}$$

Repeating a similar integration by parts, the left hand integral reads:

$$\int\varphi(\varphi^{q-1})_x = \frac{1}{q-1}\varphi^q - \varphi^q, \tag{17}$$

so that:

$$c\int_{-\infty}^{\infty}(\varphi^q)_x\varphi' = cq\left(2\varphi^q - \frac{1}{q-1}\varphi^q\right)_{-\infty}^{\infty} = cq\left(-2+\frac{1}{q-1}\right). \tag{18}$$

The same integration by parts is performed for the last of the integrals involved:

$$\int\varphi^2\varphi' = \varphi^2\varphi - \int 2\varphi^2\varphi' \rightarrow \int_{-\infty}^{\infty}\varphi^2\varphi' = \frac{1}{3}\left[\varphi^2\varphi\right]_{-\infty}^{\infty} = \frac{1}{3}(0-1) = -\frac{1}{3}. \tag{19}$$

The compilation of the assessed integrals yields:

$$-\lambda\int(\varphi')^2 = 0 + cq\left(-2+\frac{1}{q-1}\right) - \frac{1}{2}a + \frac{1}{3}, \tag{20}$$

$$\lambda = \frac{cq\left(2-\frac{1}{q-1}\right)+\frac{1}{2}a-\frac{1}{3}}{\int(\varphi')^2}, \tag{21}$$

which permits to conclude on the lemma postulations. □

3.1 Travelling Waves Instabilities

The equation (5) presents two stationary solutions at $\varphi = 0$ and $\varphi = 1$. The aim of this section is to study the instability properties of an heteroclinic orbit connecting both stationary solutions considering that $a = 1$. Firstly, the instabilities are shown to be inherent to the high order operator and introduce difficulties at studying the convergence between the heteroclinic orbit and the stationary solution to P (1). Secondly, this section aims to prove, and sharply assess, the existence of a TW speed for which the TW is positive in an inner region in contrast with an outer region of oscillations or instabilities.

The analysis of instabilities in the proximity of the stationary solutions is based on a theorem introduced in [18] for the Kuramoto-Sivashinsky equation and extended to the Cahn-Hilliard equation in [15] and to a sixth order operator in [17]. The theorem is partially modified to account for the problem P particularities and the methodology followed and is split into four different lemmas whose proofs permit to show the nonlinear instability of the heteroclinic TW orbit in the proximity of the stationary solutions.

Lemma 3.2. *Consider the Banach spaces H^4, H^4_ρ (see appendix for ρ definition) and L^1, then:*

$\|u\|_{L^1} \leq C_1\|u\|_{H^4}$ *and* $\|u\|_{L^1} \leq C_2\|u\|_\rho$

Proof. In accordance with the Sobolev norm defined in the appendix (see (63)):

$$\|u\|_{H^4} = \int_{-\infty}^{\infty} e^{4\chi^2} |\hat{u}(\chi,t)| d\chi \geq \inf_{\chi\in(-\infty,\infty)} \{e^{4\chi^2}\} \int_{-\infty}^{\infty} |\hat{u}(\chi,t)| d\chi = \|u\|_{L^1}, \tag{22}$$

consequently $C_1 = 1$. The next postulation is proved considering the expression (2):

$$\|u\|_{L^1} \leq \int_{\mathbb{R}} \sum_{j=0}^{4} |D^j u(y)| dy \leq \int_{\mathbb{R}} \rho(y) \sum_{j=0}^{4} |D^j u(y)| dy = \|u\|_{\rho}, \tag{23}$$

a.e. in $\mathbb{R}$, so that $C_2 = 1$

$\square$

The convergence of the TW solutions to the stationary $u = 0$ and $u = 1$ is reduced to the behaviour of the function $v(x,t) = u(x,t) - \phi(x,t)$ in the proximity of $v = 0$. Hence, the problem P (1) is expressed in terms of v and ϕ:

$$v_t = -v^{(4)} + \frac{1}{2} \sum_{k=0}^{\infty} \binom{q}{k} c(q-k) v^{q-k-1} \phi^q v_x + \frac{1}{2} \sum_{k=0}^{\infty} \binom{q}{k} c v^{q-k} \phi^{q-1} \phi_x. \tag{24}$$

Then:

$$v_t = L_0 v + G(v), \tag{25}$$

for $L_0 = -\partial_x^4$ and $G(v) = \frac{1}{2}\sum_{k=0}^{\infty} \binom{q}{k} c(q-k) v^{q-k-1} \phi^q v_x + \frac{1}{2}\sum_{k=0}^{\infty} \binom{q}{k} c v^{q-k} \phi^{q-1} \phi_x$.

The following lemma holds:

Lemma 3.3. *$G : H^4 \to L^1$ is continuous and there exists $\rho_0 > 0$, $C_3 > 0$ and $\alpha > 1$ such that $\|G(v)_{L^1}\| < C_3 \|v\|_{H^4}^{\alpha}$, for $0 < \|v\|_{H^4} < \rho_0$.*

Proof.

$$\begin{aligned} \|G(v)\|_{L^1} \leq \|G(v)\|_{H^4} &\leq \frac{1}{2} \sum_{k=0}^{\infty} \binom{q}{k} c(q-k) \|v^{q-k-1}\|_{H^4} \|\phi^q\|_{\infty} \beta \|v\|_{H^4} \\ &+ \frac{1}{2} \sum_{k=0}^{\infty} \binom{q}{k} c \|v^{q-k-1}\|_{H^4} \|v\| \|\phi^{q-1}\|_{\infty} \|\phi_x\|_{\infty}, \end{aligned} \tag{26}$$

where $\beta > 0$ is obtained upon the Gronwall‘s inequality applied locally $\|v_x\|_{H^4} \leq \beta \|v\|_{H^4}$. Then:

$$\|G(v)\|_{H^4} \leq (\|\phi^q\|_{\infty} \beta + \|\phi^{q-1}\|_{\infty} \|\phi_x\|_{\infty}) \frac{1}{2} \sum_{k=0}^{\infty} \binom{q}{k} c(q-k) \|v\|_{H^4}^{-k-1} \|v\|_{H^4}^{q+1}. \tag{27}$$

$$\|G(v)\|_{L^1} \leq \|G(v)\|_{H^4} \leq C_3 \|v\|_{H^4}^{\alpha}, \tag{28}$$

where $\alpha = q+1$ and $C_3 = (\|\phi^q\|_{\infty}\beta + \|\phi^{q-1}\|_{\infty}\|\phi_x\|_{\infty}) \frac{1}{2}\sum_{k=0}^{\infty} \binom{q}{k} c(q-k) N$, for a sufficiently large $N \geq \sum_{k=0}^{\infty} \|v\|_{H^4}^{-k-1}$. $\square$

