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From particles to firms: on the kinetic theory of climbing up evolutionary landscapes

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This paper constitutes the first attempt to bridge the evolutionary theory in economics and the theory of active particles in mathematics. It seeks to present a kinetic model for an evolutionary formalization of economic dynamics. The new derived mathematical representation intends to formalize the processes of learning and selection as the two fundamental drivers of evolutionary environments [G. Dosi, M.-C. Pereira and M.-E. Virgillito, The footprint of evolutionary processes of learning and selection upon the

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statistical properties of industrial dynamics, *Ind. Corp. Change*, **26** (2017) 187–210]. To coherently represent the aforementioned properties, the kinetic theory of active particles [N. Bellomo, A. Bellouquid, L. Gibelli and N. Outada, *A Quest Towards a Mathematical Theory of Living Systems* (Birkhäuser-Springer, 2017)] is here further developed, including the complex interaction of two hierarchical functional subsystems. Modeling and simulations enlighten the predictive ability of the approach. Finally, we outline the potential avenues for future research.

Keywords: Evolutionary dynamics; idiosyncratic learning; market selection; active particles; kinetic theory.

AMS Subject Classification: 82D99, 91D10

1. Introduction

This paper constitutes the first attempt to bridge the evolutionary theory in economics and the theory of active particles in mathematics. It seeks to present a kinetic model for an evolutionary formalization of economic dynamics. The new derived mathematical representation intends to formalize the processes of *learning* and *selection* as the two fundamental drivers of evolutionary environments.[12] Within the framework of an evolutionary interpretation of industrial dynamics, the interplay between these two engines shapes the dynamics of market shares and collectively the patterns of change of industry-level variables such as average productivity. The learning process derives from the accumulation of knowledge which evolves together with the patterns of innovation and imitation, the changes in techniques of production, and in general the arrival of new technological paradigms. The accumulated capabilities become the marker of each individual firm. The selection process is the outcome of market interaction, whereby more competitive firms gain market shares at the expense of less competitive ones.

In economics, a promising account of such dynamics is by means of Agent-Based Models (ABMs), a family of formalizations typically explored via computer simulations, studying the emergence of aggregate statistical regularities in the evolutionary process stemming from the interaction among a multitude of agents, characterized by a quite rich phenomenological structure in terms of what they do and how they interact. Take as an example the model in Refs. 12 and 13 which compares different learning regimes in their impact on the overall functioning of industries. That paper attempts at understanding the interplay between cumulative learning and selection processes to account for the statistical regularities characterizing firms and industry evolution, e.g. persistent productivity differentials, tent-shaped growth rates, persistent entry-exit, scaling law of the variance of growth versus size relationship. We do find that the coupling of learning and selection is an extremely powerful generative mechanism of the stylized facts above. We address also the equivalence between Polya Urns and replicator dynamics representations of the selection process.

This paper explores an alternative route and provides a first systematic bridge between the evolutionary approach in economics and the kinetic theory of active particles in mathematics. Unlike ABMs where interactions among agents occur by means of *behavioral rules*, the kinetic theory of active particles specifies the probability distributions which govern the form and the intensity of the interaction process and, as such, the overall drivers of the dynamics.

Drawing upon Ref. 1 this paper substantially extends the modeling approach. First by using a complete discrete time framework to better adhere to the description of economic processes, second by introducing a *hierarchy in the topology of the interactions* with two functional subsystems (FSs), a first one independently evolving and a second one endogenous to the interaction among entities. In so doing, we mean to couple the overall hill climbing process driven by innovation and knowledge accumulation with the interactive competitive dynamics among heterogeneous entities. As in Ref. 12, we study the interplay between alternative learning regimes in terms of degrees of cumulativeness and the ensuing selection outcomes.

Our paper goes beyond the modeling approach based on established mathematical tools. Indeed, new mathematical tools have been developed on the ground of the evolutionary theory of economic change, thus enriching its interpretative power.

The remaining of the paper is organized as follows: Sec. 2 briefly discusses learning and selection in economic systems deepening the dialogue between economics and mathematics. Section 3 first presents a generalized structure of kinetic theory for two FSs regulated by a hierarchical relationship, and second specifies the interaction mechanisms underlying the processes of learning and selection. Section 4 develops a battery of simulation exercises matching the model outputs with stylized facts on industrial structures and dynamics. Section 5 concludes by discussing further avenues of research focusing on the quest of a theory of complex evolving systems in social sciences.[3,4,6,20,27]

2. Learning and Selection in the Evolutionary Theory of Economic Change

Evolutionary theories of economic change identify the processes of idiosyncratic learning by individual firms and of market selection as the two main drivers of the dynamics of industries. The interplay between these two engines shapes the dynamics of market shares and entry-exit and, collectively, of productivities and size distributions. Firm-specific learning (what in the empirical literature is sometimes broadly called the within effect) stands for various processes of idiosyncratic innovation, imitation, and changes in techniques of production. Selection (what is usually denominated the between effect) is the outcome of processes of market interaction where more “competitive” firms — on whatever criteria — gain market shares at the expense of less competitive ones, some firms die, and others enter.

One of the basic intuition in our whole interpretation is that both learning and competition entail *correlation mechanisms* which yield as such systematic departures from Gaussian stochastic processes.

In modern capitalism, business firms are a central locus of the efforts to advance technologies, develop new products and operate new production processes. Thus, the knowledge and the procedures underlying each technology are to a good extent embodied in organizational routines and other "quasi genetic action patterns" of organizations. Indeed, an emerging capability-based theory of the firm places the "primitives" of the nature of business firms in their problem-solving features, that is their abilities to address practical and cognitive problems, ranging from, say, the production of a car to the identification of a malaria-curing molecule.

The approach — fully acknowledging ubiquitous forms of human bounded rationality, grossly imperfect processes of learning and diverse social distributions of cognitive labor — attempts to identify the distinctive capabilities of organizations as emergent from their distinctive ensembles of organizational routines. And, dynamically, the approach tries to account for the processes by which organizational knowledge is acquired, maintained, augmented and sometimes lost. Learning regimes might be independent from the relative position of the firms in the landscape, implying that good and bad firms are exposed in probability to the acquisition of similar degrees of knowledge. Alternatively, learning regimes might be proportional to the position of the firm in the landscape, wherein better performing firms acquire more knowledge than the rest.

