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MULTIPLE SOLUTIONS OF BOUNDARY VALUE PROBLEMS ON TIME SCALES FOR A φ-LAPLACIAN OPERATOR

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Abstract. We establish the existence and multiplicity of solutions for some boundary value problems on time scales with a φ-Laplacian operator. For this purpose, we employ the concept of lower and upper solutions and the Leray–Schauder degree. The results extend and improve known results for analogous problems with discrete p-Laplacian as well as those for boundary value problems on time scales.

Keywords: dynamic equations on time scales, nonlinear boundary value problems, upper and lower solutions, Leray–Schauder degree, multiple solutions.

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1. INTRODUCTION

In this work, we investigate the existence of solutions $u : [\rho(0), \sigma(T)]_{\mathbb{T}} \to \mathbb{R}$ to the following problem on time scales

$$(\varphi(u^{\Delta}(t)))^{\nabla} = f(t, u(t)), \quad t \in [0,T]_{\mathbb{T}}, \tag{1.1}$$

under Dirichlet, Neumann, or periodic boundary conditions:

$$u(\rho(0)) = u(\sigma(T)) = 0, \tag{1.2}$$

$$u^{\Delta}(\rho(0)) = u^{\Delta}(T) = 0, \tag{1.3}$$

$$u(\rho(0)) = u(\sigma(T)), \quad u^{\Delta}(\rho(0)) = u^{\Delta}(T), \tag{1.4}$$

respectively.

Here, $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ (*time scale*), $[0,T]_{\mathbb{T}} := [0,T] \cap \mathbb{T}$ denotes the interval with respect to the time scale $\mathbb{T}$, the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\varphi(0) = 0$, $f : [\rho(0), \sigma(T)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and T is a positive real number.

By a *solution* of (1.1) under the boundary condition (1.2), (1.3) or (1.4) we mean a function $u : [\rho(0), \sigma(T)]_{\mathbb{T}} \to \mathbb{R}$ of class C^1 such that $\varphi(u^{\Delta})$ is ∇-differentiable on $[0,T]_{\mathbb{T}}$ and $(\varphi(u^{\Delta}))^{\nabla}$ continuous on $[0,T]_{\mathbb{T}}$, which satisfies the respective boundary condition and $(\varphi(u^{\Delta}(t)))^{\nabla} = f(t, u(t))$ for all $t \in [0,T]_{\mathbb{T}}$.

The theory of time scales was introduced by Stefan Hilger in his PhD thesis in 1988 (see [15] and the subsequent paper [16]) in order to unify the discrete and the continuous calculus. Since then, a great variety of results were obtained for dynamic equations where the domain of the unknown function is a time scale $\mathbb{T}$. For instance, the time scale $\mathbb{R}$ corresponds to the continuous case and, hence, results for ordinary differential equations are retrieved. On the other hand, if the time scale is $\mathbb{Z}$, then the results apply to difference equations. However, the generality of the set $\mathbb{T}$ yields many different situations in which the time scales formalism proves to be useful, e.g. the study of hybrid discrete-continuous dynamical systems.

Existence of solutions for boundary value problems on time scales can be investigated by various methods: fixed point theorems [20, 22], more general topological arguments [6,14], variational methods [12,13,21], lower and upper functions [6,9,20,22], etc. In particular, in [22] the existence of solutions for a periodic boundary value problem on time scales of the form

$$\begin{cases} -y^{\Delta\nabla}(t) + q(t)y(t) = f(t, y(t)), \quad t \in [a,b]_{\mathbb{T}}\,, \\ y(\rho(a)) = y(b), \\ y^{\Delta}(\rho(a)) = y^{\Delta}(b), \end{cases}$$

is studied, where $q(t) \gneq 0$, $f : [\rho(a), b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous and $a - \rho(a) \geq \sigma(b) - b$, by means of the method of upper and lower solutions and the Schauder fixed point theorem. Moreover, using a monotone method existence and uniqueness results were obtained.

In [6], the periodic problem

$$\begin{cases} D\varphi(Dx_k) + f_k(x_k) = 0, \quad 2 \leq k \leq n-1, \\ x_1 = x_n, \\ Dx_1 = Dx_{n-1}, \end{cases}$$

where $f_k(x)$ is a continuous function and D is the standard discrete difference, namely $Dx_k = x_{k+1} - x_k$, was studied combining the method of upper and lower solutions with Brouwer degree theory. The interest in this class of problems is due to the fact that they involve the discrete φ-Laplacian operator and can be regarded as a problem on time scales with $\mathbb{T} = \{x_1, \ldots, x_n\}$.

Another relevant reference for periodic boundary value problems on time scales is [20], where the problem

$$\begin{cases} x^{\Delta\Delta}(t) = f(t, x(\sigma(t))), \quad t \in [a,b]_{\mathbb{T}}\,, \\ x(a) = x(\sigma^2(b)), \\ x(a) = x^{\Delta}(\sigma(b)) \end{cases}$$

is studied by means of the Schauder fixed point theorem and the method of upper and lower solutions. Existence of solutions was proved and a monotone iterative method was developed.

Motivated by [6, 20, 22], in this work we study existence and multiplicity results for problem (1.1) under different boundary value conditions. We distinguish several aspects of these results.

On the one hand, the problems in the present paper consist of equations on time scales, involving φ-Laplacian operators, for which the literature is scarce. Recall that such operators, among which the most prominent is the p-Laplacian given by $\varphi(x) := |x|^{p-2}x$, have deserved a lot of attention in the last decades and found several applications. There are also popular examples of *bounded* φ-Laplacians, such as the mean curvature operator $\varphi(x) := \frac{x}{\sqrt{1+x^2}}$ or *singular* φ-Laplacians, like the relativistic operator $\varphi(x) := \frac{x}{\sqrt{1-x^2}}$. Many of the well known results and methods that are valid for the standard semilinear case, with φ as the identity map, cannot be extended in an obvious way when φ is an arbitrary homeomorphism and, thus, a lot of technical issues may appear when dealing with the corresponding boundary value problems.

On the other hand, we generalize previous results concerning boundary value problems for difference equations obtained in [6, 12, 13, 21] and for dynamic equations in time scales proved in [20, 22]. However, differently to the latter two references, our results are formulated assuming a different *a priori* condition on the time scale. Moreover, we extend techniques applied both for the resonant case (see e.g. [20]) and the non-resonant case (see [1, 22]) to the context of φ-Laplacian operators. Specifically, we adapt the method of lower and upper solutions (see [10] for a complete survey on this method) and the Leray–Schauder degree theory, from the continuous calculus to time scales, in order to prove existence of solutions for boundary value problems. To this end, we apply Mawhin's continuation method in the spirit of [19], conveniently adjusted to the present situation. Our approach follows the techniques employed for example in [4, 22]. If, moreover, the function $f(t, u)$ satisfies a one-sided growth condition then a monotone iterative method can be developed, converging to extremal solutions of the problem. This proof is based on well known methods for semilinear equations, adapted for time scales in [20]. Furthermore, we obtain a version of the so-called Three Solutions Theorem. In the continuous and semilinear case, this result can be traced back to [17], although a more general statement and a proof by topological degree methods can be found in [2] and [3] (see also [10] for further references).

