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RETRACTABILITY OF SOLUTIONS TO THE YANG–BAXTER EQUATION AND p -NILPOTENCY OF SKEW BRACES

E. ACRI, R. LUTOWSKI, AND L. VENDRAMIN

ABSTRACT. Using Bieberbach groups we study multipermutation involutive solutions to the Yang–Baxter equation. We use a linear representation of the structure group of an involutive solution to study the unique product property in such groups. An algorithm to find subgroups of a Bieberbach group isomorphic to the Promislow subgroup is introduced and then used in the case of structure group of involutive solutions. To extend the results related to retractability to non-involutive solutions, following the ideas of Meng, Ballester-Bolínches and Romero, we develop the theory of right p -nilpotent skew braces. The theory of left p -nilpotent skew braces is also developed and used to give a short proof of a theorem of Smoktunowicz in the context of skew braces.

INTRODUCTION

In order to construct solutions to the celebrated Yang–Baxter equation, Drinfeld introduced in [22] set-theoretic solutions, i.e. pairs (X, r) where X is a set and $r: X \times X \rightarrow X \times X$ is a bijective map such that

$$(r \times \text{id})(\text{id} \times r)(r \times \text{id}) = (\text{id} \times r)(r \times \text{id})(\text{id} \times r).$$

To address this problem by means of combinatorial methods, one considers non-degenerate solutions, i.e. solutions (X, r) where the bijective map r can be written as

$$r(x, y) = (\sigma_x(y), \tau_y(x)), \quad x, y \in X,$$

for permutations $\sigma_x: X \rightarrow X$ and $\tau_x: X \rightarrow X$. An example of a non-degenerate solution is that of Lyubashenko, where the map r is given by $r(x, y) = (\sigma(y), \tau(x))$ for σ and τ commuting permutations of X .

The first papers on set-theoretic solutions are those of Etingof, Schedler and Soloviev [23] and Gateva–Ivanova and Van den Bergh [31]. Both papers considered involutive solutions, i.e. solutions (X, r) where $r^2 = \text{id}$.

In [45], Rump observed that each radical ring R produces an involutive solution. (A *radical ring* R is a ring such that the Jacobson circle operation $x \circ y = x + y + xy$ turns R into a group.) Then he introduced a new algebraic structure that generalizes radical rings and provides an algebraic framework

Key words and phrases. Bieberbach group, Yang–Baxter equation, set-theoretic solution, multipermutation solution, unique product property, skew brace.

to study involutive solutions. This new structure showed connections between the Yang–Baxter equation and ring theory, flat manifolds, orderability of groups, Garside theory, see for example [5, 18, 19, 21, 27, 30, 46].

As a tool to construct involutive solutions, Etingof, Schedler and Soloviev introduced retractable solutions [23]. Such solutions are those that induce a smaller solution after identifying certain elements of the underlying set. Multipermutation solutions are then those solutions that can be retracted to the trivial solution over the set with only one element after a finite number of steps. This means that multipermutation solutions generalize those solutions of Lyubashenko. Several papers study multipermutation involutive solutions, see for example [4, 7, 13, 28, 29, 47, 48, 50].

Almost all of the ideas used in the theory of involutive solutions can be transported to non-involutive solutions. The algebraic framework now is provided by skew braces [32]. This rich structure shows that the Yang–Baxter equation is related to different topics such as Hopf–Galois extensions, regular subgroups and nil-rings. For that reason, the theory of (skew) braces is intensively studied, see for example [2, 6, 8, 9, 11, 12, 14, 16, 20].

This paper explores the retractability problem for solutions and their applications to the theory of (skew) braces. In the case of finite involutive solutions, this is done by using in different ways the fact that the structure group of the solution is a Bieberbach group. Since different methods are needed to obtain similar results for non-involutive solutions, we follow the ideas of Meng, Ballester–Bolinches and Romero [41] and develop the theory of right p -nilpotent skew left braces. We also study left p -nilpotent skew left braces and (again following Meng, Ballester–Bolinches and Romero) we give a short proof of a theorem of Smoktunowicz [49, Theorem 1.1] related to left nilpotency in the context of skew left braces, see [15, Theorem 4.8].

The paper is organized as follows. Section 1 is devoted to preliminaries on set-theoretic solutions to the Yang–Baxter equation and the theory of skew braces. In Section 2 we recall a faithful linear representation of the structure group of a finite involutive solution constructed by Etingof, Schedler and Soloviev. Structure groups of finite involutive solutions are Bieberbach groups, and with the faithful linear representation constructed we compute explicitly the holonomy group of this Bieberbach group. These results are then applied to the retractability problem of involutive solutions. In Section 3 we study the unique product property for structure groups of involutive solutions. We prove that all structure groups of involutive solutions of size ≤ 7 that are not multipermutation solutions do not have the unique product property. In Section 4 we present an algorithm that detects subgroups of an arbitrary Bieberbach group that are isomorphic to the Promislow group; this algorithm is then used in Theorem 4.7 to prove that all but eight structure groups of involutive solutions of size ≤ 8 that are not multipermutation solutions do not have the unique product property. To extend some of our results to non-involutive solutions different methods are needed. Following the ideas of Meng, Ballester–Bolinches and Romero [41], we introduce right

p -nilpotency of skew left braces of nilpotent type and use this concept to explore retractable non-involutive solutions in Section 5. Finally, in Section 6, again following [41], we study left p -nilpotent skew left braces.

1. PRELIMINARIES

1.1. Set-theoretic solutions to the Yang–Baxter equation. A *set-theoretic solution* to the Yang–Baxter equation (YBE) is a pair (X, r) , where X is a set and $r: X \times X \rightarrow X \times X$ is a bijective map that satisfies

$$(r \times \text{id})(\text{id} \times r)(r \times \text{id}) = (\text{id} \times r)(r \times \text{id})(\text{id} \times r).$$

The solution (X, r) is said to be *finite* if X is a finite set. By convention, we write $r(x, y) = (\sigma_x(y), \tau_y(x))$. We say that (X, r) is non-degenerate if the maps σ_x and τ_x are permutations of X .

Convention 1.1. *By a solution we will mean a non-degenerate solution to the YBE.*

The solution (X, r) is said to be *involutive* if $r^2 = \text{id}$. The *structure group* $G(X, r)$ of (X, r) is defined in [23, 40, 52] as the group with generators $x \in X$ and relations

$$x \circ y = u \circ v \quad \text{whenever } r(x, y) = (u, v).$$

If (X, r) is finite involutive, the group $G(X, r)$ is torsion-free [31]. Moreover, $G(X, r)$ is a Garside group [17]; see [21] or [10] for other proofs.

If (X, r) is involutive, its *permutation group* is the group $\mathcal{G}(X, r)$ generated by the permutations σ_x for $x \in X$. Clearly, $\mathcal{G}(X, r)$ acts on X and $\mathcal{G}(X, r)$ is finite if X is finite. The permutation group of a non-involutive solution was defined by Soloviev in [52].

If (X, r) is an involutive solution and $x, y \in X$, following [23], we say that $x \sim y$ if and only if $\sigma_x = \sigma_y$. Then \sim is an equivalence relation over X that induces a solution $\text{Ret}(X, r)$ over the set X/\sim . We define inductively $\text{Ret}^1(X, r) = \text{Ret}(X, r)$ and $\text{Ret}^{n+1}(X, r) = \text{Ret}(\text{Ret}^n(X, r))$ for $n \geq 1$. An involutive solution (X, r) is said to be *irretractable* if $\text{Ret}(X, r) = (X, r)$ and it is a *multipermutation solution* if there exists n such that $\text{Ret}^n(X, r)$ has only one element. We refer to [34, 39, 51] for some results related to the retractability of non-involutive solutions.