The previous lemma can be similarly proved for $G : H_\rho \to L^1$, $r > 0$, $C_4 > 0$ and $\alpha > 1$ such that $\|G(v)\|_{L^1} < C_4 \|v\|_{\rho}^{\alpha}$, for $0 < \|v\|_{\rho} < r$. Note that $C_4 = (\|\phi^q\|_{\infty}\beta + \|\phi^{q-1}\|_{\infty}\|\phi_x\|_{\infty}) \frac{1}{2}\sum_{k=0}^{\infty} \binom{q}{k} c(q-k) N_1$, for a sufficiently large $N_1 \geq \sum_{k=0}^{\infty} \|v\|_{\rho}^{-k-1}$.

The bounding properties of the abstract evolution operator $G_{F_0,t}$ are provided in the appendix (see Lemma 7.2). In addition, $G_{F_0,t}$ is associated with a strongly continuous semi-group acting in the linear fourth order operator L_0 that is shown to be bounded in the next lemma.

Lemma 3.4. *L_0 generates a strongly continuous semigroup e^{tL_0} such that:*

$$\int_0^1 \|e^{tL_0}\|_{L^1 \to H^4} = C_5 < \infty, \qquad \int_0^1 \|e^{tL_0}\|_{L^1 \to H_\rho^4} = C_6 < \infty \tag{29}$$

Proof. Any evolution in L_0 is expressed as:

$$\hat{u}(\chi, t) = e^{-t\chi^4}\hat{u}_0(\chi). \tag{30}$$

Then

$$\|u\|_{H^4} \leq \|e^{-t\chi^4}\|_{H^4}\,\|u_0\|_{H^4} \leq \sup_{\chi\in(-\infty,\infty)}\{e^{4\chi^2-t\chi^4}\}\|u_0\|_{L^1}, \tag{31}$$

so that

$$\|e^{-t\chi^4}\|_{L^1\to H^4}\|u_0\|_{L^1} \leq \|e^{-t\chi^4}\|_{H^4}\|u_0\|_{H^4} \leq \sup_{\chi\in(-\infty,\infty)}\{e^{4\chi^2-t\chi^4}\}\|u_0\|_{L^1}. \tag{32}$$

Then

$$\|e^{-t\chi^4}\|_{L^1\to H^4} \leq \sup_{\chi\in(-\infty,\infty)}\{e^{4\chi^2-t\chi^4}\} = e^{4t^{-1}}, \quad 0 < t \leq 1. \tag{33}$$

Upon integration with $t \in (0, 1]$, a finite value for C_5 is obtained.

The constant C_6 obtained operating analogously:

$$\begin{aligned}\|u\|_\rho &= \int_{\mathbb{R}} \rho(\chi)\sum_{j=0}^{4}|D^j\hat{u}(\chi)|d\chi \leq 2\kappa\int_{\mathbb{R}^+} e^{a_0\chi^{4/3}}|\hat{u}(\chi)|d\chi \\ &\leq 2\kappa\int_{\mathbb{R}^+} e^{a_0\chi^{4/3}-t\chi^4}|\hat{u}_0(\chi)|d\chi \leq 2\kappa\sup_{\chi\in(0,\infty)}\{e^{a_0\chi^{4/3}-t\chi^4}\}\|u_0\|_{L^1} = e^{\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{27}}\right)a_0^{3/2}t^{-1/2}}\|u_0\|_{L^1}.\end{aligned} \tag{34}$$

In addition,

$$\|e^{-t\chi^4}\|_{L^1\to H^4_\rho}\|u_0\|_{L^1} \leq \|u\|_\rho, \tag{35}$$

so that

$$\|e^{-t\chi^4}\|_{L^1\to H^4_\rho} \leq e^{\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{27}}\right)a_0^{3/2}t^{-1/2}}. \tag{36}$$

A finite value for C_6 is obtained upon integration in the last expression for $t \in (0, 1]$. □

The next objective is to study the spectrum of L (4) in H^4. For this purpose, the following lemma enunciates:

Lemma 3.5. *The spectrum of L (see expression (4)) in H^4 in the proximity of the stationary solutions is such that there exists, at least, one eigenvalue (γ) with $Re(\gamma) > 0$.*

Proof. This lemma can be proved making use of Evans functions that permit to locate any positive eigenvalue associated to the linearized operator in the proximity of the equilibrium $\varphi(\infty)$ and $\varphi(-\infty)$. Note that the roots of Evans functions correspond to the roots of the characteristic polynomial [16]. Hence, the eigenvalues are computed with the characteristic polynomial in the proximity of the linearized wave moving to the stationary solutions $\varphi = 0$, $\varphi = 1$. Further, the effect of any of the involved parameters (mainly the travelling wave speed) is considered with an asymptotic homotopy approach supported by a computational exercise. Firstly the equation (5) is transformed into the standard matrix representation:

$$\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix}' = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ a-\varphi_1 & \lambda + cq\varphi_1^{q-1} & 0 & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix} \tag{37}$$

Or equivalently:

$$F' = A(a, c, \lambda)F. \tag{38}$$

The characteristic polynomial of A is:

$$P(\gamma) = \gamma^4 - (\lambda + cq\varphi_1^{q-1})\gamma - (a - \varphi_1) = 0. \tag{39}$$

In the asymptotic approximation $\varphi_1 < \epsilon \to 0$, it is easily check that $P(\gamma)$ has at least one positive real root. Indeed, for $0 < \gamma << 1$, $P(\gamma) < 0$ while for $\gamma >> 1$, $P(\gamma) > 0$. Furthermore, the roots

