

# On the Laplacian and Signless Laplacian Spectra of Complete Multipartite Graphs

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## ABSTRACT

Let  $G$  be a finite simple graph with vertex set  $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$  and edge set  $E(G)$ . The adjacency matrix of  $G$  is an  $(n \times n)$ -matrix  $A(G) = [a_{ij}]$  where  $a_{ij} = 1$  if  $v_i v_j \in E(G)$  and  $a_{ij} = 0$  elsewhere, and the degree matrix of  $G$  is a diagonal  $(n \times n)$ -matrix  $D(G) = [d_{ij}]$  where  $d_{ii} = \deg_G(v_i)$  and  $d_{ij} = 0$  for  $i \neq j$ . The Laplacian matrix of  $G$  is  $L(G) = D(G) - A(G)$  and the signless Laplacian matrix of  $G$  is  $Q(G) = D(G) + A(G)$ . The study of spectrum of Laplacian and signless Laplacian matrix of graph are interesting topic till today. In this paper, we determine the Laplacian and signless Laplacian spectra of complete multipartite graphs.

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## 1. INTRODUCTION

Let  $G$  be a finite graph without loop and multiple edges of order  $n$  ( $n \geq 1$ ). Let its vertex set is  $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$  and edge set is  $E(G)$ . The adjacency matrix of  $G$  is an  $(n \times n)$ -matrix  $A(G) = [a_{ij}]$  where  $a_{ij} = 1$  if  $v_i v_j \in E(G)$  and  $a_{ij} = 0$  elsewhere [1]. The degree matrix of  $G$  is a diagonal  $(n \times n)$ -matrix  $D(G) = [d_{ij}]$  where  $d_{ii} = \deg_G(v_i)$  and  $d_{ij} = 0$  for  $i \neq j$  [2]. The Laplacian matrix  $L(G)$  of  $G$  is defined by  $L(G) = D(G) - A(G)$  and the signless Laplacian matrix  $Q(G)$  of  $G$  is defined by  $Q(G) = D(G) + A(G)$  [3].

Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  where  $\lambda_1 > \lambda_2 > \dots > \lambda_k$  are distinct eigenvalues from a matrix of graph  $G$  and let  $m(\lambda_1), m(\lambda_2), \dots, m(\lambda_k)$  are algebraic multiplicities of  $\lambda_i, i = 1, 2, \dots, k$ . The spectrum of graph  $G$  is  $(2 \times k)$ -matrix that contains  $\lambda_1, \lambda_2, \dots, \lambda_k$  for the first row and  $m(\lambda_1), m(\lambda_2), \dots, m(\lambda_k)$  for the second row and denoted by  $Spec(G)$  [3]. The Laplacian spectrum of  $G$  is the spectrum of Laplacian matrix of  $G$  and denoted by  $Spec_L(G)$ . The signless Laplacian spectrum of  $G$  is the spectrum of signless Laplacian matrix of graph  $G$  and denoted by  $Spec_Q(G)$ .

Some results on Laplacian and signless Laplacian spectrum of graphs have been reported, for examples the properties of Laplacian spectra of a graph and Laplacian integral graphs [4, 5], the Laplacian spectrum of complex networks [6], the Laplacian spectrum of graph obtained from  $K_r$  by adhering the root of isomorphic trees  $T$  to every vertex of  $K_r$  [7], the Laplacian spectrum of weakly quasi-threshold graphs [8], the signless Laplacian spectrum of coronae [9], the Laplacian spectrum of non-commuting of dihedral group [10], the Laplacian spectrum of some graphs [11], the Laplacian spectra of graphs and complex networks [12], the (signless) Laplacian spectral of the line graphs of lollipop graphs [13], the Laplacian spectrum of neural networks [14], and the Laplacian spectra of product graphs [15]. Several studies relating to Laplacian and

signless Laplacian matrix have been reported such as [16-23]. In this paper, we determine the Laplacian and signless Laplacian spectra of complete multipartite graphs.