1.2. Skew left braces. We refer to [32] for the theory of skew left braces. A *skew left brace* is a triple $(A, +, \circ)$, where $(A, +)$ and (A, \circ) are groups such that $a \circ (b + c) = a \circ b - a + a \circ c$ holds for all $a, b, c \in A$. We write a' to denote the inverse of the element $a \in A$ with respect to the circle operation. A skew left brace A such that $a \circ b = a + b$ for all $a, b \in A$ is said to be *trivial*. If \mathcal{X} is a property of groups, a skew left brace is said to be of \mathcal{X} -type if its additive group belongs to the class \mathcal{X} . Skew left braces of abelian type are those braces introduced by Rump in [45] to study involutive solutions.

Convention 1.2. *Skew left braces of abelian type will be called left braces.*

If A is a skew left brace, the map $\lambda: (A, \circ) \rightarrow \text{Aut}(A, +)$, $a \mapsto \lambda_a$, where $\lambda_a(b) = -a + a \circ b$, is a group homomorphism. By definition,

$$a \circ b = a + \lambda_a(b), \quad a + b = a \circ \lambda_a^{-1}(b), \quad \lambda_a(a') = -a.$$

Moreover:

$$\begin{aligned} a * (b + c) &= a * b + b + a * c - b, \\ (a \circ b) * c &= a * (b * c) + b * c + a * c, \end{aligned}$$

where $a * b = \lambda_a(b) - b$.

The connection between skew left braces and the YBE is the following: If A is a skew left brace, then the map

$$r_A: A \times A \rightarrow A \times A, \quad r_A(a, b) = (\lambda_a(b), \lambda_a(b)' \circ a \circ b)$$

is a solution of the YBE. Moreover, $r_A^2 = \text{id}$ if and only if A is of abelian type.

If (X, r) is a solution, then the group $G(X, r)$ has a unique skew left brace structure such that

$$r_{G(X,r) \times G(X,r)}(\iota \times \iota) = (\iota \times \iota)r,$$

where $\iota: X \rightarrow G(X, r)$ is the canonical map (which in general is not injective). Moreover, the skew left brace $G(X, r)$ satisfies a universal property: if A is a skew left brace and $f: X \rightarrow A$ is a map such that $r_A(f \times f) = (f \times f)r$, then there exists a unique skew left brace homomorphism $\varphi: G(X, r) \rightarrow A$ such that $\varphi \iota = f$ and $r_A(\varphi \times \varphi) = (\varphi \times \varphi)r_{G(X,r)}$. Similar results appear in a differently language in [23, 40, 52].

Note that the multiplicative group of the skew left brace $G(X, r)$ is the structure group of (X, r) defined in Subsection 1.1.

If (X, r) is an involutive solution, then the permutation group $\mathcal{G}(X, r)$ is a left brace with additive structure given by

$$\lambda_a + \lambda_b = \lambda_a \lambda_{\lambda_a^{-1}(b)}$$

for $a, b \in A$, see for example [3]. An analog result for non-involutive solutions is proved in [2].

A *left ideal* of a skew left brace is a subgroup of the additive group that is stable under the action of λ . It follows that a left ideal of a skew left brace is a subgroup of the multiplicative group of the skew left brace. An *ideal* of a skew left brace is a left ideal that is normal as a subgroup of the additive group and normal as a subgroup of the multiplicative group. A non-zero skew left brace is *simple* if it has only two ideals. The *socle* of a skew left brace A is the ideal $\text{Soc}(A) = \ker \lambda \cap Z(A, +)$, where $Z(A, +)$ denotes the center of the additive group of A .

Notation 1.3. For a finite set X , $\pi(X)$ is the set of prime divisors of $|X|$.

For subsets X and Y of a skew left brace A , we write $X * Y$ to denote the subgroup of $(A, +)$ generated by elements of the form $x * y$, where $x \in X$ and $y \in Y$.

Lemma 1.4. *Let A be a finite skew left brace of nilpotent type and $p \in \pi(A)$. Each Sylow p -subgroup of $(A, +)$ is a left ideal of A .*

Proof. See for example [15, Lemma 4.10]. \square

Lemma 1.4 only works for skew left braces of nilpotent type:

Example 1.5. *Let $G = \{g_j : j \in \mathbb{Z}/6\mathbb{Z}\}$. The operations*

$$g_i + g_j = g_{i+(-1)^i j}, \quad g_i \circ g_j = g_{i+j}$$

turns G into a skew left brace with multiplicative group isomorphic to the cyclic group C_6 and non-nilpotent additive group isomorphic to \mathbb{S}_3 . The Sylow 2-subgroups of $(G, +)$ are not left ideals of G .

Let G be a group and $p \in \pi(G)$ be such that $|G| = p^k m$, where p does not divide m . A Hall p' -subgroup of G is a subgroup of order m .

Lemma 1.6. *Let A be a finite skew left brace of nilpotent type. For each $p \in \pi(A)$, the Hall p' -subgroup of $(A, +)$ given by*

$$A_{p'} = \sum_{q \in \pi(A) \setminus \{p\}} A_q,$$

where each A_q is the q -Sylow subgroup of $(A, +)$, is a normal subgroup of $(A, +)$ and it is a left ideal of A .

Proof. Since $(A, +)$ is nilpotent, $A_{p'}$ is a normal subgroup of $(A, +)$. Moreover, $A_{p'}$ is a left ideal of A by Lemma 1.4 and the fact that the sum of left ideals is a left ideal. \square

A skew left brace A is said to be *right nilpotent* if $A^{(n)} = 0$ for some $n \geq 1$, where $A^{(1)} = A$ and $A^{(n+1)} = A^{(n)} * A$ for $n \geq 1$. Each $A^{(n)}$ is an ideal of A . In [45], Rump introduced the sequence

$$(1.1) \quad 0 = \text{Soc}_0(A) \subseteq \text{Soc}_1(A) \subseteq \cdots \subseteq \text{Soc}_n(A) \subseteq \cdots,$$

and used it to study right nilpotent braces of abelian type and retractability of involutive solutions. In the context of skew left braces, (1.1) is defined recursively as follows: $\text{Soc}_0(A) = 0$ and for each $n \geq 1$, $\text{Soc}_{n+1}(A)$ is the ideal of A containing $\text{Soc}_n(A)$ and such that $\pi(\text{Soc}_{n+1}(A)) = \text{Soc}(\pi(A))$, where $\pi: A \rightarrow A/\text{Soc}_n(A)$ is the canonical map.

For a skew left brace A and $x, y \in A$, we write $[x, y]_+ = x + y - x - y$ to denote the additive commutator of x and y .

Lemma 1.7. *Let A be skew left brace. Then*

$\text{Soc}_{n+1}(A) = \{x \in A : x * a \in \text{Soc}_n(A) \text{ and } [x, a]_+ \in \text{Soc}_n(A) \text{ for all } a \in A\}$
for all $n \in \mathbb{N}$.

Proof. It is straightforward. \square

Lemma 1.8. *Let A be a skew left brace of nilpotent type. Then A is right nilpotent if and only if $A = \text{Soc}_n(A)$ for some $n \in \mathbb{N}$.*

Proof. It follows from [15, Lemmas 2.15 and 2.16]. \square

A skew left brace is said to be *left nilpotent* if $A^n = 0$ for some n , where $A^1 = A$ and $A^{n+1} = A * A^n$ for $n \geq 1$. Each A^n is a left ideal of A . We refer to [11, 15, 41, 45, 48, 49, 51] for results on left nilpotent skew left braces.

2. BIEBERBACH GROUPS

We refer to [53] for the theory of Bieberbach groups. A group G is said to be an n -dimensional *Bieberbach group* if it is torsion free and contains an abelian normal subgroup $A \simeq \mathbb{Z}^n$ of finite index such that $C_G(A) = A$, where

$$C_G(A) = \{g \in G : ga = ag \text{ for all } a \in A\}.$$

Thus G fits into the exact sequence

$$0 \rightarrow A \rightarrow G \xrightarrow{p} P \rightarrow 1,$$

where $P = G/A$ is a finite group. The condition $C_G(A) = A$ is equivalent to the faithfulness of the action $h: P \rightarrow \text{Aut}(A)$, $h(y)(a) = xax^{-1}$, where $a \in A$ and $x \in G$ is such that $p(x) = y$ induced by the conjugation action of G over A . In the theory of Bieberbach groups, P is known as the *holonomy group* of G , the map h as the *holonomy representation* of G and A as the *translation subgroup* of G .

