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Detour spectrum and detour energy of conjugate graph complement of dihedral group

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Abstract. Study of graph from a group has become an interesting topic until now. One of the topics is spectra of a graph from finite group. Spectrum of a finite graph is defined as collection of all distinct eigenvalues and their algebraic multiplicity of its matrix. The most related topic in the study of spectrum of finite graph is energy. Energy of a finite graph is defined as sum of absolute value of all its eigenvalues. In this paper, we study the spectrum and energy of detour matrix of conjugate graph complement of dihedral group. The main result is presented as theorems with complete proof.

1. Introduction

Several graphs from some group have been studied by researchers, such as Cayley graph [1,2], Schreier coset graph [3], identity graph [4], commuting [5,6] and non-commuting graph[7-9], subgroup graph [10,11], power graph [12], inverse graph [13,14] and conjugate graph [15] of a group. For non-abelian finite group G, two elements x and y in G are said to be conjugate to each other if there exists an element z in G that satisfies $x = zyz^{-1}$. Let $[e], [x_1], [x_2], \dots, [x_p]$ are all conjugacy classes of G. The conjugate graph of group G contains all elements of G as its vertex set and two distinct vertices will be adjacent if they are representatives of the same conjugacy class [15]. So, the vertex y will be adjacent to x_i if $y \in [x_i]$. In this paper, conjugate graph of a group G will be denoted by $C(G)$ and the complement of $C(G)$ will be denoted by $\overline{C(G)}$. Two distinct vertices of $\overline{C(G)}$ are adjacent if and only if they are not adjacent in $C(G)$. The cardinality of the vertex set of $\overline{C(G)}$ and the edge set of $\overline{C(G)}$ will be denoted by $p(\overline{C(G)})$ and $q(\overline{C(G)})$, respectively. For a graph G, $p(G)$ is called the order of G and $q(G)$ is called the size of G [16].

Detour matrix of graph G of order p that denoted by $DD(G)$ is a $(p \times p)$-matrix $DD(G) = (D_{ij})$ where D_{ij} is the length of the longest path $v_i - v_j$ in G [17]. Since $DD(G)$ is a symmetric matrix, all of its eigenvalues λ_i (i = 1, 2, ..., p) are real and can be labeled as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_p$. Let $\lambda_{i_1} > \lambda_{i_2} > \lambda_{i_3} > \cdots > \lambda_{i_n}$ are the distinct eigenvalues of $DD(G)$, then the spectrum of $DD(G)$ can be written as

$$spec_{DD}(G) = \begin{pmatrix} \lambda_{i_1} & \lambda_{i_2} & \dots & \lambda_{i_n} \\ m(\lambda_{i_1}) & m(\lambda_{i_2}) & \dots & m(\lambda_{i_n}) \end{pmatrix},$$

where $m\left(\lambda_{i_j}\right)$ is the algebraic multiplicity of the eigenvalue λ_{i_j}. The energy of $DD(G)$ will be denoted by $E_{DD}(G)$ and be defined as $E_{DD}(G) = \sum_{i=1}^{p}|\lambda_i|$ [17,18].

The concept of spectrum was introduced by Bigg [19], the concept of detour matrix was introduced by Harary [20] and the concept of energy was introduced by Gutman [21]. The researches about detour spectrum of graphs have been conducted, such as detour spectrum of several graphs [17] and of commuting and non-commuting graphs of dihedral group [22]. Several kinds of energy of graph has been studied, for instance in [18,23-33]. Finally, the survey about kinds of energy of graph can be seen in Meenakshi and Lavanya [34]. Since the study of detour spectrum and energy of conjugate graph complement of dihedral group has not been done yet, we do this study.

2. Result

First, we show some properties of conjugate graph complements of dihedral group.

THEOREM 1: Let $C(D_{2n})$ be conjugate graph of dihedral group D_{2n} of order $2n$, where $n \geq 3$ and n is positive integer. The number of edge in complement of conjugate graph of D_{2n} is

(i) $q(\overline{C(D_{2n})}) = \frac{3n^2-2n+1}{2}$ for odd n.

(ii) $q(\overline{C(D_{2n})}) = \frac{3n^2-2n+2}{2}$ for even n.