Graph  $G$  is said to be complete  $n$ -partite if vertex set of  $G$  can be partitioned into  $n$  partite sets  $V_1, V_2, \dots, V_n$  such that if  $u \in V_i$  and  $v \in V_j$  then  $uv \in E(G)$  for  $i \neq j$ . A complete  $n$ -partite graph for some integer  $n \geq 2$  is called a complete multipartite graph[1]. If  $|V_i| = \alpha_i$  for all  $i$  ( $i = 1, 2, \dots, n$ ) then graph  $G$  be denoted by  $K(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$ . If  $\alpha_i = t$  for all  $i$  then the complete  $n$ -partite graph  $K(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$  be denoted by  $K_n(t)$ . So, the complete  $n$ -partite graph  $K_n(1)$  is the complete graph  $K_n$ . If  $\alpha_i = t$  for  $i = 1, 2, \dots, n-1$  and  $\alpha_n = s$  then we denoted  $K_{n-1}(t)(s)$  instead of  $K(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$ .

## 2. RESEARCH METHOD

We use the library research method to determine Laplacian and signless Laplacian spectra of complete multipartite graphs. We begin by observing specific cases of Laplacian and signless Laplacian spectra of complete multipartite graphs. The next step is analyzing the characteristic polynomial and spectrum pattern of these specific cases to obtain general pattern. The last is present the result as a theorem with its proof.

## 3. RESULTS

Now, we present our results on the Laplacian spectra for several complete multipartite graphs.

**Theorem 1:** The Laplacian spectrum of complete bipartite graph  $K_2(m)$  for  $m \geq 2$  and  $m \in \mathbb{N}$  is

$$\text{spec}_L(K_2(m)) = \begin{bmatrix} 2m & m & 0 \\ 1 & 2(m-1) & 1 \end{bmatrix}$$

**Proof:** By eliminating matrix  $(L(K_2(m)) - \lambda I)$  to an upper triangular matrix, we have the characteristic polynomial of Laplacian matrix  $L(K_2(m))$  is  $p(\lambda) = \lambda(\lambda - m)^{2(m-1)}(\lambda - 2m)$ . Therefore, eigenvalues of  $L(K_2(m))$  are  $2m, m$  and  $0$ , and their algebraic multiplicities are  $1, 2(m-1)$  and  $1$ , respectively. So, we have the complete proof.  $\square$

**Theorem 2:** The Laplacian spectrum of complete tripartite graph  $K_3(m)$  for  $m \geq 2$  and  $m \in \mathbb{N}$  is

$$\text{spec}_L(K_3(m)) = \begin{bmatrix} 3m & 2m & 0 \\ 2 & 3(m-1) & 1 \end{bmatrix}$$

**Proof:** The characteristic polynomial of Laplacian matrix  $L(K_3(m))$  is  $p(\lambda) = \lambda(\lambda - 2m)^{3(m-1)}(\lambda - 3m)^2$ .  $\square$

For more general case, we have the following theorem.

**Theorem 3:** The Laplacian spectrum of complete multipartite graph  $K_n(m)$  for  $m \geq 2$  and  $m \in \mathbb{N}$  is

$$\text{spec}_L(K_n(m)) = \begin{bmatrix} nm & (n-1)m & 0 \\ n-1 & n(m-1) & 1 \end{bmatrix}$$

**Proof:** It is enough to show the characteristic polynomial of  $L(K_n(m))$ . The degree of all vertices in complete multipartite  $K_n(m)$  is  $(n-1)m$ . So, the degree matrix of  $K_n(m)$  is  $D(K_n(m)) = (n-1)mI$ , where  $I$  is identity matrix of order  $nm$ . According to the definition of Laplacian matrix, we have  $L(K_n(m)) = D(K_n(m)) - A(K_n(m))$ . Applying Gauss-Jordan elimination method for  $L(K_n(m)) - \lambda I$ , we will have a diagonal matrix with no zero in its main diagonal. By multiplying all elements in the main diagonal, we have the characteristic polynomial of  $L(K_n(m))$  is  $p(\lambda) = \lambda(\lambda - (n-1)m)^{n(m-1)}(\lambda - nm)^{n-1}$ . It completes the proof.  $\square$

