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Some Results for the (Signless) Laplacian Resolvent

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Abstract

The recently introduced concept of resolvent energy of a graph [6,7] is based on the adjacency matrix. We now consider the analogous resolvent energies based on the Laplacian and signless Laplacian matrices, and determine some of their basic properties.

1 Introduction

All graphs in this note are simple; i.e., undirected, with no loops, and with no multiple edges. Let G be such a graph of order n, and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and $q_1 \geq q_2 \geq \cdots \geq q_n$ be its adjacency, Laplacian, and signless Laplacian eigenvalues, respectively. We denote by $M_k(G)$, $M_k(L(G))$, and $M_k(Q(G))$ the k-th adjacency, Laplacian, and signless Laplacian spectral moments of G, i.e.,

$$M_k(G) = \sum_{i=1}^{n} \lambda_i^k \quad , \quad M_k(L(G)) = \sum_{i=1}^{n} \mu_i^k \quad , \quad M_k(Q(G)) = \sum_{i=1}^{n} q_i^k \, .$$

Inspired by the definition of resolvent energy [6, 7], we define the *Laplacian resolvent energy* $RL(G)$ of G as:

$$RL(G) = \sum_{i=1}^{n} \frac{1}{(n+1) - \mu_i} \, .$$

Since $0 \leq \mu_i/(n+1) < 1$ for each $i \in \{1, \ldots, n\}$, we obtain the following expression for $RL(G)$, which is similar to the existing ones for the Estrada index [5], the resolvent Estrada index [3], and the resolvent energy [6, 7]:

$$RL(G) = \frac{1}{n+1} \sum_{k=0}^{\infty} \frac{M_k(L(G))}{(n+1)^k} \, . \tag{1}$$

2 Preliminaries

2.1 Deng's transformation I

A *semistar vertex* v of a graph G is any vertex of G of degree at least 2 having exactly one neighbor of degree at least 2. The only neighbor of v of degree at least 2 will be denoted by v^*. If v is a semistar vertex of a graph G, then we define $\mathcal{S}_v(G)$ as the graph obtained from G by contracting the edge vv^* and adding a pendent vertex adjacent to the vertex arising from the contraction; the vertex that results of the contraction of the edge vv^* is labeled by v, whereas the added pendent vertex is labeled by v^*. The operator $\mathcal{S}$ was introduced by Hanyuan Deng in [2] under the name *Transformation I*.

Example 1. *The graph G of Figure 1 has two semistar vertices: vertices 1 and 4. Moreover, $1^* = 4$ and $4^* = 1$. The graph $\mathcal{S}_1(G)$ is depicted in Figure 2.*

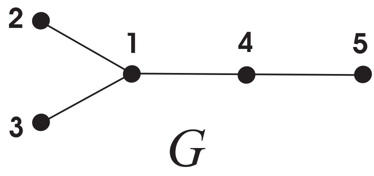

Fig. 1. A graph with two semistar vertices: 1 and 4 .

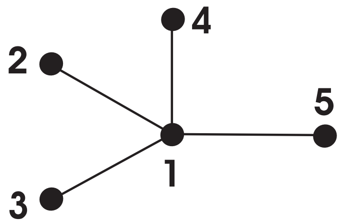

Fig. 2. The graph $\mathcal{S}_1(G)$

Remark 2. *Let T be a tree on n vertices and with exactly ℓ leaves.*

1. *If $n \geq 2$, then the semistar vertices of T are the leaves of the tree that arises from T by removing all its leaves. Hence, if T is not a star, then T has at least two semistar vertices.*

2. *If v is a semistar vertex of T, then $\mathcal{S}_v(T)$ has exactly one more leaf than T. Therefore, by applying the operator $\mathcal{S}$ to T exactly $n - \ell - 1$ times, we obtain the star S_n with n vertices.*

The importance of $\mathcal{S}$ in [2] is due to the result below, which implies that the Estrada index attains its maximum among trees of n vertices at the star S_n.

Theorem 1. [2] *Let G be a graph with a semistar vertex v. Then,*

$$M_{2k}(G) \leq M_{2k}(\mathcal{S}_v(G))\,.$$

Moreover, if $d_G(v) \geq 3$ and $k \geq 2$, then the inequality is strict.