PROOF: (i) For odd n, all of conjugacy classes of dihedral group D_{2n} are [1] = {1}, $[r] = \{r, r^{n-1}\}$, $[r^2] = \{r^2, r^{n-2}\}$, ..., $\left[r^{(n-1)/2}\right] = \left\{r^{(n-1)/2}, r^{(n-1)/2+1}\right\}$ and $[s] = \{ s, sr, sr^2, sr^3, \dots, sr^{n-1}\}$. According to definition of conjugate graph, $C(D_{2n})$ will contains a complete graph K_1, $(n - 1)/2$ complete graphs K_2 and a complete graph K_n. Thus, $q(C(D_{2n})) = (n^2-1)/2$. Then, we have

$$q(\overline{C(D_{2n})}) = \frac{2n(2n-1)}{2} - \frac{n^2-1}{2} = \frac{3n^2-2n+1}{2}.$$

(iii) For even n, all of conjugacy classes of dihedral group D_{2n} are [1] = {1}, $\left[r^{n/2}\right] = \left\{r^{n/2}\right\}$, $[r] = \{r, r^{n-1}\}$, $[r^2] = \{r^2, r^{n-2}\}$, ..., $[r^{n/2-1}] = \{r^{n/2-1}, r^{n/2+1}\}$, $[s] = \{s, sr^2, sr^4, \dots, sr^{n-2}\}$ and $[sr] = \{sr, sr^3, sr^5, \dots, sr^{n-1}\}$. According to definition of conjugate graph, $C(D_{2n})$ will contains two complete graphs K_1, $(n - 2)/2$ complete graphs K_2 and two complete graphs $K_{n/2}$. Thus, $q(C(D_{2n})) = (n^2-4)/4$. Then, we have

$$q(\overline{C(D_{2n})}) = \frac{2n(2n-1)}{2} - \frac{n^2-4}{4} = \frac{7n^2-4n+4}{4}. \blacksquare$$

THEOREM 2: Detour matrix of complement of conjugate graph of dihedral group D_{2n} for odd n is $(2n \times 2n)$-matrix

$$DD\left(\overline{C(D_{2n})}\right) = \begin{pmatrix} A & B \\ B & C \end{pmatrix},$$

where

$A = \left(a_{ij}\right)$ is an $(n \times n)$-matrix with $a_{ij} = 2n-2$ if $i \neq j$ and $a_{ij} = 0$ elsewhere,

$B = \left(b_{ij}\right)$ is an $(n \times n)$-matrix with $b_{ij} = 2n-1$ for all i and j, and

$C = \left(c_{ij}\right)$ is an $(n \times n)$-matrix with $c_{ij} = 2n-1$ if $i \neq j$ and $c_{ij} = 0$ elsewhere.

PROOF: According to the proof (i) of Theorem 1, the conjugacy classes of dihedral group are $[1] = \{1\}, [r] = \{r, r^{n-1}\}, [r^2] = \{r^2, r^{n-2}\}, \dots, [r^{(n-1)/2}] = \{r^{(n-1)/2}, r^{(n-1)/2+1}\}$ and $[s] = \{s, sr, sr^2, \dots, sr^{n-1}\}$. They will be a complete graph in $C(D2n)$, respectively. Therefore, in $\overline{C(D_{2n})}$, vertex 1 is adjacent to r^i and sr^i $(i = 1, 2, \dots, n-1)$, vertex r^i is adjacent to sr^j $(i = 1, 2, \dots, n-1$ and $j = 0, 1, 2, \dots, n-1)$ and vertex sr^i is not adjacent to sr^j $(i, j = 0, 1, 2, \dots, n-1)$. Then we can establish the longest path between two distinct vertices in $\overline{C(D_{2n})}$ as follow.

(i) For r^i and r^j, $1 \leq i < j \leq n$, we can construct a path P: $r^i, sr^i, r^{i+1}, sr^{i+1}, \dots, r^{j-1}, sr^{j-1}, r^{j+1}, sr^{j+1}, r^{j+2}, sr^{j+2}, \dots, r^n, sr^n, r^{i-1}, sr^{i-1}, r^{i-2}, sr^{i-2} \dots, r^2, sr^2, r, sr, r^j$. This path P contains all element of D_{2n} except sr^j. Hence, the length of P is $2n-2$.

(ii) For r^i and sr^j, $1 \le i, j \le n$, we can construct a path P: r^i, sr^i, r^{i+1}, sr^{i+1}, ..., r^{j-1}, sr^{j-1}, r^{j+1}, sr^{j+1}, r^{j+2}, sr^{j+2}, ..., r^n, sr^n, r^{i-1}, sr^{i-1}, r^{i-2}, sr^{i-2}..., r^2, sr^2, r, sr, r^j, sr^j. Thus, path P contains all element of D_{2n}. Hence, the length of P is $2n - 1$.

