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Q-spectral and L-spectral radius of subgroup graphs of dihedral group

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Abstract. Research on Q-spectral and L-spectral radius of graph has been attracted many attentions. In other hand, several graphs associated with group have been introduced. Based on the absence of research on Q-spectral and L-spectral radius of subgroup graph of dihedral group, we do this research. We compute Q-spectral and L-spectral radius of subgroup graph of dihedral group and their complement, for several normal subgroups. Q-spectrum and L-spectrum of these graphs are also observed and we conclude that all graphs we discussed in this paper are Q-integral dan L-integral.

1. Introduction

For finite simple graph G of order p, its signless Laplacian matrix is defined by $Q(G) = D(G) + A(G)$ and its Laplacian matrix is defined by $L(G) = D(G) – A(G)$, where $D(G)$ is the vertex degree of G and $A(G)$ is adjacency matrix of G. The Q-polynomial of $Q(G)$ is $p_Q(q) = det(Q(G) - qI)$ and the L-polynomial of $L(G)$ is $p_L(\lambda) = det(L(G) - \lambda I)$, where I is identity matrix of dimension p. The largest eigenvalue of $Q(G)$ and $L(G)$ are named Q-spectral and L-spectral radius of G, respectively. The set of all distinct Q-eigenvalues with their multiplicities is called Q-spectrum and the set of all distinct L-eigenvalues with their multiplicities is called L-spectrum.

Q-spectral and L-spectral radius have received a great deal of attention and several researches have been reported. Some researches on Q-spectral radius and its sharp bound for various graphs can be seen in [1-4]. Sharp bound of L-spectral radius of graphs has also been studied, such as in [5-12]

Graphs associated with a finite group have been introduced, for example commuting graph [13], non-commuting graph [14], conjugate graph [15] and inverse graph [16], and seem to be an interesting area of research. Researches on signless Laplacian and Laplacian spectra of graphs associated with group have been conducted, such as [17-19]. In [20], Anderson et al. introduced the concept of subgroup graph of given subgroup H of a group G as a directed graph and denoted by $\Gamma_H(G)$. When the subgroup H is normal in G, then $\Gamma_H(G)$ is an undirected simple graph [21].

We are interested in doing research on Q-spectral and L-spectral radius of graph associated with group. This paper is aimed to determine Q-spectral and L-spectral radius of subgroup graphs of dihedral group and their complements. The Q-spectrum and L-spectrum of these subgroup graphs are also observed.

2. Literature Review

A graph *G* contained a finite non-empty set *V*(*G*) of vertices together with a possibly empty set *E*(*G*) of edges. The cardinality of *V*(*G*) is called the order of *G*, while the cardinality of *E*(*G*) is called the size of *G*. An empty graph is a graph of size 0. Two vertices u and v in *G* are adjacent if $uv \in E(G)$. The degree of vertex u in *G* is defined as the number of vertices that adjacent with u and denoted by $deg(u)$.

Let K_n denoted a complete graph with *n* vertices and $K_{m,n}$ denoted a complete bipartite graph with partition sets V_1 and V_2 where $|V_1| = m$ and $|V_2| = n$. Then, $K_{m,n}$ has order $m + n$ and size mn [22]. For more general, a complete multipartite graph with k partition sets $V_1, V_2, \dots, V_k$ ($k > 1$) where $|V_i| = n_i$ for $1 \le i \le k$ is denoted by $K_{n_1,n_2,\dots,n_k}$. Graph $K_{n_1,n_2,\dots,n_k}$ has order $n = \sum_{i=1}^{k} n_i$. The union $G = G_1 \cup G_2$ of two graphs G_1 and G_2 with $V(G_1) \cup V(G_2) = \emptyset$ is a graph that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ [23]. The graph $\overline{K_n}$ is the empty graph of order *n* [24]. The graph $\overline{K_{m,n}}$ is $K_m \cup K_n$. Since $\bar{\bar{G}} = G$ [22] then $\overline{K_m \cup K_n} = K_{m,n}$.