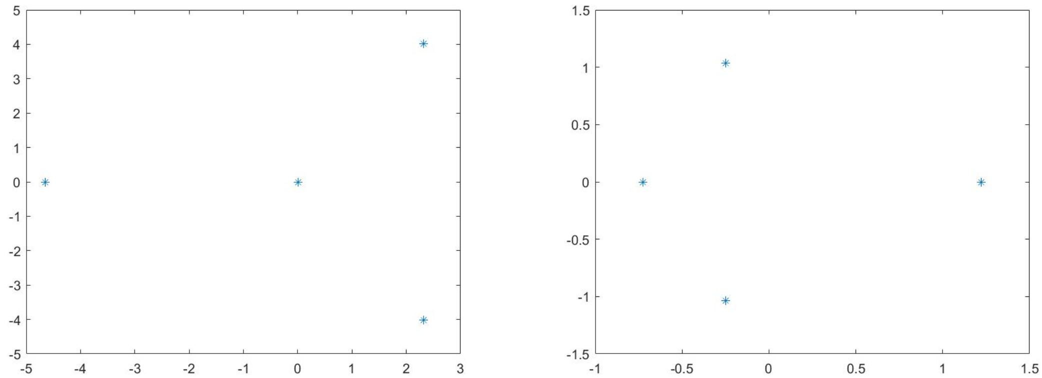

Figure 1: Complex plane (horizontal axis corresponds to the Real part and vertical axis Imaginary) $P(\gamma)$ roots evolution for $a = 1$, $\lambda = 1$ (left) and $\lambda = 10$ (right) with $\varphi_1 < \epsilon \to 0$. Note the existence of at least one root with $Re(\gamma) > 0$.

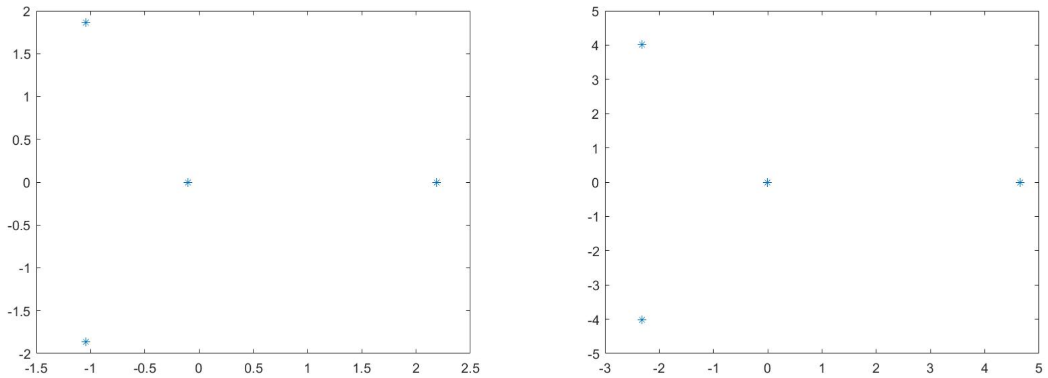

Figure 2: Complex plane (horizontal axis corresponds to the Real part and vertical axis Imaginary) $P(\gamma)$ roots evolution for $a = 1$, $\lambda = 100$ (left) and $\lambda = 1000$ (right) with $\varphi_1 < \epsilon \to 0$. Note the existence of at least one root with $Re(\gamma) > 0$.

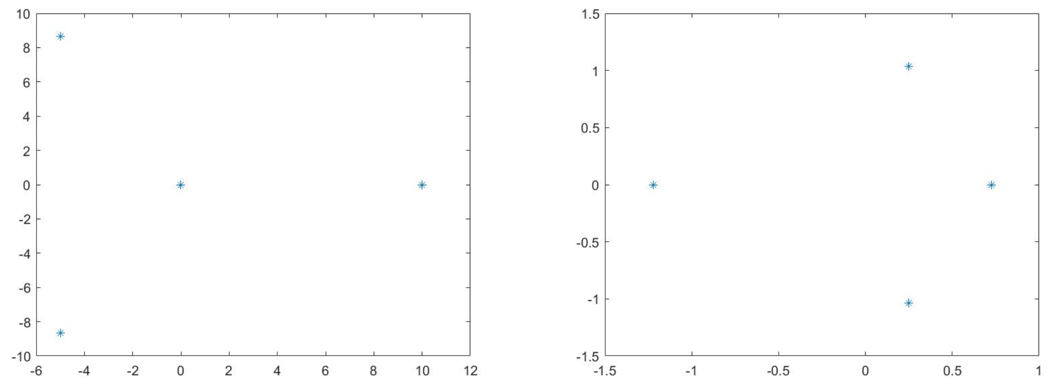

Figure 3: Complex plane (horizontal axis corresponds to the Real part and vertical axis Imaginary) $P(\gamma)$ roots evolution for $a = 1$, $\lambda = -1$ (left) and $\lambda = -10$ (right) with $\varphi_1 < \epsilon \to 0$. Note the existence of at least one root with $Re(\gamma) > 0$.

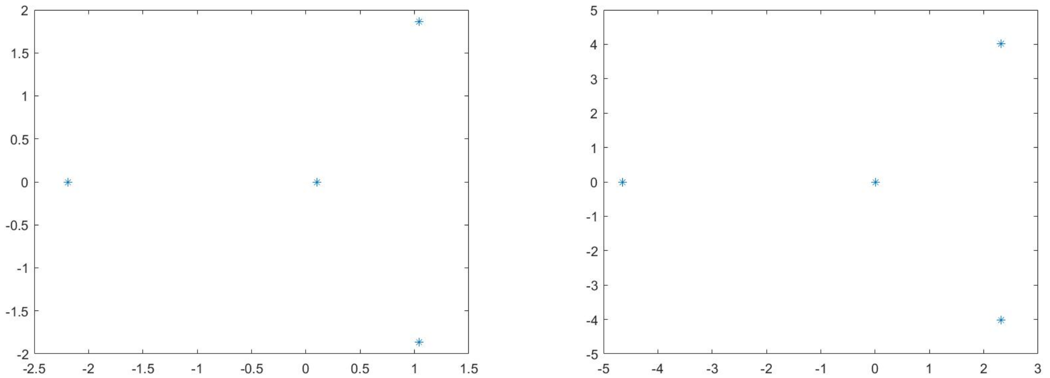

Figure 4: Complex plane (horizontal axis corresponds to the Real part and vertical axis Imaginary) $P(\gamma)$ roots evolution for $a = 1$, $\lambda = -100$ (left) and $\lambda = -1000$ (right) with $\varphi_1 < \epsilon \to 0$. Note the existence of at least one root with $Re(\gamma) > 0$.