Idiosyncratic capabilities and, dynamically, idiosyncratic patterns of learning by firms are the general rule. In turn, such persistently heterogeneous firms are nested in competitive environments, which shape their individual economic fate and, collectively, the evolution of the forms of industrial organization. Differences in products and in processes of production — and as a consequence costs and prices — are central features of the competitive process in which firms are involved in different ways. Let us call Schumpeterian competition the process through which heterogeneous firms compete on the basis of the products and services they offer and obviously their prices, and get selected — with some firms growing, some declining, some going out of business, some new ones always entering. Such processes of competition and selection are continuously fueled by the activities of innovation, adaptation, imitation by incumbent firms and by entrants.

In turn, the processes of industrial evolution leave statistical footprints in terms of industrial structures and firm dynamics. So, for example, different learning regimes impact on the selection process, resulting in more or less concentrated markets. Thanks to massive infusions of micro-data over the last 20 years, one has begun to identify a few robust statistical properties characterizing industrial structures, their changes, and performance indicators. In particular, such stylized facts include persistent heterogeneity in productivity and skewed size distributions.

In the following, we shall present a model which begins to address those patterns, under minimal phenomenological specifications of behaviors and interactions. Knowledge is represented as an independent evolving field which differently hits each firm. It is a metaphor of a growing pool of innovative opportunities to which all firms might have access. The learning dynamics is idiosyncratic and each firm accrues its own capabilities. Selection occurs over an evolving landscape via a replicator dynamics which favors the most innovative firms and selects out the least innovative ones.

3. Modeling by Means of the Kinetic Theory of Active Particles

The kinetic theory of active particles[5] represents a new and powerful avenue to formalize the evolutionary traits of industrial dynamics. This approach shares with the classical kinetic theory[10] the representation of a large system of interacting entities by a probability distribution function over their individual states, e.g. at the microscopic scale. The dynamics is obtained by equating the time derivative of the probability distribution to the difference between the inlet and the outlet flows in the elementary volume of the space of the microscopic states.

This theory has been introduced in Ref. 1 to model the dynamics of socio-economic systems constituted by a large number of interacting entities, called kinetic theory of *active particles*, in short *a-particles*. An overview of the applications and covered domains in which the kinetic theory of a-particles has been employed is presented in Ref. 2, see also Ref. 11. Applications on biological systems, crowd dynamics, and social systems are presented and discussed in Ref. 5 showing how a mathematical description of living entities can be achieved.

Unlike the kinetic theory of classical particles, the microscopic state is not only identified by the position and velocity of the particles, but it also includes a vector of additional variables, called *activities*, which models the forms of interactions. The whole system can be subdivided into groups of interest called FSs. Additionally, interactions, which in the kinetic theory[10] are governed by basic principles of classical mechanics, in the a-particles approach are modeled by stochastic links, wherein actors/agents are identified by probability distributions. In so doing, interactions do not simply involve individual entities but also collections of them.

Irreversibility of the interaction processes and potential state-dependent parameters fuel the nonlinear nature of the approach, increasing the level of complexity and calling for a computational analysis. Indeed, diverse types of behaviors of agents and more generally system complexity[4] might be appropriately modeled. Theoretical tools of evolutionary games[16,21–23,26] are simple domains of application, but, as we shall see here, the applicability goes well beyond.

In the following, inspired by Ref. 2, our mathematical derivation will avoid mean-field approximations to let extreme behaviors to emerge. Complementary but different approaches closer to a kinetic theory allowing for Boltzmann and Fokker–Plank equations are in Refs. 15, 17 and 24, while mathematical tools of statistical-stochastic dynamics and game theory are in Refs. 18 and 25.

The sequential steps of the derivation of the model include:

(1) Representation of the FSs involved in the dynamics, where FSs are constituted by *active particles*, and where each FS expresses one or more functions defined as *activities*.

(2) Derivation of a mathematical structure suitable to describe the dynamics of the dependent variables derived in the first step.
(3) Specification of individual interactions by inserting them into the general mathematical structure derived in the second step.

3.1. *Functional subsystems*

In order to characterize the dynamics of learning and selection we introduce two FSs which are nested into a hierarchical structure:

- **Subsystem 1. Evolutionary landscape**
  It represents the dynamics of learning to which firms are subject to. It is meant to capture the arrival of new technologies, new ideas, new organizational practices. It evolves independently from firm interactions, and it follows a continuous growth process. In economic terms, it represents the evolution of the technological frontier.
- **Subsystem 2. Endogenous system of interactions**
  It comprises two distinct levels of interactions: one which determines the advancement of knowledge of each individual firm through the action of the first subsystem, the second which entails the competition in the market arena among heterogeneous firms in terms of knowledge level.

Formally, we suppose that the system of firms expresses two components of the activity, namely $w \in [0, \infty)$ and $v \in [0, 1]$, which correspond, respectively, to the *level of knowledge* and to the *market shares*, where v is divided by the overall size of the market. The overall system of endogenous interactions moves according to the shape of the landscape $\varphi(\xi)$, where $\xi \in D_\xi$ is the activity variable modeling the learning action. This support constantly increases the domain of w and modifies the probability distribution over this variable so that a scaling of w implies an analogous scaling of ξ. As illustrated in Fig. 1, the growth process of the support

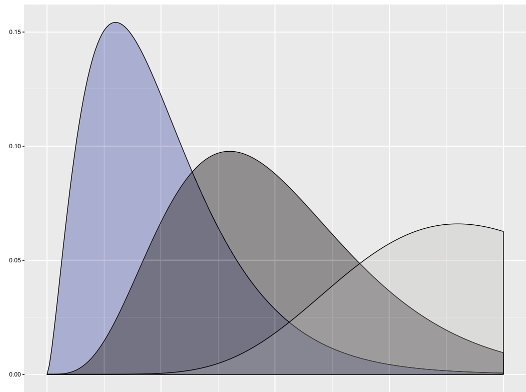


Fig. 1. An example of an evolutionary landscape.

is not a translation (constant shift) but is instead a nonlinear process acting in probability, meaning that, according to the specification, it might differently hit firms occupying different knowledge positions.

