Physics Letters B 811 (2020) 135978

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Vortices in fracton type gauge theories

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ARTICLE INFO

Article history:
Received 6 October 2020
Received in revised form 12 November 2020
Accepted 23 November 2020
Available online 26 November 2020
Editor: A. Ringwald

ABSTRACT

We consider a vector gauge theory in $2+1$ dimensions of the type recently proposed by Radzihovsky and Hermele [1] to describe fracton phases of matter. The theory has $U(1)\times U(1)$ vector gauge fields coupled to an additional vector field with a non conventional gauge symmetry. We added to the theory scalar matter in order to break the gauge symmetry. We analyze non trivial configurations by reducing the field equations to first order self dual (BPS) equations which we solved numerically. We have found vortex solutions for the gauge fields which in turn generate for the extra vector field non-trivial configurations that can be associated to magnetic dipoles.

1. Introduction

The study of non-trivial solutions in quantum field theories has historically played an essential role in describing non-perturbative phenomena usually linked to topological properties of these theories, both in High Energy applications [2] and in condensed matter systems [3]. Vortices play a fundamental role in the descriptions of many properties of Superconductors and have been also investigated in connection with symmetry-breaking phase transitions in the early universe as cosmic strings, one dimensional stable objects that can play the role of cosmological sources of gravitational waves, a subject that is actively investigated at present (see [4] and references therein). They can also arise in superstring theory as cosmologically-stretched fundamental strings (see [5] and references therein).

In the last year, there has been a growing interest in the study of a new class of quantum states of matter in which quasiparticles called "fractons" were introduced in quantum spin-liquid models [6]. Afterwards, topological quantum order was studied in Majorana fermion models in which only composites of such elementary excitations were free to move in certain directions [10]. Later on, a connection in the low energy limit between fracton phases and tensor gauge theories was studied in ref. [7]. Since then, interest in the subject grew in various directions of condensed matter and quantum field theories physics including studies on gravity and elasticity areas (for reviews see [8,9] and references therein).

More recently Radzihovsky and Hermele (RH) have considered a description of fracton phases in $2+1$ dimension in terms of gauge vector fields [1]. The model discussed by these authors consists of $U(1)\times U(1)$ (conventional) vector gauge fields coupled to an additional vector field in such a way that the resulting Lagrangian is invariant under a deformed gauge transformation.

In this work we will consider a theory where the $U(1)\times U(1)$ vector gauge fields in the RH model are minimally coupled to scalar matter implementing the Higgs mechanism. We will show that also this model having an additional vector field has magnetic like vortex solutions of the Nielsen-Olesen type, which in turn generate a non trivial configuration for the extra vector field of the model. In addition, proceeding as in the original simpler $U(1)$ case, we will be able to reduce the second order field equations to first order self-dual equations [11,12]. The well known Nielsen-Olesen ansatz leads to radial equations that can be solved numerically. The solution corresponds to stable vortex magnetic fields associated to the $U(1)\times U(1)$ gauge field sector and an additional magnetic field associated to the extra vector field.

2. The model

We shall consider a $d=2+1$ dimensional $U(1)\times U(1)$ gauge theory with gauge fields A_i^a, A_0^a with $i=1,2$ spatial and $a=1,2$ "flavor" indices. There is also an additional vector field (V_0,V_i). The corresponding Lagrangian density L_G is the one introduced in [1] (without external sources),

$$L_G = -\sum_a \frac{1}{4} F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(\partial_t V_1 + \partial_1 V_0 - A_0^{(1)})^2 + \frac{1}{2}(\partial_t V_2 + \partial_2 V_0 - A_0^{(2)})^2 - \frac{1}{2}(\epsilon_{ij}\partial_i V_j + \mathcal{A})^2. \quad (1)$$

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https://doi.org/10.1016/j.physletb.2020.135978

We assume the standard summation convention for space-time indices with a metric $(-++)$ but we write explicitly sums involving flavor indices. Here $F_{ij} = \partial_i A_j - \partial_j A_i$ and

$$\mathcal{A} = \sum_a \epsilon_{ia} A_i^a . \tag{2}$$

We will also introduce scalar matter minimally coupled to the fields A_μ^a together with a scalar potential to implement gauge symmetry breaking and the Higgs mechanism

$$L_S = -\sum_a D_\mu \phi^{a\dagger} D^\mu \phi^a - V[\phi^a] \tag{3}$$

with the covariant derivatives $(\mu = 0, 1, 2)$ defined as

$$D_\mu \phi^a = (\partial_\mu - i g^a A_\mu^a)\phi^a \tag{4}$$

and

$$V[\phi^a] = \sum_a \lambda^a (|\phi^a|^2 - (\eta^a)^2)^2 . \tag{5}$$

In principle the potential could include a mixing $\tilde\lambda |\phi^{(1)}|^2 |\phi^{(2)}|^2$ but for simplicity we will assume $\tilde\lambda = 0$.

