New Insights on Syllogistic and Cut

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Abstract

There is a quite intentional resemblance between the Cut Rule and Aristotle's Syllogism. In this paper some deep connections between Sequent Calculus and Syllogistics will be investigated. Taking into consideration Álvarez & Correia's axiomatization of Syllogistics, currently the most complete available in the literature, I will show how this ancient logical system can be put into correspondence with the structural features of a special Sequent Calculus system, **SS**. On the grounds of this discovery I will present some improvements of the expressive power of Álvarez & Correia's system. As for the philosophical consequences of the correspondence, I will give answers to several concerns Manuel Correia had on his system. A somewhat new philosophical relevance of the Cut-Elimination Theorem will be highlighted in the end.

Keywords: Sequent Calculus, Cut-Elimination Theorem, Theory of Oppositions, Substructural logics.

1.1. Introduction

There is a rather obvious resemblance between the Sequent Calculus rule dubbed 'Cut' and Aristotle's perfect syllogism. These are not big news: the rule Gentzen called 'Schnitt' was originally formulated by Herz in the twenties and labeled by him precisely: 'syllogismus'. Nonetheless, there's more to the picture than meets the eye, and the purpose of this paper is to show what that is. In the following sections I will show an interesting correspondence between Syllogistics and Sequent Calculus that exploits in a novel way the symmetry between Cut and

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the Perfect Syllogism. This will lead to a full formalization of Syllogistics by means of a substructural logic called **SS**, with peculiar properties and an insightful philosophical interpretation.

The pivot for this correspondence is the Axiom of Linkage introduced in Álvarez and Correia (2012), so this paper may be seen also as an effort to deepen the mathematical interest of their proposal. In fact, as I will show in due course, the correspondence with Sequent Calculus ties up many of the loose ends already recognized by Correia in some presentations of his work.

In section 1.2 I present the Syllogistics along the lines of Álvarez and Correia (2012). In section 1.3 the correspondence with Sequent Calculus is presented and the system **SS** is defined and their most salient mathematical features explained. The philosophical consequences of the overall proposal are introduced and commented in section 1.4. Appendix 1.6.1 shows some deviations from the Aristotelian standards that open up the possibility of a further extension of the system.

1.2. The Three Laws for Syllogistics

Syllogistics is a logic system developed both by Aristotle and the Peripatetics and several other logicians over history[2]. In 2012, Enrique Álvarez and Manuel Correia presented the currently most concise and complete axiomatization of Syllogistics, integrating mediate and immediate inferences with complete and incomplete terms[3].

Before Álvarez and Correia's system (hereafter called **AC**), the best known axiomatization for Syllogistics consisted of six axioms that govern most, but not all of the valid syllogisms. For instance, these six axioms are unfit for predicting the validity of syllogisms with indefinite terms such as 'not-A', and they fail to recover the validity of immediate inferences (conversions). The system **AC** provides a unified solution to all these problems (for details see Álvarez and Correia (2012) and Correia (2017)).

In what follows I present a slightly simpler version of **AC** and explain its behavior. The minor differences with the original system will be explicated along the way.

[2] I intentionally avoid the use of the name "Aristotelian Logic" for Syllogistics, for this system is *not* the perfect and complete discovery of Aristotle, *pace* Kant.

[3] Note that 'axiom' here should not be understood in the Hilbertian sense of unproved propositional-or-first-order formula, but as a general rule stating a sufficient and necessary formal feature for a valid syllogism.

1.2.1. The System AC

Terms, propositions, quantities

A *term* is a variable that stands for a predicable expression in natural language. If A is a term, we say that it is *definite* and call non-A its *conjugate*. I call the 'non-' compound of this term the *conjugate-operator* and the 'A' compound the *base of conjugation* or simply the *base*. The conjugate of a definite term is an *indefinite* term. The conjugate of an indefinite term is its base.

The interpretation of a conjugate term is evident.

A *proposition* is a sentence involving two terms in one of the following four relations. Where α and β are terms (definite or indefinite), a proposition involving them can be:

1. All α are β
2. No α are β
3. Some α are β
4. Some α are not β

I call these *kinds* of propositions. Following scholastic mnemonics I will call these kinds A-, E-, I- and O- respectively. Please note that in the case of the latter form the word 'not' belongs to the structure of the proposition and is not (for the time being) a conjugate-operator over β.