(iii) For sr^i and sr^j, $1 \le i < j \le n$, we can construct a path P: sr^i, r^i, r^{i+1}, sr^{i+1}, ..., r^{j-1}, sr^{j-1}, r^{j+1}, sr^{j+1}, r^{j+2}, sr^{j+2}, ..., r^n, sr^n, r^{i-1}, sr^{i-1}, r^{i-2}, sr^{i-2}..., r^2, sr^2, r, sr, r^j, sr^j. Thus, path P contains all element of D_{2n}. Hence, the length of P is $2n - 1$.

From (i)-(iii), giving label to the rows and the columns of $DD\left(\overline{C(D_{2n})}\right)$ in appropriate way, we will reach the desired proof. ■

For any two distinct vertices in $\overline{C(D_{2n})}$ for even n, the longest path has the length $2n - 1$. It is stated as the following theorem.

THEOREM 3: Detour matrix of conjugate graph complement of dihedral group D_{2n} for even n is $(2n \times 2n)$-matrix $DD\left(\overline{C(D_{2n})}\right) = (D_{ij})$ where $D_{ij} = 2n - 1$ if $i \ne j$ and $D_{ij} = 0$ elsewhere.

PROOF: All of conjugacy class of dihedral group D_{2n} for even n are $[1] = \{1\}$, $[r^{n/2}] = \{r^{n/2}\}$, $[r] = \{r, r^{n-1}\}$, $[r^2] = \{r^2, r^{n-2}\}$, ..., $[r^{n/2-1}] = \{r^{n/2-1}, r^{n/2+1}\}$, $[s] = \{s, sr^2, sr^4, \ldots, sr^{n-2}\}$ and $[sr] = \{sr, sr^3, sr^5, \ldots, sr^{n-1}\}$. Each conjugacy class will be a complete graph. So, in $\overline{C(D_{2n})}$, it will be a complete 5-partite graph where $V_1 = \{1, r^{n/2}\}$, $V_2 = \{r, r^2, \ldots, r^{n/2-1}\}$, $V_3 = \{r^{n-1}, r^{n-2}, \ldots, r^{n/2+1}\}$, $V_4 = \{s, sr^2, sr^4, \ldots, sr^{n-2}\}$ and $V_5 = \{sr, sr^3, \ldots, sr^{n-1}\}$ are its partition sets with $|V_1| = 2$, $|V_2| = |V_3| = n/2 - 1$ and $|V_4| = |V_5| = n/2$. Cycle W: 1, s, r, sr^2, r^2, sr^4, ..., $r^{n/2-1}$, sr^{n-2}, $r^{n/2}$, sr, r^{n-1}, sr^3, r^{n-2}, ..., $r^{n/2+1}$, sr^{n-1}, 1 is one of the Hamiltonian cycles in $\overline{C(D_{2n})}$. Hence, $\overline{C(D_{2n})}$ is a Hamiltonian graph. And for every two distinct vertices in $\overline{C(D_{2n})}$ for even n, we can always find its Hamiltonian path. Consequently, the longest path between two distinct vertices in complement of conjugate graph of dihedral group D_{2n} for even n has the length $2n - 1$. ■

Based on Theorem 2 and Theorem 3, we can determine the characteristics polynomial of detour matrix $DD\left(\overline{C(D_{2n})}\right)$. The characteristics polynomial of detour matrix $DD\left(\overline{C(D_{2n})}\right)$ is defined by $\rho(\lambda) = \det(DD\left(\overline{C(D_{2n})}\right) - \lambda I)$, where I is identity matrix of order $(2n \times 2n)$ [35]. To compute $\det(DD\left(\overline{C(D_{2n})}\right) - \lambda I)$, we can eliminate matrix $DD\left(\overline{C(D_{2n})}\right) - \lambda I$ using Gaussian elimination method to get an upper triangular matrix U. Then, $\det(DD\left(\overline{C(D_{2n})}\right) - \lambda I)$ is equal to the product of all entry in the main diagonal of U. We present the following lemma for odd and even n. The lemma will be very useful in determining detour spectrum and energy of $\overline{C(D_{2n})}$.

LEMMA 1: Let $\overline{C(D_{2n})}$ be a complement of conjugate graph of dihedral group D_{2n} for positive integer n and $n \ge 3$. The characteristics polynomial $\rho(\lambda)$ of detour matrix $DD\left(\overline{C(D_{2n})}\right)$ is

(i) $\rho(\lambda) = (\lambda^2 - A\lambda - (A/2)^2 - B)(\lambda + (2n-2))^{n-1}(\lambda + (2n-1))^{n-1}$ where $A = (4n^2 - 7n + 3)$ and $B = (4n^4 - 4n^3 + (5n^2 - 2n + 1)/4)$ for odd n, and

(ii) $\rho(\lambda) = (\lambda - (2n-1)^2)(\lambda + (2n-1))^{2n-1}$ for even n.