We emphasize the fact that, in this work, the above mentioned ideas are extended to a more general setting, which includes different boundary conditions and a φ-Laplacian operator. Moreover, for Neumann boundary conditions the results are most likely not only new but also original in the sense that the problem has not been studied employing this particular technique.

The article is organized as follows. In Section 2, we introduce some notation and preliminaries concerning the classic theory of real times scales, the concept of upper and lower solutions and set the operators required for the continuation approach. In Section 3, we prove existence and multiplicity results for periodic boundary conditions, that can be extended to Neumann and Dirichlet boundary conditions in a straightforward manner. Furthermore, under a one-sided Lipschitz condition, we define monotone sequences that converge to extremal solutions of the problem. Our main results are extensions of the results in [6, 20] and [22].

2. NOTATION AND PRELIMINARIES

Let us firstly recall some basic definitions and results concerning time-scales. Further, general details can be found, for example, in [7, 8].

A time scale $\mathbb{T}$ is a nonempty closed subset of the real line $\mathbb{R}$. For $t \in \mathbb{T}$ we define the *forward jump* operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the *backward jump* operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\},$$
$$\rho(t) := \sup \{s \in \mathbb{T} : s < t\}.$$

For convenience, if $\mathbb{T}$ is bounded from above or from below we define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, respectively.

We say that a point $t \in \mathbb{T}$ is *right scattered*, *left scattered*, *right dense*, *left dense* if $\sigma(t) > t, \rho(t) < t, \sigma(t) = t, \rho(t) = t$, respectively. a point $t \in \mathbb{T}$ is *isolated* if it is right scattered and left scattered. We define the sets $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$ which are derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum in m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Similarly, if $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$.

Finally, we define the *forward graininess* function $\mu : \mathbb{T} \to [0, +\infty)$ by

$$\mu(t) := \sigma(t) - t \quad \text{for all } t \in \mathbb{T},$$

and the *backward graininess* function $\nu : \mathbb{T} \to [0, +\infty)$ by

$$\nu(t) := t - \rho(t) \quad \text{for all } t \in \mathbb{T}.$$

We endow $\mathbb{T}$ with the topology inherited from $\mathbb{R}$. a function $u : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* on $\mathbb{T}$ if it is continuous at right-dense points of $\mathbb{T}$ and its left-side limit exists at left-dense points. Then, $u : \mathbb{T} \to \mathbb{R}$ is called continuous on $\mathbb{T}$ if is continuous at each right-dense point and each left-dense point. Finally, we say that the function u is *delta differentiable* at $t \in \mathbb{T}^\kappa$ if there exists a number (denoted by $u^\Delta(t)$) with the property that given any $\epsilon > 0$ there is a neighbourhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|(u(\sigma(t)) - u(s)) - u^\Delta(t)\,(\sigma(t) - s)\right| \leq \epsilon\,|\sigma(t) - s|$$

for all $s \in U$. Thus, we call $u^\Delta(t)$ the *delta derivative* of u at t. Moreover, we say that u is *delta differentiable* on $\mathbb{T}^\kappa$ provided that $u^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $u^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of u on $\mathbb{T}^\kappa$.

Similarly, if $u : \mathbb{T} \to \mathbb{R}$ we say that the function u is *nabla differentiable* at $t \in \mathbb{T}_\kappa$ if there exists a number (denoted by $u^\nabla(t)$) with the property that given any $\epsilon > 0$ there is a neighbourhood U of t on $\mathbb{T}$ such that

$$\left|(u(\rho(t)) - u(s)) - u^\nabla(t)\,(\rho(t) - s)\right| \leq \epsilon\,|\rho(t) - s|$$

for all $s \in U$. Thus, we call $u^{\nabla}(t)$ the *nabla derivative* of u at t. Note that for $\mathbb{T} = \mathbb{R}$, we have $u^{\Delta} = u^{\nabla} = u'$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$ we have that $u^{\Delta}(t) = \Delta u(t) = u(t+1) - u(t)$ and $u^{\nabla}(t) = \nabla u(t) = u(t) - u(t-1)$.

For fixed $T > 0$, let $C := C([\rho(0), \sigma(T)]_{\mathbb{T}}, \mathbb{R})$ be the Banach space of continuous functions on $[\rho(0), \sigma(T)]_{\mathbb{T}}$ endowed with the uniform norm

$$\|u\|_{\infty} = \sup_{[\rho(0),\sigma(T)]_{\mathbb{T}}} |u(t)|$$

and let $C^1 := C^1([\rho(0), \sigma(T)]_{\mathbb{T}}, \mathbb{R})$ denote the Banach space of all continuous functions on $[\rho(0), \sigma(T)]_{\mathbb{T}}$ that are Δ-differentiable with continuous Δ-derivatives on $[\rho(0), T]_{\mathbb{T}}$ endowed with the usual norm

$$\|u\|_1 = \sup_{[\rho(0),\sigma(T)]_{\mathbb{T}}} |u(t)| + \sup_{[\rho(0),T]_{\mathbb{T}}} \left|u^{\Delta}(t)\right|.$$

A function $U : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $u : \mathbb{T} \to \mathbb{R}$ provided that $U^{\Delta}(t) = u(t)$ holds for all $t \in \mathbb{T}^{\kappa}$. Then, the Δ-integral from t_0 to t of the function u is defined by

$$\int_{t_0}^{t} u(s)\Delta s = U(t) - U(t_0), \quad \text{for all } t \in \mathbb{T}.$$

A function $V : \mathbb{T} \to \mathbb{R}$ we call a ∇-antiderivative of $u : \mathbb{T} \to \mathbb{R}$ provided that $V^{\nabla}(t) = u(t)$, for $t \in \mathbb{T}_{\kappa}$. We then define the ∇-integral from t_0 to t of the function u by

$$\int_{t_0}^{t} u(s)\nabla s = V(t) - V(t_0), \quad \text{for all } t \in \mathbb{T}.$$

In particular, it is well known that a continuous function has always a Δ-antiderivative and a ∇- antiderivative, which are unique up to a constant term. For the details on basic notions related to time scales, we refer the readers to the books [7, 8].

We introduce the following operators:

– the *Nemytskii operator* $N_f : C^1 \to C$,

$$N_f(u)(t) = f(t, u(t)),$$

– the Δ-*integration operator* $H_{\Delta} : C \to C^1$,

$$H_{\Delta}(u)(t) = \int_{\rho(0)}^{t} u(s)\Delta s,$$

– the ∇-*integration operator* $H_{\nabla} : C \to C^1$,

$$H_{\nabla}(u)(t) = \int_{\rho(0)}^{t} u(s)\nabla s,$$

and the following continuous linear projectors onto the subset of constant functions of C:

$$Q : C \to C, \quad Q(u)(t) \equiv \frac{1}{T - \rho(0)} \int_{\rho(0)}^{T} u(s) \nabla s,$$

$$P : C \to C, \quad P(u)(t) \equiv u(\rho(0)).$$

Moreover, given $u : [\rho(0), \sigma(T)]_{\mathbb{T}} \to \mathbb{R}$ continuous, we shall denote

$$u_m := \min_{[\rho(0),\sigma(T)]_{\mathbb{T}}} u,$$
$$u_M := \max_{[\rho(0),\sigma(T)]_{\mathbb{T}}} u.$$

Remark 2.1. Let $\mathbb{T}$ be a time scale. The following equivalences hold:

(i) $0 \in ([\rho(0), \sigma(T)]_{\mathbb{T}})_k \Leftrightarrow \sigma(\rho(0)) = 0$,
(ii) $T \in ([\rho(0), \sigma(T)]_{\mathbb{T}})^k \Leftrightarrow \rho(\sigma(T)) = T$.