2.2 Deng's transformation II

Let G_1 and G_2 be two graphs and, by renaming the vertices if necessary, assume without loss of generality that $V(G_1) \cap V(G_2) = \emptyset$. If $v \in V(G_1)$ and $w \in V(G_2)$, denote by $G_1 \circ_{v,w} G_2$ the graph obtained from the disjoint union of G_1 and G_2 by *identifying* the vertices v and w; i.e., by replacing v and w by a new vertex adjacent to the neighbors of v in G_1 and to the neighbors of w in G_2.

Let P be an induced path of G of length ℓ. Let $v \in V(P)$ be a vertex that is not an endpoint of P. If v is the only vertex of P having some neighbor in $G - V(P)$, then v is said to be a *semipath vertex* of G (of span ℓ). If so, we denote by $\mathcal{P}_v(G)$ the graph $(G - (V(P) - \{v\})) \circ_{v,w} P_\ell$ where w is an endpoint of P_ℓ. The operator $\mathcal{P}$ was introduced in [2] under the name of *Transformation II*.

Example 3. *The graph G depicted in Figure 1 has 1 as a semipath vertex of span 4 because it is a vertex of the path P induced by $\{2, 1, 4, 5\}$, the vertex 1 is not an endpoint of P, and 1 is the only vertex of P having a neighbor not in P. The corresponding graph $\mathcal{P}_1(G)$ is depicted in Figure 3. In the graph G, 1 is also a semipath vertex of span 3.*

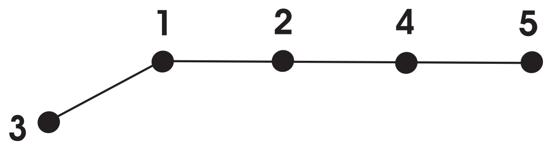

Fig. 3. The graph $\mathcal{P}_1(G)$ corresponding to the path P induced by $\{2, 1, 4, 5\}$

Let T be a tree on n vertices with exactly r vertices of degree at least 3.

- Let T' be the tree without vertices of degree 2, such that T arises from T' by some (eventually empty) sequence of edge subdivisions. The semipath vertices of T are the vertices of T of degree at least 3 which are adjacent to at least two leaves in T'.
- Hence, if T is not a path, then T' is also not a path, and T has some semipath vertex v.

- Moreover, $\mathcal{P}_v(T)$ has exactly one less vertex of degree at least 3 than T. Therefore, by applying the operator $\mathcal{P}$ to T exactly r times, we obtain the path P_n.

The relevance of the operator $\mathcal{P}$ in [2] is its effect on the spectral moments, which implies that the Estrada index attains its minimum among the trees on n vertices at the path P_n.

Theorem 2. [2] *Let G be a graph with a semipath vertex v. Then,*

$$M_{2k}(\mathcal{P}_v(G)) \leq M_{2k}(G)\,.$$

Moreover, if $k \geq 2$, then the inequality is strict.

2.3 Similarity between the Laplacian and signless Laplacian matrices of bipartite graphs

We denote by $Q(G)$ the signless Laplacian matrix of the graph G. It is known that the spectra of the Laplacian and signless Laplacian matrices of a bipartite graph coincide.

Theorem 3. [4] *If G is bipartite, then $L(G)$ and $Q(G)$ are similar matrices.*

The above theorem has the following immediate consequence.

Corollary 4. *If G is a bipartite graph, then*

$$RL(G) = \frac{1}{n+1} \sum_{k=0}^{\infty} \frac{M_k(Q(G))}{(n+1)^k}\,.$$

2.4 Signless Laplacian spectral moments and semiedge walks

A *semiedge walk* of length k in a graph G is a sequence $v_1, v_2, \ldots, v_{k+1}$ of vertices of G such that, for each $i \in \{1, \ldots, k\}$, either v_i is adjacent to v_{i+1} or $v_i = v_{i+1}$. Such a semiedge walk is said to be *closed* if $v_1 = v_{k+1}$. We denote by $CSEW_k(G)$ the set of all closed semiedge walks of length k in G. The following result relates the signless Laplacian spectral moments to the number of closed semiedge walks.

Theorem 4. [1] *For each non-negative integer k, the k-th signless Laplacian spectral moment is equal to the number of closed semiedge walks of length k; i.e, $M_k(Q(G)) = |CSEW_k(G)|$.*

2.5 Laplacian resolvent energy of a bipartite graph and adjacency spectral moments of its complete subdivision

If G is a graph, the *complete subdivision* $S(G)$ of G is the graph obtained from G by subdividing each of its edges exactly once. The following result relates the Laplacian spectrum of a bipartite graph G to the adjacency spectrum of $S(G)$.