Let *G* is a graph of order *p*. Let the adjacency matrix of *G* is *A*(*G*) and the degree matrix of *G* is *D*(*G*). Then the matrix *Q*(*G*) = *D*(*G*) + *A*(*G*) is named the signless Laplacian matrix of *G* [25,26] and *L*(*G*) = *D*(*G*) – *A*(*G*) is named the Laplacian matrix of *G* [27]. The *Q*-polynomial of *Q*(*G*) is $p_Q(q) = det(Q(G) - qI)$ [28] and the *L*-polynomial of *L*(*G*) is $p_L(\lambda) = det(L(G) - \lambda I)$, where I is identity matrix of dimension *p* [2]. The roots of characteristics equation associated with a matrix are called eigenvalues [29]. The eigenvalues of *Q*(*G*) are called *Q*-eigenvalues of *G* and the eigenvalues of *L*(*G*) are called *L*-eigenvalues of *G*. Since *Q*(*G*) and *L*(*G*) are real and symmetric matrices then their eigenvalues are real and nonnegative [10,30] and can be arranged as $q_p \ge q_{p-1} \ge \cdots \ge q_2 \ge q_1$ and $\lambda_p \ge \lambda_{p-1} \ge \cdots \ge \lambda_2 \ge \lambda_1$, respectively. The largest eigenvalue q_p of *Q*(*G*) is called *Q*-spectral radius of *G* [31] and the largest eigenvalue λ_p of *L*(*G*) is called *L*-spectral radius of *G* [5].

Let $q_t > q_{t-1} > \cdots > q_2 > q_1$ are *t* distinct *Q*-eigenvalues with the corresponding multiplicities $m_t, m_{t-1}, \dots, m_2, m_1$. Then, *Q*-spectrum of *G* is defined by

$$spec_Q(G) = \begin{bmatrix} q_t & q_{t-1} & \cdots & q_2 & q_1 \\ m_t & m_{t-1} & \cdots & m_2 & m_1 \end{bmatrix}.$$

If every *Q*-eigenvalues of *G* are integer then *G* is called *Q*-integral [28]. *L*-spectrum of *G* is defined in similar manner, and if every *L*-eigenvalues of *G* are integer then *G* is called *L*-integral [32].

The following are the results of previous research that will be used in this paper.

Result 1 [2]. *Q*-polynomial of complete multipartite graph $K_{n_1,n_2,\dots,n_k}$ of order *n* is

$$p_Q(q) = (-1)^n \left(\sum_{i=1}^{k} \frac{n_i}{n-2n_i-q} + 1\right) \prod_{i=1}^{k} (n - 2n_i - q)(n - n_i - q)^{(n_i-1)}.$$

Q-polynomial in Result 1 can be expressed as

$$p_Q(q) = \prod_{i=1}^{k} (q - n + n_i)^{(n_i-1)} \prod_{i=1}^{k} (q - n + 2n_i) \left(1 - \sum_{i=1}^{k} \frac{n_i}{q-n+2n_i}\right) \text{ [28,33]}$$

Result 2 [34]. *Q*-eigenvalues of K_n are $2(n-1)$ and $n-2$ with their multiplicities are 1 and $n-1$, respectively.

Result 3 [35]. *Q*-polynomial of bipartite graphs is equal to *L*-polynomial.

Result 4 [36]. *L*-eigenvalues of complete graph K_n are *n* and 0 with multiplicities $n - 1$ and 1, respectively.

Result 5 [37]. Let $C = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ is a block symmetric matrix of order 2. The eigenvalues of *C* are those of $A + B$ together with those of $A - B$.

3. Main Results

Based on Anderson et al. [20] and Kakeri and Erfanian [21], if *G* is a group and *H* is its normal subgroup then the subgroup graph $\Gamma_H(G)$ of *G* and its complement $\overline{\Gamma_H(G)}$ are undirected simple graphs. So, we focus on the normal subgroup of dihedral group along this paper.

The dihedral group D_{2n} ($n \geq 3$) has $2n$ elements that consist of n rotations $1, r, r^2, r^3, \ldots, r^{n-1}$ and n reflection $s, sr, sr^2, sr^3, \ldots, sr^{n-1}$. The order of r is n ($|r| = n$) and the order of sr^i is 2 ($|sr^i| = 2$) for $i = 1, 2, \ldots, n$. By using its generator, we can write $D_{2n} = \langle r, s\rangle = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$. It is well known that $sr \neq rs$ and $sr^i = r^{-i}s$. Hence, composition of two reflections is a rotation. For odd n, all normal subgroups of D_{2n} are $\langle 1\rangle$, $\langle r^d\rangle$ for all d dividing n and D_{2n} itself. For even n, all normal subgroups of D_{2n} are $\langle 1\rangle$, $\langle r^d\rangle$ for all d dividing n, $\langle r^2, s\rangle$, $\langle r^2, rs\rangle$ and D_{2n} itself.

By definition of subgroup graph, we have $\Gamma_{D_{2n}}(D_{2n})$ is complete graph of order $2n$, for $n \geq 3$. So, $\overline{\Gamma_{D_{2n}}(D_{2n})}$ is empty graph of order $2n$. The fact leads us to our first result.