In this paper we shall consider time scales $\mathbb{T}$ such that $0 \in ([\rho(0), \sigma(T)]_{\mathbb{T}})_\kappa$ and $T \in ([\rho(0), \sigma(T)]_{\mathbb{T}})^\kappa$. This is equivalent to say that $0 \in \mathbb{T}$ is left scattered or right dense and $T \in \mathbb{T}$ is right scattered or left dense, that is:

$$\nu(0) > 0 \text{ or } \mu(0) = 0 \text{ and } \nu(T) = 0 \text{ or } \mu(T) > 0. \tag{2.1}$$

For example, $\mathbb{T} = \mathbb{Z}, \mathbb{R}$ or $\mathbb{Z}^- \cup [0, +\infty)$.

The following lemma is a straightforward adaptation of a result in [4] to the time scales context.

Lemma 2.2. *For each $h \in C$, there exists a unique $Q_\varphi = Q_\varphi(h) \in [h_m, h_M]$ such that*

$$\int_{\rho(0)}^{\sigma(T)} \varphi^{-1}(h(t) - Q_\varphi(h)) \Delta t = 0.$$

Moreover, the function $Q_\varphi : C \to \mathbb{R}$ is continuous and maps bounded sets into bounded sets.

Next, we define lower and upper solutions for problem (1.1) as follows.

Definition 2.3. a lower solution α (resp. upper solution β) of (1.1) is a function $\alpha \in C^1$ such that $\varphi(\alpha^\Delta)$ is ∇-differentiable on $[0,T]_{\mathbb{T}}$, $\left(\varphi\left(\alpha^\Delta(t)\right)\right)^\nabla$ is continuous on $[0,T]_{\mathbb{T}}$ (resp. $\beta \in C^1$, $\varphi(\beta^\Delta)$ is ∇-differentiable on $[0,T]_{\mathbb{T}}$, $\left(\varphi\left(\beta^\Delta(t)\right)\right)^\nabla$ is continuous on $[0,T]_{\mathbb{T}}$) and

$$\left(\varphi\left(\alpha^\Delta(t)\right)\right)^\nabla \geq f(t,\alpha(t)) \quad (resp.\ \left(\varphi\left(\beta^\Delta(t)\right)\right)^\nabla \leq f(t,\beta(t))) \tag{2.2}$$

for all $t \in [0,T]_{\mathbb{T}}$. In addition, we shall assume

(i) for the Dirichlet boundary condition (1.2):

$$\alpha(\rho(0)) \leq 0,\ \alpha(\sigma(T)) \leq 0$$
$$(\text{resp. } \beta(\rho(0)) \geq 0,\ \beta(\sigma(T)) \geq 0).$$

(ii) For the Neumann boundary condition (1.3):

$$\alpha^\Delta(T) \leq 0 \leq \alpha^\Delta(\rho(0))$$
$$(\text{resp. } \beta^\Delta(\rho(0)) \leq 0 \leq \beta^\Delta(T)).$$

(iii) For the periodic boundary condition (1.4):

$$\alpha(\rho(0)) = \alpha(\sigma(T)), \quad \alpha^\Delta(\rho(0)) \geq \alpha^\Delta(T)$$
$$(\text{resp. } \beta(\rho(0)) = \beta(\sigma(T)), \quad \beta^\Delta(\rho(0)) \leq \beta^\Delta(T)).$$

Such lower (upper) solution is called proper if it is not a solution of the equation with the respective boundary condition. If furthermore the inequality (2.2) is strict for all $t \in [0,T]_{\mathbb{T}}$, then it is called a strict lower (resp. upper) solution.

For convenience, for each pair α, β as before such that $\alpha(t) \leq \beta(t)$ for all t we associate a function $\gamma : [\rho(0), \sigma(T)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ given by

$$\gamma(t,x) := \begin{cases} x & \text{if } \alpha(t) \leq x \leq \beta(t), \\ \beta(t) & \text{if } x > \beta(t), \\ \alpha(t) & \text{if } x < \alpha(t). \end{cases}$$

3. THE PERIODIC PROBLEM

In this section, we consider problem (1.1) under the periodic boundary condition (1.4). To this end, we shall define an appropriate fixed point operator, which is similar to the one introduced in [18]. In order to transform problem (1.1), (1.4) into a fixed point problem we employ Lemma 2.2. The proof is similar to the continuous case and shall not be repeated here.

Lemma 3.1. *$u \in C^1$ is a solution of* (1.1), (1.4) *if and only if u is a fixed point of the operator M_f defined on C^1 by*

$$u \mapsto M_f(u) := P(u) + Q(N_f(u)) + H_\Delta\big(\varphi^{-1}\big[H_\nabla(N_f(u) - Q(N_f(u))) \\ - Q_\varphi(H_\nabla(N_f(u) - Q(N_f(u))))\big]\big).$$

Here φ^{-1}, with a slight abuse of notation, is understood as the operator $\varphi^{-1} : C \to C$ defined as $\varphi^{-1}(v)(t) := \varphi^{-1}(v(t))$. Moreover, by the Arzelà–Ascoli theorem, M_f is completely continuous.

Remark 3.2. Note that if u is a solution of (1.1), then the following equivalence holds: $u^\Delta(\rho(0)) = u^\Delta(T) \Leftrightarrow Q(N_f(u)) = 0$.

3.1. UPPER AND LOWER SOLUTIONS AND A MODIFIED PROBLEM

Let α, β be lower and upper solutions of (1.1), (1.4) such that $\alpha(t) \le \beta(t)$ for all $t \in [\rho(0), \sigma(T)]_\mathbb{T}$, and consider the modified problem

$$\begin{cases} (\varphi(u^\Delta(t)))^\nabla = F(t, u(t)), \quad t \in [0,T]_\mathbb{T}, \\ u(\rho(0)) = u(\sigma(T)), \\ u^\Delta(\rho(0)) = u^\Delta(T), \end{cases} \tag{3.1}$$

where $F : [\rho(0), \sigma(T)]_\mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(t,x) := f(t, \gamma(t,x)) + \eta \frac{x - \gamma(t,x)}{1 + |x - \gamma(t,x)|}$$

for some arbitrary $\eta > 0$. The fixed point operator associated to problem (3.1) is given by

$$M_F(u) = P(u) + Q(N_F(u)) + H_\Delta\big(\varphi^{-1}\big[H_\nabla(N_F(u) - Q(N_F(u))) \\ - Q_\varphi(H_\nabla(N_F(u) - Q(N_F(u))))\big]\big).$$

Theorem 3.3. *Suppose that* (1.1), (1.4) *has a lower solution α and an upper solution β such that $\alpha(t) \le \beta(t)$ for all $t \in [\rho(0), \sigma(T)]_\mathbb{T}$. If u is a solution of* (3.1)*, then $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in [\rho(0), \sigma(T)]_\mathbb{T}$ and hence u is a solution of* (1.1), (1.4).

