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THE FRACTIONAL INTEGRAL OPERATORS ON MORREY SPACES OVER *Q*-HOMOGENEOUS METRIC MEASURE SPACE

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ABSTRACT

This paper establishes necessary and sufficient condition for the boundedness of the fractional integral operator $I_\alpha f$ on Morrey spaces over metric measure spaces which satisfies the Q-homogeneous and its corollary.

Key words: Morrey Space Classic; Metric Measure Space; Q-Homogeneous.

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1. INTRODUCTION

We consider to a topological space $X := (X, \delta, \mu)$, endowed with complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a function (metric) $\delta: X \times X \to [0, \infty)$ satisfying the following conditions.

1. $\delta(x, y) = 0$ if and only if $x = y$;
2. $\delta(x, y) > 0$ for all $x \neq y, x, y \in X$;
3. $\delta(x, y) = \delta(y, x)$;
4. $\delta(x, y) \leq \{\delta(x, z) + \delta(z, y)\}$

for every $x, y, z \in X$. We have an assumptions that the balls $B(a, r) := \{x \in X: \delta(x, a) < r\}$ are measurable, for $a \in X, r > 0$, and $0 \leq \mu\big(B(a, r)\big) < \infty$. For every neighborhood V of $x \in X$, there exists $r > 0$, such that $B(x, r) \subset V$. We also assume that $\mu(X) = \infty$, $\mu\{a\} =$

0 and $B(a,r_2)\backslash B(a,r_1) = \emptyset$, for all $a \in X, 0 < r_1 < r_2 < \infty$. The triple (X,δ,μ) will be called metric measure space [7].

X is called Q-homogeneous $(Q > 0)$ such that $C_0\, r^Q \leq \mu(B) \leq C_1 r^Q$ where C_0 and C_1 are positive constants [8].

Eridani [6,7] proved the boundedness theorem on Lebesgue spaces in K_α and classic Morrey spaces over quasi metric space where

$$K_\alpha := \int_X \frac{f(y)}{\mu\left(B\left(x,\delta(x,y)\right)\right)^{1-\alpha}} d\mu(y)$$

with $0 < \alpha < 1$.

The result of [7] can be adapted to the operator K_α with doubling condition. Let $0 < \alpha < \beta$, we consider the fractional integral operator I_α given by

$$I_\alpha f(x) := \int_X \frac{f(y)}{\delta(x,y)^{\beta-\alpha}} d\mu(y)$$

for suitable f on X

The boundedness theorem of I_α on homogeneous classic Morrey spaces can be proved using Q-Homogeneous. In this paper, we will prove the generalization of the boundedness theorem from [6,7].

2. PRELIMINARIES

The following theorem is the inequality for the operator K_α from $\mathcal{L}^p(X,\mu)$ to $\mathcal{L}^q(X,\nu)$ for the case of Euclidean spaces.

Theorem 2.1 [6] Let (X,δ,μ) be a space of homogeneous type. Suppose that $1 < p < q < \infty$ and $0 < \alpha < \frac{1}{p}$. Assume that ν is another measure on X. Then K_α is bounded from $\mathcal{L}^p(X,\mu)$ to $\mathcal{L}^q(X,\nu)$ if and only if

$$\nu(B) \leq C\mu(B)^{q(\frac{1}{p}-\alpha)}$$

for all balls B in X.

Eridani and Meshki [7] proved the boundedness results of K_α from $\mathcal{L}^p(X,\mu)$ to the classic Morrey spaces $\mathcal{L}^{p,\lambda}(X,\nu,\mu)$ which is defined as a set of functions $f \in \mathcal{L}^p_{lok}(X,\nu)$ such that

$$\left\|f{:}\,\mathcal{L}^{p,\lambda}(X,\nu,\mu)\right\| = sup_B \left(\frac{1}{\mu(B)^\lambda}\int_B |f(y)|^p d\nu(y)\right)^{\frac{1}{p}} < \infty.$$

with ν is another measure on X, where $1 \leq p < \infty$ and $\lambda \geq 0$. Their theorem can be stated as the following theorem.

Theorem 2.2 [7] Let (X,δ,μ) be a space of homogeneous type and let $1 < p < q < \infty$. Suppose that $0 < \alpha < \frac{1}{p}$, $0 < \lambda_1 < 1 - \alpha p$ and $\frac{\lambda_2}{q} = \frac{\lambda_1}{p}$. Then K_α is bounded from $\mathcal{L}^{p,\lambda_1}(X,\nu,\mu)$ to $\mathcal{L}^{q,\lambda_2}(X,\nu,\mu)$ if and only if there is a positive constant C such that

$$\nu(B) \leq C\mu(B)^{q(\frac{1}{p}-\alpha)}$$

3. MAIN RESULT

In this section, we formulate the main results of the paper. We begin with the case of β-homogeneous over metric measure space.