The total Lagrangian density L is then given by

$$L = L_G + L_S . \tag{6}$$

The theory is invariant under "deformed" gauge transformations [1],

$$\begin{array}{ll} \vec{A}_i^a \to A_i^a + \partial_i \alpha^a & A_0^a \to A_0^a + \partial_0 \alpha^a , \\ V_i \to V_i + \partial_i \beta - \alpha_i & V_0 = V_0 - \partial_0 \beta , \end{array} \tag{7}$$

together with

$$\phi^a \to \exp(i g^a \alpha^a)\phi^a . \tag{8}$$

In this work we will be interested only in static, purely magnetic configurations, so that the energy density can be written as

$$E = \sum_a \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2}(\epsilon_{ij}\partial_i V_j + \mathcal{A})^2 + \sum_a (D_i \Phi^a)^\dagger (D_i \Phi^a) + V[\phi^a]. \tag{9}$$

Euler-Lagrange equations are then

$$\partial_i F_{ik}^a = \epsilon_{ka}(\epsilon_{ij}\partial_i V_j + \mathcal{A}) + i g (\phi^a D_k \phi^{a^\dagger} - \phi^{a\dagger} D_k \phi) \tag{10}$$

$$D_k D_k \phi^a = -\frac{\delta V}{\delta \phi^{a\dagger}} \tag{11}$$

$$\epsilon_{ij}\partial_k \partial_i V_j = \partial_k \mathcal{A}. \tag{12}$$

Instead of solving these second order field equations, we shall follow the standard Bogomolny procedure [11] and we rewrite the energy density as

$$\begin{aligned} E = \sum_a \Bigg(& \frac{1}{2} |D_i \Phi^a - i\gamma^a \varepsilon_{ij} D_j \Phi^a|^2 \\ & + \frac{1}{4}\left(F_{ij}^a - \gamma^a g^a \varepsilon_{ij}(\phi^a \phi^{a\dagger} - (\eta^a)^2)\right)^2 \\ & + \frac{1}{2}(\epsilon_{ij}\partial_i V_j + \mathcal{A})^2 + \sum_a \left(\lambda^a - \frac{(g^a)^2}{2}\right)(\Phi^a \Phi^{a\dagger} - (\eta^a)^2)^2 \\ & - \gamma^a \frac{g^a}{2}(\eta^a)^2 \varepsilon_{ij} F_{ij}^a \Bigg) \end{aligned} \tag{13}$$

where $\gamma^a = \pm 1$ and we have discarded total derivatives which vanish after integration for appropriate boundary conditions (in this case we require finite energy in $R^{(2)}$ which implies vanishing of the scalar covariant derivatives at infinity). So, if

$$\lambda^a = \frac{(g^a)^2}{2} \tag{14}$$

the minimal value of the energy $\mathcal{E}$

$$\mathcal{E} = \int d^2 x E \tag{15}$$

is reached when the three squared terms in eq. (13) vanish

$$D_i \Phi^a - i\gamma^a \varepsilon_{ij} D_j \Phi^a = 0 \tag{16}$$

$$F_{ij}^a - \gamma^a g^a \varepsilon_{ij}(\Phi^a \Phi^{a\dagger} - (\eta^a)^2) = 0 \tag{17}$$

$$\epsilon_{ij}\partial_i V_j + \mathcal{A} = 0. \tag{18}$$

If eqs. (16)-(18) are satisfied the energy $\mathcal{E}$ is

$$\mathcal{E} = \int d^2 x E = -\sum_a \gamma_a g_a (\eta^a)^2 \int d^2 x B^a = 2\pi \sum_a (\eta^a)^2 |m^a| \tag{19}$$

where m^a is the winding number associated to the quantized magnetic flux.

Now, the simplicity and convenience of the self dual equations are apparent. Equations for $(A_i^{(1)}, \Phi^{(1)})$ a $(A_i^{(2)}, \Phi^{(2)})$ are first order and decoupled. After solving them, we can use $A_i^{(1)}, A_i^{(2)}$ as sources for V_i. On the other hand, the energy can be calculated explicitly and their stability is ensured because they satisfy the Bogomolny bound. The self-dual equations are valid only when the relation Eq. (14) is valid. It is simple to see that this relation implies the equality between vector and scalar masses of the theory. It is also well established the connection between the existence of self dual equations and $N = 2$ supersymmetry for several models [12,13]. In the original Ginzburg-Landau theory of superconductivity (which has a single $U(1)$ sector) relation Eq. (14) signals the boundary between Type I and Type II superconductors.