The group G can be seen as a discrete subgroup of the isometries of a finite-dimensional euclidean space, that is $G \subseteq \mathcal{O}_n(\mathbb{R}) \times \mathbb{R}^n$ for some n . In this case, the *translation subgroup* A can be seen as $G \cap \mathbb{R}^n$, see [24, page 533].

In [31, Theorem 1.6], Gateva-Ivanova and Van den Bergh proved that if (X, r) is a finite involutive solution, then the structure group of $G(X, r)$ is a Bieberbach group of dimension $|X|$. The holonomy group of $G(X, r)$ will be computed in Theorem 2.2. First we need a faithful representation of $G(X, r)$ that allows us to deal with these groups as subgroups of the isometries of an euclidean space. The following result goes back to Etingof, Schedler and Soloviev, see [23].

Theorem 2.1. *Let (X, r) be a finite involutive solution of size n . Then there exists an injective group homomorphism $G(X, r) \rightarrow \mathcal{O}_n(\mathbb{R}) \times \mathbb{R}^n$. In particular, $G(X, r)$ is isomorphic to a subgroup of $\mathbf{GL}(n+1, \mathbb{Z})$.*

Proof. Let \mathbb{S}_X denote the group of permutations of X and let \mathbb{Z}^X be the free abelian group spanned by $\{t_x : x \in X\}$. Let $M_X = \mathbb{S}_X \times \mathbb{Z}^X$ be the semidirect product associated with the action of \mathbb{S}_X on \mathbb{Z}^X . By Propositions 2.3 and 2.4 of [23], the map $X \rightarrow M_X$, $x \mapsto (\sigma_x, t_x)$, extends to an injective group homomorphism $G(X, r) \rightarrow M_X$. Using permutation matrices we see \mathbb{S}_X as a subgroup of $\mathcal{O}_n(\mathbb{Z}) \subseteq \mathcal{O}_n(\mathbb{R})$. Then, since $\mathbb{Z}^X \simeq \mathbb{Z}^n \subseteq \mathbb{R}^n$, it follows that M_X is isomorphic to a subgroup of the semidirect product $\mathcal{O}_n(\mathbb{R}) \times \mathbb{R}^n$. Since the multiplication of $\mathcal{O}_n(\mathbb{R}) \times \mathbb{R}^n$ is given by

$$(A, a)(B, b) = (AB, a + Ab),$$

after identifying each $(A, a) \in M_X$ with the matrix $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \in \mathbf{GL}(n+1, \mathbb{Z})$, the claim follows. \square

Notice that under this identification, we can see at the socle of $G(X, r)$ as the translation subgroup, i.e. it is the set of elements of $\mathcal{O}_n(\mathbb{R}) \times \mathbb{R}^n$ of the form $(I, a) \in M_X \subset \mathbf{GL}(n+1, \mathbb{Z})$. Furthermore,

$$C_{G(X,r)}(\text{Soc}(G(X, r))) = \text{Soc}(G(X, r)).$$

Since $\text{Soc}(G(X, r))$ is abelian,

$$\text{Soc}(G(X, r)) \subseteq C_{G(X,r)}(\text{Soc}(G(X, r))).$$

Now for every $(I, x) \in \text{Soc}(G(X, r))$ and $(A, a) \in C_{G(X,r)}(\text{Soc}(G(X, r)))$ we have

$$(A, a)(I, x)(A^{-1}, -A^{-1}a) = (I, Ax) = (I, x),$$

so $Ax = x$ holds for all elements of the set $\{x \in \mathbb{R}^n : (I, x) \in \text{Soc}(G(X, r))\}$. But by the first Bieberbach theorem (see [53, Theorem 2.1]) this set spans \mathbb{R}^n , hence $(A, a) \in C_{G(X,r)}(\text{Soc}(G(X, r)))$ if and only if $A = I$. Thus the only elements of the group that centralizes the socle are exactly the elements of the socle.

We know from [31, Theorem 1.6] that the structure group of a solution is Bieberbach. As a direct consequence of the first Bieberbach Theorem, the socle is the subgroup of pure translations and it is torsion-free and maximal normal abelian subgroup of finite index. So, the holonomy group is exactly the permutation group of the solution. The holonomy representation h is the action by conjugation of $G(X, r)$ over the socle that descends to a faithful representation. We summarize this result in the following theorem for the sequel.

Theorem 2.2. *Let (X, r) be a finite involutive solution. Then $G(X, r)$ is a Bieberbach group with holonomy group isomorphic to $\mathcal{G}(X, r)$.*

2.1. Applications to the YBE. Multipermutation solutions are related to orderability of groups. Jespers and Okniński proved in [35, Proposition 4.2] that the structure group of a finite involutive multipermutation solution is poly- \mathbb{Z} and hence left orderable. Independently in [18, Theorem 2] Chouraqui, interested in studying left orderability of structure groups of involutive solutions, proved the same result. It was proved later in [5, Theorem 2.1] that a finite involutive solution is multipermutation if and only if its structure group is left orderable. A group G is said to be *diffuse* if for each finite non-empty subset A of G there exists an element $a \in A$ such that for all $g \in G$, $g \neq 1$, either $ga \notin A$ or $g^{-1}a \notin A$. In [39, Theorem 7.12] it is proved that structure groups of finite non-degenerate involutive solutions are left orderable if and only if they are diffuse. We collect all these facts in the following theorem.

Theorem 2.3. *Let (X, r) be a finite involutive solution. The following statements are equivalent:*

- (1) (X, r) is a multipermutation solution.
- (2) $G(X, r)$ is poly- \mathbb{Z} .
- (3) $G(X, r)$ is left orderable.
- (4) $G(X, r)$ is diffuse.

As an application of Theorem 2.3 we obtain the following particular case of a theorem proved by Cedó, Jespers and Okniński in [13] and by Cameron and Gateva-Ivanova in [29]. For a direct proof (without the finiteness assumption), see [44, Proposition 10].

Corollary 2.4. *Let (X, r) be a finite involutive solution. If $\mathcal{G}(X, r)$ is cyclic, then (X, r) is a multipermutation solution.*

Proof. Since X is finite, the group $G(X, r)$ is finitely generated. It is torsion-free and $\text{Soc}(G(X, r))$ is an abelian normal subgroup such that

$$G(X, r)/\text{Soc}(G(X, r)) \simeq \mathcal{G}(X, r)$$

is cyclic. This implies that $G(X, r)$ is left orderable [42, Lemma 13.3.1] and hence (X, r) is a multipermutation solution by Theorem 2.3. \square

Diffuse groups allow us to obtain a generalization of Corollary 2.4:

Theorem 2.5. *Let (X, r) be a finite involutive solution such that all Sylow subgroups of $\mathcal{G}(X, r)$ are cyclic. Then (X, r) is a multipermutation solution.*

Proof. By Theorem 2.2, the structure group $G(X, r)$ is a Bieberbach group with holonomy group isomorphic to $\mathcal{G}(X, r)$. Since all Sylow subgroups of $\mathcal{G}(X, r)$ are cyclic, all Bieberbach groups with holonomy group isomorphic to $\mathcal{G}(X, r)$ are diffuse by [37, Theorem 3.5]. In particular, $G(X, r)$ is diffuse and hence the claim follows from Theorem 2.3. \square

The converse of Theorem 2.5 does not hold:

Example 2.6. *Let $X = \{1, 2, 3, 4\}$ and $r(x, y) = (\varphi_x(y), \varphi_y(x))$, where*

$$\varphi_1 = \varphi_2 = \text{id}, \quad \varphi_3 = (34), \quad \varphi_4 = (12)(34).$$

Then (X, r) is an involutive multipermutation solution. One easily checks that $\mathcal{G}(X, r) \simeq C_2 \times C_2$.