A term in a proposition can be taken *universally* or *particularly*. For a given term, the fact of being taken in one way or the other determines its *quantity*. The assignment of quantity values follows the following rule, corresponding to Álvarez & Correia's *Axiom of Quantity*:

(A1) *Axiom of Quantity*:

1. The term α is taken universally in the propositions 'All α are β' and 'No α are β'.
2. The term α is taken particularly in the propositions 'Some α are β' and 'Some α are not β'.
3. The term β is taken universally in the propositions 'No α are β' and 'Some α are not β'.
4. The term β is taken particularly in the propositions 'All α are β' and 'Some α are β'.
5. To take universally (/particularly) a conjugate term is to take particularly (/universally) its base.

Using superscripts U and P to denote the universal and particular quantity respectively, the content of (A1:1-4) may be resumed as follows:

1. All α^U are β^P
2. No α^U are β^U
3. Some α^P are β^P
4. Some α^P are not β^U

A proposition is called *universal* (*/particular*) if its first term is universally (/particularly) taken.

Inferences, syllogisms

An *inference* is a tuple of more than one proposition. In every inference the rightmost proposition is called its *conclusion* and the rest of them the *premisses*.

An inference is called a *syllogism* if it conforms to the following rule:

(A0) *Formation Rule*: The inference has three propositions and three terms, either in conjugate or non-conjugate form, appearing twice but never on the same proposition.

In a syllogism the term that appears only in the premisses is called *the middle term*. It can be proved rather easily that the middle term always exists and is unique.

A syllogism is *valid* if and only if it conforms to the following axioms:

(A2) *Axiom of Particularity*: There is no more than one particular premisse; and the conclusion is particular if and only if one of the premisses is particular.

(A3) *Axiom of Linkage*: The middle term is taken once universally and once particularly in the premisses, and the quantity of the terms of the conclusion is the same as in the premisses.

1.2.2. Comments

Both (A2) and (A1:1-4) were already known by the syllogistic tradition. The clause (A1:5), relating quantity and conjugates, and (A3) have been presented in Álvarez and Correia (2012) as their main contribution to the theory of the Syllogism (although primitive formulations of the latter may be found in works of previous authors, for example, in (De Morgan, 1880, §42)). As Manuel Correia has observed in some presentations of this work, the key part of their proposal is (A1:5), which was primarily suggested by his collaborator, Enrique Álvarez.

This system validates all the syllogisms that are alike to be valid in the natural interpretation of the syllogistic system. As far as I know, the authors proved this through a brute-force checking. A proper weakening of (A3) also yields the syllogisms with problems of existential import.

A technical virtue of this system is that it avoids much of the terminology originally necessary to present the Syllogistics system properly. For instance, the *quality* feature of propositions is disposable, as well as the distinction between major and minor terms and premisses.

Given this last observation, the following improvement may be introduced. Let's replace the original definition of *inference* by this one: An *inference* is a pair $\langle P, C\rangle$ with P a multiset of propositions called *the premisses* and C a proposition called the *conclusion*.

This new definition identifies, for instance, the valid syllogisms 'All A are B, All B are C, All A are C' and 'All B are C, All A are B, All A are C', which were in fact treated as different by the traditional syllogistic logicians.

In order to reflex the symmetry on the premisses I will adopt the tree-like notation for depicting syllogisms:

$$\frac{\text{All } A \text{ are } B \qquad \text{All } B \text{ are } C}{\text{All } A \text{ are } C}$$

Another virtue of the system is that it governs the so-called *conversions* or *immediate inferences*; that is, inferences with only one premisse and no middle term. The tradition recognized three sorts of these inferences:

1. *(Simple) Conversion* (Ω): the conclusion is of the same kind of proposition than the premisse but with the terms exchanged. *Example:*

$$\frac{\text{All } A \text{ are } B}{\text{All } B \text{ are } A}\,\Omega$$

2. *Contraposition* (Σ): the conclusion is the same kind of proposition than the premisse but with the terms exchanged and conjugated. *Example:*

$$\frac{\text{All } A \text{ are } B}{\text{All non-}B \text{ are non-}A}\,\Sigma$$

3. *Obversion* (Θ): the conclusion has the same quantity than the premisse without being of the same kind, it takes the same terms in the same order, but the second one is conjugated. *Example:*

$$\frac{\text{All } A \text{ are } B}{\text{No } A \text{ are non-}B}\,\Theta$$

The definition of obversion can be more easily stated if one considers the *quality* of the propositions, but I preferred this most complicated version to stress out that it is strictly not necessary to introduce that terminology. For these inferences, (A3) must be read as follows:

(A3') *Axiom of Linkage*: The quantity of the terms of the conclusion is the same as in the premisses.