Let us now provide additional details on the scaling problem. Supposing that at time $t = 0$, the domain of w is bounded by the minimal and maximal values $w_m(t = 0)$ and $w_M(t = 0)$, respectively, a scaled variable can be introduced:

$$u = \frac{w - w_m(t = 0)}{w_M(t = 0) - w_m(t = 0)} \quad \Rightarrow \quad D_u(t = 0) = [0, 1]. \tag{3.1}$$

We define the dynamics of the *evolutionary landscape* as the variation over time of the domains of the activity variables u and ξ which move from the initial state to higher values.

In order to position each firm, a discrete equally spaced distribution is used for all variables:

$$C_u = \{u_1, \ldots, u_n, \ldots, u_m\}, \quad h = \frac{1}{m-1}, \quad m > n, \tag{3.2}$$

where the initial conditions for the level of knowledge, learning action, and market share are referred to the collocation $\{u_1, \ldots, u_n\}$. Nodes for $i > n$ mark the evolution of the system, while u_m represents a limit value of the knowledge which can be reached during each time interval.

We can now proceed to define the variables describing the state of the system as follows:

- $\mathbf{f} = \{f_i(t) = f(t; u_i)\}$ represents the fraction of firms, *called i-firms*, for each i-level, where $i > n \Rightarrow f_i(t = 0) = 0$. The evolution dynamics is such that, for $t > 0$, f_i can reach positive values for $i > 0$.
- $\varphi = \gamma_0\{\varphi_i\}$ defines the set of the learning actions over the i-firms, where γ_0 is a positive defined constant modeling the rate of the action and φ_i defines the learning action for each i-knowledge level.
- $\mathbf{v} = \{v_i(t)\}$ defines the set of market shares of each i-firm.

The distributions $\mathbf{f}$ and φ satisfy the probability condition

$$\sum_{i=1}^{m} f_i(t) = \sum_{\nu=1}^{m} \varphi_\nu(t) = 1 \quad \text{for all } t \geq 0. \tag{3.3}$$

3.2. *Interaction dynamics, learning and selection*

Let us now detail the relevant *interactions* to define the dynamics of the system:

(1) The model entails a hierarchical structure: the idiosyncratic learning process, i.e. the dynamics of $\mathbf{f}$, governed by the interaction between the first and the second FSs over the variable u, is not influenced by the market shares, namely by the variable v. The dynamics of the latter, conversely, depends on the variable $\mathbf{f}$.

(2) The hierarchical structure entails that both the shape of the probability distribution over u and the collocation C_u evolve over time, while the collocation of the learning action follows the evolution of C_u.

Learning

Let us consider the derivation of a general mathematical structure suitable to describe the dynamics of the probability distribution $\mathbf{f}$ based on the selection of non trivial interactions. According to the kinetic theory of active particles, the number of firms $\mathbf{f} = \{f_i(t) = f(t; u_i)\}$ in the elementary volume of the space of microscopic states should remain unaltered, when subject to firm activities. In fact, given the interactive structure, firms might move from one state to another. Some firms might enter new states by advancing or regressing along the collocation. These movements define inflow and outflow of firms for each position. Interactions are supposed to be stochastic, according to the following hypotheses:

- η_0 denotes the rate of interactions between firms and learning actions, supposed to be constant.
- The dynamics is sensitive only to the action at the same level in the collocation and $\mathcal{A}_h(h \to i)$ is the probability that an h-firm shifts in the knowledge level from h to i, independently from the level of market shares, due to the interaction with an h-action (see Fig. 2).

The following constraint applies to the components of $\mathcal{A}_h$, being the latter a probability distribution:

$$\sum_{i=1}^{m} \mathcal{A}_h(h \to i) = 1, \quad \text{for all inputs.} \tag{3.4}$$

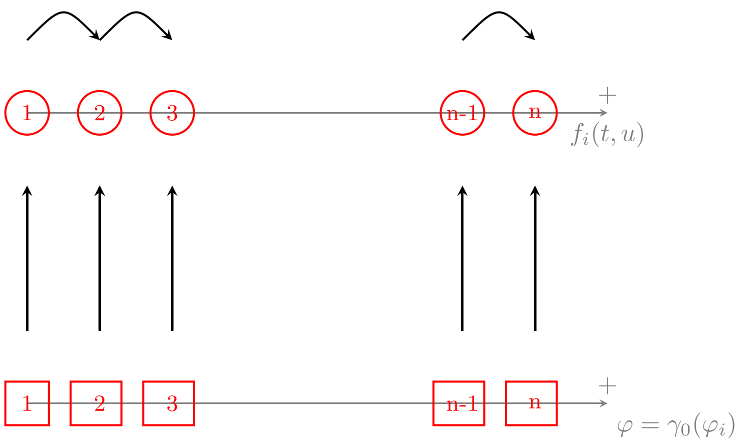


Fig. 2. The learning process derived from the interaction with the landscape. The squares represent the technological opportunities hitting each firm (circle). The vertical arrows represent the learning actions over the landscape $\varphi = \gamma_0(\varphi_i)$, while the horizontal arrows represent the shift of the position in the knowledge space $f_i(t, u)$.

The inflow into the elementary volume of the space of the activity variables corresponds to the number (rate) of firms adopting the ith microscopic state, while the outflow corresponds to the number of firms losing such state. Therefore, the condition of invariance of the volume of the microscopic state entails that

$$\begin{aligned}\partial_t f_i &= \mu_0 \left(\sum_{h=1}^{m} \mathcal{A}_h(h \to i) f_h \varphi_h - f_i \sum_{h=1}^{m} \varphi_h \right) \\ &= \mu_0 \left(\sum_{h=1}^{m} \mathcal{A}_h(h \to i) f_h \varphi_h - f_i \right), \end{aligned} \tag{3.5}$$

where $\mu_0 = \eta_0\, \gamma_0$ and where the notation ∂_t is used to distinguish the time derivative of a probability distribution with respect to the time derivative of a deterministic variable.