Theorem 3.1 Let (X, δ, μ) be a β-homogeneous metric measure space, ν be a measure on X, $1 < p < q < \infty$, $1 < \alpha < \beta$. Then I_α is bounded from $\mathcal{L}^p(X, \mu)$ to $\mathcal{L}^q(X, \nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X,

$$\nu(B) \leq C\mu(B)^{q\left(\frac{1}{p}-\frac{\alpha}{\beta}\right)}$$

Proof:(Necessity) If $x, y \in B(a,r)$ then $\delta(x,a) < r$ and $\delta(y,a) < r$ thus $\delta(x,y) \leq \delta(x,a) + \delta(y,a) < 2r$ thus

$$\frac{1}{(2r)^{\beta-\alpha}} \leq \frac{1}{\delta(x,y)^{\beta-\alpha}}$$

the above inequality implies.

$$\frac{\mu(B)}{r^{\beta-\alpha}} = \int_B \frac{d\mu(y)}{(2r)^{\beta-\alpha}} \leq \int_B \frac{d\mu(y)}{\delta(x,y)^{\beta-\alpha}} = \int_X \frac{\chi_B(y)d\mu(y)}{\delta(x,y)^{\beta-\alpha}} = CI_\alpha\chi_B(x)$$

$$r^\alpha \leq CI_\alpha\chi_B(x)$$

$$\|I_\alpha\chi_B : \mathcal{L}^q(\nu)\| \leq C\|\chi_B : \mathcal{L}^p(\mu)\| \leq C\left(\int_X \chi_B(t)d\mu(t)\right)^{\frac{1}{p}} \leq C\mu(B)^{\frac{1}{p}}$$

$$\left(\int_B |r^\alpha|^q d\nu(x)\right)^{\frac{1}{q}} \leq C\left(\int_B \left|I_\alpha\chi_B(t)\right|^q d\nu(t)\right)^{\frac{1}{q}} \leq C\|I_\alpha\chi_B : \mathcal{L}^q(\nu)\| \leq C\mu(B)^{\frac{1}{p}}$$

Thus

$$r^\alpha\nu(B)^{\frac{1}{q}} \leq C\mu(B)^{\frac{1}{p}}$$

$C_0\, r^\beta \leq \mu(B) \leq C_1 r^\beta$ thus

$$\mu(B)^{\frac{\alpha}{\beta}} \leq Cr^\alpha$$

$$\mu(B)^{\frac{\alpha}{\beta}}\nu(B)^{\frac{1}{q}} \leq Cr^\alpha\nu(B)^{\frac{1}{q}} \leq C\mu(B)^{\frac{1}{p}}$$

$$\nu(B)^{\frac{1}{q}}\mu(B)^{\frac{\alpha}{\beta}-\frac{1}{p}} \leq C$$

Thus

$$\nu(B)^{\frac{1}{q}} \leq C\mu(B)^{\frac{1}{p}-\frac{\alpha}{\beta}}$$

or alternatively

$$\nu(B) \leq C\mu(B)^{q\left(\frac{1}{p}-\frac{\alpha}{\beta}\right)}$$

Sufficiency: Let $f \geq 0$. We define

$$S(s) := \int_{\delta(a,y)<s} f(y)d\mu(y)$$

for every $s \in [0, r]$. Suppose that $S(r) < \infty$, then $2^m < S(r) \leq 2^{m+1}$, for some $m \in \boldsymbol{Z}$. Let

$$s_j := sup\{t : S(t) \leq 2^j\}, j \leq m, and\ s_{m+1} := r.$$

Then $(s_j)_{j=-\infty}^{m+1}$ is non-decreasing sequence, $S(s_j) \le 2^j, S(t) \ge 2^j \, for\ t > s_j$ and

$$2^j \le \int_{s_j \le \delta(a,y) \le s_{j+1}} f(y) d\mu(y)$$

If $\rho := \lim_{j\to-\infty} s_j$, then

$$\delta(a,x) < r \Leftrightarrow \delta(a,x) \in [0,\rho] \cup \bigcup_{j=-\infty}^{m} (s_j, s_{j+1}],$$

if $S(r) = \infty$ then $m = \infty$. Thus

$$0 \le \int_{\delta(a,y)<\rho} f(y) d\mu(y) \le S(s_j) \le 2^j$$

for every j, thus

$$\int_{\delta(a,y)<\rho} f(y) d\mu(y) = 0$$

from these observations, we have

$$\begin{aligned}
\int_{\delta(a,x)<r} \left(I_\alpha f(x)\right)^q d\nu(x) &\le \sum_{j=-\infty}^{m} \int_{s_j \le \delta(a,x) \le s_{j+1}} \left(I_\alpha f(x)\right)^q d\nu(x) \\
&\le \sum_{j=-\infty}^{m} \int_{s_j \le \delta(a,x) \le s_{j+1}} \left(\int_{\delta(a,y) \le s_{j+1}} \frac{f(y) d\mu(y)}{\delta(x,y)^{\beta-\alpha}} \right)^q d\nu(x) \\
&\le \sum_{j=-\infty}^{m} \int_{s_j \le \delta(a,x) \le s_{j+1}} \left(\sum_{k=0}^{\infty} \left(\frac{1}{s_{j+1}}\right)^{\beta-\alpha} \int_{\delta(a,y) \le s_{j+1}} f(y) d\mu(y) \right)^q d\nu(x) \\
&\le \left(\sum_{j=-\infty}^{m} \left(\frac{1}{s_{j+1}}\right)^{\beta-\alpha} \int_{\delta(a,y) \le s_{j+1}} f(y) d\mu(y) \right)^q \nu(B)
\end{aligned}$$