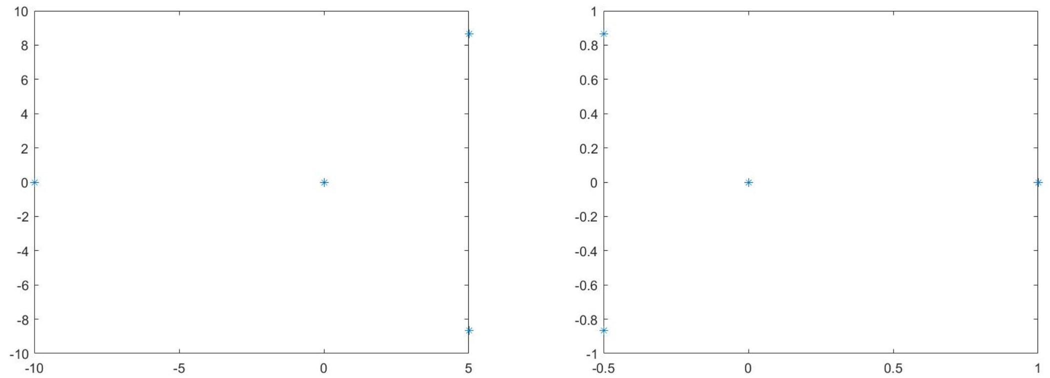

Figure 5: Complex plane (horizontal axis corresponds to the Real part and vertical axis Imaginary) $P(\gamma)$ roots evolution for $a = 1$, $\lambda = 1$ (left) and $\lambda = 1000$ (right) with $\varphi_1 \to a$. Note the existence of at least one root with $Re(\gamma) > 0$.

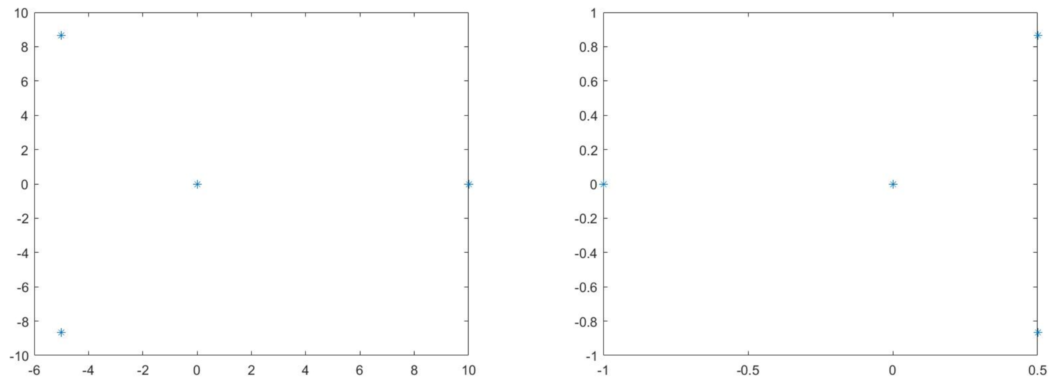

Figure 6: Complex plane (horizontal axis corresponds to the Real part and vertical axis Imaginary) $P(\gamma)$ roots evolution for $a = 1$, $\lambda = -1$ (left) and $\lambda = -1000$ (right) with $\varphi_1 \to a$. Note the existence of at least one root with $Re(\gamma) > 0$.

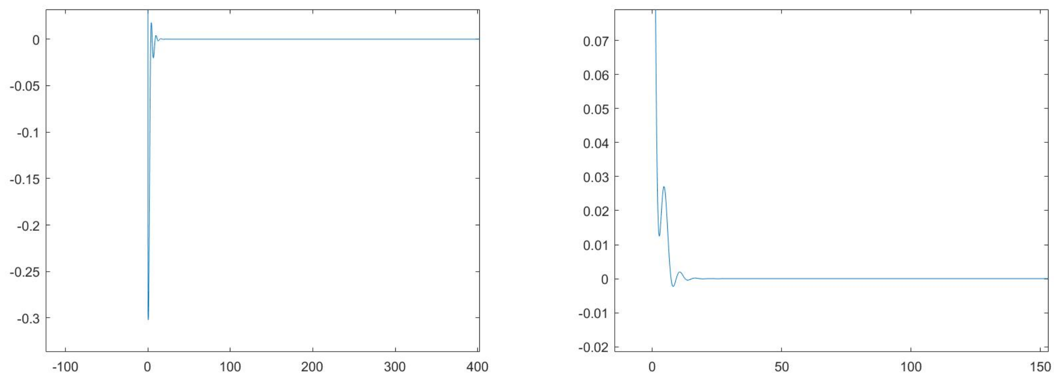

Figure 7: Horizontal axis corresponds to ξ and vertical to φ. For $\lambda = 1.56$, $\varphi(min(m)) < 0$ (left) while for $\lambda = 1.57$, $\varphi(min(m)) > 0$ (right).

of the characteristic polynomial can be represented for different values in each of the involved coefficients (see Figures 1, 2, 3 and 4).

In the proximity of the stationary solution $\varphi_1 = a$, there exist one real and positive root which can be easily shown to be $\gamma = \left(\lambda + cqa^{q-1}\right)^{\frac{1}{3}}$. In the same manner, Figures 5, 6 represent the root evolutions for different values in the involved parameters.

□

3.2 Travelling wave characteristic propagation speed

The instabilities lemmas shown in Section 3.1 permit to confirm the non-existence of a minimal TW-speed with a monotone decay at infinity. Thus, the search of an appropriate TW-speed can be translated into finding a suitable value for which the first TW minimum is positive for $\xi > 0$. Nonetheless, the positivity of the TW for all $\xi \in \mathbb{R}$ does not hold as the natural instabilities shown impedes the formulation of a maximum principle.

The sharp estimation of a TW-speed, at which the first TW minimum is positive, has been done via a numerical algorithm in Matlab with the bvp4c function. The numerical analysis has been done over a sufficiently large ξ interval $[-1000, 1000]$ to avoid the influence of the pseudo-boundary conditions given by the stationary solutions.

Lemma 3.6. *Define the set m as:*

$$m = \{\xi > 0, \varphi' = 0, \varphi'' > 0\}, \tag{40}$$

which expresses a set of elements locating the minimums of $\varphi(\xi)$ beyond the TW-front (i.e. in the TW-tail).