**Theorem 4:** The Laplacian spectrum of complete bipartite graph  $K(m, m+1)$  for  $m \geq 2$  and  $m \in \mathbb{N}$  is

$$\text{spec}_L(K(m, m+1)) = \begin{bmatrix} 2m+1 & m+1 & m & 0 \\ 1 & m-1 & m & 1 \end{bmatrix}$$

**Proof:** We can compute that characteristic polynomial of Laplacian matrix  $L(K(m, m+1))$  is

$$p(\lambda) = \lambda(\lambda - m)^m(\lambda - (m+1))^{m-1}(\lambda - (2m+1)). \square$$

**Theorem 5:** The Laplacian spectrum of complete multipartite graph  $K_n(1)(2)$  for  $n \in \mathbb{N}$  is

$$\text{spec}_L(K_n(1)(2)) = \begin{bmatrix} n+2 & n & 0 \\ n & 1 & 1 \end{bmatrix}$$

**Proof:** It is easy to check that characteristic polynomial of Laplacian matrix  $L(K_n(1)(2))$  is

$$p(\lambda) = \lambda(\lambda - n)(\lambda - (n+2))^n. \square$$

The following two theorems are special case for the next theorem.

**Theorem 6:** The Laplacian spectrum of complete multipartite graph  $K_n(2)(3)$  for  $n \in \mathbb{N}$  is

$$\text{spec}_L(K_n(2)(3)) = \begin{bmatrix} 2n+3 & 2n+1 & 2n & 0 \\ n & n & 2 & 1 \end{bmatrix}$$

**Proof:** The characteristic polynomial of Laplacian matrix  $L(K_n(2)(3))$  is

$$p(\lambda) = \lambda(\lambda - 2n)^2(\lambda - (2n + 1))^n(\lambda - (2n + 3))^n. \square$$

**Theorem 7:** The Laplacian spectrum of complete multipartite graph  $K_n(3)(4)$  for  $n \in \mathbb{N}$  is

$$\text{spec}_L(K_n(3)(4)) = \begin{bmatrix} 3n + 4 & 3n + 1 & 3n & 0 \\ n & 2n & 3 & 1 \end{bmatrix}$$

**Proof:** The characteristic polynomial of Laplacian matrix  $L(K_n(3)(4))$  is

$$p(\lambda) = \lambda(\lambda - 3n)^3(\lambda - (3n + 1))^{2n}(\lambda - (3n + 4))^n. \square$$

From Theorem 6 and Theorem 7, if we set  $n = 1$  then we have  $K_1(2)(3) = K(2)(3)$  and  $K_1(3)(4) = K(3)(4)$ . We see that the Laplacian spectra for these two graphs can be computed using Theorem 4. For more general result, we have the following theorem.

**Theorem 8:** The Laplacian spectrum of complete multipartite graph  $K_n(m)(m + 1)$  for  $m, n \in \mathbb{N}$  and  $m \geq 2$  is

$$\text{spec}_L(K_n(m)(m + 1)) = \begin{bmatrix} nm + m + 1 & nm + 1 & nm & 0 \\ n & n(m - 1) & m & 1 \end{bmatrix}$$

**Proof:** We can observe that characteristic polynomial of Laplacian matrix  $L(K_n(1)(2))$  is

$$p(\lambda) = \lambda(\lambda - mn)^m(\lambda - (mn + 1))^{(m-1)n}(\lambda - (mn + m + 1))^n$$

The following theorem can be used to determine the Laplacian spectrum of complete multipartite graph  $K(1, 2, 3, \dots, n)$  for  $n \in \mathbb{N}$  and  $n \geq 2$ .