This general transition probability can be specified according to the learning regime one wants to depict. In the following, we shall look at two alternative learning regimes.

- Independent learning: *The learning action acts on the knowledge level by shifting firms to the next high level i.e.* $\mathcal{A}_{(i-1)}((i-1) \to i)$, *or* $\mathcal{A}_i(i \to (i+1))$, *by a transition probability independent from the collocation* i. This assumption yields:

$$\mathcal{A}_i(i \to (i+1)) = \alpha, \quad \text{and} \quad \mathcal{A}_i(i \to i) = 1 - \alpha\cdot \tag{3.6}$$

  By substituting into the mathematical structure (3.5), we get

$$\partial_t f_i = \mu_0 \Gamma_{1i}(\mathbf{f}, \varphi) = \mu_0 \alpha f_{i-1} \varphi_{i-1} - \mu_0 \alpha f_i \varphi_i, \tag{3.7}$$

  where absence of learning actions, namely $\partial_t f_i = 0$ for all $i = 1, \ldots, n$ implies also $\partial_t \varphi_i = 0$ for all levels of knowledge. The evolution of the learning action reads:

$$\partial_t \varphi_i = \gamma_0 \Gamma_{1i}(\mathbf{f}, \varphi) = \gamma_0 \alpha f_{i-1} \varphi_{i-1} - \gamma_0 \alpha f_i \varphi_i. \tag{3.8}$$

- Cumulative learning: *The learning action acts on the knowledge level by shifting firms to the next high level i.e.* $\mathcal{A}_{(i-1)}((i-1) \to i)$, *or* $\mathcal{A}_i(i \to (i+1))$, *by a transition probability which increases with the collocation* i.

  This assumption yields:

$$\mathcal{A}_i(i \to (i+1)) = \alpha \frac{i}{n+i} \quad \text{and} \quad \mathcal{A}_i(i \to i) = 1 - \alpha \frac{i}{n+i}\cdot \tag{3.9}$$

  Substitution into the mathematical structure (3.5) yields:

$$\partial_t f_i = \mu_0 \Gamma_{1i}(\mathbf{f}, \varphi) = \mu_0 \alpha \frac{i}{n+i} f_{i-1} \varphi_{i-1} - \mu_0 \alpha \frac{i}{n+i} f_i \varphi_i, \tag{3.10}$$

  while the evolution of the learning action reads:

$$\partial_t \varphi_i = \gamma_0 \Gamma_{1i}(\mathbf{f}, \varphi) = \gamma_0 \alpha \frac{i}{n+i} f_{i-1} \varphi_{i-1} - \gamma_0 \alpha \frac{i}{n+i} f_i \varphi_i\cdot \tag{3.11}$$

Selection

Let us now present the selection process according to which firms with higher values of knowledge increase their market shares at the expense of less-knowledgeable firms. The interaction can be specified according to the following assumptions:

- i-firms gain market shares whenever they interact with h-firms, being $u_h < u_i$, while they lose market shares whenever they interact with h-firms, being $u_h > u_i$. Gains cannot exceed the level $v = 1$, while losses cannot go below $v = 0$.
- The frequency and the intensity of interactions are determined by an exogenous, positively defined parameter β.
- The gain dynamics depends on β, on the distance $u_i - u_h$, on the available market shares v_h of the losers, and on the number f_i of the gainers in each fitness class.
- The loss dynamics depends on β, on the distance $u_h - u_i$, on the available market shares v_i of the losers, and on the number f_h of the gainers in each fitness class.
- We introduce the formulation $H_{ij} = H(u_i - u_j)$, where H is the Heaviside function, a dichotomous function which maps non-negative values into 1 ($H(x) = 1$ if $x \geq 0$) and negative values into 0 ($H(x) = 0$ if $x < 0$).

The assumptions above, while resembling a replicator dynamics, modulate the selection pressure in the model because of the presence of a reinforcing effect related to the number of firms in each fitness class. A synthetic representation is provided in Fig. 3.

We then specify the following m-dimensional differential system, presenting relative gains and losses, being both terms normalized, where ∂_t stands as above for the time derivative of stochastic variables:

$$\partial_t v_i = \beta(1 - v_i)\frac{\sum_{h=1}^m H_{ih} \cdot f_i \cdot (u_i - u_h) \cdot v_h}{\sum_{k=1}^m (1 - v_k) \sum_{h=1}^m H_{kh} \cdot f_k \cdot (u_k - u_h) \cdot v_h} - \beta v_i \frac{\sum_{h=1}^m H_{hi} \cdot f_h \cdot (u_h - u_i) \cdot v_i}{\sum_{k=1}^m v_k \sum_{h=1}^m H_{hk} \cdot f_h \cdot (u_h - u_k) \cdot v_k}. \tag{3.12}$$

3.3. *The model*

The resulting model is obtained by taking the system of the three differential equations. Dividing all equations by μ_0, which can be inserted into the time scale, we get

$$\begin{cases} \partial_t f_i = f_{i-1}\varphi_{i-1} - f_i \varphi_i, \\ \partial_t \varphi_i = \varepsilon_1 \dfrac{1}{n + i - 1} f_{i-1}\varphi_{i-1} - \varepsilon_1 \dfrac{1}{n + i} f_i \varphi_i, \\ \partial_t v_i = \varepsilon_2 (1 - v_i) \dfrac{\sum_{h=1}^m H_{ih} \cdot f_i \cdot (u_i - u_h) \cdot v_h}{\sum_{k=1}^m (1 - v_k) \sum_{h=1}^m H_{kh} \cdot f_k \cdot (u_k - u_h) \cdot v_h} \\ \qquad - \varepsilon_2 v_i \dfrac{\sum_{h=1}^m H_{hi} \cdot f_h \cdot (u_h - u_i) \cdot v_i}{\sum_{k=1}^m v_k \sum_{h=1}^m H_{hk} \cdot f_h \cdot (u_h - u_k) \cdot v_k}, \end{cases} \tag{3.13}$$