Using the fact that

$$\int_{\delta(a,y) \le s_{j+1}} f(y) d\mu(y) \le S(s_{j+1}) \le 2^{j+2} \le C \int_{s_{j-1} \le \delta(a,y) \le s_j} f(y) d\mu(y)$$

then, by using Holder's inequality, we obtain

$$\le C\nu(B) \left(\sum_{j=-\infty}^{m} \left(\int_{s_{j-1} \le \delta(a,y) \le s_j} \left(f(y)\right)^P d(\mu) y \right)^{\frac{1}{p}} \left(\int_{s_{j-1} \le \delta(a,y) \le s_j} 1^q d(\mu) y \right)^{\frac{1}{q}} \frac{1}{s_j^{\beta-\alpha}} \right)^q$$

$$\le C\nu(B) \left(\left(\int_{s_{j-1} \le \delta(a,y) \le s_j} \left(f(y)\right)^P d(\mu) y \right)^{\frac{1}{p}} \sum_{j=-\infty}^{m} \mu(B(x,r))^{1-\frac{1}{p}} \frac{1}{s_j^{\beta-\alpha}} \right)^q$$

$$= C\nu(B) r^{q\left(\alpha - \frac{\beta}{p}\right)} \left(\left(\int_{s_{j-1} \le \delta(a,y) \le s_j} \left(f(y)\right)^P d(\mu) y \right)^{\frac{1}{p}} \right)^q$$

$$\leq C\mu(B)^{q\left(\frac{1}{p}-\frac{\alpha}{\beta}\right)} r^{q\left(\alpha-\frac{\beta}{p}\right)} \left(\left(\int_{s_{j-1} \leq \delta(a,y) \leq s_j} (f(y))^P d(\mu) y \right)^{\frac{1}{p}} \right)^q$$

$$= C \left(\left(\int_{s_{j-1} \leq \delta(a,y) \leq s_j} (f(y))^P d(\mu) y \right)^{\frac{1}{p}} \right)^q$$

Thus

$$\|I_\alpha f : \mathcal{L}^q(\nu)\| \leq C\|f : \mathcal{L}^p(\mu)\|$$

Next, using the modified condition for measure ν, we obtain the following result.

Theorem 3.2 Let (X, δ, μ) be a Q-homogeneous metric measure space, ν be a measure on X, $1 < p < q < \infty$, $1 < \alpha < \beta - \frac{Q}{p'}$. Then I_α is bounded from $\mathcal{L}^p(X, \mu)$ to $\mathcal{L}^q(X, \nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X,

$$\nu(B) \leq Cr^{\left(\beta-\alpha-\frac{Q}{p'}\right)q}$$

with $p' = \frac{p}{p-1}$.

Proof. (Necessity) Suppose that I_α is bounded from $\mathcal{L}^p(X, \mu)$ to $\mathcal{L}^q(X, \nu)$ thus

$$\|I_\alpha f : \mathcal{L}^q(X, \nu)\| \leq C\|f : \mathcal{L}^p(X, \mu)\|$$

Hence,

$$\left(\int_X |I_\alpha f|^q d\nu \right)^{1/q} \leq C \left(\int_X |f(x)|^p d\mu \right)^{1/p}$$

$f := \chi_B$ where $a \in X$, $r > 0$ then

$$\left(\int_B |I_\alpha \chi_B|^q d\nu \right)^{1/q} \leq C \left(\int_B |\chi_B|^p d\mu \right)^{1/p}$$

$$\left(\int_B \left(\int_B \frac{\chi_B}{\delta(x,y)^{\beta-\alpha}} d\mu(y) \right)^q d\nu \right)^{1/q} \leq C\mu(B)^{1/p}$$

$$r^{\alpha-\beta}\mu(B)\nu(B)^{1/q} \leq C\mu(B)^{1/p}$$

$$\nu(B)^{1/q} \leq C\mu(B)^{\frac{1}{p}-1} r^{\beta-\alpha}$$

Because $p' = \frac{p}{p-1}$ and $C_0 r^Q \leq \mu(B) \leq C_1 r^Q$ then

$$\nu(B)^{1/q} \leq Cr^{-\frac{Q}{p'}} r^{\beta-\alpha}$$

$$\nu(B) \leq Cr^{q\left(\beta-\alpha-\frac{Q}{p'}\right)}$$

Sufficiency. Let $f \geq 0$. For $x, a \in X$, next we consider the notation

$$E_1(x) := \left\{y: \delta(a,y) < \frac{\delta(a,x)}{2a_1}\right\};$$

$$E_2(x) := \left\{y: \frac{\delta(a,x)}{2a_1} \leq \delta(a,y) \leq 2a_1\delta(a,x)\right\};$$