Together with (A2) and under the dispositions of (A1), these three axioms validate all the expected immediate inferences:

1. *Simple Conversion*: I- and E-propositions.
2. *Contraposition*: A- and O-propositions.
3. *Obversion*: The four kinds of propositions.

1.3. Sequent Calculus and the Perfect Syllogism

In this section I will be dealing with the standard apparatus for Sequent Calculus. For the sake of brevity, minor details on what is a sequent and how a derivation should be constructed is assumed to be known. I use '$\vdash$' as the turnstile symbol and consider sequents with *series*, not *multisets* of formulas at their left and right-hand sides.

1.3.1. Linkage and Cut

Recall the *bArbArA* syllogism, the first of Aristotle's *Perfect Syllogisms* (25b 30):

$$\frac{\text{All } A \text{ are } B \qquad \text{All } B \text{ are } C}{\text{All } A \text{ are } C}$$

Using this notation the analogy with the CUT rule appears quite naturally:

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Specially if one reduces the contexts and relabel the formulas in a convenient way:

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

The most evident analogy here is the relation between the middle term and the cut formula: both appear only in the premisses of the arguments. But it is worth noting also that:

1. The formulas of the conclusion are on the same side of the sequent than on the premisses.
2. The cut formula is once on the left and once on the right side of the sequents.

These are the exact same dispositions of (A3) regarding the quantities of the terms in a syllogism. This suggests the following correspondence between **AC** and Sequent Calculus:

1. *Terms* are *Formulas*.
2. *Propositions* are *Sequents*.
3. A *universally taken term* is a *formula on the left-side of the sequent*.
4. A *particularly taken term* is a *formula on the right-side of the sequent*.
5. A *syllogism* is an *instance of the* CUT *rule*.

If this is correct, then the following correspondence between kinds of propositions and sequents also holds:

1. 'All α are β' is '$\alpha \vdash \beta$'.
2. 'No α are β' is '$\alpha, \beta \vdash$'.
3. 'Some α are β' is '$\vdash \alpha, \beta$'.
4. 'Some α are not β' is '$\beta \vdash \alpha$'.

Up to this point the sequent $\alpha \vdash \beta$ is underdetermined by the propositions 'All α are β' and 'Some β are α'. This is unpleasant, as for example the correct instance of CUT

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

Determines two valid syllogisms (*bArbArA,bOcArdO/bArOcO*) but four invalid ones:

$$\frac{\text{Some } B \text{ are not } A \qquad \text{Some } C \text{ are not } B}{\text{Some } C \text{ are not } A}$$

$$\frac{\text{Some } B \text{ are not } A \qquad \text{Some } C \text{ are not } B}{\text{All } A \text{ are } C}$$

(Counter-interpretation: A =animals, B =born in the desert, C =camels)

$$\frac{\text{All } A \text{ are } B \qquad \text{Some } C \text{ are not } B}{\text{All } A \text{ are } C}$$

(Counter-interpretation: A =Albanian ports, B =on the Baltic sea, C =Chilean Ports)

$$\frac{\text{All } A \text{ are } B \qquad \text{All } B \text{ are } C}{\text{Some } C \text{ are not } A}$$

(Counter-interpretation: A, B and C any three co-extensive predicables)

This is where the relevance of (A2) is highlighted: as a valid syllogism can have only one particular premisse and the conclusion will be particular if and only if one of the premisses is too, one cannot read the conclusion as an O-proposition if both premisses are A-propositions, nor can we have an A-conclusion if one of the premisses is of the O-kind. This immediately rules out these four invalid syllogisms and in fact completes the symmetry between the valid syllogisms and the proper instances of CUT. (A2) appears then not only as an axiom for the inner validity of the syllogisms but also as a translation rule between Sequent Calculus and **AC**[4]. This significant role of (A2) will appear over and over again in what follows.