Let us apply Theorem 2.5 to finite left braces. The following result of Rump appears in [45, Proposition 7] without the finiteness assumption:

Lemma 2.7. *Let A be a finite left brace. Then (A, r_A) is an involutive solution such that $\mathcal{G}(A, r_A) \simeq A/\text{Soc}(A)$.*

Proof. We only need to prove that $A/\text{Soc}(A) \simeq \mathcal{G}(A, r_A)$. The permutation group $\mathcal{G}(A, r_A) = \{\lambda_a : a \in A\}$ is a left brace where the additive structure is given by $\lambda_a + \lambda_b = \lambda_a \lambda_{\lambda_a^{-1}(b)}$ for $a, b \in A$. This implies that the map $\lambda: (A, \circ) \rightarrow \text{Aut}(A, +)$, $a \mapsto \lambda_a$, is a left brace homomorphism and hence

$$A/\text{Soc}(A) \simeq \lambda(A) = \{\lambda_a : a \in A\} = \mathcal{G}(A, r_A)$$

by the first isomorphism theorem. \square

As an application of Theorem 2.5 we obtain the following result related to the structure of left braces:

Theorem 2.8. *Let A be a finite left brace. If all Sylow subgroups of the multiplicative group of A are cyclic, then A is right nilpotent.*

Proof. If (A, \circ) has Sylow cyclic subgroups, then $(A/\text{Soc}(A), \circ)$ has cyclic Sylow subgroups. By Lemma 2.7, $A/\text{Soc}(A) \simeq \mathcal{G}(A, r_A)$ as left braces. In particular, $\mathcal{G}(A, r_A)$ has cyclic Sylow subgroups and therefore (A, r_A) is a multipermutation solution by Theorem 2.5. Now the claim follows from [11, Proposition 6]. \square

The following consequence of Theorem 2.8 is immediate:

Corollary 2.9. *Let A be a non-trivial finite left brace. If all Sylow subgroups of the multiplicative group are cyclic, then A is not simple.*

It is natural to ask whether Theorems 2.5 and 2.8 can be proved for groups with abelian Sylow subgroups. The following example answers this question negatively.

Example 2.10. *There exists a unique simple left brace of size 72, see [12, Remark 4.5] and [38, Proposition 4.3]. The multiplicative group of this left brace is isomorphic to $\mathbb{A}_4 \times \mathbb{S}_3$ and therefore all of its Sylow subgroups are abelian. Since the socle of this left brace is trivial, the canonical solution to the YBE associated with this left brace is not a multipermutation solution (moreover, it is irretractable).*

In Section 5, using the techniques of [41] and skew left braces of nilpotent type we will generalize the results of this section to non-involutive solutions.

Example 2.11. *Let $G = \{g_j : j \in \mathbb{Z}/8\mathbb{Z}\}$. The operations*

$$g_i + g_j = g_{i+(-1)^i j}, \quad g_i \circ g_j = g_{i+j}$$

turns G into a skew left brace with multiplicative group isomorphic to the cyclic group C_8 of eight elements and nilpotent (non-abelian) additive group isomorphic to the dihedral group \mathbb{D}_8 of eight elements. A direct calculation shows that G is right nilpotent.

3. GROUPS WITH THE UNIQUE PRODUCT PROPERTY

This section is devoted to study the unique product property in structure groups of involutive solutions. Recall that a group G has the *unique product property* if for all finite non-empty subsets A and B of G there exists $x \in G$ that can be written uniquely as $x = ab$ with $a \in A$ and $b \in B$. We refer to [42] for more information related to the unique product property.

It is natural to ask when $G(X, r)$ has the unique product property, see [39, Section 8]. If (X, r) is a multipermutation solution, then $G(X, r)$ has the unique product property since $G(X, r)$ is left orderable.

TABLE 3.1. The number of (not multipermutation) involutive solutions.

n	1	2	3	4	5	6	7	8
solutions	1	2	5	23	88	595	3456	34528
not multipermutation	0	0	0	2	4	41	161	2375

All involutive solutions of size ≤ 8 were constructed by Etingof, Schedler and Soloviev in [23]. There are 38698 solutions and among them only 2583 are not multipermutation solutions, see Table 3.1. Our aim is to know when the structure group of a not multipermutation involutive solution does not have the unique product property. We start with the following observation made by Jespers and Okniński:

Proposition 3.1. *Let $X = \{1, 2, 3, 4\}$ and $r(x, y) = (\sigma_x(y), \tau_y(x))$ be the irretractable involutive solution given by*

$$\begin{aligned} \sigma_1 &= (34), & \sigma_2 &= (1324), & \sigma_3 &= (1423), & \sigma_4 &= (12), \\ \tau_1 &= (24), & \tau_2 &= (1432), & \tau_3 &= (1234), & \tau_4 &= (13). \end{aligned}$$

The structure group $G(X, r)$ with generators x_1, x_2, x_3, x_4 and relations

$$\begin{aligned} x_1x_2 &= x_2x_4, & x_1x_3 &= x_4x_2, & x_1x_4 &= x_3^2, \\ x_2x_1 &= x_3x_4, & x_2^2 &= x_4x_1, & x_3x_1 &= x_4x_3. \end{aligned}$$

does not have the unique product property.

Proof. See [36, Example 8.2.14]. □

To prove Proposition 3.1 Jespers and Okniński found a subgroup of the structure group isomorphic to the Promislow subgroup. This idea motivates the results of this section.

Proposition 3.2. *Let $X = \{1, 2, 3, 4\}$ and $r(x, y) = (\sigma_x(y), \tau_y(x))$ be the irretractable involutive solution given by*

$$\begin{aligned} \sigma_1 &= (12), & \sigma_2 &= (1324), & \sigma_3 &= (34), & \sigma_4 &= (1423), \\ \tau_1 &= (14), & \tau_2 &= (1243), & \tau_3 &= (23), & \tau_4 &= (1342). \end{aligned}$$

Then the group $G(X, r)$ with generators x_1, x_2, x_3, x_4 and relations

$$\begin{aligned} x_1^2 &= x_2x_4, & x_1x_3 &= x_3x_1, & x_1x_4 &= x_4x_3, \\ x_2x_1 &= x_3x_2, & x_2^2 &= x_4^2, & x_3^2 &= x_4x_2. \end{aligned}$$

does not have the unique product property.

Proof. Let $x = x_1x_2^{-1}$ and $y = x_1x_3^{-1}$ and

$$(3.1) \quad S = \{x^2y, y^2x, xyx^{-1}, (y^2x)^{-1}, (xy)^{-2}, y, (xy)^2x, (xy)^2, (xyx)^{-1}, yxy, y^{-1}, x, xyx, x^{-1}\}.$$

To prove that $G(X, r)$ does not have the unique product property it is enough to prove that each $s \in S^2 = \{s_1 s_2 : s_1, s_2 \in S\}$ admits at least two different decompositions of the form $s = ab = uv$ for $a, b, u, v \in S$. To perform these calculations we use the injective group homomorphism $G \rightarrow \mathbf{GL}(5, \mathbb{Z})$ of Theorem 2.1,

$$\begin{aligned} x_1 &\mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & x_2 &\mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ x_3 &\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & x_4 &\mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

This faithful representation of $G(X, r)$ allows us to compute all possible products of the form $s_1 s_2$ for all $s_1, s_2 \in S$. By inspection, each element of S^2 admits at least two different representations. \square

Remark 3.3. *The solutions of Propositions 3.1 and 3.2 are the only two involutive solutions of size four that are not multipermutation solutions. Therefore structure groups of involutive solutions of size four that are not multipermutation solutions do not have the unique product property.*

Remark 3.4. *The set (3.1) appears in the work of Promislow [43].*

Remark 3.5. *The technique used to prove Proposition 3.2 could be used to prove Proposition 3.1.*

Proposition 3.6. *Let $G(X, r)$ be the structure group of a not multipermutation involutive solution of size ≤ 7 . Then $G(X, r)$ does not have the unique product property.*

Proof. The proof is a case-by-case analysis using the technique used to prove Proposition 3.2 and the list of solutions of size ≤ 7 of [23]. In several cases, the elements x and y that realize the set (3.1) were found after a random search. \square

In principle, the argument used to prove Propositions 3.2, 3.6 and 3.7 could be used for solutions of size eight. The following solution appeared in [54] as a counterexample to a conjecture of Gateva-Ivanova related to the retractability of square-free solutions, see [26, 2.28(1)].