$$E_3(x) := \{y: \delta(a,y) > a_1\delta(a,x)\}.$$

Thus

$$\int_X \left(I_\alpha f(x)\right) d\nu(x)$$
$$\leq C\int_X \left(\int_{E_1(x)} |f(y)|\delta(x,y)^{\alpha-\beta} d\mu(y)\right)^q d\nu(x)$$
$$+ C\int_X \left(\int_{E_2(x)} |f(y)|\delta(x,y)^{\alpha-\beta} d\mu(y)\right)^q d\nu(x)$$
$$+ C\int_X \left(\int_{E_3(x)} |f(y)|\delta(x,y)^{\alpha-\beta} d\mu(y)\right)^q d\nu(x) = S_1 + S_2 + S_3$$

If $y \in E_1(x)$, then $\delta(a, x) < 2a_1a_0\delta(a,x)$. Thus obviously

$$S_1 = \int_{\delta(a,x)<r} \left(\int_{E_1(x)} |f(y)|\delta(x,y)^{\alpha-\beta} d\mu(y)\right)^q d\nu(x)$$

$$\leq C\int_B \left(\int_{\delta(a,y)<\delta(a,x)} |f(y)|\delta(x,y)^{\alpha-\beta} d\mu(y)\right)^q d\nu(x)$$

$$\leq C\int_B \delta(a,x)^{q(\alpha-\beta)} \left(\int_{\delta(a,y)<\delta(a,x)} |f(y)| d\mu(y)\right)^q d\nu(x)$$

Thus we have

$$\int_{\delta(a,x)\geq t} \delta(a,x)^{q(\alpha-\beta)} d\nu(x) = \sum_{n=0}^{\infty} \int_{B\left(a,2^{k+1}t\right)\setminus B\left(a2^k t\right)} \left(\delta(a,x)^{q(\alpha-\beta)} d\nu(x)\right)$$

$$\leq C\sum_{n=0}^{\infty} (2^k t)^{q(\alpha-\beta)} \nu(B), = Ct^{q(\alpha-\beta)}\nu(B)$$

which implies

$$\int_{\delta(\alpha,x)\leq t} 1^{(1-p')} d\mu(x) \leq C\mu(B)$$

Thus

$$\sup_{a \in X, t>0} \left(\int_{\delta(\alpha,x) \geq t} \delta(\alpha,x)^{q(\alpha-\beta)} dv(x) \right)^{\frac{1}{q}} \left(\int_{\delta(\alpha,x) \leq t} 1^{(1-p')} d\mu(x) \right)^{\frac{1}{p'}}$$

$$\leq \left(Ct^{q(\alpha-\beta)} v(B) \right)^{\frac{1}{q}} C\mu(B)^{\frac{1}{p'}}$$

$$\leq Ct^{(\alpha-\beta)} Ct^{\left(\beta-\alpha-\frac{Q}{p'}\right)q\frac{1}{q}} t^{Q\left(\frac{p-1}{p}\right)} = C < \infty$$

Now, using theorem C in [9], we have

$$S_1 \leq C \left(\int_B |f(y)|^p d\mu(y) \right)^{q/p} \leq C \|f\|^q_{\mathcal{L}^p(X,\mu)}$$

Next, we observe that if $\delta(a,y) > 2a_1\delta(a,x)$, then $\delta(a,y) \leq a_1\delta(a,x) + a_1\delta(a,y) \leq \delta(a,y)/2 + a_1\delta(x,y)$. Thus $\delta(a,y)/2a_1 \leq \delta(x,y)$. Implies, using the condition $v(B) \leq Cr^{\left(\beta-\alpha-\frac{Q}{p'}\right)q}$, then

$$S_3 \leq C \int_{B(a,r)} \left(\int_{\delta(a,y)>\delta(a,x)} \frac{|f(y)|}{\delta(a,y)^{\beta-\alpha}} d\mu(y) \right)^q dv(x)$$

$$\leq C \int_{B(a,r)} \left(\sum_{k=0}^{\infty} \int_{B\left(a,2^{k+1}\delta(a,x)\right) \setminus B\left(a,2^k\delta(a,x)\right)} \frac{|f(y)|}{\delta(a,y)^{\beta-\alpha}} d\mu(y) \right)^q dv(x)$$

$$\leq C \int_{B(a,r)} \left[\sum_{k=0}^{\infty} \left(\int_{B\left(a,2^{k+1}\delta(a,x)\right)} |f(y)|^p \, d\mu(y) \right)^{\frac{1}{p}} \right.$$