Theorem 5. [10] *Let G be a bipartite graph with n vertices and m edges. If the nonzero Laplacian eigenvalues of G are $\mu_1, \ldots, \mu_h$, then the adjacency spectrum of $S(G)$ consists of the numbers $\pm\sqrt{\mu_i}$ for each $i \in \{1, \ldots, h\}$ and $m+n-2h$ zeros.*

The following is an immediate consequence.

Corollary 5. *If G is a bipartite graph, then, for each nonnegative integer k,*

$$M_k(L(G)) = \frac{1}{2}\, M_{2k}(S(G))\,.$$

3 Some results for the Laplacian resolvent

We define a partial order $\trianglelefteq$, analogous to the quasi-order defined in Section 4.3 of [8], as follows. For any two graphs G and H,

$$H \trianglelefteq G \quad \text{if and only if} \quad M_k(L(H)) \leq M_k(L(G)) \text{ for every } k.$$

It is clear from Eq. (1) that $H \trianglelefteq G$ implies $RL(H) \leq RL(G)$.

Remark 6. *K_n denotes the complete graph on n vertices, and $n\,K_1$ denotes the graph on n vertices with no edges. Let G be a graph of order n, and F a spanning forest of G. Since adding edges never decreases and eventually increases each of the Laplacian eigenvalues* [9]*, it also never decreases and eventually increases each of the Laplacian spectral moments. Hence, $n\,K_1 \trianglelefteq F \trianglelefteq G \trianglelefteq K_n$ and consequently*

$$\frac{n}{n+1} = RL(n\,K_1) \leq RL(F) \leq RL(G) \leq RL(K_n) = \frac{n^2}{n+1}\,.$$

We now prove the analogue of Theorem 1 for signless Laplacian spectral moments. If v is a vertex of a graph G, denote by $N_G[v]$ its *closed neighborhood* $N_G(v) \cup \{v\}$.

Lemma 7. *Let G be a graph with a semistar vertex v. Then,*

$$M_k(Q(G)) \leq M_k(Q(\mathcal{S}_v(G)))$$

holds for all k non-negative integers. Moreover, if $k \geq 4$, then the inequality is strict.

Proof. By Theorem 4, it suffices to show that $|CSEW_k(G)| < |CSEW_k(\mathcal{S}_v(G))|$. In order to do so, we define a mapping

$$\theta_k : CSEW_k(G) \rightarrow CSEW_k(\mathcal{S}_v(G))$$

and show that

(i) θ_k is injective and

(ii) if $k \geq 4$, then θ_k is not surjective.

We define the mapping θ_k as follows: If $W \in CSEW_k(G)$, let $\theta_k(W)$ be the sequence obtained from W by replacing by v each occurrence of v^* immediately preceded or immediately followed by a vertex not in $N_G[v]$. Notice that the sequence $\theta_k(W)$ belongs to $CSEW_k(\mathcal{S}_v(G))$ because $N_G[v^*] \subseteq N_{\mathcal{S}_v(G)}[v]$ and because occurrences of v^* in W which are not immediately preceded or immediately followed in W by a vertex not in $N_G[v]$ can only be immediately preceded or immediately followed in W by v, and $v \in N_{\mathcal{S}_v(G)}[v^*]$.

The transformation θ_k is injective because W can be recovered from $\theta_k(W)$ by replacing by v^* those occurrences of v immediately preceded or immediately followed by a vertex not in $N_G[v]$ (because θ_k replaces with v some occurrences of v^*, and both v and v^* belong to $N_G[v]$).

Assume now that $k \geq 4$. Since v and v^* have degree at least 2 each, there is some vertex $u \in N_G[v^*] \setminus \{v\}$ and some vertex $w \in N_G[v] \setminus \{v^*\}$. Since $u, v, w, w, \ldots, w, w, v, u$ (with $k-3$ occurrences of w) belongs to $CSEW_k(\mathcal{S}_v(G))$ but not to the image of θ_k (because neither $N_G[v]$ nor $N_G[v^*]$ contains both u and w), θ_k is not surjective, which completes the proof. ■

We now prove the analogue of Theorem 2 for the signless Laplacian spectral moments of bipartite graphs.