Proof. Let u be a solution of (3.1), we shall see that $\alpha(t) \le u(t)$ for all $t \in [\rho(0), \sigma(T)]_\mathbb{T}$. Assume, by contradiction, that the function $z := \alpha - u$ attains a positive maximum in $[\rho(0), \sigma(T)]_\mathbb{T}$. Let

$$\max_{[\rho(0),\sigma(T)]_\mathbb{T}} (\alpha(t) - u(t)) = \alpha(t_0) - u(t_0) = z(t_0) > 0$$

and for simplicity assume, without loss of generality, that $z(t) < z(t_0)$ for all $t \in (t_0, \sigma(T)]_\mathbb{T}$. In particular we observe that, due to the boundary condition, $t_0 > \rho(0)$.

We now distinguish between three cases, according to the position of the point t_0.

(a) $t_0 \in (\rho(0), \sigma(T))_{\mathbb{T}}$. We claim that

$$(\varphi(\alpha^{\Delta}))^{\nabla}(t_0) \leq (\varphi(u^{\Delta}))^{\nabla}(t_0).$$

Indeed, if t_0 is left scattered, then it is readily seen that

$$\alpha^{\Delta}(t_0) \leq u^{\Delta}(t_0),$$
$$\alpha^{\Delta}(\rho(t_0)) \geq u^{\Delta}(\rho(t_0))$$

and the claim follows. Next, suppose that t_0 is left dense and observe, in the first place, that $z^{\Delta}(t_0) = 0$, since otherwise $z^{\Delta}(t_0) < 0$ and z decreases strictly in a neighbourhood of t_0, a contradiction.

If the claim is not true, then the function $\varphi(\alpha^{\Delta}) - \varphi(u^{\Delta})$ is strictly increasing over some nonempty interval $[t_1, t_0)_{\mathbb{T}}$. From the monotonicity of φ and the fact that $z^{\Delta}(t_0) = 0$, we conclude that $z^{\Delta}(s) < 0$ for all $s \in [t_1, t_0)_{\mathbb{T}}$, which contradicts the maximality of $z(t_0)$.

Because $\alpha(t_0) > u(t_0)$, using the definition of lower solution from (2.2), and the fact that $0 \leq t_0 \leq T$ we get the following contradiction:

$$\begin{aligned}(\varphi(\alpha^{\Delta}))^{\nabla}(t_0) &\leq (\varphi(u^{\Delta}))^{\nabla}(t_0) = F(t_0, u(t_0)) \\ &= f(t_0, \alpha(t_0)) + \eta \frac{u(t_0) - \alpha(t_0)}{1 + |u(t_0) - \alpha(t_0)|} \\ &< f(t_0, \alpha(t_0)) \leq (\varphi(\alpha^{\Delta}))^{\nabla}(t_0).\end{aligned} \tag{3.2}$$

(b) If $t_0 = \sigma(T) = T$ then, on the one hand, using the boundary conditions we obtain:

$$z(\rho(0)) = z(T), \quad z^{\Delta}(\rho(0)) \geq z^{\Delta}(T).$$

On the other hand, z achieves its maximum at $\rho(0)$, then $z^{\Delta}(\rho(0)) \leq 0$. Due to condition (2.1), T is left dense and hence $z^{\Delta}(T) \geq 0$; thus we conclude that

$$z^{\Delta}(\rho(0)) = z^{\Delta}(T) = 0.$$

Now we must distinguish between two possibilities.

(i) If 0 is left scattered, then $z(0) = z(\rho(0))$ and the function z attains its maximum also at $t = 0$. But, since $0 \in (\rho(0), \sigma(T))$, a contradiction is obtained as in case (a).

(ii) If 0 is left dense, then $z^{\Delta}(0) = 0$, that is $u^{\Delta}(0) = \alpha^{\Delta}(0)$.
We claim that $(\varphi(\alpha^{\Delta}))^{\nabla}(0) \leq (\varphi(u^{\Delta}))^{\nabla}(0)$. Indeed, otherwise we deduce as before, using condition (2.1), that $\varphi(u^{\Delta}(t)) < \varphi(\alpha^{\Delta}(t))$ over an interval $(0, \delta)_{\mathbb{T}}$. By the monotonicity of φ, this implies $z^{\Delta} > 0$ on $(0, \delta)_{\mathbb{T}}$, which contradicts the fact that z achieves its maximum at 0. Hence, a contradiction is obtained exactly as in (3.2), with $t_0 = 0$.

(c) If $t_0 = \sigma(T) > T$ then $z^{\Delta}(T) \geq 0$. From the boundary conditions we deduce, as before, that $z^{\Delta}(T) = 0$ and hence $z(T) = z(\sigma(T))$. Thus z attains its maximum also in T and a contradiction yields as in case (a).

Summarizing, we proved that $\alpha(t) \leq u(t)$ for all $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$. Similarly, it can be shown that $u(t) \leq \beta(t)$ for all $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$ and the conclusion follows. □

3.2. EXISTENCE RESULT

In order to establish the existence of a solution to (3.1), let us consider the following family of problems defined for $\lambda \in [0,1]$:

$$\begin{cases} (\varphi(u^\Delta))^\nabla = \lambda N_F(u) + (1-\lambda)Q(N_F(u)), \\ u(\rho(0)) = u(\sigma(T)), \\ u^\Delta(\rho(0)) = u^\Delta(T). \end{cases} \tag{3.3}$$

So, for each $\lambda \in [0,1]$, the nonlinear operator associated to (3.3) is the operator $M(\lambda,\cdot)$, where M is defined on $[0,1] \times C^1$ by

$$\begin{aligned} M(\lambda,u) &= P(u) + Q(N_F(u)) \\ &\quad + H_\Delta(\varphi^{-1}\left[\lambda H_\nabla(N_F(u) - Q(N_F(u))) - Q_\varphi(\lambda H_\nabla(N_F(u) - Q(N_F(u))))\right]). \end{aligned} \tag{3.4}$$

Again, it is seen that M is completely continuous and, moreover, system (3.3) is equivalent to the fixed point problem:

$$u = M(\lambda, u).$$

The following result gives a priori bounds for the possible solutions of the family of boundary value problems (3.3). This result is based on the works by Bereanu and Mawhin [4,5] for the continuous case $\mathbb{T} = \mathbb{R}$.

Lemma 3.4. *Assume there exist $R, \varepsilon > 0$ such that*

$$\int_{\rho(0)}^{T} f(t,\gamma(t,u(t)))\nabla t > \varepsilon \quad \text{if } u_m \geq R,$$

$$\int_{\rho(0)}^{T} f(t,\gamma(t,u(t)))\nabla t < -\varepsilon \quad \text{if } u_M \leq -R$$

and fix $\eta := \frac{\varepsilon}{T-\rho(0)}$. Then there exists $\Theta > 0$ such that if $(\lambda,u) \in [0,1] \times C^1$ verifies $u = M(\lambda,u)$, then $\|u\|_1 < \Theta$.