Proposition 3.7. *Let $X = \{1, \dots, 8\}$ and $r(x, y) = (\varphi_x(y), \varphi_y(x))$ be the irretractable involutive solution given by*

$$\begin{aligned} \varphi_1 &= (78), & \varphi_2 &= (56), & \varphi_3 &= (25)(46)(78), & \varphi_4 &= (17)(38)(56), \\ \varphi_5 &= (24), & \varphi_6 &= (17)(24)(38), & \varphi_7 &= (13), & \varphi_8 &= (13)(25)(46). \end{aligned}$$

Then $G(X, r)$ does not have the unique product property.

Proof. Let $x = x_4 x_2^{-1} x_1 x_3^{-1}$ and $y = x_1 x_2^{-1} x_3 x_1^{-1} x_4 x_1^{-1}$. (These elements were found after a random search.) The injective group homomorphism $G(X, r) \rightarrow \mathbf{GL}(9, \mathbb{Z})$ of Theorem 2.1 allows us to use the set (3.1) to prove that $G(X, r)$ does not have the unique product property. \square

TABLE 3.2. Some solutions that retract to the solution of Proposition 3.1.

x	σ_x	τ_x	σ_x	τ_x
1	(3745)	(3648)	(3745)(68)	(3648)(57)
2	(3745)	(3648)	(3745)(68)	(3648)(57)
3	(1826)	(1527)	(1826)(57)	(1527)(68)
4	(1826)	(1527)	(1826)(57)	(1527)(68)
5	(13872465)	(16542873)	(1465)(2387)	(1654)(2873)
6	(17842563)	(13562478)	(1784)(2563)	(1478)(2356)
7	(13872465)	(16542873)	(1465)(2387)	(1654)(2873)
8	(17842563)	(13562478)	(1784)(2563)	(1478)(2356)
1	(12)(4675)	(12)(3685)	(12)(35)(4867)	(12)(3857)(46)
2	(12)(4675)	(12)(3685)	(12)(35)(4867)	(12)(3857)(46)
3	(1435)(2786)	(1578)(2643)	(16582437)	(17652843)
4	(1587)(2634)	(1345)(2876)	(17562834)	(15682347)
5	(1823)(56)	(1724)(56)	(16582437)	(17652843)
6	(1823)(56)	(1724)(56)	(17562834)	(15682347)
7	(1587)(2634)	(1345)(2876)	(1325)(46)(78)	(1426)(35)(78)
8	(1435)(2786)	(1578)(2643)	(1325)(46)(78)	(1426)(35)(78)

There are solutions of size eight where our technique does not seem to work. One of these solutions appears in the following example:

Example 3.8. Let $X = \{1, \dots, 8\}$ and $r(x, y) = (\sigma_x(y), \tau_y(x))$, where

$$\begin{aligned} \sigma_1 = \sigma_2 &= (3745), & \tau_1 = \tau_2 &= (3648), \\ \sigma_3 = \sigma_4 &= (1826), & \tau_3 = \tau_4 &= (1527), \\ \sigma_5 = \sigma_7 &= (13872465), & \tau_5 = \tau_7 &= (16542873), \\ \sigma_6 = \sigma_8 &= (17842563), & \tau_6 = \tau_8 &= (13562478). \end{aligned}$$

Then (X, r) is an involutive solution that retracts to the solution of Proposition 3.1. In particular, (X, r) is not a multipermutation solution.

Table 3.2 shows four involutive solutions that retract to the solution of Proposition 3.1 and where our technique does not seem to work; the solution of Example 3.8 is the first entry of Table 3.2. In Table 3.3 one finds four involutive solutions that retract to the solution of Proposition 3.2 and where our technique does not seem to work. We do not know whether the structure groups of the solutions of Tables 3.2 and 3.3 have the unique product property.

4. FINDING PROMISLOW SUBGROUPS

In this section we explain the general theory we will use to find subgroups isomorphic to the Promislow group in a given Bieberbach group. The Promislow group was the first example of a torsion-free group that does not have the unique product property, see [43].

TABLE 3.3. Some solutions that retract to the solution of Proposition 3.2.

x	σ_x	τ_x	σ_x	τ_x
1	(12)(78)	(14)(67)	(12)(35)(46)(78)	(14)(28)(35)(67)
2	(1584)(2673)	(1265)(3784)	(1324)(5867)	(1243)(5786)
3	(34)(56)	(23)(58)	(17)(28)(34)(56)	(17)(23)(46)(58)
4	(1485)(2376)	(1562)(3487)	(1423)(5768)	(1342)(5687)
5	(34)(56)	(23)(58)	(17)(28)(34)(56)	(17)(23)(46)(58)
6	(1485)(2376)	(1562)(3487)	(1423)(5768)	(1342)(5687)
7	(12)(78)	(14)(67)	(12)(35)(46)(78)	(14)(28)(35)(67)
8	(1584)(2673)	(1265)(3784)	(1324)(5867)	(1243)(5786)
1	(13687542)	(13867524)	(1652)	(1854)
2	(17)(2583)(46)	(1278)(35)(46)	(17645328)	(12835647)
3	(18657243)	(16857423)	(3874)	(2367)
4	(1476)(28)(35)	(17)(28)(3456)	(14635827)	(17825346)
5	(18657243)	(16857423)	(1652)	(1854)
6	(1476)(28)(35)	(17)(28)(3456)	(17645328)	(12835647)
7	(13687542)	(13867524)	(3874)	(2367)
8	(17)(2583)(46)	(1278)(35)(46)	(14635827)	(17825346)

Lemma 4.1. *Let P be the Promislow group*

$$\langle x, y \mid x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2} \rangle.$$

Then $A = \langle x^2, y^2, (xy)^2 \rangle$ is a normal free abelian subgroup of P of rank 3 with P/A isomorphic to the Klein group. Furthermore, P is torsion-free and not left orderable.

Proof. See for example [42, Lemma 13.3.3]. □

Let $\Gamma \subseteq \mathbf{GL}(n, \mathbb{Z}) \times \mathbb{Z}^n$ be a Bieberbach group defined by the following short exact sequence

$$(4.1) \quad 0 \longrightarrow L \longrightarrow \Gamma \xrightarrow{\pi} \Gamma/L \longrightarrow 0.$$

Here $L \subseteq \mathbb{Z}^n$ is taken such that $\{I\} \times L$ is the maximal normal abelian subgroup of Γ , where I denotes the identity matrix in $\mathbf{GL}(n, \mathbb{Z})$ and π is the canonical map, i.e. $\pi(A, a) = A$.

We say that elements x, y of a group G satisfy (P) if and only if

$$(P) \quad x^2y = yx^{-2} \quad \text{and} \quad y^2x = xy^{-2}$$

holds in G .

Lemma 4.2. *Let $\alpha = (A, a)$ and $\beta = (B, b)$ be elements of Γ that generate a subgroup isomorphic to P . Then the following statements hold:*

- (1) $A \neq I$ and $B \neq I$.
- (2) A and B satisfy (P).