**Theorem 9:** The characteristic polynomial of  $L(K(1, 2, 3, \dots, n))$  for  $n \in \mathbb{N}$  and  $n \geq 2$  is

$$p(\lambda) = \lambda \left( \lambda - \left( \frac{n(n+1)}{2} - n \right) \right)^{n-1} \left( \lambda - \left( \frac{n(n+1)}{2} - (n-1) \right) \right)^{n-2} \cdots \left( \lambda - \left( \frac{n(n+1)}{2} \right) \right)^{n-1}$$

**Proof:** Using Gaussian elimination method to Laplacian matrix  $L(K(1, 2, 3, \dots, n))$  will leads to the desired result.  $\square$

**Theorem 10:** The signless Laplacian spectrum of complete bipartite graph  $K_2(m)$  for  $m \in \mathbb{N}$  and  $m \geq 2$  is

$$\text{spec}_L(K_2(m)) = \begin{bmatrix} 2m & m & 0 \\ 1 & 2(m-1) & 1 \end{bmatrix}$$

**Proof:** By the fact that Laplacian spectrum and signless Laplacian spectrum of bipartite graphs are always equal, then the proof follows from Theorem 1.  $\square$

**Theorem 11:** The signless Laplacian spectrum of complete tripartite graph  $K_3(m)$  for  $m \in \mathbb{N}$  and  $m \geq 2$  is

$$\text{spec}_L(K_3(m)) = \begin{bmatrix} 4m & 2m & m \\ 1 & 3(m-1) & 2 \end{bmatrix}$$

**Proof:** We can observe that characteristic polynomial of signless Laplacian matrix  $Q(K_3(m))$  is

$$p(\lambda) = (\lambda - 4m)(\lambda - 2m)^{3(m-1)}(\lambda - m)^2. \square$$

**Theorem 12:** The signless Laplacian spectrum of complete bipartite graph  $K(m, m+1)$  for  $m \geq 2$  and  $m \in \mathbb{N}$  is

$$\text{spec}_L(K(m, m+1)) = \begin{bmatrix} 2m + 1 & m + 1 & m & 0 \\ 1 & m - 1 & m & 1 \end{bmatrix}$$

**Proof:** By the fact that Laplacian spectrum and signless Laplacian spectrum of bipartite graphs are always equal, then the proof follows from Theorem 4.  $\square$

The following theorem is presented without proof. It is easy to be observed from the characteristic polynomial of the Laplacian matrix of  $K_n(1)(2)$  for  $n \in \mathbb{N}$ .

**Theorem 13:** The Laplacian spectrum of complete multipartite graph  $K_n(1)(2)$  for  $n \in \mathbb{N}$  is

$$\text{spec}_L(K_n(m)) = \begin{bmatrix} \frac{3n}{2} + \sqrt{\frac{13n-6}{4}} & n & \frac{3n}{2} - \sqrt{\frac{13n-6}{4}} \\ 1 & n & 1 \end{bmatrix}$$

#### 4. CONCLUSION

We have determined the Laplacian and signless Laplacian spectra of several complete multipartite graphs. Because there are many kinds of complete multipartite graphs according to the cardinality of each partition sets, so the further studies can be done to determine the Laplacian and signless Laplacian spectra of

another complete multipartite graphs. Further studies also can be done to determine another spectrum of these graphs.