$$\left. \times \left(\int_{B\left(a,2^{k+1}\delta(a,x)\right) \setminus B\left(a,2^k\delta(a,x)\right)} \delta(a,y)^{(\alpha-\beta)p'} d\mu(y) \right)^{\frac{1}{p'}} \right]^q dv(x)$$

$$\leq C\|f\|^q_{\mathcal{L}^p(X,\mu)} \int_{B(a,r)} \left(\sum_{k=0}^{\infty} \left(2^k\delta(a,x) \right)^{\alpha-\beta} \left(\mu B\left(a, 2^{k+1}\delta(a,x)\right) \right)^{\frac{1}{p'}} \right)^q dv(x)$$

$$\leq C\|f\|^q_{\mathcal{L}^p(X,\mu)} \int_{B(a,r)} \left(\sum_{k=0}^{\infty} \left(2^k\delta(a,x) \right)^{\alpha-\beta} r^{\frac{Q}{p'}} \right)^q dv(x)$$

$$= C\|f\|^q_{\mathcal{L}^p(X,\mu)} r^{(\alpha-\beta)q} r^{\frac{Qq}{p'}} v(B)$$

$$= C\|f\|^q_{\mathcal{L}^p(X,\mu)}$$

Hence, we conclude that

$$S_3 \leq C\|f\|^q_{\mathcal{L}^p(X,\mu)}$$

To estimate S_2, we consider two cases. First assumption is that$\alpha < \beta - \frac{Q}{p}$. The hypothesis on the theorem $\alpha > 0$ which implies $0 < \alpha < \beta - \frac{Q}{p}$. Given $p^* = \frac{pQ}{p(\beta-\alpha-Q)+Q}$ then $q \leq p^*$. First assumption $q < p^*$and suppose that

$$F_k \coloneqq \{x: 2^k \leq \delta(a,x) < s^{k+1}\};$$

$$G_k \coloneqq \left\{y: \frac{2^{k-2}}{a_1} \leq \delta(a,y)\backslash a_1 2^{k+2}\right\}.$$

Assume that $\frac{p^*}{q}$, using Holder's inequality, we obtain

$$S_2 = \int_X \left(\int_{E_2(x)} |f(y)|\, \delta(x,y)^{\alpha-\beta} d\mu(y)\right)^q d\nu(x)$$

$$= C\sum_{k\in Z}\int_{F_k}\left(\int_{E_2(x)} |f(y)|\delta(x,y)^{\alpha-\beta}\, d\mu(y)\right)^q d\nu(x)$$

$$\leq \sum_{k\in Z}\left(\int_{F_k}\left(\int_{E_2(x)} |f(y)|\delta(a,x)^{\alpha-\beta}\, d\mu(y)\right)^{p^*} d\nu(x)\right)^{\frac{q}{p^*}} \times \left(\int_{F_k} 1^{\frac{p^*}{p^*-q}} d\nu(x)\right)^{\frac{p^*-q}{p^*}}$$

$$\leq C\sum_{k\in Z}\nu(B)^{\frac{p^*-q}{p^*}}\left(\int_X \left(I_\alpha\left(|f|\chi_{G_k}\right)\right)^{p^*} d\nu(y)\right)^{\frac{q}{p^*}}$$

$$\leq C\sum_{k\in Z}\nu(B)^{\frac{p^*-q}{p^*}}\left(\int_{G_k} |f(y)|^p\, d\mu(y)\right)^{\frac{q}{p}}$$

Where

$$\frac{p^*-q}{p^*} = 1 - \frac{q}{p^*}$$

$$= 1 - \frac{q(p(\beta-\alpha) - pQ + Q)}{pQ}$$

$$= 1 - \frac{Q(pq + p - q)}{pQ} + q - \frac{q}{p} = 0$$

$$\leq C\left(\int_{G_k}|f(y)|^p\,d\mu(x)\right)^{\frac{q}{p}}$$

$$\leq C\|f\|^q_{\mathcal{L}^p(X,\mu)}$$

if $q = p^*$, thus, we have

$$S_2 = \int_X\left(\int_{E_2(x)}|f(y)|\,\delta(x,y)^{\alpha-\beta}d\mu(y)\right)^q dv(x)$$

$$= C\sum_{k\in Z}\int_{F_k}\left(\int_{E_2(x)}|f(y)|\delta(x,y)^{\alpha-\beta}\,d\mu(y)\right)^{p^*} dv(x)$$

$$\leq C\sum_{k\in Z}\left(\int_X\left(I_\alpha\left(|f|\chi_{G_k}\right)\right)dv(y)\right)^{p^*}$$