We will look for axially symmetric configurations for $(A_i^{(1)}, \Phi^{(1)})$ and $(A_i^{(2)}, \Phi^{(2)})$, so we make the following ansatz in polar coordinates (r, φ)

$$A_\varphi^a = -A_x^a r \sin\varphi + A_y^a r \cos\varphi = \frac{1}{g_a} a_\varphi(r) \tag{20}$$

$$A_r^a = A_x^a \cos\varphi + A_y^a \sin\varphi = 0 \tag{21}$$

$$\Phi^a = \eta^a f^a(r) e^{i m_a \varphi}. \tag{22}$$

Then, the first two equations become

$$\partial_r f^a = -\frac{\gamma^a}{r}(m^a - a_\varphi^a) f^a \tag{23}$$

$$\frac{1}{r}\partial_r a_\varphi^a = (\eta^a)^2 \gamma^a (g^2)^a ((f^a)^2 - 1). \tag{24}$$

Finite energy requires the following boundary conditions

$$\begin{array}{ll} a_\varphi^a(0) = 0 , & a_\varphi^a(\infty) = m^a \\ f^a(0) = 0 , & f^a(\infty) = 1 . \end{array} \tag{25}$$

It is easy to check that consistency requires $\gamma^a/m^a < 0$. It will also be convenient to redefine

$$\rho = |g^1 \eta| r \quad \tilde{a}_\varphi^a = a_\varphi^a - m^a \tag{26}$$

then

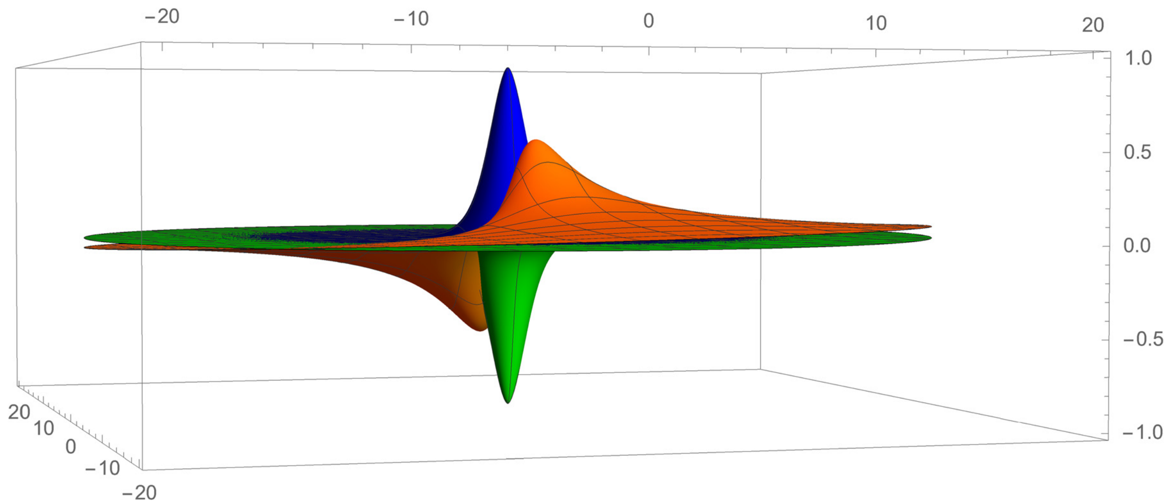


Fig. 1. We show the magnetic field B^1 associated to a vortex with winding number $m^{(1)} = 1$, (blue), $m^{(2)} = -1$ (green) and the magnetic field associated to the field a_i (orange). Parameters have been chosen so that $g^a = 1$ and $\eta^a = 1$ for $a = 1, 2$.

$$\partial_\rho f^{(1)} = \frac{\gamma^{(1)}}{\rho}(\tilde{a}_\varphi^{(1)}) f^{(1)} \quad \frac{1}{\rho}\partial_\rho \tilde{a}_\varphi^{(1)} = \gamma^{(1)}((f^{(1)})^2 - 1)) \tag{27}$$

$$\partial_\rho f^{(2)} = \frac{\gamma^{(2)}}{\rho}(\tilde{a}_\varphi^{(2)}) f^{(2)} \quad \frac{1}{\rho}\partial_\rho \tilde{a}_\varphi^{(2)} = \gamma^{(2)}\delta^2((f^{(2)})^2 - 1)) \tag{28}$$

where $\delta = \frac{g^{(2)}\eta^{(2)}}{g^{(1)}\eta^{(1)}}$ and

$$\tilde{a}_\varphi^a(0) = m^a\,, \qquad \tilde{a}_\varphi^a(\infty) = 0$$
$$f^a(0) = 0\,, \qquad f^a(\infty) = 1 \tag{29}$$