Proof. We have the following short exact sequence:

$$0 \longrightarrow \langle \alpha^2, \beta^2, (\alpha\beta)^2 \rangle \longrightarrow P \longrightarrow C_2^2 \longrightarrow 0.$$

To prove that $A \neq I$ and $B \neq I$, let us assume that $A = I$. Let $k \in \mathbb{N}$ be such that $\beta^{2k} = (I, b')$; this is possible because $\pi(\beta) = B$ lies on a finite group. Then

$$\beta^{-2k} = \alpha^{-1} \beta^{2k} \alpha = (I, -a)(I, b')(I, a) = (I, b') = \beta^{2k},$$

a contradiction since Γ is torsion free.

To prove that A and B satisfy (P) just notice that $\pi(\alpha) = A$ and $\pi(\beta) = B$ and that α, β satisfy (P). \square

We will make use of two Laurent polynomials

$$P_1(X, Y) = 1 + X + YX^{-1} + YX^{-2}, \quad P_2(X, Y) = -1 + X^2.$$

Lemma 4.3. *Let G be a group and $A, B \in G$ be two elements that satisfy (P). Let (A, v) and (B, w) be any pair of elements of Γ that projects to A and B respectively. If*

$$\begin{bmatrix} P_1(A, B) & P_2(A, B) \\ P_2(B, A) & P_1(B, A) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} P_1(A, B) & P_2(A, B) \\ P_2(B, A) & P_1(B, A) \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

has an integral solution $x, y \in L$, then

$$\alpha = (A, x + v) \quad \text{and} \quad \beta = (B, y + w)$$

satisfy (P).

Proof. We prove that $\alpha^2\beta = \beta\alpha^{-2}$. By assumption, $A^2B = BA^{-2}$. Then, using the identification of α and β as matrices, we see that $\alpha^2\beta = \beta\alpha^{-2}$ is equivalent to $P_1(A, B)(x + v) + P_2(A, B)(y + w) = 0$, which is true by hypothesis. Similarly one proves that $\beta^2\alpha = \alpha\beta^{-2}$. \square

Proposition 4.4. *Let Γ be a group defined by a short exact sequence as (4.1). Let $\alpha, \beta \in \Gamma$ be such that $\pi(\alpha) \neq I$, $\pi(\beta) \neq I$. If α and β satisfy (P), then they generate a subgroup of Γ isomorphic P .*

Proof. Let $P = \langle \alpha, \beta \rangle$, $L_P = \langle a, b, c \rangle$ where $a = \alpha^2, b = \beta^2, c = (\alpha\beta)^2$. Then P is a Bieberbach group which fits into the short exact sequence

$$0 \longrightarrow L_P \longrightarrow P \longrightarrow C_2^2 \longrightarrow 0.$$

L_P is an abelian subgroup of Γ , hence it is free abelian and it is maximal normal abelian subgroup of P . It is enough to show that L_P is of rank 3. Let n_a, n_b, n_c be integers such that $a^{n_a} b^{n_b} c^{n_c} = 1$. Conjugation by α leaves $a^{n_a} b^{-n_b} c^{-n_c} = 1 = a^{n_a} b^{n_b} c^{n_c}$ and hence $b^{2n_b} c^{2n_c} = 1$. Now, conjugation by β gives us $c^{4n_c} = 1$. Since Γ is a torsion free group, we conclude that $n_a = n_b = n_c = 0$. \square

Remark 4.5. *Calculations of the previous proposition are easily checked using the representation from [25, Lemma 1] that we state here for completeness:*

$$\left\langle \alpha = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle \subseteq \mathbf{GL}(4, \mathbb{Q}).$$

We now present an algorithm for finding subgroups (of a Bieberbach group) that are isomorphic to the Promislow group:

Algorithm 4.6. *Let $\Gamma \subseteq \mathbf{GL}(n, \mathbb{Z}) \times \mathbb{Z}^n$ be a Bieberbach group defined by the following short exact sequence*

$$0 \longrightarrow L \longrightarrow \Gamma \xrightarrow{\pi} \Gamma/L \longrightarrow 0,$$

where $L \subseteq \mathbb{Z}^n$ is taken such that $\{I\} \times L$ is the maximal normal abelian subgroup of Γ and π is the canonical map.

- (1) Check all pairs $A, B \in G \setminus \{1\}$ that satisfy (P).
- (2) Determine preimages $(A, v) \in \pi^{-1}(A)$ and $(B, w) \in \pi^{-1}(B)$.
- (3) Check if the linear system of Lemma 4.3 has integer solutions. By Proposition 4.4, the existence of such solutions is equivalent to the existence of a subgroup isomorphic to P .

As an application, we obtain the following improvement of Proposition 3.6:

Theorem 4.7. *Let $G(X, r)$ be the structure group of a not multipermutation involutive solution of size ≤ 8 . Then $G(X, r)$ contains a subgroup isomorphic to the Promislow subgroup if and only if (X, r) is not isomorphic to the solutions of Tables 3.2 and 3.3.*

Proof. The proof is a case-by-case analysis using Algorithm 4.6 and the list of involutive solutions of [23]. \square

5. RIGHT p -NILPOTENT SKEW LEFT BRACES

Let A be a skew left brace. For subsets X and Y of A we define inductively $R_0(X, Y) = X$ and $R_{n+1}(X, Y)$ as the additive subgroup generated by $R_n(X, Y) * Y$ and $[R_n(X, Y), Y]_+$ for $n \geq 0$.

Lemma 5.1. *Let I be an ideal of a skew left brace A . Then $R_{n+1}(I, A) \subseteq R_n(I, A)$ for all $n \geq 0$.*

Proof. We proceed by induction on n . The case $n = 0$ is trivial as I is an ideal of A . Let us assume that the claim holds for some $n \geq 0$. Since by the inductive hypothesis $R_n(I, A) * A \subseteq R_{n-1}(I, A) * A \subseteq R_n(I, A)$ and

$$[R_n(I, A), A]_+ \subseteq [R_{n-1}(I, A), A]_+ \subseteq R_n(I, A),$$

it follows that $R_{n+1}(I, A) \subseteq R_n(I, A)$. \square

Proposition 5.2. *Let I be an ideal of a skew left brace A . Then each $R_n(I, A)$ is an ideal of A .*

Proof. We proceed by induction on n . The case where $n = 0$ follows from the fact that I is an ideal of A . So assume that the result holds for some $n \geq 0$. We first prove that $R_{n+1}(I, A)$ is a normal subgroup of $(A, +)$. Let $a, b \in A$ and $x \in R_n(I, A)$. Then

$$a + x * b - a = -x * a + x * (a + b) \in R_{n+1}(I, A),$$

by definition. Since moreover

$$a + (x + b - x - b) - a = (a + x - a) + (a + b - a) - (a + x - a) - (a + b - a) \in R_{n+1}(I, A)$$

by the inductive hypothesis, it follows that $R_{n+1}(I, A)$ is a normal subgroup of $(A, +)$.

We now prove that

$$(5.1) \quad \lambda_a(R_{n+1}(I, A)) \subseteq R_{n+1}(I, A)$$

for all $a \in A$. Using the inductive hypothesis and that each $\lambda_a \in \text{Aut}(A, +)$,

$$\lambda_a(x * b) = (a \circ x \circ a') * \lambda_a(b) \in R_{n+1}(I, B)$$

and

$$\begin{aligned} \lambda_a([R_n(I, A), A]_+) &\subseteq [\lambda_a(R_n(I, A)), \lambda_a(A)]_+ \\ &\subseteq [R_n(I, A), A]_+ \subseteq R_{n+1}(I, A), \end{aligned}$$

equality (5.1) follows.