There exist a λ value for which $\varphi(\min(m)) > 0$. Particularly, a sharp assessment is given $\lambda_{\varphi(\min(m))>0} = 1.568$ (see Figure 8)

Proof. A sharp assessment of $\lambda_{\varphi(\min(m))>0}$ is difficult in a general case given the problem dependence with the parameters c, q and a. Consequently, to make the problem tractable and without loss of generality, the numerical approach is performed with particular values for each of the mentioned parameters. In addition, admit the TW moves from the left to the right, i.e. given certain values for a and q, the convection term c shall satisfy the expression (6). Therefore, assume $q = 2$, $a = 1$ and $c = 1$.

The most remarkable difference between the classical Fisher-KPP order two problem [4] and the problem dealt along this Lemma relays on shifting the idea of finding a λ_{min} by finding a $\lambda_{\varphi(\min(m))>0}$. Namely, for the classical Fisher-KPP, the TW moving at λ_{min} does not oscillate as it changes from the sub-critical solution to the critical one [4]. The instabilities results derived in Section 3.1 lead to use the concept of $\lambda_{\varphi(\min(m))>0}$, for which the oscillatory behavior is observed in the TW-tail when $\xi >> 1$ (see Figure 9). Thus, $\lambda_{\varphi(\min(m))>0}$ is the TW-speed for which, in an

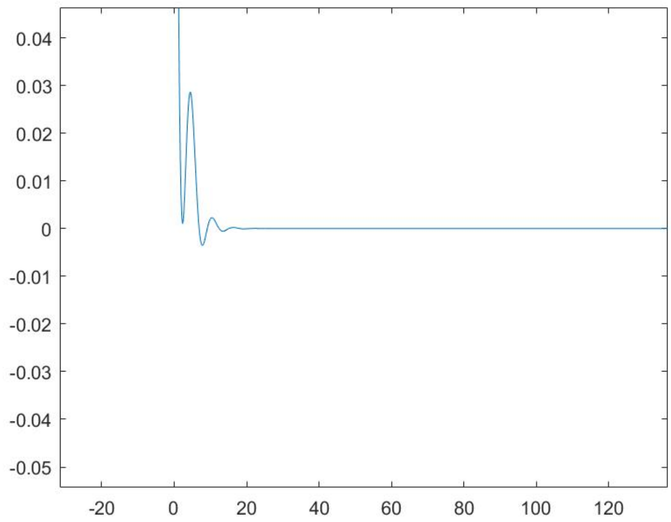

Figure 8: For $\lambda = 1.568$, $f(min(m)) = 0.0010 > 0$. This value can be considered as a sufficient sharp estimate so that for $\lambda > 1.568$, the TW is positive in an inner region and oscillatory for $\xi >> m$.

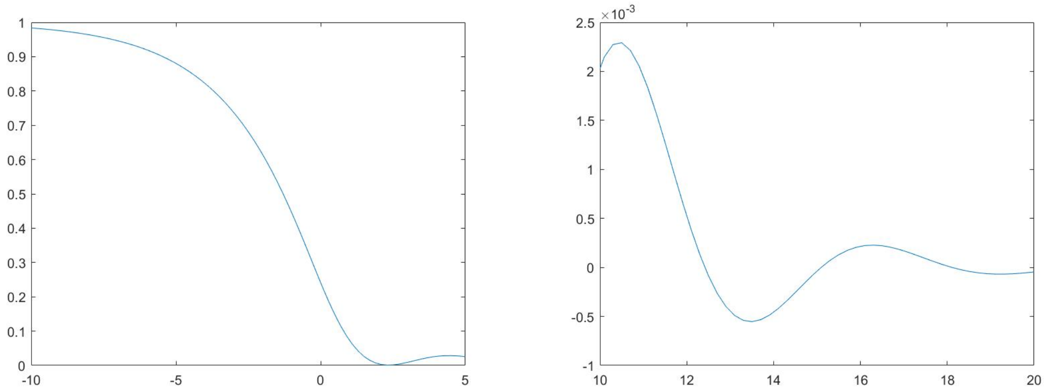

Figure 9: Horizontal axis corresponds to ξ and vertical to φ. Solution structure for $\lambda = 1.568$ (left) and unstable character for an outer region $\xi >> m$ (right).

appropriate inner region (inner compared to $\xi >> 1$, see Section 4), it is possible to express the high order TW profile in an analogous manner as done in the KPP-2 problem.

The numerical exploration of the TW profile with the bvp4c function in Matlab has led to a value of $\lambda_{\varphi(\min(m))>0} = 1.568$ (Figure 8) for which the first minimum is positive $\varphi(min(m)) = 0.0010 > 0$. $\square$

4 Local assessment to characterize positivity in a spatial region

The following lemma provides a proof of a spatial region where solutions are positive (i.e. with no oscillations leading to negative values). The intention is to find $R(t)$ such that for any $|x| > R(t)$ solutions to P (1) are oscillatory while for $|x| << R(t)$ solutions are positive and further regularity assessments can be determined (mainly the existence of a maximum principle).

Lemma 4.1. *There exists a spatio-temporal evolution $R(t)$ such that for $|x| << R(t)$ any solution* *to P (1) is positive. Such function $R(t)$ is assessed to be:*

$$R(t) = t^{1/4}|\ln t|, \tag{41}$$

where $t << 1$.

Proof. Firstly, a scaling is introduced following and idea in [26]:

$$\xi = \frac{x}{t^{1/4}}; \quad \tau = \ln t \to -\infty, \quad t \to 0^{+}. \tag{42}$$

After replacement in P (1) and introducing the function $w(\xi, \tau)$:

$$w_{\tau} = \left(\mathbf{A} - \frac{1}{4}I\right) w + w_{\xi} e^{3/4\tau} + e^{\tau} w(1-w). \tag{43}$$

Note that $\mathbf{A}$ is a spatial operator given by $\mathbf{A} = -D_{\xi}^{4} + \frac{1}{4}\xi D_{\xi} + \frac{1}{4}I$.

Admit the following stationary conditions to introduce the step-like condition:

$$\left(\mathbf{A} - \frac{1}{4}I\right) w_e = 0, \quad w_e(\infty) = 0, \; w_e(-\infty) = 1. \tag{44}$$

Operating in the proximity of the stationary conditions, a solution is given by:

$$w(\xi, \tau) = w_e(\xi) + \beta(\xi, \tau). \tag{45}$$

where $|\beta| << 1$.