Lemma 8. *Let G be a bipartite graph with semipath vertex v. Then,*

$$M_k(Q(\mathcal{P}_v(G))) < M_k(Q(G))$$

holds for all non-negative integers k. Moreover, if $k \geq 2$, the inequality is strict.

Proof. Since G is bipartite, $\mathcal{P}_v(G)$ is also bipartite. Hence, Theorem 3 and Corollary 5 imply that

$$M_k(Q(G)) = M_k(L(G)) = \frac{1}{2}\,M_{2k}(S(G))$$

and

$$M_k(Q(\mathcal{P}_v(G))) = M_k(L(\mathcal{P}_v(G))) = \frac{1}{2}\,M_{2k}(S(\mathcal{P}_v(G)))\,.$$

Since $S(\mathcal{P}_v(G)) = \mathcal{P}_v(S(G))$, in order to prove the theorem it suffices to show that

$$M_{2k}(\mathcal{P}_v(S(G)))) \leq M_{2k}(S(G))$$

and that the inequality is strict if $k \geq 2$. But this directly follows from Theorem 2. ■

Remark 9. *It seems that the condition that G is bipartite can be dropped from Lemma 8. If so, then it is highly likely that the corresponding proof can be obtained by slightly adapting the argument used in the proof of Theorem 2 given in* [2] *from closed walks to closed semiedge walks.*

By combining the above results, we are able to characterize the graphs minimizing and maximizing the Laplacian resolvent energy among trees on n vertices.

Theorem 6. *It T is a tree on n vertices, such that $T \ncong P_n$ and $T \ncong S_n$, then*

$$RL(P_n) < RL(T) < RL(S_n)$$

where P_n and S_n are the path and the star on n vertices, respectively.

Proof. Recall from Remark 2 that, since T is not a star, it is possible to apply repeatedly the operator $\mathcal{S}$ to transform T into the star on S_n. Hence, by Lemma 7, $M_k(Q(T)) \leq M_k(Q(S_n))$ for every k and the inequality is strict for each $k \geq 4$. Therefore, by Corollary 4, $RL(T) < RL(S_n)$, which concludes the proof.

Similarly, since T is not a path, it is possible to transform T into P_n by repeated application of operator $\mathcal{P}$. Hence, by Lemma 8, $M_k(Q(P_n)) < M_k(Q(T))$ for every $k \geq 2$. Therefore, by Corollary 4, $RL(P_n) < RL(T)$, which concludes the proof. ■

In fact, it is possible to characterize the path P_n as the only graph minimizing the Laplacian resolvent energy among connected graphs on n vertices.

Corollary 10. *If G is a connected graph on n vertices such that $G \ncong P_n$, then $RL(P_n) < RL(G)$.*

Proof. Since G is not a path, it has some spanning tree T which is not a path. By Theorem 6 and Remark 6, $RL(P_n) < RL(T) \leq RL(G)$. ■

4 On signless Laplacian resolvent energy

The *signless Laplacian resolvent energy* $RQ(G)$ of a graph G may be defined similarly by means of the formula

$$RQ(G) = \sum_{i=1}^{n} \frac{1}{(2n-1) - q_i}$$

where $q_1, q_2, \ldots, q_n$ are the signless Laplacian eigenvalues of G. The analogue of Eq. (1) would then be

$$RQ(G) = \frac{1}{2n-1} \sum_{k=0}^{\infty} \frac{M_k(Q(G))}{(2n-1)^k} \,. \tag{2}$$

Since adding edges to a graph never decreases and eventually increases the number of closed semiedge walks of length k, it also never decreases and eventually increases each of the signless Laplacian spectral moments (see Theorem 4). Hence, we have the following analogue of Remark 6.

Remark 11. *If G is a graph on n vertices and F is a spanning forest of G, then*

$$\frac{n}{2n-1} = RQ(nK_1) \leq RQ(F) \leq RQ(G) \leq RQ(K_n) = \frac{2n}{n+1} \,.$$

Because of Eq. (2) and Lemmas 7 and 8, we have the following analogues of Theorem 6 and Corollary 10 for the signless Laplacian resolvent energy.

Theorem 7. *It T is a tree on n vertices which is neither the path nor the star, then*

$$RQ(P_n) < RQ(T) < RQ(S_n) \,.$$

Corollary 12. *If G is a connected graph on n vertices which is not the path, then $RQ(P_n) < RQ(G)$.*

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