1.3.2. Adding indefinite terms

In order to fully capture **AC** in the Sequent framework we need to formalize the indefinite terms. An educated guess would be to consider the two rules for negation:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\neg\mathrm{R} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\neg\mathrm{L}$$

If '¬' is about to stand for the conjugate-operator, then these rules should be giving us some valid conversions. And they actually do: the four possible applications of these rules give us the four valid obversions of Syllogistics:

1. $\dfrac{A, B \vdash}{A \vdash \neg B}\neg\mathrm{R}$ stands for $\dfrac{\text{No } A \text{ are } B}{\text{All } A \text{ are non-}B}\Theta$
2. $\dfrac{A \vdash B}{\vdash \neg A, B}\neg\mathrm{R}$ stands for $\dfrac{\text{Some } B \text{ are not } A}{\text{Some } B \text{ are non-}A}\Theta$
3. $\dfrac{A \vdash B}{A, \neg B \vdash}\neg\mathrm{L}$ stands for $\dfrac{\text{All } A \text{ are } B}{\text{No } A \text{ are non-}B}\Theta$
4. $\dfrac{\vdash A, B}{\neg A \vdash B}\neg\mathrm{L}$ stands for $\dfrac{\text{Some } A \text{ are } B}{\text{Some } B \text{ are not non-}A}\Theta$

Notice how (A2) makes mandatory the particular interpretation of 2 and 4. The crucial symmetry here is that the rules for negation respect the clause (A1:5): "To take universally (/particularly) a conjugate term is to take particularly (/universally) its base."

As we want the conjugate-operator to be involutive, we should also admit the reverse of these rules instead of allowing multiple applications of the ¬ rules on the same formula. This is a pernickety care, but I am perfectionist enough to assume it.

Contraposition (Σ) is now obtained by a double application of the ¬ rules:

[4]As a matter of fact, not only (A2) is constraining the admissible syllogisms to be recovered up from the derivations, but also (A0) is playing the same role.

$$\dfrac{\dfrac{A \vdash B}{\vdash \neg A, B}\ \neg\,\mathrm{R}}{\neg B \vdash \neg A}\ \neg\,\mathrm{L}$$

And this yields us the only two valid instances of Contraposition:

$$\dfrac{\text{All } A \text{ are } B}{\text{All non-}B \text{ are non-}A}\ \Sigma \qquad \dfrac{\text{Some } B \text{ are not } A}{\text{Some non-}A \text{ are not non-}B}\ \Sigma$$

Distinguished, once again, by (A2).

1.3.3. The System SS

We are in position now to fully formalize **AC** within Sequent Calculus. The resulting system will be called **SS** (after *Sequent Syllogistics*, not the *Schutzstaffel*), described below.

Let $\mathfrak{L}$ be a set of uppercase variables. The system **SS** is the set of all derivations constructible from the following set of rules:

Where $A, B \in \mathfrak{L}$ and Γ, Δ are sub-multisets of $\mathfrak{L}$,

1. The *hypothesis rule*: $\dfrac{}{\Gamma \vdash \Delta}\ \mathrm{hyp}$ provided $\#\Gamma + \#\Delta = 2$.
2. The CUT *rule*: $\dfrac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{CUT}$
3. The two rules for $\neg$.
4. The *reverse rules* for $\neg$: $\dfrac{\Gamma \vdash \neg A, \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{rev}\,\neg\,\mathrm{R}$ and $\dfrac{\Gamma, \neg A \vdash \Delta}{\Gamma \vdash A, \Delta}\ \mathrm{rev}\,\neg\,\mathrm{L}$
5. The *exchange rules*: $\dfrac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash B, A, \Delta}\ \mathrm{Ex\,R}$ and $\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, B, A \vdash \Delta}\ \mathrm{Ex\,L}$

I allow the application of the Negation Rules over negated formulas but I will never do it.

1.3.4. Comments

The exchange rules recovers the validity of the Ω conversions. And only the E- and I-propositions validate it, as expected.

As we do not have weakness nor contraction, the number of terms in every Sequent is always 2. This will give us Cut-Elimination in a very straightforward (and almost silly) way: just replace every instance of CUT with a proper instance of the Hypothesis rule. For instance,

$$\dfrac{\dfrac{}{A, B \vdash} \qquad \dfrac{}{\vdash B, C}}{A \vdash C} \quad \rightsquigarrow \quad \dfrac{}{A \vdash C}$$

The most important feature of this system is that, as all these rules conform to (A1-3), they produce derivations where every branching corresponds to a syllogistically valid inference. It can be checked by brute force that it is also complete: thanks to the limit of formulas in the application of the *hypothesis rule*, there are only 12 interesting cases of it. Thus the number of interesting derivations (where an 'interesting' derivation is one where '$\neg$' appears at most once in each formula) is upper bounded and the checking process, although tedious, is happily finite.