Returning to (43):

$$\beta_{\tau} = \left(\mathbf{A} - \frac{1}{4}I\right) \beta + w_{e,\xi} e^{3/4\tau} + e^{\tau} w_e(1-w_e). \tag{46}$$

Admit that any solution can be expressed asymptotically as:

$$\beta(\xi, \tau) = \varphi(\xi)\theta(\xi). \tag{47}$$

After standard assessment in (46):

$$\frac{\theta'}{\theta} = \frac{\left(\mathbf{A} - \frac{1}{4}I\right)\theta + \varphi' e^{3/4\tau} + e^{\tau} w_e(1-w_e)\,/\,\theta}{\varphi} = K. \tag{48}$$

Then

$$\theta(\tau) = e^{\tau}, \tag{49}$$

where it has been considered $K = 1$ for simplicity.

Now. a solution to $\varphi(\xi)$ is obtained under the condition $w_e(\infty) = 0$:

$$\left(\mathbf{A} - \frac{1}{4}I\right) \varphi + \varphi' e^{3/4\tau} = \varphi. \tag{50}$$

Note that $\mathbf{A}$ has a discrete set of eigenfunctions in $H_{\rho}^{4} \subset L_{\rho}^{2}$ [27], consequently. any spanned solution φ converges in H_{ρ}^{4}. Note that the intentions is to look for solutions:

$$\varphi(\xi) = e^{\delta\,\xi}. \tag{51}$$

Coming to (50) and after balancing the leading terms:

$$\delta^{4} = -1, \quad \frac{1}{4}\xi + e^{3/4\tau} << 1 \;\to\; t \geq \frac{1}{4}|x|, \tag{52}$$

that provides a domain where the exponential bundle (51) holds. Now, consider the two real roots for δ:

$$\varphi_{+} = e^{\delta\xi}, \quad \xi \to -\infty\,; \quad \varphi_{-} = e^{-\delta\xi}, \quad \xi \to \infty. \tag{53}$$

Then:

$$\beta(\xi,\tau) = e^{\tau}(e^{\delta\xi} + e^{-\delta\xi}). \tag{54}$$

The solution (45) is then:

$$w(\xi,\tau) = w_e(\xi) + e^{\tau}(e^{\delta\xi} + e^{-\delta\xi}). \tag{55}$$

So that in (x,t):

$$w(x,t) = w_e\left(\frac{x}{t^{1/4}}\right) + t(e^{\delta\frac{x}{t^{1/4}}} + e^{-\delta\frac{x}{t^{1/4}}}). \tag{56}$$

Note that $|\beta| << 1$ so that for $x \to \infty$:

$$|te^{-\delta\frac{x}{t^{1/4}}}| << 1 \;\Rightarrow\; |x| >> t^{1/4}\ln t. \tag{57}$$

As $\ln t < 0$:

$$|x| << t^{1/4}|\ln t| = R(t), \tag{58}$$

that incorporates the region in (52):

$$|x| < 4t << t^{1/4}|\ln t| \tag{59}$$

for any $t \to 0^+$.

□

The expression (59) can be considered together with the TW results. According to Lemma 3.6, there exists a particular TW where the first minimum is positive, and sufficiently close to zero. Admit that such minimum is given by ξ_0, then the TW variable permits to define a spatio-temporal relation of the form $x - \lambda t = \xi_0$ where solutions are positive. Note that the expression (59) provides another relation to ensure positiveness of solutions. It is possible to check that whenever $t \to 0^+$ the expression in (59) includes the relation given by the TW positivity region.

5 Conclusions

The results presented along this study have introduced a systematic procedure to analyze high order operators. It is considered that following the proposed ideas developed, it is possible to have a comprehensive frame to explore physical models involving non-regular diffusion. Note that the principal questions related with the instability of TW solutions have been dealt along the presented analysis for the problem P (1) . In addition, a TW-speed for which positiviy holds in an inner region has been sharply obtained as $\lambda_{\varphi(\min(m))>0} = 1.568$ (Figure 8). Finally, such inner region has been determined with the objective of segregating a positive TW front from the inherent unstable behaviour located in an outer region.

6 Conflict of Interest

The author states that there is no conflict of interest.

7 Appendix. Supplementary information.

7.1 Initial Assessments. Existence and Uniqueness

The following lemma provides a-priori bounds.

Lemma 7.1. *For $F_0 \in L^1(\mathbb{R}^N)$, $\|F\|_{L^1} \le \|F_0\|_{L^1}$. In addition, for $m \in \mathbb{R}^+$ and $F_0 \in H^m(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $\|F\|_{H^m} \le \|F_0\|_{H^m}$, $\|F\|_{H^m} \le \|F_0\|_{L^1}$, for $t \ge \frac{m}{2}$. Furthermore,*

$$\|F\|_{\rho} \le \sigma\|F\|_{H^m} \le \sigma\|F_0\|_{H^m}, \quad \sigma = \max\{D^1F, D^2F, D^3F, D^4F\} \tag{60}$$

Proof. Given the homogeneous $F_t = LF$, a generic solution is expressed as $F(x,t) = e^{tL} F_0(x)$. Now, in the Fourier transformed domain (χ):

$$\hat{F}(\chi,t) = e^{t(-\chi^4 + q\hat{F}^{q-1} c\chi i)}\, \hat{F}_0(\chi), \tag{61}$$

so that

$$\|F\|_{L^1} = \int_{-\infty}^{\infty} |e^{t(-\chi^4 + q\hat{u}^{q-1} c\chi i)}|\, |\hat{F}_0(\chi)| d\chi \le \sup_{\chi\in\mathbb{R}} (e^{-\chi^4 t}) \int_{-\infty}^{\infty} |\hat{F}_0(\chi)| d\chi = \|F_0\|_{L^1}. \tag{62}$$

Admit the weighted Sobolev norm:

$$\|F\|_{H^m} = \int_{-\infty}^{\infty} e^{m\chi^2}\, |\hat{F}(\chi,t)| d\chi, \tag{63}$$

satisfying the A_p-condition [9] for $p = 1$

$$\|F\|_{H^m} = \int_{-\infty}^{\infty} e^{m\chi^2} |\hat{F}(\chi,t)| d\chi \le \sup_{\chi\in\mathbb{R}} (e^{-\chi^4 t}) \int_{-\infty}^{\infty} e^{m\chi^2} |\hat{F}_0(\chi)| d\chi = \|F_0\|_{H^m}. \tag{64}$$