$$\leq C\sum_{k\in Z}\left(\int_{G_k}|f(y)|^p\,d\mu(x)\right)^{\frac{q}{p}}$$

$$\leq C\left(\int_{G_k}|f(y)|^p\,d\mu(x)\right)^{\frac{p^*}{p}}$$

$$\leq C\|f\|^q_{\mathcal{L}^p(X,\mu)}$$

If $\alpha > \beta - \frac{Q}{p'}$, using Holder's inequality, we obtain

$$S_2 \leq \int_X\left(\int_{E_2(x)}\big(f(y)\big)^p d\mu(y)\right)^{\frac{q}{p}}\left(\int_{E_2(x)}\delta(a,x)^{(\alpha-\beta)p'}d\mu(y)\right)^{\frac{q}{p'}} dv(y)$$

thus we have

$$\int_{E_2(x)}\delta(a,x)^{(\alpha-\beta)p'}d\mu(y) \leq \int_0^\infty \mu\left(B\big(a,\delta(a,x)\big)\cap\left\{y|\delta(x,y) < \lambda^{\frac{1}{(\alpha-\beta)p'}}\right\}\right)d\lambda$$

$$\leq \int_0^{\delta(a,x)^{(\alpha-\beta)p'}}\mu\left(B\big(a,\delta(a,x)\big)\cap\left\{y|\delta(x,y) < \lambda^{\frac{1}{(\alpha-\beta)p'}}\right\}\right)d\lambda + \int_{\delta(a,x)^{(\alpha-\beta)p'}}^\infty \mu\left(B\big(a,\delta(a,x)\big)\cap\left\{y|\delta(x,y) < \lambda^{\frac{1}{(\alpha-\beta)p'}}\right\}\right)d\lambda$$

$$\leq C\delta(a,x)^{Q+(\alpha-\beta)p'} + \int_{\delta(a,x)^{(\alpha-\beta)p'}}^\infty \lambda^{\frac{1}{(\alpha-\beta)p'}}\,d\lambda = C\delta(a,x)^{Q+(\alpha-\beta)p'}$$

where the positive constant C is independent of a and x. Hence, using Holder's inequality, we obtain

$$S_2 \leq \int_X \left(\int_{E_2(x)} \delta(a,x)^{(\alpha-\beta)p'} d\mu(y) \right)^{\frac{q}{p'}} \left(\int_{E_2(x)} |f(y)|^p \, d\mu(y) \right)^{\frac{q}{p}} d\nu(x)$$

$$\leq \sum_{k \in Z} \int_{F_k} \delta(a,x)^{Q+(\alpha-\beta)p'\left(\frac{q}{p'}\right)} \left(\int_{E_2(x)} |f(y)|^p \, d\mu(y) \right)^{\frac{q}{p}} d\nu(y)$$

$$\leq C2^{k\left(\left(\beta-\alpha-\frac{Q}{p'}\right)q+\frac{Qq}{p'}+(\alpha-\beta)q\right)} \left(\int_{G_k} |f(y)|^p \, d\mu(y) \right)^{\frac{q}{p}} \leq C \left(\int_X |f(y)|^p \, d\mu(y) \right)^{\frac{q}{p}}$$

$$\leq C\|f\|^q_{\mathcal{L}^p(X,\mu)}$$

The proof is complete.

The similar results concerning the boundedness properties of the fractional integral operator I_α on the classic Morrey spaces using *Q-homogeneous metric measure space* is obtained by the following theorem.

Theorem 3.3 Let (X, δ, μ) be a Q -homogeneous metric measure space, ν be a measure on X, $1 < p < q < \infty$, $1 < \alpha < \beta - \frac{Q}{p'}, 0 < \lambda_1 < \frac{\beta p}{q}$, and $\frac{Q\lambda_1}{\beta p} = \frac{\lambda_2}{q}$. Then I_α is bounded from $\mathcal{L}^{p,\frac{Q\lambda_1}{\beta p}}(X, \mu)$ to $\mathcal{L}^{q,\frac{\lambda_2}{q}}(X, \nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X,

$$\nu(B) \leq Cr^{\left(\beta-\alpha-\frac{Q}{p'}\right)q}$$

with $p' = \frac{p}{p-1}$.

Proof: (Necessity) Suppose that $I_\alpha$is bounded from $\mathcal{L}^{p,\frac{Q\lambda_1}{\beta}}(X, \mu)$ to $\mathcal{L}^{q,\lambda_2}(X, \nu)$ which implies that

$$\left\| I_\alpha f: \mathcal{L}^{q,\lambda_2}(X, \nu) \right\| \leq C \left\| f: \mathcal{L}^{p,\frac{Q\lambda_1}{\beta}}(X, \mu) \right\|$$

Thus

$$\left(\frac{1}{\mu(B)^{\lambda_2}} \int_X |I_\alpha f|^q \, d\nu(x) \right)^{\frac{1}{q}} \leq C \left(\frac{1}{\mu(B)^{\frac{Q\lambda_1}{\beta}}} \int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