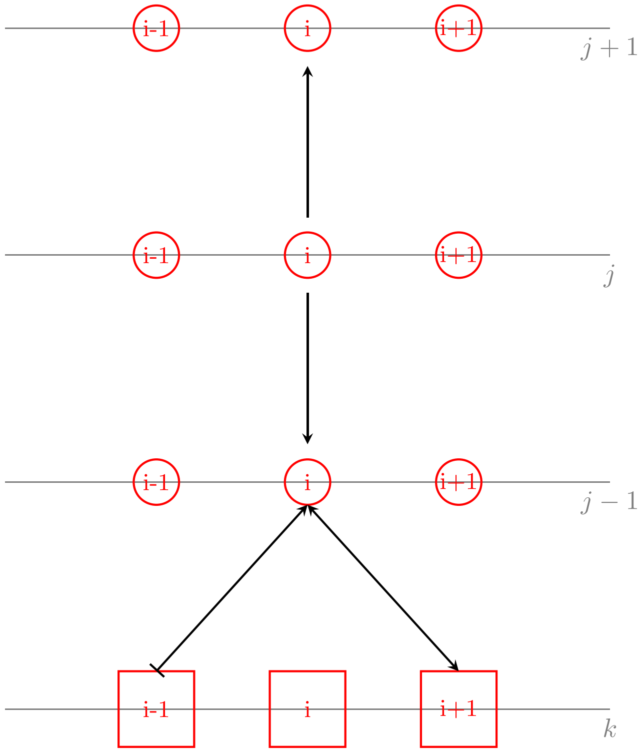


Fig. 3. The competition process. The index j stands for the fitness class. The index i characterizes the position of each firm in the corresponding fitness class. The index k stands for the learning actions.

where time has been scaled by multiplying the real time by $\alpha\mu_0$, while

$$\varepsilon_1 = \frac{\gamma_0}{\mu_0} \quad \text{and} \quad \varepsilon_2 = \frac{\beta}{\alpha\mu_0}$$

are parameters.

The structural dimension of this system is $3 \cdot m$, it should be solved for initial conditions for all components $f_{i0} = f_i(t = 0)$, $\varphi_{i0} = \varphi_i(t = 0)$, $v_{i0} = v_i(t = 0)$, given for $i = 1, \ldots, n$. Consistency with the model requires that for $i > 0$, $f_{i0} = \varphi_{i0} = v_{i0} = 0$.

The hierarchical structure of the model entails the possibility to consider a special case of *absence of learning action*. It corresponds to the case $\varphi = 0$, whereby only the dynamics of selection and variation of market shares occur, given the initial knowledge levels.

4. Simulations

In the following, we present a battery of simulation exercises discussing the model results. We shall focus on the analysis of the evolution of the variables u and v

according to different learning regimes. All simulations are developed for the same initial conditions, namely

(1) We consider $n = 10$ FSs corresponding to i-firms, each of them characterized by the knowledge level u_i, with $u_1 < u_2 < \cdots < u_n$, where $f_i(0) = f_{i0}$ is the probability that a firm belongs to the ith FS at the initial time $t = 0$. Each f_i can be interpreted as the (normalized) number of firms belonging to the ith FS with $i = 1, \ldots, n = 10$. The initial common market share of each ith FS is denoted by v_{i0}.
(2) Simulations have been undertaken using the parameter values

$$\eta_0 = \mu_0 = \gamma_0 = 1, \quad \alpha = \beta = 0.1, \quad \gamma_0 = 0.5.$$

We deem $t = 1200$ a reasonable time window to allow the model to stabilize.
(3) In terms of initial conditions, knowledge and market share distributions are supposed to linearly decay as the level of relative knowledge increases. These initial conditions are shown in Fig. 4.

Three alternative learning regimes are presented in the next subsections focusing on the following dynamics:

Case 1. Absence of learning;

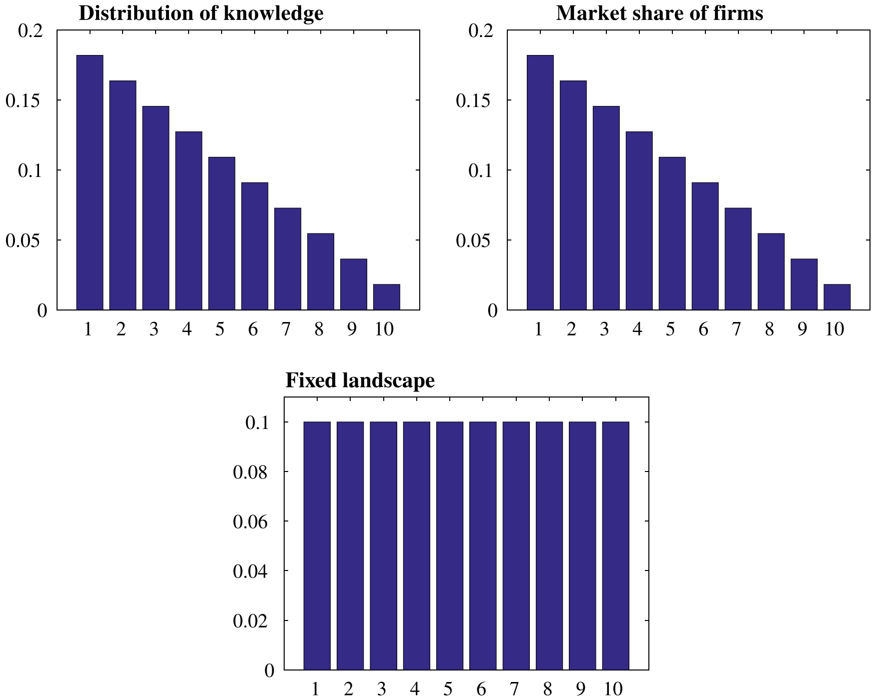


Fig. 4. Initial conditions for knowledge, market share and fixed landscape.