Theorem 1.

(a) Q-spectral radius of $\Gamma_{D_{2n}}(D_{2n})$ is $4n - 2$ and L-spectral radius of $\Gamma_{D_{2n}}(D_{2n})$ is $2n$.

(b) $Spec_Q\left(\Gamma_{D_{2n}}(D_{2n})\right) = \begin{bmatrix} 4n-2 & 2n-2 \\ 1 & 2n-1 \end{bmatrix}$ and $spec_L\left(\Gamma_{D_{2n}}(D_{2n})\right) = \begin{bmatrix} 2n & 0 \\ 2n-1 & 1 \end{bmatrix}$.

(c) Q-spectral and L-spectral radius of $\overline{\Gamma_{D_{2n}}(D_{2n})}$ are 0.

Proof. It is straightforward from Result 2 and then Result 4. ♦

The normal subgroup $\langle 1\rangle$ has only identity element of D_{2n}. Therefore, $xy \in \langle 1\rangle$ if and only if $y = x^{-1}$ in D_{2n}. We know that $(r^i)^{-1} = r^{n-i}$ and $(sr^i)^{-1} = sr^i$ for odd and even n, and in addition $(r^{n/2})^{-1} = r^{n/2}$ for even n. Because graph in this paper is simple graph, then sr^i and $r^{n/2}$ are not adjacent to themselves in $\Gamma_{\langle 1\rangle}(D_{2n})$. Hence, only r^i and r^{n-i} are adjacent in $\Gamma_{\langle 1\rangle}(D_{2n})$ for $i \neq n/2$. Now, we have the following results on subgroup graph $\Gamma_{\langle 1\rangle}(D_{2n})$, for $n \geq 3$.

Theorem 2.

(a) Q-spectral and L-spectral radius of $\Gamma_{\langle 1\rangle}(D_{2n})$ are 2.

(b) $Spec_Q\left(\Gamma_{\langle 1\rangle}(D_{2n})\right) = Spec_L\left(\Gamma_{\langle 1\rangle}(D_{2n})\right) = \begin{bmatrix} 2 & 0 \\ (n-1)/2 & (3n+1)/2 \end{bmatrix}$ for odd n and

$Spec_Q\left(\Gamma_{\langle 1\rangle}(D_{2n})\right) = Spec_L\left(\Gamma_{\langle 1\rangle}(D_{2n})\right) = \begin{bmatrix} 2 & 0 \\ (n-2)/2 & (3n+2)/2 \end{bmatrix}$ for even n.

(c) L-spectral radius of $\overline{\Gamma_{\langle 1\rangle}(D_{2n})}$ are $2n$.

(d) $Spec_L\left(\overline{\Gamma_{\langle 1\rangle}(D_{2n})}\right) = \begin{bmatrix} 2n & 2(n-1) & 0 \\ (3n-1)/2 & (n-1)/2 & 1 \end{bmatrix}$ for odd n

and $Spec_L\left(\overline{\Gamma_{\langle 1\rangle}(D_{2n})}\right) = \begin{bmatrix} 2n & 2(n-1) & 0 \\ 3n/2 & (n-2)/2 & 1 \end{bmatrix}$ for even n.

The next results are for subgroup graph $\Gamma_{\langle r\rangle}(D_{2n})$ of dihedral group D_{2n}, where $n \geq 3$.

Theorem 3.

(a) Q-spectral radius of $\Gamma_{\langle r\rangle}(D_{2n})$ is $2(n - 1)$ and L-spectral radius of $\Gamma_{\langle r\rangle}(D_{2n})$ is n.

(b) $Spec_Q\left(\Gamma_{\langle r\rangle}(D_{2n})\right) = \begin{bmatrix} 2(n-1) & n-2 \\ 2 & 2(n-1) \end{bmatrix}$ and $spec_L\left(\Gamma_{\langle r\rangle}(D_{2n})\right) = \begin{bmatrix} n & 0 \\ 2(n-1) & 2 \end{bmatrix}$.

(c) Q-spectral and L-spectral radius of $\overline{\Gamma_{\langle r\rangle}(D_{2n})}$ are $2n$.

(d) $spec_Q\left(\overline{\Gamma_{\langle r\rangle}(D_{2n})}\right) = spec_L\left(\overline{\Gamma_{\langle r\rangle}(D_{2n})}\right) = \begin{bmatrix} 2n & n & 0 \\ 1 & 2(n-1) & 1 \end{bmatrix}$.