Proof. Let $(\lambda,u) \in [0,1] \times C^1$ be such that $u = M(\lambda,u)$. Then, taking $t = \rho(0)$ we obtain

$$Q(N_F(u)) = \frac{1}{T-\rho(0)} \int_{\rho(0)}^{T} F(s,u(s))\nabla s = 0, \tag{3.5}$$

which implies

$$\varphi(u^\Delta(t)) = \lambda H_\nabla(N_F(u))(t) - Q_\varphi(\lambda H_\nabla(N_F(u))) \tag{3.6}$$

for all $t \in [\rho(0), T]_{\mathbb{T}}$.

From the definition of $F(t,x)$, we get

$$\begin{aligned} |\lambda H_\nabla(N_F(u))(t)| &\leq \int\limits_{\rho(0)}^{\sigma(T)} \left| f(s,\gamma(s,u(s))) + \eta \frac{u(s)-\gamma(s,u(s))}{1+|u(s)-\gamma(s,u(s))|} \right| \nabla s \\ &\leq \int\limits_{\rho(0)}^{\sigma(T)} |f(s,\gamma(s,u(s))| \nabla s + \eta[\sigma(T)-\rho(0)] \\ &\leq L(\omega+\eta), \end{aligned}$$

with $L := \sigma(T) - \rho(0)$ and

$$\omega := \sup_{s\in[\rho(0),\sigma(T)]_{\mathbb{T}}, \alpha(s)\leq v\leq\beta(s)} |f(s,v)| .$$

Using (3.6) and Lemma 2.2, it is seen, for some constant k, that

$$\left|\varphi(u^\Delta(t))\right| \leq k \quad t \in [\rho(0),T]_{\mathbb{T}}$$

and hence

$$\left|u^\Delta(t)\right| \leq \theta \quad t \in [\rho(0),T]_{\mathbb{T}}\,, \tag{3.7}$$

where $\theta := \max\left\{\left|\varphi^{-1}(k)\right|, \left|\varphi^{-1}(-k)\right|\right\}$.

On the other hand, using (3.5), we obtain

$$\left| \int\limits_{\rho(0)}^{T} f(t,\gamma(t,u(t)))\nabla t \right| \leq \varepsilon.$$

It follows that

$$u_M > -R, \quad u_m < R. \tag{3.8}$$

Using the inequality $u_M \leq u_m + \int_{\rho(0)}^{\sigma(T)} \left|u^\Delta(s)\right| \Delta s$, (3.6) and (3.8), we obtain

$$u_M < R + \theta L.$$

Analogously it can be shown that

$$u_m > -(R+\theta L)\,.$$

Thus,

$$\sup_{[\rho(0),\sigma(T)]_{\mathbb{T}}} |u(t)| < R + \theta L. \tag{3.9}$$

Then using (3.7) and (3.9) we conclude that

$$\|u\|_1 = \sup_{[\rho(0),\sigma(T)]_{\mathbb{T}}} |u(t)| + \sup_{[\rho(0),T]_{\mathbb{T}}} \left|u^\Delta(t)\right| < \Theta,$$

where $\Theta := R + \theta(L+1)$. □

Remark 3.5. In particular, the assumption in the previous lemma is satisfied if

$$\int_{\rho(0)}^{T} f(t,\beta(t))\nabla t > 0 > \int_{\rho(0)}^{T} f(t,\alpha(t))\nabla t$$

with $R \geq \|\alpha\|_\infty, \|\beta\|_\infty$ and $\varepsilon > 0$ sufficiently small. Observe furthermore that, from the definition (2.2) together with the boundary condition and the monotonicity of φ, one has:

$$\int_{\rho(0)}^{T} f(t,\beta(t))\nabla t \geq \varphi(\beta^\Delta(T)) - \varphi(\beta^\Delta(\rho(0))) \geq 0$$

and

$$\int_{\rho(0)}^{T} f(t,\alpha(t))\nabla t \leq \varphi(\alpha^\Delta(T)) - \varphi(\alpha^\Delta(\rho(0))) \leq 0,$$

and equality holds in each case only if β or α are solutions of (1.1), (1.4). We conclude that the conditions in the previous lemma are always fulfilled if α and β are proper.

Remark 3.6. As mentioned in the introduction, our results impose a different condition on the time scale from the ones assumed in previous works, such as a smallness condition on the graininess function μ in [20] or the condition $\nu(0) \geq \mu(T)$ in [22]. Moreover, it is worth noticing that the assumption in the previous lemma depends in fact on the time scale, because it involves the ∇-integral. An elementary example showing this dependence is $f(t,u) = a(t)g(u)$, with $a : \mathbb{R} \to \mathbb{R}_{\geq 0}$ continuous such that its set Z_a of zeros is nonempty and discrete. If g is continuous and satisfies $g(R) > 0 > g(-R)$ then the lemma is verified for $\mathbb{T} = \mathbb{R}$ with $\beta = R$ and $\alpha = -R$. However, if $\mathbb{T} = Z_a$, then the assumptions of Lemma 3.4 cannot be fulfilled. In concordance with the previous remark, this is due to the fact that, in this case, the only possible lower or upper solution is the trivial one.

We are now able to prove an existence theorem for (1.1), (1.4). Let us denote by $\deg_B$ and $\deg_{LS}$ the Brouwer and Leray–Schauder degrees respectively. The following result shows that the Leray–Schauder degree of the solution operator over large balls of the space C^1 is different from zero.

Theorem 3.7. *Suppose that* (1.1), (1.4) *has a proper lower solution* α *and a proper upper solution* β *such that* $\alpha(t) \leq \beta(t)$ *for all* $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$. *Assume that* $f : [\rho(0), \sigma(T)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *is continuous and choose* η *according to Lemma 3.4 and Remark 3.5. Then* $\deg_{LS}(I - M_F, B_\delta(0), 0) = -1$ *for* $\delta \gg 0$*, and the problem* (1.1), (1.4) *has at least one solution.*

Proof. Let M be the operator given by (3.4) and let $\delta \geq \theta$. Then for each $\lambda \in [0,1]$, the Leray–Schauder degree $\deg_{LS}(I - M(\lambda,\cdot), B_\delta(0), 0)$ is well defined, and by the homotopy invariance

$$\deg_{LS}(I - M(0,\cdot), B_\delta(0), 0) = \deg_{LS}(I - M(1,\cdot), B_\delta(0), 0).$$

On the other hand,

$$\deg_{LS}(I - M(0,\cdot), B_\delta(0), 0) = \deg_{LS}(I - (P + QN_F), B_\delta(0), 0).$$

Next, observe that the range of the mapping

$$u \mapsto P(u) + Q(N_F(u))$$

is contained in the subspace of constant functions, isomorphic to $\mathbb{R}$. Thus, using the reduction property of Leray–Schauder degree [11, 19] we obtain:

$$\begin{aligned}&\deg_{LS}(I - (P + QN_F), B_\delta(0), 0)\\&= \deg_B\Big(I - (P + QN_F)\Big|_{\overline{B_\delta(0)\cap\mathbb{R}}}, B_\delta(0)\cap\mathbb{R}, 0\Big)\\&= \deg_B(-QN_F, (-\delta,\delta), 0).\end{aligned}$$

Using the definition of F and setting $\delta > \|\alpha\|_\infty, \|\beta\|_\infty$ we obtain $QN_F(\delta) > 0$ and $QN_F(-\delta) < 0$ which, in turn, implies

$$\deg_B(-QN_F, (-\delta,\delta), 0) = -1.$$

Then, $\deg_{LS}(I - M(1,\cdot), B_\delta(0), 0) = -1$. Hence, there exists $u \in B_\delta(0)$ such that $M(1,\cdot)(u) = M_F(u) = u$, which is a solution for (3.1) and, by Theorem 3.3, (1.1)–(1.4) has at least one solution. □

Remark 3.8. If α and β in Theorem 3.7 are strict, then reasoning as in Theorem 3.3 it is seen that in fact $\alpha(t) < u(t) < \beta(t)$ for all $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$.