Since $R_{n+1}(I, A) \subseteq R_n(I, A)$ by Lemma 5.1,

$$R_{n+1}(I, A) * A \subseteq R_n(I, A) * A \subseteq R_{n+1}(I, A).$$

Hence the claim follows from [15, Lemma 1.9]. \square

Lemma 5.3. *Let A be a skew left brace, X be a subset of A and $n, m \in \mathbb{N}$. Then $R_m(X, A) \subseteq \text{Soc}_n(A)$ if and only if $X \subseteq \text{Soc}_{m+n}(A)$.*

Proof. We proceed by induction on m . The case where $m = 0$ is trivial, so assume that the result is valid for some $m \geq 0$. Note that $R_{m+1}(X, A) \subseteq \text{Soc}_n(A)$ is equivalent to $R_m(X, A) * A \subseteq \text{Soc}_n(A)$ and $[R_m(X, A), A]_+ \subseteq \text{Soc}_n(A)$. By Lemma 1.7, this is equivalent to $R_m(X, A) \subseteq \text{Soc}_{n+1}(A)$, which is equivalent to $X \subseteq \text{Soc}_{m+n+1}(A)$ by the inductive hypothesis. \square

Lemma 5.4. *A skew left brace A of nilpotent type is right nilpotent if and only if $R_n(A, A) = 0$ for some $n \in \mathbb{N}$.*

Proof. By Lemma 5.3, $R_n(A, A) = 0$ if and only if $A = \text{Soc}_n(A)$. By Lemma 1.8, the latter is equivalent to A being right nilpotent. \square

Recall that a finite group G is said to be p -nilpotent if there exists a normal Hall p' -subgroup of G . One proves that this subgroup is characteristic in G . Following [41] we define right p -nilpotent skew left braces of nilpotent type:

Definition 5.5. *Let p be a prime number. A finite skew left brace A of nilpotent type is said to be right p -nilpotent if there exists $n \geq 1$ such that $R_n(A_p, A) = 0$, where A_p is the Sylow p -subgroup of $(A, +)$.*

Proposition 5.6. *Let A be a finite skew left brace of nilpotent type and $p \in \pi(A)$. Then $A_p \subseteq \text{Soc}_n(A)$ for some $n \geq 1$ if and only if A is right p -nilpotent.*

Proof. By Lemma 5.3, $R_n(A_p, A) = 0$ if and only if $A_p \subseteq \text{Soc}_n(A)$. \square

Proposition 5.7. *A finite skew left brace A of nilpotent type is right nilpotent if and only if A is right p -nilpotent for all $p \in \pi(A)$.*

Proof. Assume first that A is right nilpotent. By Lemma 1.8, there exists $n \in \mathbb{N}$ such that $A_p \subseteq A = \text{Soc}_n(A)$ for all $p \in \pi(A)$. Hence the claim follows from Proposition 5.6. Assume now that A is right p -nilpotent for all $p \in \pi(A)$. This means that for each $p \in \pi(A)$ there exists $n(p) \in \mathbb{N}$ such that $A_p \subseteq \text{Soc}_{n(p)}(A)$. Let $n = \max\{n(p) : p \in \pi(A)\}$. Then $A_p \subseteq \text{Soc}_n(A)$ for all $p \in \pi(A)$. Since $\text{Soc}_n(A)$ is an ideal of A and A is of nilpotent type, $A = \bigoplus_{p \in \pi(A)} A_p \subseteq \text{Soc}_n(A)$. Hence A is right nilpotent by Lemma 1.8. \square

In [41], Meng, Ballester–Bolinches and Romero prove the following theorem for left braces:

Theorem 5.8. *Let A be a finite skew left brace of nilpotent type. If (A, \circ) has an abelian normal Sylow p -subgroup for some $p \in \pi(A)$, then A is right p -nilpotent.*

Our proof is very similar to that of [41]. We shall need the following lemmas:

Lemma 5.9. *Let A be a finite skew left brace of nilpotent type. If (A, \circ) has a normal Sylow p -subgroup for some $p \in \pi(A)$, then A_p is an ideal of A .*

Proof. Since the group $(A, +)$ is nilpotent, there exists a unique normal Sylow p -subgroup A_p of $(A, +)$. By Lemma 1.4, A_p is a left ideal of A . Then A_p is a Sylow p -subgroup of (A, \circ) , normal by hypothesis and hence A_p is an ideal of A . \square

Lemma 5.10. *Let A be a finite skew left brace of nilpotent type. If (A, \circ) has a normal Sylow p -subgroup for some $p \in \pi(A)$, then $\text{Soc}(A_p) = \text{Soc}(A) \cap A_p$. In particular, $\text{Soc}(A_p)$ is an ideal of A .*

Proof. By Lemma 5.9, A_p is an ideal of A . Clearly $\text{Soc}(A_p) \supseteq \text{Soc}(A) \cap A_p$, so we only need to prove that $\text{Soc}(A_p) \subseteq \text{Soc}(A) \cap A_p$. If $a \in \text{Soc}(A_p)$, then $a \in Z(A_p, +)$ and $a * b = 0$ for all $b \in A_p$. Let $c \in A$ and write $c = x + y$, where $x \in A_p$ and $y \in A_{p'}$. Since

$$a * c = a * (x + y) = a * x + x + a * y - x = x + a * y - x \in A_p \cap A_{p'} = 0$$

and $a \in Z(A, +)$, the lemma is proved. \square

Now we prove Theorem 5.8.

Proof. Let us assume that the result does not hold and let A be a counterexample of minimal size. We may assume that A is non-trivial, i.e. $\text{Soc}(A) \neq A$. By Lemma 5.9, A_p is an ideal of A .

Since $\lambda_a \in \text{Aut}(A_p, +)$, $\lambda_a(Z(A_p, +)) \subseteq Z(A_p, +)$ and hence $Z(A_p, +)$ is a left ideal of A_p .

By Lemma 5.10, $\text{Soc}(A_p)$ is an ideal of A . Furthermore, since (A_p, \circ) is abelian,

$$\begin{aligned} \text{Soc}(A_p) &= \{a \in A_p : a * b = 0 \text{ for all } b \in A_p\} \cap Z(A_p, +) \\ &= \{a \in A_p : a \circ b = a + b \text{ for all } b \in A_p\} \cap Z(A_p, +) \\ &= \{a \in A_p : b \circ a = b + a \text{ for all } b \in A_p\} \cap Z(A_p, +) \\ &= \text{Fix}(A_p) \cap Z(A_p, +). \end{aligned}$$

Since $|A_p| = p^m$ for some $m \geq 1$, the skew left brace A_p is left nilpotent by [15, Proposition 4.4] and, moreover, $Z(A_p, +)$ is a non-zero subgroup of $(A_p, +)$. Then $\text{Soc}(A_p) = \text{Fix}(A_p) \cap Z(A_p, +) \neq 0$ by [15, Proposition 2.26]. In particular, $0 \neq \text{Soc}(A_p) \subseteq \text{Soc}(A)$. By Lemma 5.10, $I = \text{Soc}(A_p)$ is a non-trivial ideal of A . Then A/I is a skew left brace of nilpotent type such that $0 < |A/I| < |A|$. The minimality of $|A|$ implies that A/I is right p -nilpotent. Hence, $R_n(A_p/I, A/I) = 0$ for some n . That is $R_n(A_p, A) \subseteq I \subseteq \text{Soc}(A)$. Now, by Lemma 5.3, $R_{n+1}(A_p, A) = 0$. Then A is right p -nilpotent, a contradiction. \square

Recall that a group G has the *Sylow tower property* if there exists a normal series $1 = G_0 \subseteq G_1 \subseteq \dots \subseteq G_n = G$ such that each quotient G_i/G_{i-1} is isomorphic to a Sylow subgroup of G . We also recall that *A-groups* are finite groups whose Sylow subgroups are abelian.