It is nonetheless worth noting that **SS** is not perfectly correct, for it has some derivations that cannot be read as instances of valid syllogistic inferences. I say "perfectly correct" because it is not that some derivations correspond to invalid syllogisms. As a matter of fact, this never happens, so the system in this sense is minimally correct. But in some cases (A2) produces insurmountable inconsistencies in the translation process. Consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B}\,\text{hyp}}{A, \neg B \vdash}\,\neg\,\text{L}}{\neg B \vdash \neg A}\,\neg\,\text{R}}{\vdash B, \neg A}\,\text{rev}\,\neg\,\text{L}$$

This is an SS-valid derivation, but there is no sequence of valid syllogistic conversions that corresponds to it. The reason is that, as the conclusion is a particular proposition, for (A2) to be respected across the derivation all sequents in it should also be particular; but they are not, for '$A, \neg B \vdash$' is in fact universal. But it still holds that every *particular application* of a rule recovers a valid syllogistic inference.

There are other derivations that raise some suspicions. For instance, consider this one:

$$\dfrac{\dfrac{}{A, A \vdash}\,\text{hyp} \qquad \dfrac{}{B \vdash A}\,\text{hyp}}{A, B \vdash}\,\text{CUT}$$

That would correspond to the syllogism

$$\dfrac{\text{No } A \text{ are } A \qquad \text{All } B \text{ are } A}{\text{No } B \text{ are } A}$$

This syllogism is inadmissible in **AC** (for they does not conform to (A0)), but its validity according to **SS** is harmless, for even Aristotle accepted the validity of inferences from false premisses:

> It is possible for the premisses of the deduction to be true, or to be false, or to be the one true, the other false. The conclusion is either true or false necessarily. From true premisses it is not possible to draw a false conclusion; but a true conclusion may be drawn from false premisses. (Aristotle, 1995, 53b 4-7)

But here's a beautiful serendipity. I was working on an early draft of this paper when I noticed that derivations like this are not only innocuous, but in fact they can be used to provide an insightful improvement of the expressive power of **AC**.

Call the following propositions and their corresponding sequents as:

1. 'No α are α' ($A, A \vdash$): *contradiction*.
2. 'Some α are α' ($\vdash A, A$): *existential claim*.
3. 'All α are α' ($A \vdash A$): *tautology*.

And allow for the weakening of (A0) in the following way:

(A0') *Permissive Formation Rule*: The inference has three propositions and no more than three terms, either in conjugate or non-conjugate form, of which one must appear at least once in every premisse.

These changes makes admissible the consideration of the syllogisms corresponding to the following (already SS-valid) derivations:

1. Syllogism with a contradiction:

$$\frac{A, A \vdash \qquad \Gamma \vdash A, \Delta}{\Gamma, A \vdash \Delta}\,\textsc{cut}$$

2. Syllogism with an existential claim:

$$\frac{\vdash A, A \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash A, \Delta}\,\textsc{cut}$$

3. Syllogism with a tautology (I):

$$\frac{A \vdash A \qquad \Gamma, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,\textsc{cut}$$

4. Syllogism with a tautology (II):

$$\frac{A \vdash A \qquad \Gamma \vdash A, \Delta}{\Gamma, A \vdash \Delta}\,\textsc{cut}$$

On closer examination this give us the following pseudo-valid inferences:

1. Syllogism with a contradiction:

$$\frac{A, A \vdash \qquad B \vdash A}{B, A \vdash} \text{ and } \frac{A, A \vdash \qquad \vdash A, B}{A \vdash B}$$

The *contrary* or *sub-contrary* of the second premisse.

2. Syllogism with an existential claim:

$$\frac{\vdash A, A \qquad B, A \vdash}{B \vdash A} \text{ and } \frac{\vdash A, A \qquad A \vdash B}{\vdash A, B}$$

The *subaltern* of each second premisse.