Fixing $\delta \gg 0$ and taking $\eta > 0$ small enough, using the addition-excision property of the Leray–Schauder degree, it is seen hat

$$\deg_{LS}(I - M_F, \Omega_{\alpha,\beta}, 0) = \deg_{LS}(I - M_F, B_\delta(0), 0) = -1,$$

where $\Omega_{\alpha,\beta} := \{u \in C^1 : \alpha < u < \beta\}$. Furthermore, as the operator M_f associated to (1.1), (1.4) is equal to M_F on $\overline{\Omega_{\alpha,\beta}}$, we deduce that $\deg_{LS}(I - M_f, \Omega_{\alpha,\beta}, 0) = -1$.

The choice of constant lower and upper solutions in Theorem 3.7 leads to the following existence result.

Corollary 3.9. *Let $f(t,x)$ be continuous on $[\rho(0), \sigma(T)]_{\mathbb{T}} \times \mathbb{R}$. If there exists $R > 0$ such that $f(t,R) > 0 > f(t,-R)$ for all $t \in [0,T]_{\mathbb{T}}$, then the problem* (1.1), (1.4) *has at least one solution.*

Proof. Let $\alpha(t) = -R$ and $\beta(t) = R$ for all $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$. It is easy to check that α and β are strict lower and upper solutions of (1.1), (1.4) such that $\alpha(t) < \beta(t)$ for all $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$. Thus, using Theorem 3.7, we deduce that (1.1), (1.4) has at least one solution. □

Example 3.10. Let $\mathbb{T}$ be any time scale such that $0 \in \mathbb{T}$ is left scattered or right dense and $10 \in \mathbb{T}$ is left dense or right scattered. We consider the following boundary value problem:

$$\begin{cases} \left(\left|u^{\Delta}\right|^{p-2} u^{\Delta}\right)^{\nabla} = e^{-u^2} + 1 + 6ue^{(u^2+t^2)}, \\ u(\rho(0)) = u(\sigma(10)), \\ u^{\Delta}(\rho(0)) = u^{\Delta}(10), \end{cases} \tag{3.10}$$

where $p \in (1, \infty)$. As $f(t, x) = e^{-x^2} + 1 + 6xe^{(x^2+t^2)}$ is continuous with $f(t, 1) > 0$ and $f(t, -1) < 0$ for all $t \in [0, 10]_{\mathbb{T}}$, then by Corollary 3.9, we deduce that (3.10) has at least one solution u with

$$-1 < u(t) < 1, \quad \text{for all } t \in [\rho(0), \sigma(10)]_{\mathbb{T}}.$$

3.3. MULTIPLICITY RESULT

In this section we establish the existence of at least three solutions to problem (1.1), (1.4).

Theorem 3.11. *For $i = 1, 2$, assume there exist α_i and β_i strict lower and upper solutions of* (1.1), (1.4)*, respectively, such that $\alpha_i(t) < \beta_i(t)$, $\alpha_1(t) < \alpha_2(t)$, $\beta_1(t) < \beta_2(t)$ for all $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$, and $\{t \in [\rho(0), \sigma(T)]_{\mathbb{T}} : \alpha_2(t) > \beta_1(t)\} \neq \emptyset$. Then* (1.1), (1.4) *has at least three different solutions u_1, u_2, u_3 such that*

$$\alpha_1(t) < u_3(t) < \beta_2(t), \quad \alpha_i(t) < u_i(t) < \beta_i(t),$$

for all $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$ and $i = 1, 2$.

Proof. Let γ_1, γ_2 and γ_3 be the functions associated to the pairs of lower and upper solutions (α_1, β_1), (α_2, β_2) and (α_1, β_2), respectively. Consider M_{F_1}, M_{F_2} and M_{F_3} the operators associated to the pairs (α_1, β_1), (α_2, β_2) and (α_1, β_2), respectively, with $\eta > 0$ small enough according to Lemma 3.4 and Remark 3.5. As before we deduce, for $\delta \gg 0$, that M_{F_i} has no fixed points in $\overline{B_\delta(0)} \backslash \Omega_i$, with

$$\begin{aligned} \Omega_1 &= \Omega_{\alpha_1,\beta_1} := \{u \in C^1 : \alpha_1 < u < \beta_1\}, \\ \Omega_2 &= \Omega_{\alpha_2,\beta_2} := \{u \in C^1 : \alpha_2 < u < \beta_2\}, \\ \Omega_3 &= \Omega_{\alpha_1,\beta_2} := \{u \in C^1 : \alpha_1 < u < \beta_2\}. \end{aligned}$$

Hence, by Remark 3.8,

$$\deg_{LS}(I - M_{F_1}, \Omega_1, 0) = \deg_{LS}(I - M_{F_2}, \Omega_2, 0) = \deg_{LS}(I - M_{F_3}, \Omega_3, 0) = -1.$$

Since $\alpha_1(t) < \beta_1(t) < \beta_2(t)$, $\alpha_1(t) < \alpha_2(t) < \beta_2(t)$ for all $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$ and $\{t \in [\rho(0), \sigma(T)]_{\mathbb{T}} : \alpha_2(t) > \beta_1(t)\} \neq \emptyset$, we have

$$\Omega_1 \cap \Omega_2 = \emptyset, \quad \Omega_1 \cup \Omega_2 \subset \Omega_3 \text{ and } \Omega_3 \backslash \overline{\Omega_1 \cup \Omega_2} \neq \emptyset.$$

Moreover, for $i = 1, 2$ and $u \in \Omega_i$ it is clear that $\gamma_i(u) = \gamma_3(u) = u$ and thus $M_{F_i}(u) = M_{F_3}(u)$. Hence, from the addition-excision property of the Leray–Schauder degree we obtain

$$\begin{aligned}&\deg_{LS}(I - M_{F_3}, \Omega_3 \backslash \overline{\Omega_1 \cup \Omega_2}, 0)\\ &= \deg_{LS}(I - M_{F_3}, \Omega_3, 0) - \deg_{LS}(I - M_{F_2}, \Omega_2, 0) - \deg_{LS}(I - M_{F_1}, \Omega_1, 0) = 1.\end{aligned}$$

Then (1.1), (1.4) has at least three distinct solutions u_1, u_2, u_3 such that

$$\alpha_1(t) < u_3 < \beta_2(t), \quad \alpha_i(t) < u_i(t) < \beta_i(t),$$

for all $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$ and $i = 1, 2$. □

Example 3.12. Consider the φ-Laplacian version of the forced pendulum equation

$$\begin{cases} (\varphi(u^{\Delta}(t))^{\nabla} + \sin(u(t)) = p(t), \quad t \in [0, T]_{\mathbb{T}}, \\ u(\rho(0)) = u(\sigma(T)), \\ u^{\Delta}(\rho(0)) = u^{\Delta}(T), \end{cases}$$

where p is continuous. If $-1 < p(t) < 1$ for all $t \in [0, T]_{\mathbb{T}}$, then the problem has at least two geometrically distinct solutions u, v, that is, such that $v \neq u + 2k\pi$. Indeed, from the previous theorem, we deduce the existence of distinct u_1, u_2, u_3, with

$$-\frac{3\pi}{2} < u_1(t) < -\frac{\pi}{2}, \qquad \frac{\pi}{2} < u_2(t) < \frac{3\pi}{2}, \qquad -\frac{3\pi}{2} < u_3(t) < \frac{3\pi}{2}.$$

It may happen that $u_2 = u_1 + 2\pi$, but in this case $u_3 \neq u_1, u_1 + 2\pi$.