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## REFERENCES

- [1] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs and Digraphs*, 6th ed. Florida: CRC Press, 2016.
- [2] Abdussakir, N. N. Azizah, and F. F. Novandika, *Teori graf: Topik dasar untuk tugas akhir/skripsi*. Malang: UIN Malang Press, 2009.
- [3] Abdussakir, R. R. Elvierayani, and M. Nafisah, "On the spectra of commuting and non commuting graph on dihedral group," *Cauchy-Jurnal Mat. Murni dan Apl.*, vol. 4, no. May, pp. 176–182, 2017.
- [4] R. Grone, R. Merris, and V. S. Sunder, "The Laplacian spectrum of a graph," *SIAM J. Matrix Anal. Appl.*, vol. 11, no. 2, pp. 218–238, 1990.
- [5] R. Grone and R. Merris, "The Laplacian spectrum of a graph II," *SIAM J. Discret. Math.*, vol. 7, no. 2, pp. 221–229, 1994.
- [6] A. Jamakovic and P. Van Mieghem, "The Laplacian spectrum of complex networks," in *Proceedings of the European Conference on Complex Systems, Oxford, September 25-29, 2006*, 2006.
- [7] S. Yin, "Investigation on spectrum of the adjacency matrix and Laplacian matrix of graph  $G_1$ ," *WSEAS Trans. Syst.*, vol. 7, no. 4, pp. 362–372, 2008.
- [8] R. B. Bapat, A. K. Lal, and S. Pati, "Laplacian spectrum of weakly quasi-threshold graphs," *Graphs Comb.*, vol. 24, no. 4, pp. 273–290, 2008.
- [9] S. Y. Cui and G. X. Tian, "The spectrum and the signless Laplacian spectrum of coronae," *Linear Algebra Appl.*, vol. 437, no. 7, pp. 1692–1703, 2012.
- [10] R. R. Elvierayani and Abdussakir, "Spectrum of the Laplacian matrix of non-commuting graph of dihedral group  $D_{2n}$ ," in *Proceeding of International Conference The 4th Green Technology*, 2013, pp. 321–323.
- [11] D. M. Cardoso, E. A. Martins, M. Robbiano, and V. Trevisan, "Computing the Laplacian spectra of some graphs," *Discret. Appl. Math.*, vol. 160, no. 18, pp. 2645–2654, 2012.
- [12] Y. Chen, R. Pan, and X. Zhang, "The Laplacian spectra of graphs and complex networks," *J. Univ. Sci. Technol. China*, vol. 40, no. 12, pp. 1236–1244, 2010.
- [13] G. Guo and G. Wang, "On the (signless) Laplacian spectral characterization of the line graphs of lollipop graphs," *Linear Algebra Appl.*, vol. 438, no. 12, pp. 4595–4605, 2013.
- [14] S. C. De Lange, M. A. De Reus, and M. P. van den Heuvel, "The Laplacian spectrum of neural networks," *Front. Comput. Neurosci.*, vol. 7, no. 1, pp. 1–12, 2014.
- [15] S. Barik, R. B. Bapat, and S. Pati, "On the Laplacian spectra of product graphs," *Appl. Anal. Discret. Math.*, vol. 91, pp. 39–58, 2015.
- [16] W. N. Anderson and T. D. Morley, "Eigenvalues of the Laplacian of a graph," *Linear Multilinear Algebr.*, vol. 18, no. 2, pp. 141–145, 1985.
- [17] B. Mohar, "Laplace eigenvalues of graphs—a survey," *Discrete Math.*, vol. 109, no. 1–3, pp. 171–183, 1992.
- [18] R. Merris, "Laplacian matrices of graphs: a survey," *Linear Algebra Appl.*, vol. 197–198, no. C, pp. 143–176, 1994.
- [19] R. Merris, "Laplacian graph eigenvectors," *Linear Algebra Appl.*, vol. 278, no. 1–3, pp. 221–236, 1998.
- [20] X. D. Zhang, "Bipartite graphs with small third Laplacian eigenvalue," *Discrete Math.*, vol. 278, no. 1, pp. 241–253, 2004.
- [21] B. Jiao, Y. Chun, and T. Y. Qiang, "Signless laplacians of finite graphs," *Int. Conf. Apperceiving Comput. Intell. Anal. ICACIA 2010 - Proceeding*, vol. 423, pp. 440–443, 2010.
- [22] H. Deng and H. Huang, "On the main signless Laplacian eigenvalues of a graph," *Electron. J. Linear Algebr.*, vol. 26, no. 10, pp. 381–393, 2013.
- [23] J. M. Guo, J. Y. Ren, and J. S. Shi, "Maximizing the least signless Laplacian eigenvalue of unicyclic graphs," *Linear Algebra Appl.*, vol. 519, pp. 136–145, 2017.