3. Syllogism with a tautology yields an *equivalent* of the second premisse.

That is, we have recovered a major part of the Theory of Oppositions within our framework.[5] The remaining oppositions can be obtained by composition of these ones in quite obvious ways. The *contradictory* of an I-proposition, for instance, is obtained by:

$$\frac{B, B \vdash \qquad \dfrac{A, A \vdash \qquad \vdash A, B}{A \vdash B}}{A, B \vdash}$$

The morale here is illuminating: the Three Axioms of **AC** not only govern the mediate and immediate inferences with definite and indefinite terms, but also the Theory of Oppositions, an important part of traditional logic.

This insight is not mentioned in Correia's work, so, as far as I can tell, it has been presented for the first time here.

1.4. Philosophical harvest

In a conference given at the University of Chile in 2015, Manuel Correia presented **AC** and commented some open problems within his project.

For a start, he manifested some suspicions on (A2), based on his already successful disposal of the Axiom of Negativity (from two negative premisses no conclusion follows). He considered to be an unpleasant asymmetry that only one of these two Axioms were disposable, given their similarities.

On the other hand, he explained that **AC** already captures some propositional inferences (*Modus Ponendo Ponens*, *Tollendo Tollens* and *Tollendo Ponens*), up to

[5]Notice that the only valid inference with $A \vdash A$ read as 'Some A are not A' is

$$\frac{A \vdash A \qquad A \vdash B}{A \vdash B}$$

And this is an existential invalid inference that, as far as I know, has not a special name. Still, interestingly it recovers a mixed aspect of the invalidity of subalternation and contrariety:

$$\frac{\text{All } A \text{ are } B}{\text{Some } B \text{ are not } A}$$

a proper extension of the notions involved in (A1-3) (cf. Correia (2017)). This suggests that the whole (classical) propositional logic may be successfully captured by this extension of **AC**, thus contradicting the modern prejudice that Syllogistics is an incomplete fragment of predicate logic (an apparently interesting result for Correia's philosophical agenda). But within his project the classically valid inference:

$$\frac{A \qquad \neg A}{B}$$

has the problem that the conclusion term B appears not in the premisses and thus it cannot be said to be governed by (A3) in a straightforward way.

These concerns may be resumed in three questions:

1. If (A2) is necessary, why is it so?
2. Is it possible to extend the three axioms (A1-3) over modern mathematical logic?
3. What is the relation between the instances of explosion and (A3)?

The correspondence between **SS** and **AC** suggests the answers to these questions. As we saw in section 1.3, (A2) is mandatory for the proper translation from **SS** back to **AC**, and thus its full meaning can only be acknowledged once the correspondence has been established. Correia's suspicions were justified, for he had not considered **SS** and thus was unable to fully comprehend the role that (A2) was playing within his system.

On the same line, the fact that axioms (A1) and (A3) correspond to structural properties of Sequent Calculus suggests that both modern logic and Syllogistics are governed by the same geometrical properties, which may be properly characterized as a certain kind of symmetry. This symmetry reveals itself as grounding both the CUT rule and the Axiom of Linkage without any conceptual priority, thus showing that the common notion should be seek beyond the realms of Syllogistics or Sequent Calculus.

As for Explosion, there are some things to be noticed. To begin with, one must recognize that the conformity with the rule (A0) aligns Syllogistics with Relevant Logics, because it explicitly states the condition of relevance between terms of the propositions. If this is correct, then one shall expect, rather than regret, the failure of Explosion[6]. As a matter of fact, **AC** can be fully recognized as a Paraconsistent Logic, for **SS** invalidates the Meta Explosion scheme (cf. Barrio et al. (2019)):

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash \neg A, \Delta'}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}$$

[6]That Aristotle's logic is classical is far less an expectation of the aristotelians than it is of classical logicians.

When $B \notin \Gamma \cup \Gamma' \cup \Delta \cup \Delta'$.

Besides Correia's problems, system **SS** is philosophically interesting on its own. When considered in isolation, it is a rather silly system, with poor structural properties and a trivial Cut-Elimination theorem; and yet, once one considers its correspondence with **AC**, its blatant philosophical relevance is highlighted. I see this as abductive evidence in favor of the development of weird substructural logics for more than mere mathematical curiosity.