3.4. MONOTONE ITERATIVE METHODS

Here, we shall extend the results from [22] to a problem for a φ-Laplacian on time scales. To this end, let us define the sector between two elements u, v in the Banach space C^1 as follows:

$$[u, v]_1 = \left\{ w \in C^1 : u \leq w \leq v \right\}.$$

Definition 3.13. A function u^* is a *maximal solution* (resp. u_* is a *minimal solution*) of (1.1), (1.4) in $[\alpha, \beta]_1$ if it is a solution with $\alpha \leq u^* \leq \beta$ on $[\rho(0), \sigma(T)]_{\mathbb{T}}$ (resp. $\alpha \leq u_* \leq \beta$) and every solution w of (1.1), (1.4) verifies $w \leq u^*$ on $[\rho(0), \sigma(T)]_{\mathbb{T}}$ (resp. $u_* \leq w$).

The following result shows the existence of extremal solutions of (1.1), (1.4) under a one-sided growth condition on f.

Theorem 3.14. *Let $f : [\rho(0), \sigma(T)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be continuous and let $\alpha(t)$ and $\beta(t)$ be lower and upper solutions of* (1.1), (1.4) *respectively, with $\alpha \leq \beta$ on $[\rho(0), \sigma(T)]_{\mathbb{T}}$. Assume that there exists $M > 0$ such that for all $t \in [0, T]_{\mathbb{T}}$ and any $u, w \in \mathbb{R}$ with $\alpha(t) \leq w \leq u \leq \beta(t)$ it is verified that*

$$f(t, u) - f(t, w) \leq M(u - w). \tag{3.11}$$

Then problem (1.1), (1.4) *has a maximal solution u^* and a minimal solution u_* in the sector $[\alpha, \beta]_1$ with $u_* \leq u^*$.*

Proof. For any function $z(t)$ which satisfies $\alpha(t) \le z(t) \le \beta(t)$ for $t \in [\rho(0), \sigma(T)]_{\mathbb{T}}$, consider the following boundary value problem on time scales

$$\begin{cases} (\varphi(u^{\Delta}(t)))^{\nabla} = \widetilde{F}(t, z(t)) := f(t, z(t)) + M(u(t) - z(t)), \quad t \in [0, T]_{\mathbb{T}}, \\ u(\rho(0)) = u(\sigma(T)), \\ u^{\Delta}(\rho(0)) = u^{\Delta}(T), \end{cases} \tag{3.12}$$

where (3.12) is equivalent to

$$\begin{aligned} u = \mathcal{T}(z) := u(\rho(0)) + Q(N_{\widetilde{F}}(z)) \\ + H_{\Delta}\left(\varphi^{-1}\left[H_{\nabla}(N_{\widetilde{F}}(z) - Q(N_{\widetilde{F}}(z))) - Q_{\varphi}(H_{\nabla}(N_{\widetilde{F}}(z) - Q(N_{\widetilde{F}}(z))))\right]\right). \end{aligned}$$

Then $u(t)$ is a solution of (1.1)–(1.4) if and only if $\mathcal{T}(u) = u$.

We claim that if $w, u \in [\alpha, \beta]_1$ are such that $w \le u$, then $\mathcal{T}(w) \le \mathcal{T}(u)$. Indeed, let the functions u and w be such that $\alpha \le w \le u \le \beta$ and denote $u_1 := \mathcal{T}(u)$ and $w_1 := \mathcal{T}(w)$. By contradiction, suppose there exists a point where the function $v := u_1 - w_1$ is negative and fix $m \in [\rho(0), \sigma(T)]_{\mathbb{T}}$ such that

$$\min_{[\rho(0), \sigma(T)]_{\mathbb{T}}} v(t) = v(m) < 0.$$

For simplicity, we may assume that $v(m) < v(t)$ for all $t \in [\rho(0), m)_{\mathbb{T}}$ and, consequently, $m < \sigma(T)$. We distinguish between the following cases.

(i) Assume that $m \in (\rho(0), \sigma(T))_{\mathbb{T}}$ is a left-dense point. Since the function v achieves its minimum at m, then it follows as in the proof of Theorem 3.3 that $v^{\Delta}(m) = 0$ and hence $\varphi(u_1^{\Delta}(m)) = \varphi(w_1^{\Delta}(m))$. On the other hand, there exists $t_1 < m$ such that $v(t) \le 0$ for all $t \in [t_1, m]_{\mathbb{T}}$ and $v(t_1) > v(m)$. Taking into account the definition of the operator $\mathcal{T}$, it follows that

$$(\varphi(u_1^{\Delta}(t)))^{\nabla} = f(t, u(t)) + M(u_1(t) - u(t))$$

and

$$(\varphi(w_1^{\Delta}(t)))^{\nabla} = f(t, w(t)) + M(w_1(t) - w(t))$$

for all $t \in [t_1, m]_{\mathbb{T}}$. By (3.11), we obtain

$$\begin{aligned} &\int_t^m (\varphi(u_1^{\Delta}(s)))^{\nabla} \nabla s - \int_t^m (\varphi(w_1^{\Delta}(s)))^{\nabla} \nabla s \\ &= \int_t^m (f(s, u(s)) - f(s, w(s)) + M(u_1(s) - u(s) - w_1(s) + w(s))) \nabla s \\ &\le \int_t^m M(u_1(s) - w_1(s)) \nabla s \le 0 \quad \text{for all } t \in [t_1, m]_{\mathbb{T}}. \end{aligned}$$

Hence $-\varphi(u_1^\Delta(t)) + \varphi(w_1^\Delta(t)) \leq 0$, which implies that $w_1^\Delta \leq u_1^\Delta$ on $[t_1, m]_\mathbb{T}$. Thus, $v^\Delta(t) = (u_1 - w_1)^\Delta(t) \geq 0$ for all $t \in [t_1, m]_\mathbb{T}$, so v is nondecreasing on $[t_1, m]_\mathbb{T}$, contradicting the fact that $v(t_1) > v(m)$.