Admit $F_0 \in L^1(\mathbb{R}^N)$:

$$\|F\|_{H^m} = \int_{-\infty}^{\infty} e^{m\chi^2} |\hat{F}(\chi,t)| d\chi \le \sup_{\chi\in\mathbb{R}} (e^{m\chi^2}\, e^{-\chi^4 t}) \int_{-\infty}^{\infty} |\hat{F}_0(\chi)| d\chi. \tag{65}$$

By standard operations:

$$\|F\|_{H^m} \le \left(\frac{m}{2t}\right)^{1/2} \|F_0\|_{L^1}, \tag{66}$$

equivalently $\|F\|_{H^m} \le \|F_0\|_{L^1}$, $t \ge \frac{m}{2}$. Eventually:

$$\|F\|_\rho = \int_{\mathbb{R}} \rho(y) \sum_{j=0}^{4} |D^j F(y)| dy \le \int_{\mathbb{R}} e^{my^2} \sum_{j=0}^{4} |D^j F(y)| dy \le \sigma \int_{\mathbb{R}} e^{my^2} |F(y)| dy \le \sigma \|F\|_{H^m}. \tag{67}$$

$\square$

Admit the existence of an operator $G_{F_0,t} \in H^4_\rho(\mathbb{R})$ as a map:

$$G_{F_0,t} : H^4_\rho \to H^4_\rho, \tag{68}$$

given by:

$$G_{F_0,t}(F) = g(x,t) * F_0(x) + \int_0^t \left[c \cdot \nabla g(x,t-s) * F^q(s) + g(x,t-s) * F(x,s)(a - F(x,s))\right] ds, \tag{69}$$

where $g(x,t) = e^{\Delta^2 t}$ is the infinitesimal representation of the homogeneous operator. The following lemma holds:

Lemma 7.2. *The operator $G_{F_0,t}$ is bounded in $H^4_\rho(\mathbb{R})$ with the norm (2).*

Proof. Previously, note that:

$$c_0 \|F_0\|_\rho \le \|F\|_\rho, \tag{70}$$

indeed

$$\begin{aligned} \|F\|_\rho &= \int_{\mathbb{R}} \rho(\chi) \sum_{j=0}^{4} |D^j \hat{F}(\chi)| d\chi = \int_{\mathbb{R}} \rho(\chi) \sum_{j=0}^{4} |D^j \left[e^{t(-\chi^4 + q\hat{F}^{q-1} c\chi i)}\, \hat{F}_0\right]| d\chi \\ &\ge \int_{\mathbb{R}} \rho(\chi) \sum_{j=0}^{4} |D^j \left[e^{t(-\chi^4 + q\hat{F}^{q-1} c\chi i)}\right]| \sum_{j=0}^{4} |D^j \hat{F}_0| d\chi \ge c_0 \int_{\mathbb{R}} \rho(\chi) \sum_{j=0}^{4} |D^j \hat{F}_0| d\chi = c_0 \|F_0\|_\rho, \end{aligned} \tag{71}$$

such that

$$c_0 = \inf_{\chi \in B_R} \{ \sum_{j=0}^{4} |D^j \left[e^{t(-\chi^4 + q\hat{u}^{q-1} c\chi i)} \right] | \} > 0 \tag{72}$$

for $B_R = \{\chi,\ |\chi| < R\}$ for $R > 0$.

Then:

$$\begin{aligned} \|G_{F_0,t}(F)\|_\rho &= \|G_{F_0,t}\|_\rho \, \|F\|_\rho \\ &\leq \left[\|g\|_\rho \, \frac{1}{c_0\, t} + \int_0^t \left[\|c \cdot \nabla g\|_\rho \, \|F_0^{q-1}\|_{H^m} + \|g\|_\rho \ \left| \, \|a\|_\rho - c_0 \|F_0\|_\rho \right| \right] ds \right] \ t \, \|F\|_\rho \end{aligned} \tag{73}$$

According to (64) and (67): $\|F^{q-1}\|_\rho \leq \|u^{q-1}\|_{H^m} \leq \|u_0^{q-1}\|_{H^m}$. Then

$$\|G_{F_0,t}(u)\|_\rho \leq \left[\|g\|_\rho \, \frac{1}{c_0\, t} + \int_0^t \left[\|c \cdot \nabla g\|_\rho \, \|F_0^{q-1}\|_{H^m} + \|g\|_\rho \ \left| \, \|a\|_\rho - c_0 \|F_0\|_\rho \right| \right] ds \right] \ t, \tag{74}$$

that is bounded given any $t > 0$. □

7.2 Uniqueness

Uniqueness is shown if a unique fix point $F(x,t) = G_{F_0,t}(F(x,t))$ exists.

$$\begin{aligned} \|G_{u_0,t}(F_1) - G_{F_0,t}(F_2)\|_\rho &\leq \int_0^t \|c \cdot \nabla g(x,t-s) * (F_1^q - F_2^q) + g(x,t-s) * [F_1(a-F_1) - F_2(a-F_2)]\|_\rho ds \\ &\leq \delta \int_0^t \int_t^s \{ \|F_1^q - F_2^q\|_\rho + \|F_1(a-F_1) - F_2(a-F_2)\|_\rho \} dr ds, \end{aligned} \tag{75}$$

where

$$\delta = \sup\{ \|g(x,t-s-r)\|_\rho \, , \|c \cdot \nabla g(x,t-s-r)\|_\rho ; \ \ \forall t > 0, \ x \in \mathbb{R} \} \tag{76}$$

for any of s, r.

Admit the following balancing function:

$$b(\varepsilon, s) = \left[\begin{array}{c} \frac{F_1(\varepsilon,s)^q - F_2(\varepsilon,s)^q}{F_1(\varepsilon,s) - F_2(\varepsilon,s)} \ , \ \ F_1 \not\equiv F_2 \\ qF_1^{q-1} \ otherwise \end{array} \right]. \tag{77}$$

For particular values of ε and $s = \beta$, the previous function is bounded:

$$0 \leq b(\varepsilon, s) \leq a_0(\|F_0\|_\infty, m, \beta). \tag{78}$$

so that $\|F_1^q - F_2^q\|_\rho \leq A_0 \|F_1 - F_2\|_\rho$, where $A_0 = \|a_0\|_\rho$. Now, the reaction terms can be shown to be bounded by

$$\|[F_1(a-F_1) - F_2(a-F_2)]\|_\rho \leq Q \int_{\mathbb{R}} \rho(y) \sum_{i=0}^{4} |D^i [F_1 - F_2]| dy = Q \|F_1 - F_2\|_\rho. \tag{79}$$

where Q is a suitable constant. Then:

$$\|G_{F_0,t}(F_1) - G_{F_0,t}(F_2)\|_\rho \leq M(Q + A_0) \int_0^\tau \int_\tau^\beta \|F_1 - F_2\|_\rho dq ds = M(Q + A_0) \tau(\beta - \tau) \|F_1 - F_2\|_\rho. \tag{80}$$

For any local time τ, uniqueness holds for $F_1 \swarrow F_2$ leading to a contractive map $G_{F_0,t}$ such that $G_{F_0,t}(F_1) \swarrow F_1$ in H_ρ^4.