$f := \chi_B$ where $a \in X$ and $r > 0$ then

$$\left(\frac{1}{\mu(B)^{\lambda_2}}\int_X |I_\alpha \chi_B(\mathrm{x})\;|^q\, d\nu(x)\right)^{\frac{1}{q}} \leq C\left(\frac{1}{\mu(B)^{\frac{Q\lambda_1}{\beta}}}\int_X |\chi_B(\mathrm{x})\;|^p\, d\mu(x)\right)^{\frac{1}{p}}$$

$$\left(\frac{1}{\mu(B)^{\lambda_2}}\int_B \left(\int_B \frac{\chi_B}{\delta(x,y)^{\beta-\alpha}}\, d\mu(y)\right)^q d\nu(x)\right)^{\frac{1}{q}} \leq C\mu(B)^{\frac{-Q\lambda_1}{p\beta}}\mu(B)^{\frac{1}{p}}$$

$$\mu(B)^{\frac{-\lambda_2}{q}} r^{\alpha-\beta}\,\mu(B)\nu(B)^{\frac{1}{q}} \leq C\mu(B)^{\frac{-Q\lambda_1}{p\beta}}\mu(B)^{\frac{1}{p}}$$

Because $p' = \frac{p}{p-1}, \frac{Q\lambda_1}{\beta p} = \frac{\lambda_2}{q}$ and $C_0 r^Q \leq \mu(B) \leq C_1 r^Q$ then

$$\nu(B)^{\frac{1}{q}} \leq C\mu(B)^{\frac{1}{p'}} r^{\alpha-\beta}$$

$$\nu(B)^{\frac{1}{q}} \leq Cr^{-\frac{Q}{p'}} r^{\beta-\alpha}$$

$$\nu(B) \leq Cr^{\left(\beta-\alpha-\frac{Q}{p'}\right)q}$$

Sufficiency. Given arbitrary ball B on X. Suppose that $B \coloneqq B(a,r)$ and $\tilde{B} \coloneqq (a,2r)$ and $f \in \mathcal{L}^{p,\frac{Q\lambda_1}{\beta}}(\mu)$. we write

$$f = f_1 + f_2 \coloneqq f_{\chi_{\tilde{B}}} + f_{\chi_{\tilde{B}}c}$$

$$\|f_1 \colon \mathcal{L}^p(\mu)\| = \left(\int_B |f(x)|^p\, d\mu(x)\right)^{\frac{1}{p}}$$

$$= \mu(B)^{\frac{Q\lambda_1}{\beta p}}\left(\frac{1}{\mu(B)^{\frac{Q\lambda_1}{\beta}}}\int_B |f(x)|^p\, d\mu\right)^{\frac{1}{p}}$$

$$\leq \mu(B)^{\frac{Q\lambda_1}{\beta p}}\left\|f \colon \mathcal{L}^{p,\frac{Q\lambda_1}{\beta}}(X,\mu)\right\|$$

if $f_1 \in \mathcal{L}^p(X,\mu)$, and using Theorem 3.2, it is obvious that

$$\left(\frac{1}{\mu(B)^{\lambda_2}}\int_B |I_\alpha f_1|^q\, d\nu(x)\right)^{\frac{1}{q}} \leq \mu(B)^{-\frac{\lambda_2}{q}}\left(\int_B |I_\alpha f_1|^q\, d\nu(x)\right)^{\frac{1}{q}}$$

$$\leq \mu(B)^{-\frac{\lambda_2}{q}}\|I_\alpha f_1 \colon \mathcal{L}^q(\nu)\|$$

$$\leq C\mu(B)^{-\frac{\lambda_2}{q}}\|f_1 \colon \mathcal{L}^p(\mu)\|$$

$$\leq C\mu(B)^{-\frac{\lambda_2}{q}}\mu(B)^{-\frac{Q\lambda_1}{\beta p}}\left\|f:\mathcal{L}^{p,\frac{Q\lambda_1}{\beta}}(X,\mu)\right\|$$

$$\leq C\left\|f:\mathcal{L}^{p,\frac{Q\lambda_1}{\beta}}(X,\mu)\right\|$$

further we will prove,

$$|I_\alpha f_2(x)| = \left|\int_{\delta(x,y)\geq r}\frac{f(y)}{\delta(x,y)^{\beta-\alpha}}d\mu(y)\right|$$

$$\leq \int_{\delta(x,y)\geq r}\frac{|f(y)|}{\delta(x,y)^{\beta-\alpha}}d\mu(y)$$

$$\leq \sum_{k=0}^{\infty}\int_{2^k r\leq\delta(x,y)\leq 2^{k+1}r}\frac{|f(y)|}{\delta(x,y)^{\beta-\alpha}}d\mu(y)$$

$$\leq \sum_{k=0}^{\infty}\left(\frac{1}{2^k r}\right)^{\beta-\alpha}\int_{\delta(x,y)\leq 2^{k+1}r}|f(y)|\,d\mu(y)$$