Proof.

(a) Subgroup graph $\Gamma_{\langle r\rangle}(D_{2n})$ is disconnected with two components and each component is a complete graph of order n. Hence, $\deg(v) = n - 1$, for all $v \in \Gamma_{\langle r\rangle}(D_{2n})$. Therefore, $Q\left(\Gamma_{\langle r\rangle}(D_{2n})\right) = \begin{bmatrix} A & O \\ O & A \end{bmatrix}$, where $A = [a_{ij}]$ is matrix of order n with $a_{ij} = n - 1$ for $i = j$ and $a_{ij} = 1$ otherwise and O is zero matrix of order n. Using Result 5 on $\begin{bmatrix} A & O \\ O & A \end{bmatrix}$ and then Result 2 on $A + O$ and $O - A$, we have the Q-eigenvalues are $2(n - 1)$ and $n - 2$ with their multiplicities are 2 and $2(n - 1)$, respectively. In other hand, $L\left(\Gamma_{\langle r\rangle}(D_{2n})\right) = \begin{bmatrix} B & O \\ O & B \end{bmatrix}$, where $B = [b_{ij}]$ is matrix of order n with $b_{ij} = n - 1$ for $i = j$ and $b_{ij} = -1$ otherwise and O is zero matrix of order n. With similar fashion, we have the L-eigenvalues are n and 0 with their multiplicities are $2(n - 1)$ and 2, respectively. It completes the proof.

(b) From the proof of (a), Q-polynomial and L-polynomial of $\Gamma_{\langle r\rangle}(D_{2n})$ are $p_Q(q) = (q - (2n-2))^2(q-(n-2))^{2n-2}$ and $p_L(\lambda) = (\lambda - n)^2\lambda^{2n-2}$. So, we have the desired proof.

(c) Since $\Gamma_{\langle r\rangle}(D_{2n}) = K_n \cup K_n$, then $\overline{\Gamma_{\langle r\rangle}(D_{2n})} = K_{n,n}$. By Result 1, $p_Q(q) = (q-2n)(q-n)^{2n-2}q$. Because $\overline{\Gamma_{\langle r\rangle}(D_{2n})}$ is complete bipartite graph, by Result 3 we have $p_L(\lambda) = (\lambda - 2n)(\lambda - n)^{2n-2}\lambda$. So, $2n$ is the largest eigenvalue and the poof is complete.

(d) It is clear from (c). ♦

Normal subgroup $\langle r^2\rangle$ of dihedral group D_{2n}, where $n \geq 4$ and n is even, is $\langle r^2\rangle = \{1, r^2, r^4, \dots, r^{n-2}\}$ and $r^i r^j, sr^i sr^j \in \langle r^2\rangle$ if and only if i and j both even or both odd, for $1 \leq i, j \leq n - 2$. Therefore, subgroup graph $\Gamma_{\langle r^2\rangle}(D_{2n})$ has four components and each component is complete graph $K_{n/2}$. So, we have the following results.

Theorem 4.

(a) Q-spectral radius of $\Gamma_{\langle r^2\rangle}(D_{2n})$ is $n - 2$ and L-spectral radius of $\Gamma_{\langle r^2\rangle}(D_{2n})$ is $n/2$, for even n and $n \geq 4$.

(b) $spec_Q(\Gamma_{\langle r^2\rangle}(D_{2n})) = \begin{bmatrix} n-2 & \frac{n-4}{2} \\ 4 & 2(n-2) \end{bmatrix}$ and $spec_L(\Gamma_{\langle r^2\rangle}(D_{2n})) = \begin{bmatrix} \frac{n}{2} & 0 \\ 2(n-2) & 4 \end{bmatrix}$.

(c) Q-spectral radius of $\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}$ is $3n$ and L-spectral radius of $\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}$ is $2n$, where n is even and $n \geq 4$.

(d) $spec_Q(\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}) = \begin{bmatrix} 3n & \frac{3n}{2} & n \\ 1 & 2(n-2) & 3 \end{bmatrix}$ and $spec_L(\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}) = \begin{bmatrix} 2n & \frac{3n}{2} & 0 \\ 3 & 2(n-2) & 1 \end{bmatrix}$.

Proof.

(a) The Q-polynomial of $\Gamma_{\langle r^2\rangle}(D_{2n})$ is

$$p_Q(q) = (-1)^{\frac{n}{2}}(q-(n-2))^4\left(q - \left(\frac{n-4}{2}\right)\right)^{2(n-2)}$$

and L--polynomial of $\Gamma_{\langle r^2\rangle}(D_{2n})$ is

$$p_L(\lambda) = (-1)^{\frac{n}{2}}\left(\lambda - \frac{n}{2}\right)^{2(n-2)}\lambda^4.$$

(b) It is clear from (a).