References

[1] Okubo A., Levin S.A. (2001). *The Basics of Diffusion. In: Diffusion and Ecological Problems: Modern Perspectives.* Interdisciplinary Applied Mathematics, vol 14. Springer, New York, NY. https://doi.org/10.1007/978-1-4757-4978-6 .

[2] Cohen, D.S., Murray, J.D. (1981). A generalized diffusion model for growth and dispersal in a population. *J. Math. Biology* 12, 237–249. https://doi.org/10.1007/BF00276132

[3] Coutsias, E.A. (1980). *Some effects of spatial nonuniformities in chemically reacting systems.* California Institute of Technology.

[4] Kolmogorov, A.N., Petrovskii I.G. and Piskunov N.S. (1937). Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem. *Byull. Moskov. Gos. Univ.*, Sect. A, 1.

[5] Fisher, R.A. (1937). The advance of advantageous genes. *Ann. Eugenics,* 7. pp 355–369.

[6] Aronson, D. (1980). Density-dependent interaction-diffusion systems. *Proc. Adv. Seminar on Dynamics and Modeling of Reactive System*, Academic Press, New York.

[7] Aronson, D. and Weinberger, H. (1975). Nonlinear diffusion in population genetics, combustion and nerve propagation. *Partial Differential Equations and Related Topic.* Pub., New York, 5–49.

[8] Aronson, D. and Weinberger, H. (1978). Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.* 30, pp 33–76.

[9] Goldshtein, V. Ukhlov A (2009). Weighted Sobolev Spaces and embeddings Theorems. *Transactions of the american mathematical society.* 361, 3829-3850.

[10] Bonheure, D. and Hamel, F. (2017). One-dimensional symmetry and Liouville type results for the fourth order Allen-Cahn equation in $\mathbb{R}^N$. *Chin. Ann. Math. Ser. B 38*, 149–172.

[11] Audrito, A. and Vázquez, J.L. (2017) The Fisher–KPP problem with doubly nonlinear "fast" diffusion *Nonlinear Analysis*, 157, pp 212-248.

[12] Rottschäfer, V. and Doelman A. (1998). On the transition from the Ginzburg-Landau equation to the extended Fisher-Kolmogorov equation. *Physica D.* 118. 261 - 292.

[13] Peletier, L.A. and Troy, W.C.(2001). Spatial Patterns. Higher order models in Physics and Mechanics. *Progress in non linear differential equations and their applications.* Volume 45. Université Pierre et Marie Curie.

[14] Dee, G.T. and Van Sarloos, W. (1998). Bistable systems with propagating fronts leading to pattern formation. *Physical Review Letter* Volume 60.

[15] Hongjun, G. and Changchun, L. (2004) Instabilities of traveling waves of the convective-diffusive Cahn-Hilliard equation. *Chaos, Solitons and Fractals*, 20. 253-258.

[16] Alexander J, Gardner R, Jones C. (1990) A topological invariant arising in the stability analysis of travelling waves. *J. Reine Angew. Math.* 410, 167–212. (doi:10.1515/crll.1990.410.167)

[17] Li, Zhenbang and Liu, Changchun (2012) On the Nonlinear Instability of Traveling Waves for a Sixth-Order Parabolic Equation. *Abstr. Appl. Anal.* Article ID 739156, 17 pages. doi:10.1155/2012/739156

[18] Strauss, W and Wang, G. (2002). Instabilities of travelling waves of the Kuramoto-Sivashinsky equation. *Chin Ann Math B.* 23. 267 - 76.

[19] Bonheure, D. Sánchez, L. (2006). *Heteroclinics Orbits for some classes of second and fourth order differential equations. Handbook of differential equations.* 3(06), 103-202.

[20] Montaru, A. (2014) Wellposedness and regularity for a degenerate parabolic equation arising in a model of chemotaxis with nonlinear sensitivity. *Discrete and Continuous Dynamical Systems - B,19,1*, pp 231,256.

[21] Galaktionov, V.A. (2001). On a spectrum of blow-up patterns for a higher-order semilinear parabolic equation. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences.*

[22] Rauprich O, Matsushita M, Weijer CJ, Siegert F, Esipov SE, Shapiro JA. (1996) Periodic phenomena in proteus mirabilis swarm colony development. *J. Bacteriol.* 178, 6525–6538. (doi:10.1128/jb.178.22.6525-6538.1996)

[23] Niemela JJ, Ahlers G, Cannell DS. (1990) Localized traveling-wave states in binary-fluid convection. *Phys. Rev. Lett.* 64, 1365–1368. (doi:10.1103/PhysRevLett.64.1365).

[24] Durham AC and Ridgway EB. (1976) Control of chemotaxis in physarum polycephalum. *J. Cell. Biol.* 69, 218–223. (doi:10.1083/jcb.69.1.218)

[25] Enright, H. and Muir P.H. (1993). *A Runge-Kutta type boundary value ODE solver with defect control.* Teh.Rep. 267/93, University of Toronto, Dept. of Computer Sciences. Toronto. Canada.

[26] Galaktionov, V. (2012). Towards the KPP–Problem and Log-Front Shift for Higher-Order Nonlinear PDEs I. Bi-Harmonic and Other Parabolic Equations. *Cornwell University arXiv:1210.3513.*

[27] Egorov, Y. Galaktionov, V. Kondratiev V. Pohozaev, S. (2004). Global solutions of higher-order semilinear parabolic equations in the supercritical range. *Adv. Differ. Equat.* 9, pp 1009-1038.

[28] Ladyzhenskaya, O. (1998). Some results on modifications of three-dimensional Navier-Stokes equations. *Nonlinear Analysis and Continuum Mechanics.* pp. 73–84.