PROOF: (i) If n is odd, we determine the characteristics polynomial $\rho(\lambda)$ of $DD\left(\overline{C(D_{2n})}\right)$ in Theorem 2 by eliminating $DD\left(\overline{C(D_{2n})}\right) - \lambda I$ using Gaussian elimination method to an upper triangular matrix U. It follows that $\rho(\lambda)$ is a product along main diagonal of U. (ii) If n is even, then $DD\left(\overline{C(D_{2n})}\right) = (2n-1)(J - I)$, where J is all one square and I is identity matrix whose order is the same as the order of $DD\left(\overline{C(D_{2n})}\right)$. Hence, the characteristics polynomial $\rho(\lambda)$ of $DD\left(\overline{C(D_{2n})}\right)$ is $\rho(\lambda) = (\lambda - (2n-1)^2)(\lambda + (2n-1))^{2n-1}$. ■

THEOREM 4: The spectrum of detour matrix of conjugate graph complement of dihedral group D_{2n} for odd positive integer n and $n \ge 3$ is

$$spec_{DD}\left(\overline{C(D_{2n})}\right) = \begin{pmatrix} \frac{A}{2}+\frac{1}{2}\sqrt{2A^2+4B} & -(2n-1) & -(2n-2) & \frac{A}{2}-\frac{1}{2}\sqrt{2A^2+4B} \\ 1 & n-1 & n-1 & 1 \end{pmatrix},$$

where $A = (4n^2 - 7n + 3)$ and $B = (4n^4 - 4n^3 + (5n^2 - 2n + 1)/4)$.

PROOF: Let n be odd, letting $\rho(\lambda) = 0$ for Lemma 1(i), we have its eigenvalues are $\lambda_1 = (A + \sqrt{2A^2 + 4B})/2$, $\lambda_2 = -(2n-1)$, $\lambda_3 = -(2n-2)$ and $\lambda_4 = (A - \sqrt{2A^2 + 4B})/2$. From Lemma 1(i) we also have $m(\lambda_1) = m(\lambda_4) = 1$ and $m(\lambda_2) = m(\lambda_3) = n - 1$. It completes the proof. ■

THEOREM 5: The spectrum of detour matrix of conjugate graph complement of dihedral group D_{2n} for even positive integer n and $n \geq 3$ is

$$spec_{DD}\left(\overline{C(D_{2n})}\right) = \begin{pmatrix} (2n-1)^2 & -(2n-1) \\ 1 & 2n-1 \end{pmatrix}$$

PROOF: From Lemma 1(ii), it is clear that the eigenvalues of $DD\left(\overline{C(D_{2n})}\right)$ are $\lambda_1 = (2n-1)^2$ and $\lambda_2 = -(2n-1)$ and we have their algebraic multiplicity are $m(\lambda_1) = 1$ and $m(\lambda_2) = 2n - 1$, respectively. ■

COROLLARY 1: The energy of detour matrix of conjugate graph complement of dihedral group D_{2n} for odd positive integer n and $n \geq 3$ is $E_{DD}\left(\overline{C(D_{2n})}\right) \geq 2(n-1)(4n-3)$

PROOF: Based on Theorem 4, we have

$$\begin{aligned} E_{DD}\left(\overline{C(D_{2n})}\right) &= \left|\frac{A}{2}+\frac{1}{2}\sqrt{2A^2+4B}\right| + (n-1)(2n-1) + (n-1)(2n-2) + \left|\frac{A}{2}-\frac{1}{2}\sqrt{2A^2+4B}\right| \\ &\geq (n-1)(4n-3) + \left|\frac{A}{2}+\frac{1}{2}\sqrt{2A^2+4B}+\frac{A}{2}-\frac{1}{2}\sqrt{2A^2+4B}\right| \\ &= (n-1)(4n-3) + |A| \\ &= (n-1)(4n-3) + (4n^2-7n+3) \\ &= 2(n-1)(4n-3). \ ■ \end{aligned}$$

COROLLARY 2: The energy of detour matrix of conjugate graph complement of dihedral group D_{2n} for even positive integer n and $n \geq 3$ is $E_{DD}\left(\overline{C(D_{2n})}\right) = 2(2n-1)^2$.

PROOF: According to definition of energy, it is clear from Theorem 5 that $E_{DD}\left(\overline{C(D_{2n})}\right) = 2(2n-1)^2$. ■

3. Conclusion

In this paper, we have discussed the detour spectra and detour energy of conjugate graph complement of dihedral group D_{2n}. Given that the kinds of energy of a graph are so numerous, further research may be undertaken to determine the other energies of the conjugate graph complement of dihedral group D_{2n}.

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