(c) Complement of subgroup graph $\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}$ is complete multipartite $K_{n/2,n/2,n/2,n/2}$ of order $2n$. By using Result 1, then Q-polynomial of $\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}$ is

$$p_Q(\lambda) = (\lambda - 3n)\left(\lambda - \frac{3n}{2}\right)^{2(n-2)}(\lambda - n)^3.$$

And we have L-polynomial of $\overline{\Gamma_{\langle r^2\rangle}(D_{2n})}$ is

$$p(\lambda) = (\lambda - 2n)^3\left(\lambda - \frac{3n}{2}\right)^{2(n-2)}\lambda.$$

(d) It is clear from (c). ♦

The normal subgroup $\langle r^2, s\rangle$ of D_{2n} for even n and $n \geq 4$ is $\langle r^2, s\rangle = \{1, r^2, r^4, \dots, r^{n-2}, s, sr^2, sr^4, \dots, sr^{n-2}\}$ and $(s^k r^i)(s^k r^j) \in \langle r^2, s\rangle$ if and only if i and j both even or both odd, for $1 \leq i, j \leq n - 2$ and $k = 0, 1$. Therefore, subgroup graph $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ has two components and each component is complete graph K_n of order n. Then, subgroup graph $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ is isomorphic to $\Gamma_{\langle r\rangle}(D_{2n})$. The following results are obvious.

Theorem 5.

(a) Q-spectral radius of $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ is $2(n - 1)$ and L-spectral radius of $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ is n.

(b) $spec_Q\left(\Gamma_{\langle r^2,s\rangle}(D_{2n})\right) = \begin{bmatrix} 2(n-1) & n-2 \\ 2 & 2(n-1) \end{bmatrix}$ and $spec_L\left(\Gamma_{\langle r^2,s\rangle}(D_{2n})\right) = \begin{bmatrix} n & 0 \\ 2(n-1) & 2 \end{bmatrix}$.

(c) Q-spectral and L-spectral radius of $\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}$ are $2n$.

(d) $spec_Q\left(\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}\right) = spec_L\left(\overline{\Gamma_{\langle r^2,s\rangle}(D_{2n})}\right) = \begin{bmatrix} 2n & n & 0 \\ 1 & 2(n-1) & 1 \end{bmatrix}$.

For even n and $n \geq 4$, we also can observe that subgroup graph $\Gamma_{\langle r^2,rs\rangle}(D_{2n})$ is isomorphic to $\Gamma_{\langle r^2,s\rangle}(D_{2n})$ and the following result is obvius.

Theorem 6.

(a) Q-spectral radius of $\Gamma_{\langle r^2,rs\rangle}(D_{2n})$ is $2(n-1)$ and L-spectral radius of $\Gamma_{\langle r^2,rs\rangle}(D_{2n})$ is n.

(b) $spec_Q\left(\Gamma_{\langle r^2,rs\rangle}(D_{2n})\right) = \begin{bmatrix} 2(n-1) & n-2 \\ 2 & 2(n-1) \end{bmatrix}$ and $spec_L\left(\Gamma_{\langle r^2,rs\rangle}(D_{2n})\right) = \begin{bmatrix} n & 0 \\ 2(n-1) & 2 \end{bmatrix}$.

(c) Q-spectral and L-spectral radius of $\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}$ are $2n$.

(d) $spec_Q\left(\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}\right) = spec_L\left(\overline{\Gamma_{\langle r^2,rs\rangle}(D_{2n})}\right) = \begin{bmatrix} 2n & n & 0 \\ 1 & 2(n-1) & 1 \end{bmatrix}$.

4. Conclusion

We have computed Q-spectral and L-spectral radius of subgroup graphs of dihedral group D_{2n} and their complement. According to our results, we can conclude that $\Gamma_{D_{2n}}(D_{2n})$ and $\Gamma_{\langle r\rangle}(D_{2n})$ and their complement are Q-integral and L-integral, for all n and $n \geq 3$. For even n and $n \geq 4$, the subgroup graphs $\Gamma_{\langle r^2\rangle}(D_{2n})$, $\Gamma_{\langle r^2,s\rangle}(D_{2n})$, $\Gamma_{\langle r^2,rs\rangle}(D_{2n})$ and their complement also Q-integral and L-integral.

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