Case 2. Independent learning;

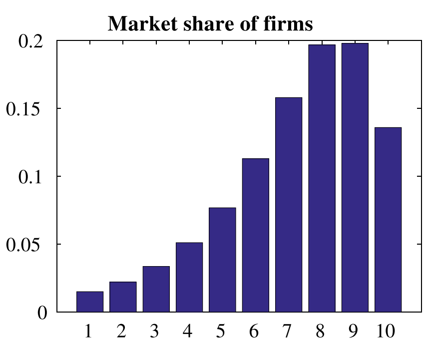
Case 3. Cumulative learning.

In Case 1 (absence of learning) the dynamics occurs on the initial 10 knowledge levels, implying a fixed landscape. However, in Cases 2 and 3 the landscape will move from the initial 10 positions toward an increasing number of knowledge levels. The chosen number of evolving subsystems allows to characterize the distribution asymptotically.

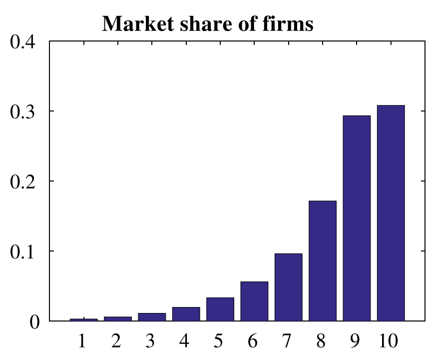
4.1. *CASE* 1. *Absence of learning*

We start by considering the extreme case wherein the learning process is shut-down. In this case, since no idiosyncratic learning occurs, firms do not improve or worsen their knowledge level, therefore the number of firms in each FS does not change over time and the landscape is fixed.

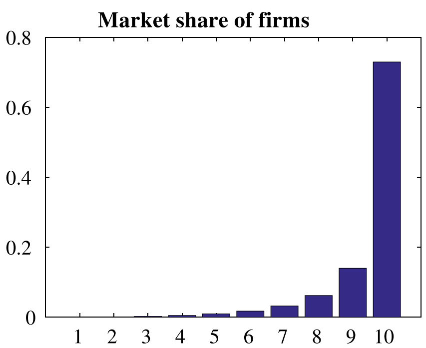
The market share in each FS is modified by the interaction between high-knowledge firms, which accrue market shares, and low-knowledge firms which lose shares. Figure 5 shows the tendency toward a monopolistic market structure whereby high-knowledge firms end up taking the whole market.



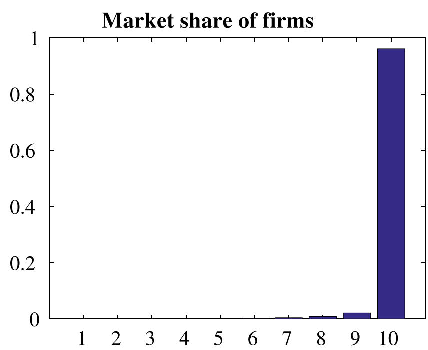
(a) $t = 1$



(b) $t = 1$



(c) $t = 5$



(d) $t = 10$

Fig. 5. CASE 1 — Market shares over time — Fixed landscape.

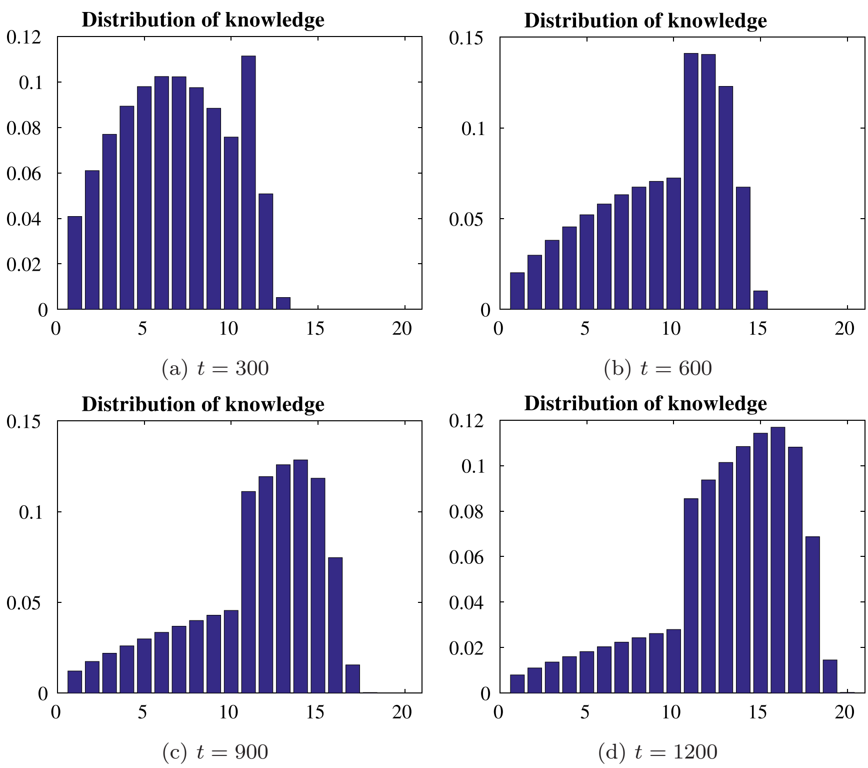
The robustness of our results are confirmed for a broad range of parameter values. The latter only affect the speed of the transition dynamics in the rate of convergence toward a monopoly. The rate of convergence gets slower the higher the initial number of high-knowledge firms.

The monopoly tendency indeed confirms the correct model specification: as predicted by the Fisher–Price theorem, any replicator dynamics yields the sole survival of the fittest. And the rate of growth of average fitness in the transient depends on its variance across micro-entities.