And yet one more word on **SS**'s Cut-Elimination. Albeit this result is quite uninteresting from a mathematical point of view, I think that it has an interesting philosophical significance: it shows that one learns nothing from syllogisms. Logic is, so to speak, strictly analytical. This follows the spirit of logic as a toolbox for thinking, the *organon* of the Peripatetics. Being the syllogism the only rule of inference of Syllogistics, the possibility of its elimination reveals a fine aspect of Aristotle's logical subtlety.

1.5. Conclusions

In Álvarez and Correia (2012) is presented what is currently the most complete axiomatization of Syllogistics. In this paper I showed that the key concepts involving that system can be put into correspondence with the structural features of a special Sequent Calculus system, **SS**. On the grounds of this discovery I proved that the system of Álvarez and Correia (2012) not only covers mediate and immediate inferences with definite and indefinite terms, but also can be extended to recover the Theory of Oppositions, a major part of ancient logic that was neglected by the original proposal. As for the philosophical interpretation of the system, I showed that most of the concerns Manuel Correia had on his system can be answered within this new framework. In the end, a somewhat new philosophical relevance of the Cut-Elimination Theorem was highlighted, on the grounds of its relation with Syllogistics.

1.6. Appendix

1.6.1. Beyond the Aristotelian realm

System **SS** can be extended beyond the realm of standard Syllogistics. In this appendix I will present some preliminary results on these extensions.

One natural question about **SS** is whether one can introduce connectives such as conjunction and disjunction. As a matter of fact, either the rules of weakening or additive left-hand conjunction introduction and right-hand disjunction introduction are compatible with the **AC** correspondence, for they produce truth-preserving syllogisms with complex terms. For instance, given the validity of the corresponding syllogism for

$$\frac{A \vdash B \qquad B, C \vdash}{A, C \vdash}$$

One can weaken (before or after the cut) on both sides harmlessly, given that the resulting proposition preserve its kind:

$$\frac{\dfrac{A \vdash B}{A, D \vdash B} \qquad \dfrac{B, C \vdash}{B, C, E \vdash}}{A, C, D, E \vdash}$$

And this is still a valid syllogism:

$$\frac{\dfrac{\text{All } A \text{ are } B}{\text{All } A \textit{ and } D \text{ are } B} \qquad \dfrac{\text{No } B \text{ are } C}{\text{No } B \text{ are } C \textit{ and } E}}{\text{No } A \textit{ and } D \text{ are } C \textit{ and } E}$$

Where the conclusion is valid up to simple conversions (Ω).

The problem with weakening on the empty side of a sequent or the usual rules for right-hand conjunction introduction or left-hand disjunction introduction is that the kind of the proposition is altered and thus some derivation steps are no longer syllogistically valid:

$$\frac{\vdash A, B}{C \vdash A, B}$$

$$\frac{\text{Some } A \text{ are } B}{\text{Some } A \textit{ or } B \text{ are not } C}$$

But all the counter-examples to these common rules that I've encountered have to do with the ambivalence of $\alpha \vdash \beta$ as either an A-proposition or an O-proposition. For instance, weakening on the left for an E-proposition gives a beautiful *ex falso quodlibet* syllogism:

$$\frac{B, C \vdash}{B, C \vdash E}$$

$$\frac{\text{No } B \text{ are } C}{\text{All } B \textit{ and } C \text{ are } E}$$

As a matter of fact, in the fragment of **SS** where the left-hand side of the sequent cannot be empty, everything seems to be working fine and the complete additive group of rules for conjunction and disjunction can be introduced. Note, however, that within this subsystem there are no particular propositions. This reinforces the suspicion that Particularity seems to be playing a major role in the inheritance of the weaknesses of Syllogistics by Sequent Calculus.

Bibliography

Aristotle (1995). Prior analytics. In Barnes, J., editor, *The Complete Works of Aristotle. One volume, digital edition*, pages 39–113. Princeton.

Barrio, E. A., Pailos, F., and Szmuc, D. (2019). A hierarchy of classical and paraconsistent logics. *Journal of Philosophical Logic*.

Correia, M. (2017). La lógica aristotélica y sus perspectivas. *Pensamiento*, 73(275):5–19.

De Morgan, A. (1880). *Syllabus of a proposed system of logic*. Harvard University Press.

Álvarez, E. and Correia, M. (2012). Syllogistics with indefinite terms. *History and Philosophy of Logic*, 33(4):297–306.