(ii) Assume that $m \in (\rho(0), \sigma(T))_\mathbb{T}$ is left-scattered. From the minimality of v at m it is seen that

$$v^\Delta(\rho(m)) \leq 0, \;\; v^\Delta(m) \geq 0.$$

Then

$$\varphi(u_1^\Delta(\rho(m))) \leq \varphi(w_1^\Delta(\rho(m))) \tag{3.13}$$

and

$$\varphi(u_1^\Delta(m)) \geq \varphi(w_1^\Delta(m)). \tag{3.14}$$

Using (3.13) and (3.14) we obtain:

$$\left(\varphi(u_1^\Delta)\right)^\nabla(m) \geq \left(\varphi(w_1^\Delta)\right)^\nabla(m).$$

Using again the definition of the operator $\mathcal{T}$ and (3.11) it follows that

$$0 \leq \left(\varphi(u_1^\Delta)\right)^\nabla(m) - \left(\varphi(w_1^\Delta)\right)^\nabla(m) \leq M(u_1(m) - w_1(m)) < 0,$$

a contradiction.

(iii) Assume that $m = \rho(0)$, then v achieves its absolute minimum also at $\sigma(T)$. Observe also that $v^\Delta(T) = v^\Delta(\rho(0)) \geq 0$. Next, consider the following two possibilities:

1. If T is right-scattered, then $v^\Delta(T) \leq 0$ and we conclude that $v^\Delta(T) = 0$. Consequently, $v(T) = v(\sigma(T))$ and hence v achieves its absolute minimum also at T.
2. If T is right-dense, then v achieves its absolute minimum at $t = T$. Due to condition (2.1), it follows that $v^\Delta(T) \leq 0$ which, in turn, implies $v^\Delta(T) = 0$.

In both situations, a contradiction follows as in cases (i) and (ii) replacing m by T.

We conclude that $\mathcal{T}(u) - \mathcal{T}(w) = u_1 - w_1 \geq 0$ on $[\rho(0), \sigma(T)]_\mathbb{T}$, that is, the operator $\mathcal{T}$ is monotone nondecreasing over the sector $[\alpha, \beta]_1$.

Next, define

$$\alpha_0 = \alpha, \; \alpha_{n+1} = \mathcal{T}(\alpha_n) \quad \text{for } n \geq 0,$$

and

$$\beta_0 = \beta, \; \beta_{n+1} = \mathcal{T}(\beta_n), \quad \text{for } n \geq 0.$$

Similarly, it can be shown that

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \ldots \leq \beta_n \leq \ldots \leq \beta_2 \leq \beta_1 \leq \beta_0 = \beta.$$

In other words, the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are bounded and monotone. Using the Arzelà–Ascoli theorem, it is seen that $\mathcal{T}$ is completely continuous. This, together with the boundedness of the defining sequences, implies that there exist some subsequences and functions $u_* \leq u^*$ such that

$$\alpha_{n_k} \to u_* \quad \text{and} \quad \beta_{n_k} \to u^*$$

uniformly on $[\rho(0),\sigma(T)]_{\mathbb{T}}$. But the monotonicity of $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ implies that

$$\alpha_n \to u_* \quad \text{and} \quad \beta_n \to u^*,$$

uniformly and it is readily verified that u_* and u^* are extremal solutions of (1.1)–(1.4). □

An immediate consequence of Theorem 3.14 is the following.

Corollary 3.15. *Assume that $f(t,x)$ is continuous, nonincreasing in $x \in \mathbb{R}$, and that there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of* (1.1), (1.4) *such that $\alpha(t) \le \beta(t)$ for all $t \in [\rho(0),\sigma(T)]_{\mathbb{T}}$. Then the problem* (1.1), (1.4) *has at least one solution in the sector $[\alpha,\beta]_1$.*

Corollary 3.16. *Assume that $\frac{\partial f(t,x)}{\partial x}$ exists and is continuous on $[\rho(0),\sigma(T)]_{\mathbb{T}} \times \mathbb{R}$ and that there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of* (1.1), (1.4) *such that $\alpha(t) \le \beta(t)$ for all $t \in [\rho(0),\sigma(T)]_{\mathbb{T}}$. Then the problem* (1.1), (1.4) *has at least one solution in the sector $[\alpha,\beta]_1$.*

Proof. It is readily verified that condition (3.11) holds, with

$$M := \max\left\{ \left|\frac{\partial f(t,x)}{\partial x}\right| : t \in [\rho(0),\sigma(T)]_{\mathbb{T}},\ a \le x \le b \right\},$$

where $a = \min_{[\rho(0),\sigma(T)]_{\mathbb{T}}} \alpha(t)$ and $b = \max_{[\rho(0),\sigma(T)]_{\mathbb{T}}} \beta(t)$. □

Example 3.17. Consider the following boundary value problem:

$$\begin{cases} \left(\left|u^{\Delta}\right|^{p-2} u^{\Delta}\right)^{\nabla} = \frac{\sin(u+1)-1+4ue^{u^2 t}}{1+t^2}, \\ u(-\frac{1}{4}) = u(\frac{1}{2}), \\ u^{\Delta}(-\frac{1}{4}) = u^{\Delta}(\frac{1}{4}), \end{cases} \tag{3.15}$$

where $\mathbb{T}$ is a time scale such that $-\frac{1}{4} \in \mathbb{T}$ is left scattered or right dense and $\frac{1}{2} \in \mathbb{T}$ right scattered or left dense and $p \in (1,\infty)$. It is easy to check that the functions $\alpha(t) = -1$ and $\beta(t) = 1$ are respectively a lower and an upper solution of (3.15). Thus, by Corollary 3.16, we deduce that (3.15) has at least one solution u such that $-1 \le u(t) \le 1$ for all $t \in [-\frac{1}{4},\frac{1}{2}]_{\mathbb{T}}$.

Remark 3.18. All the results of the previous section hold for Dirichlet and Neumann boundary conditions. In order to verify this, it suffices to define the fixed point operators

(i) for the Dirichlet boundary condition (1.2):

$$M_f(u) = H_{\Delta}\left(\varphi^{-1}\left[H_{\nabla}(N_f(u)) - Q_{\varphi}(H_{\nabla}(N_f(u)))\right]\right),$$

(ii) for the Neumann boundary condition (1.3):

$$M_f(u) = P(u) + Q(N_f(u)) + H_{\Delta}\left(\varphi^{-1}\left[H_{\nabla}(N_f(u) - Q(N_f(u)))\right]\right).$$

In the Dirichlet case, it is seen that the range of the corresponding operator M_F is bounded; thus, by Schauder's Theorem the problem $(\varphi(u^\Delta(t)))^\nabla = F(t, u(t))$ has at least one solution satisfying (1.2) which, in turn, implies the existence of a solution of (1.1), (1.2) between α and β.

If furthermore α and β are strict, then the addition-excision property of the Leray–Schauder degree implies that $\deg_{LS}(I - M_F, B_\delta(0), 0) = 1$ for $\delta \gg 0$. and the multiplicity of solutions is proved as in Theorem 3.11.

For the Neumann conditions, the results (and their proofs) are completely analogous to the periodic case. We remark that the difference between the Dirichlet and Neumann/Periodic conditions relies on the fact that, in the first case, the associated operator is invertible. In other words, the problem $\varphi(u^\Delta)^\nabla = h$ has, for each h, a unique solution satisfying (1.2). This is clearly not the case under conditions (1.3) and (1.4), for which the operator $\varphi(u^\Delta)^\nabla$ is a (nonlinear) zero-index Fredholm operator, namely, the problem $\varphi(u^\Delta)^\nabla = h$ has family of solutions $\{u + c : c \in \mathbb{R}\}$ satisfying the respective boundary conditions if and only if $Qh = 0$.

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