Corollary 5.11. *Let A be a finite skew left brace of nilpotent type. Assume that (A, \circ) has the Sylow tower property and that all Sylow subgroups of (A, \circ) are abelian. Then A is right nilpotent.*

Proof. Assume that the result is not true and let A be a counterexample of minimal size. Since (A, \circ) has the Sylow tower property, there exists a normal Sylow p -subgroup A_p of (A, \circ) . Then A_p is a non-zero ideal of A and one proves that

$$0 \neq \text{Soc}(A_p) = \text{Soc}(A) \cap A_p \subseteq \text{Soc}(A).$$

The group $(A/\text{Soc}(A), \circ)$ has abelian Sylow subgroups and has the Sylow tower property. Since A is a non-trivial skew left brace, $0 < |A/\text{Soc}(A)| < |A|$, and therefore $A/\text{Soc}(A)$ is right nilpotent by the minimality of $|A|$. By [15, Proposition 2.17], A is right nilpotent, a contradiction. \square

There are examples of right nilpotent left braces where the multiplicative group contains a non-abelian Sylow subgroup or does not have the Sylow tower property:

Example 5.12. *The operation $a \circ b = a + 3^a b$ turns $\mathbb{Z}/8$ into a right nilpotent left brace with multiplicative group isomorphic to the quaternion group. This example appears in [1].*

Example 5.13. Let $G = \mathbb{A}_4 \times \mathbb{S}_3$. Each Sylow subgroups of G is abelian, so it follows from [12, Theorem 2.1] that there exists a left brace with multiplicative group isomorphic to G . The group G does not have the Sylow tower property. The database of left braces of [32] shows that there are only four left braces with multiplicative group isomorphic to G , all with additive group isomorphic to $C_6 \times C_6 \times C_2$. However, only one of these four braces is not right nilpotent.

As a corollary, we obtain a generalization of Theorem 2.8:

Corollary 5.14. Let A be a finite skew left brace of nilpotent type. If all Sylow subgroups of the multiplicative group of A are cyclic, then A is right nilpotent.

Proof. Since all Sylow subgroups of (A, \circ) are cyclic, the group (A, \circ) is supersolvable and hence it has the Sylow tower property. Then the claim follows from Corollary 5.11. \square

6. LEFT p -NILPOTENT SKEW LEFT BRACES

Let A be a skew left brace. For subsets X and Y of A we define inductively $L_0(X, Y) = Y$ and $L_{n+1}(X, Y) = X * L_n(X, Y)$ for $n \geq 0$.

Definition 6.1. Let p be a prime number. A finite skew left brace A of nilpotent type is said to be left p -nilpotent if there exists $n \geq 1$ such that $L_n(A, A_p) = 0$, where A_p is the Sylow p -subgroup of $(A, +)$.

Lemma 6.2. Let A be a skew left brace such that its additive group is the direct product of the left ideals B and C . Then $A * (B + C) = A * B + A * C$. Moreover, if $A = \bigoplus_{i=1}^n B_i$ where the B_i are left ideals, then

$$A * \sum_{i=1}^n B_i = \sum_{i=1}^n A * B_i.$$

Proof. Let $a \in A$, $b \in B$ and $c \in C$. Then

$$a * (b + c) = a * b + b + a * c - b = a * b + a * c$$

holds for all $a \in A$, $b \in B$ and $c \in C$. The second part follows by induction. \square

Proposition 6.3. Let A be a finite skew left brace of nilpotent type. Then A is left nilpotent if and only if A is left p -nilpotent for all $p \in \pi(A)$.

Proof. For each $p \in \pi(A)$ there exists $n(p) \in \mathbb{N}$ such that $L_{n(p)}(A, A_p) = 0$. Let $\bar{n} = \max\{n(p) : p \in \pi(A)\}$. Then $L_{\bar{n}}(A, A_p) = 0$ for all $p \in \pi(A)$. Since A is of nilpotent type, the group $(A, +)$ is isomorphic to the direct sum of the A_p for $p \in \pi(A)$. Then Lemma 6.2 implies that

$$L_{\bar{n}}(A, A) = \sum_{p \in \pi(A)} L_{\bar{n}}(A, A_p) = 0.$$

The other implication is trivial. \square

We now recall some notation about commutators. Given a skew left brace A , the group (A, \circ) acts on $(A, +)$ by automorphisms. If in the semidirect product $(A, +) \rtimes (A, \circ)$ we identify a with $(0, a)$ and b with $(b, 0)$, then

$$\begin{aligned} [a, b] &= (0, a)(b, 1)(0, a)^{-1}(b, 1)^{-1} = (0, a)(b, 1)(0, a')(-b, 1) \\ &= (\lambda_a(b), a)(-\lambda_{a'}(b), a') = (\lambda_a(b) - b, 1) \\ &= (a * b, 1) \end{aligned}$$

Under this identification, we write $[X, Y] = X * Y$ for any pair of subsets $X, Y \subseteq A$. Then the iterated commutator satisfies

$$[X, \dots, X, Y] = [X, [X, \dots, [X, Y] \dots]] = L_n(X, Y),$$

where the subset X appears n times.

The following theorem was proved in [41] by Meng, Ballester–Bolinches and Romero for left braces:

Theorem 6.4. *Let A be a finite skew left brace of nilpotent type. The following statements are equivalent:*

- (1) A is left p -nilpotent.
- (2) $A_{p'} * A_p = 0$.
- (3) The group (A, \circ) is p -nilpotent.

Proof. We first prove that (1) implies (2). Since A is left p -nilpotent, there exists $n \in \mathbb{N}$ such that $L_n(A_{p'}, A_p) \subseteq L_n(A, A_p) = 0$. Since $(A_{p'}, \circ)$ acts by automorphisms on $(A_p, +)$ and this is a coprime action, it follows from [33, Lemma 4.29] that

$$L_1(A_{p'}, A_p) = A_{p'} * A_p = A_{p'} * (A_{p'} * A_p) = L_2(A_{p'}, A_p).$$

By induction one then proves that $A_{p'} * A_p = L_n(A_{p'}, A_p) = 0$.

We now prove that (2) implies (3). It is enough to prove that $(A_{p'}, \circ)$ is a normal subgroup of (A, \circ) . By using Lemma 6.2,

$$A_{p'} * A = A_{p'} * (A_p + A_{p'}) = (A_{p'} * A_p) + (A_{p'} * A_{p'}) \subseteq A_{p'}.$$

since A_p' is a left ideal of A and $A_{p'} * A_p = 0$. Then $A_{p'}$ is an ideal of A by Lemma 1.6 and [15, Lemma 1.9]. In particular, $(A_{p'}, \circ)$ is a normal subgroup of (A, \circ) .

Finally we prove that (3) implies (1). We need to prove that $L_n(A_p, A_p) = 0$ for some n . Since (A, \circ) is p -nilpotent, there exists a normal p -complement that is a characteristic subgroup of (A, \circ) . This group is $A_{p'}$ and hence $A_{p'}$ is an ideal of A . Then $A_{p'} * A_p \subseteq A_{p'} \cap A_p = 0$. We now prove that $L_n(A, A_p) = L_n(A_p, A_p)$ for all $n \geq 0$. The case where $n = 0$ is trivial, so assume that the result holds for some $n \geq 0$. By the inductive hypothesis,

$$L_{n+1}(A, A_p) = A * L_n(A, A_p) = A * L_n(A_p, A_p).$$

Thus it is enough to prove that $A * L_n(A_p, A_p) \subseteq A_p * L_n(A_p, A_p)$. Let $a \in A$ and $b \in L_n(A_p, A_p)$. Write $a = x \circ y$ for $x \in A_p$ and $y \in A_{p'}$. Then

$$a * b = (x \circ y) * b = x * (y * b) + y * b + x * b = x * b \in A_p * L_n(A_p, A_p)$$

since $A_{p'} * A_p = 0$. The skew left brace A_p is left nilpotent by [15, Proposition 4.4], so there exists $n \in \mathbb{N}$ such that $L_n(A_p, A_p) = 0$. \square

The following theorem was proved by Smoktunowicz for left braces, see [49, Theorem 1.1]. For skew left braces a proof appears in [15, Theorem 4.8].

Theorem 6.5. *Let A be a finite skew left brace of nilpotent type. Then A is left nilpotent if and only if the multiplicative group of A is nilpotent.*

Proof. As it was observed in [41], Proposition 6.3 and Theorem 6.4 prove the theorem. \square

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