4.2. *CASE* 2. *Independent learning*

Let us now consider the case in which the learning process affects the individual knowledge of each firm. We suppose the learning dynamics to act independently from the position of each individual firm, according to a fix transition probability defined α (see Eq. (3.6)). The resulting learning process entails now a completely different dynamics of the distribution in the knowledge space.

As shown in Fig. 6 as time goes by, the knowledge distribution changes shape, while the underlying landscape evolves. In fact, we start by considering $m = 20$ levels of learning u_i, with $u_1 < u_2 < \cdots < u_m$.



(a) $t = 300$ (b) $t = 600$ (c) $t = 900$ (d) $t = 1200$

Fig. 6. CASE 2 — Knowledge distribution over time — Evolving landscape.

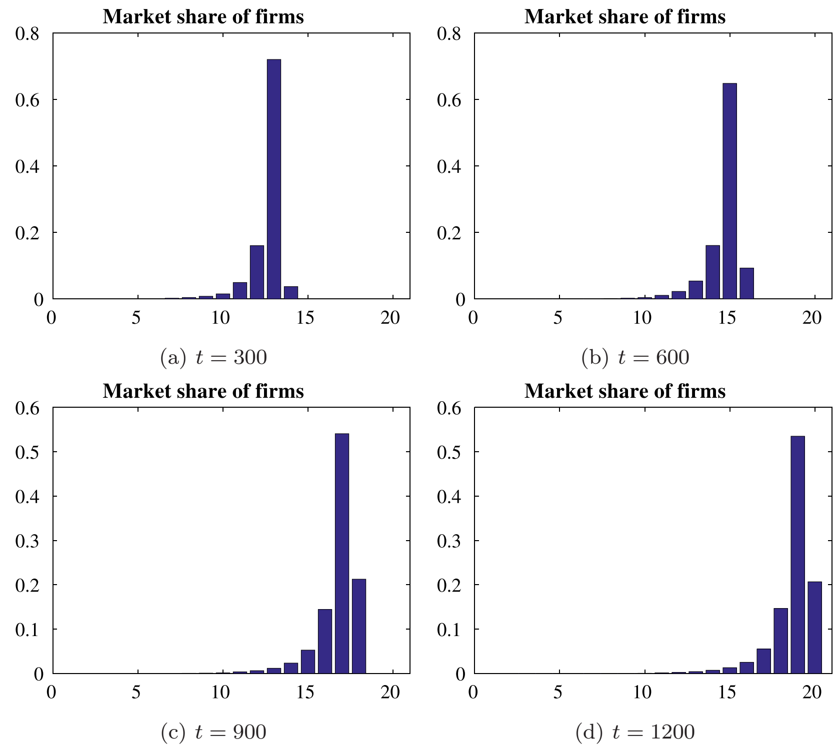


Fig. 7. CASE 2 — Market shares over time — Evolving landscape.

Initially only the first $n = m/2$ knowledge levels are occupied. Due to the changing landscape, firms gradually move towards higher levels spanning the whole knowledge range of 20 levels.

Figure 6 clearly shows that not all firms learn at the same rate. Interestingly, the shape resembles a long left-tail distribution, with knowledge concentrating in the second half of the landscape. The distribution of u is in line with the persistent heterogeneity in productivity observed in reality, which tends to increase as long as the landscape evolves.

How does the "size distribution" evolve? It does so by following the learning dynamics. In fact, differently from the monopolistic tendency of Case 1, now market shares in Fig. 7 present a left-skewed distribution, with the highest market share at fifty percent. Indeed, the evolution of the landscape is reflected in the selection dynamics severely acting against low-knowledge firms, which tend to be selected out by the competition process. Market concentration is reflected into a relatively skewed distribution.

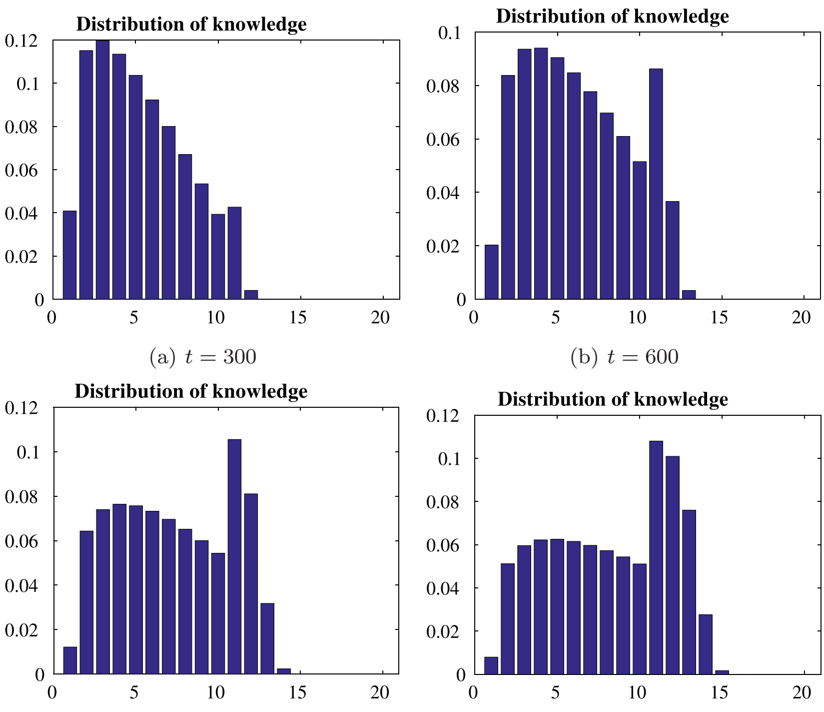
4.3. *CASE 3. Cumulative learning*

The third case that we consider entails processes of cumulative learning. This learning regime, also known as Matthew effect or self-reinforcing mechanism, implies that who knows more will get in probability more knowledge, while those who know less will get less knowledge. This case is strongly in line with the learning regimes occurring in the information economies, wherein economies of scale and near-zero marginal costs allow an almost "infinite" knowledge growth (more in Ref. 14).