$$\leq C\sum_{k=0}^{\infty}\left(\int_{\delta(x,y)\leq 2^{k+1}r}|f(x)|^p\,d\mu(y)\right)^{\frac{1}{p}}\left(\int_{\delta(x,y)\leq 2^{k+1}r}1^q\,d\mu(y)\right)^{\frac{1}{q}}\frac{1}{2^k r^{\beta-\alpha}}$$

$$\leq C\mu(B)^{\frac{Q\lambda_1}{\beta p}}\left\|f:\mathcal{L}^{p,\frac{Q\lambda_1}{\beta p}}(X,\mu)\right\|\sum_{k=0}^{\infty}\mu(B(x,2^{k+1}r))^{\frac{1}{q}}\frac{1}{(2^k r)^{\beta-\alpha}}$$

$$= \mu(B)^{\frac{Q\lambda_1}{\beta p}}r^{\alpha-\beta}r^{\frac{Q}{p'}}\left\|f:\mathcal{L}^{p,\frac{Q\lambda_1}{\beta p}}(X,\mu)\right\|$$

Then

$$\left(\frac{1}{\mu(B)^{\lambda_2}}\int_B|I_\alpha f_2(x)|^q d\nu(x)\right)^{\frac{1}{q}} = C\nu(B)^{\frac{1}{q}}\mu(B)^{\frac{-\lambda_2}{q}}\mu(B)^{\frac{Q\lambda_1}{\beta p}}r^{\alpha-\beta}r^{\frac{Q}{p'}}\left\|f:\mathcal{L}^{p,\frac{Q\lambda_1}{\beta p}}(X,\mu)\right\|$$

$$= C\left\|f:\mathcal{L}^{p,\frac{Q\lambda_1}{\beta p}}(X,\mu)\right\|$$

The proof is complete.

The condition $\frac{Q\lambda_1}{\beta p}=\frac{\lambda_2}{q}$ is interchangeable to the condition $\nu(B)\leq Cr^{q(\beta-\alpha-\frac{Q}{p'})}$ Yet, the following theorem is hold obviously.

Theorem 3.4 Let (X,δ,μ) be a Q -homogeneous metric measure space, ν be a measure on X, $1 < p < q < \infty$, $1 < \alpha < \beta - \frac{Q}{p'}, 0 < \lambda_1 < \frac{\beta p}{q}$, and $\nu(B) \le Cr^{\left(\beta-\alpha-\frac{Q}{p'}\right)q}$ with $p' = \frac{p}{p-1}$. Then I_α is bounded from $\mathcal{L}^{p,\frac{Q\lambda_1}{\beta p}}(X,\mu)$ to $\mathcal{L}^{q,\frac{\lambda_2}{q}}(X,\nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X,

$$\frac{Q\lambda_1}{\beta p} = \frac{\lambda_2}{q}$$

When $Q = \beta$, the previous theorem implies the following corollary.

Corollary 3.5 Let (X,δ,μ) be a β-homogeneous metric measure space, ν be a measure on X, $0 < \lambda_1 < \frac{\beta p}{q}$, $1 < p < \frac{\beta}{\alpha}$, and $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$. Then I_α is bounded from $\mathcal{L}^{p,\frac{\lambda_1}{p}}(X,\mu)$ to $\mathcal{L}^{q,\frac{\lambda_2}{q}}(X,\nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X,

$$\nu(B) \le C\mu(B)^{q\left(\frac{1}{p}-\frac{\alpha}{\beta}\right)}$$

Corollary 3.6 Let (X,δ,μ) be a β-homogeneous metric measure space, ν be a measure on X, $0 < \lambda_1 < \frac{\beta p}{q}$, $1 < p < \frac{\beta}{\alpha}$, and $\nu(B) \le C\mu(B)^{q\left(\frac{1}{p}-\frac{\alpha}{\beta}\right)}$. Then I_α is bounded from $\mathcal{L}^{p,\frac{\lambda_1}{p}}(X,\mu)$ to $\mathcal{L}^{q,\frac{\lambda_2}{q}}(X,\nu)$ if and only if there is a constant $C > 0$ such that for every ball B on X,

$$\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$$

4. CONCLUSIONS

Through our work we have been able to extend the known results for the classical fractional integral operator I_α to the boundedness of with measure μ and ν on Morrey spaces over Q-homogeneous metric measure space. Our results not only cover the known results for I_α, but also enrich the class of funtions of α, λ_1 and λ_2 for which the operator I_α is bounded from the classical Morrey space $\mathcal{L}^{p,\frac{Q\lambda_1}{\beta}}(\mu)$ to $\mathcal{L}^{q,\lambda_2}(\nu)$, on Q-homogeneous and the corollary I_α is bounded from the classical Morrey space $\mathcal{L}^{p,\lambda_1}(\mu)$ to $\mathcal{L}^{q,\lambda_2}(\nu)$, on β-homogeneous.

COMPETING INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper

AUTHOR’S CONTRIBUTIONS

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