As in the previous case, we start by considering $m = 20$ FSs of firms, each of them characterized by knowledge level u_i, with $u_1 < u_2 < \cdots < u_m$. Initially only the first $n = m/2$ knowledge levels are occupied. Once the landscape evolves over time firms move from the bottom to the upper part of the support.

Figure 8 shows the knowledge distribution. This case represents a sort of leader-laggard dynamics in which only few high-level knowledge firms are able to reach the highest positions, while the remaining part of marginal firms stacks in lower ones. This results into a bimodal knowledge distribution.

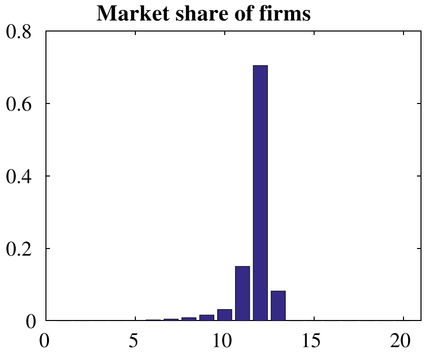
In terms of selection, Fig. 9 describes a strongly concentrated market wherein only two firms own almost the eighty and the twenty percent of the overall market, respectively. The rest of the firms get out of the market, with almost zero shares. Forms of tight oligopoly are typical of industries with high front up costs and low marginal ones, cumulative learning (e.g. the ICT industry), protections by strong forms of knowledge appropriation as intellectual property rights (e.g. the pharmaceutical industry).



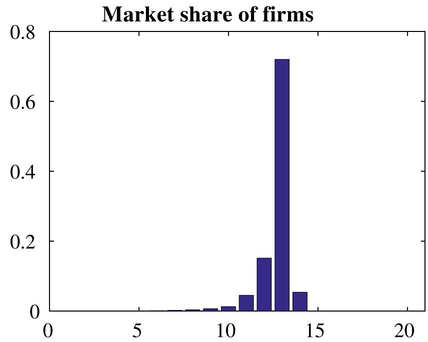
(a) $t = 300$ (b) $t = 600$

(c) $t = 900$ (d) $t = 1200$

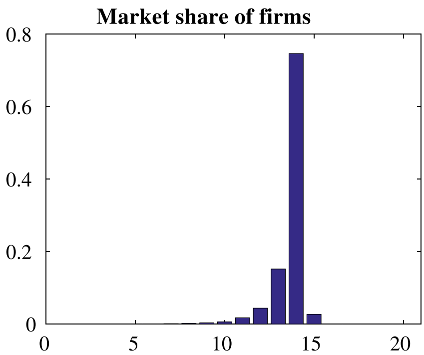
Fig. 8. CASE 3 — Knowledge distribution over time — Evolving landscape.



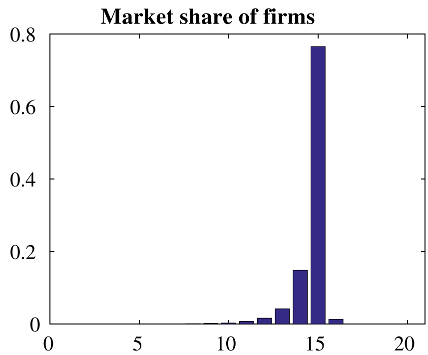
k (a) $t = 300$



(b) $t = 600$



(c) $t = 900$



(d) $t = 1200$

Fig. 9. CASE 3 — Market shares over time — Evolving landscape.

4.4. *On the role of mathematical models*

Our battery of simulation exercises does not claim to be exhaustive in covering the all possible specifications. Our aim has been to identify a number of configurations able to enlighten the predictive ability of the model and its consistency with empirical evidence. Indeed, the three cases have shown a qualitative and quantitative description of the model dynamics which has been then interpreted within the framework of the evolutionary theory of economic change. Potentially, other configurations might be studied as well.

Finally, simulations enable to explore how the behavior of the system depends on parameters. The sensitivity analysis we conducted indicates that parameters exert an influence on quantitative results but do not modify the qualitative shapes of the dynamics and, in particular, the asymptotic trends.

5. Conclusions

This paper has proposed the first systematic attempt to link the evolutionary theory of industry dynamics and the kinetic theory of active particles. Moving from particles to firms, it has studied how the process of learning has differentiated effects over selection. Our modeling structure allows to recover a wide array of market configurations, from monopoly, to oligopoly, to less concentrated ones. In all studied regimes, we do find a persistent heterogeneous distribution in the knowledge space, coherently with the evidence.

The essence of the approach relies in the tension between micro-scale interactions and description of macro-scale collective behaviors. Additionally, the specification of a hierarchical interactive structure allows to disentangle the role of each mechanism of the system. Our exercise at this stage was not intended to provide any normative implication. However, acting on the micro-scale dynamics essentially means influencing the learning process in its relationship with selection. The latter is reflected into the market arena, shaping the survival of more or less productive firms. Clearly, the interplay between micro-scale individual learning and micro-scale selection defines the evolving aggregate state of industries.

The study of the co-evolution of the two processes is crucial to define the roles and instruments of industrial policies. In fact, our model entails the possibility to include higher hierarchical layers influencing the evolution of the landscape. For example, the role of the State as promoter of national systems of innovation might be studied. Other model extensions entail the introduction of an imitation factor among firms and the study of the emergence of agglomerations. By characterizing the ensuing network structure, one might analyze the resiliency and stability of alternative configurations.

Finally, let us go back to a topic flagged in Sec. 1, namely the development of a *mathematical theory of complex evolving systems* whose quest has informed the whole paper. The general framework recalls the so-called *science of behavioral systems*[20] or the *science of living systems*.[5] Within that, in this paper, we address some foundational aspects of the *evolutionary theory of economic change* and in particular, the formalization of the coupled dynamics between learning and selection within the mathematical framework of the kinetic theory of collective learning.[9]

The mathematical theory here developed provides a new approach to formalize by means of the kinetic theory of active particles the interplay between micro-level entities and the ensuing macro-dynamic, indeed refining upon a multiscale vision of living systems,[7,8] possibly a vindication of the challenge by the sixth Hilbert problem.[19]

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