One can then show that

$$\partial^2_{\rho^2}\tilde{a}_\varphi^{(1)} - \frac{1}{\rho}\partial_\rho \tilde{a}_\varphi^{(1)}(1 + 2\gamma^a \tilde{a}_\varphi^{(1)}) - 2\tilde{a}_\varphi^{(1)} = 0 \tag{30}$$

$$\partial^2_{\rho^2}\tilde{a}_\varphi^{(2)} - \frac{1}{\rho}\partial_\rho \tilde{a}_\varphi^{(2)}(1 + 2\gamma^a \tilde{a}_\varphi^{(2)}) - 2\delta^2\tilde{a}_\varphi^{(2)} = 0. \tag{31}$$

It is obvious that, if $m^{(1)} = m^{(2)}$ then $\tilde{a}^{(2)}(\rho^*) = \tilde{a}^{(1)}(\rho)$ with $\rho^* = \delta\rho$.

The equation (18) for V_i can be now written in terms $a_\varphi^{(1)}$ and $a_\varphi^{(2)}$,

$$\tilde{B} \equiv \epsilon_{ij}\partial_i V_j = -A_x^{(2)} + A_y^{(1)} = \frac{a_\varphi^{(1)}\cos\varphi + a_\varphi^{(2)}\sin\varphi}{r} \tag{32}$$

so that once we have solved the equations for a_φ^a, we can easily obtain the solution for $\tilde{B}$.

We have found numerical solutions of Eqs. (29)–(31) by using a relaxation method. We have analyzed different topological sectors with different winding numbers $(m^{(1)}, m^{(2)})$ and fluxes B^a

$$\Phi^a = \int d^2x B^a = \int d^2x F^a_{12} = \frac{2\pi}{g^a} m^a. \tag{33}$$

We show in Fig. 1 a solution for the case in which the topological numbers $m^{(1)} = 1$ and $m^{(2)} = -1$, and where for simplicity we have set $g^{(1)} = g^{(2)} = 1$ and $\eta^{(1)} = \eta^{(2)}$. The upper peak (in blue) corresponds to the magnetic field associated with the vortex with winding number $m^{(1)} = 1$, and the lowest one to the magnetic field of the vortex with $m^{(2)} = -1$. In the same plot (in orange) we show the $\tilde{B}$ field defined in Eq. (32), which present a double peak structure. We remark that the particular (mirror) symmetry of the figures originates from our choices for η^a and m^a but more generic cases can be considered without additional computational effort.

Notice that the sources of this generalized magnetic field are the vector potentials of the $U(1) \times U(1)$ sector via the term $\sum_a \epsilon_{ia} A_i^a$. Thus, both $U(1)$ gauge fields contribute to the $\tilde{B}$ field. Nevertheless, it is enough to have only one of these gauge fields different from zero to produce a non-zero $\tilde{B}$ field. Indeed in Fig. 2 we display contour plots of the $\tilde{B}$ field for two different choices of gauge fields of the $U(1) \times U(1)$ sector. Panel (*a*) corresponds to the contour plot of $\tilde{B}$ associated to the Fig. 3, this is $(m^{(1)}, m^{(2)}) = (1, -1)$. Panel (*b*), corresponds to a contour plot of $\tilde{B}$ for the case $(m^{(1)}, m^{(2)}) = (1, 0)$, where only the $A^{(1)}$ acts as a source for $\tilde{B}$. Notice that not only the intensity of the field changes depending on the choice of m^a but also figure in the panel (b) is rotated with respect to the one in panel (*a*).

Looking in more detail to panel (*b*) in Fig. 2, the contour plot looks qualitatively very similar to those of the magnetic field produced by a magnetic dipole placed outside the (x, y) plane, at a certain height in a z axis in three spatial dimensions, as represented schematically in Fig. 3 for the $A^{(1)}$ field. In this figure we display the $B^{(1)}$ magnetic field tube (in light blue), and the two effective magnetic dipoles $\vec{\mu}$ (represented as orange arrows) associated to the $\tilde{B}$ field. Notice that the direction of the dipole is correlated with the flavor of the gauge field ($a = 1$ in this case). Had we chosen, the other flavor $a = 2$, the orientation of the dipole would be different. In fact, panel (*b*) of Fig. 2 results from the superposition of these cases. Associated to the generalized gauge transformation of the V_i field, a conserved (and gauge invariant) density j_0^m was identified in Ref. [1],

$$j_0^m = \epsilon_{ij}\partial_i V_j + \mathcal{A} - r_i \epsilon_{ij} J^m_{0j} \tag{34}$$

with

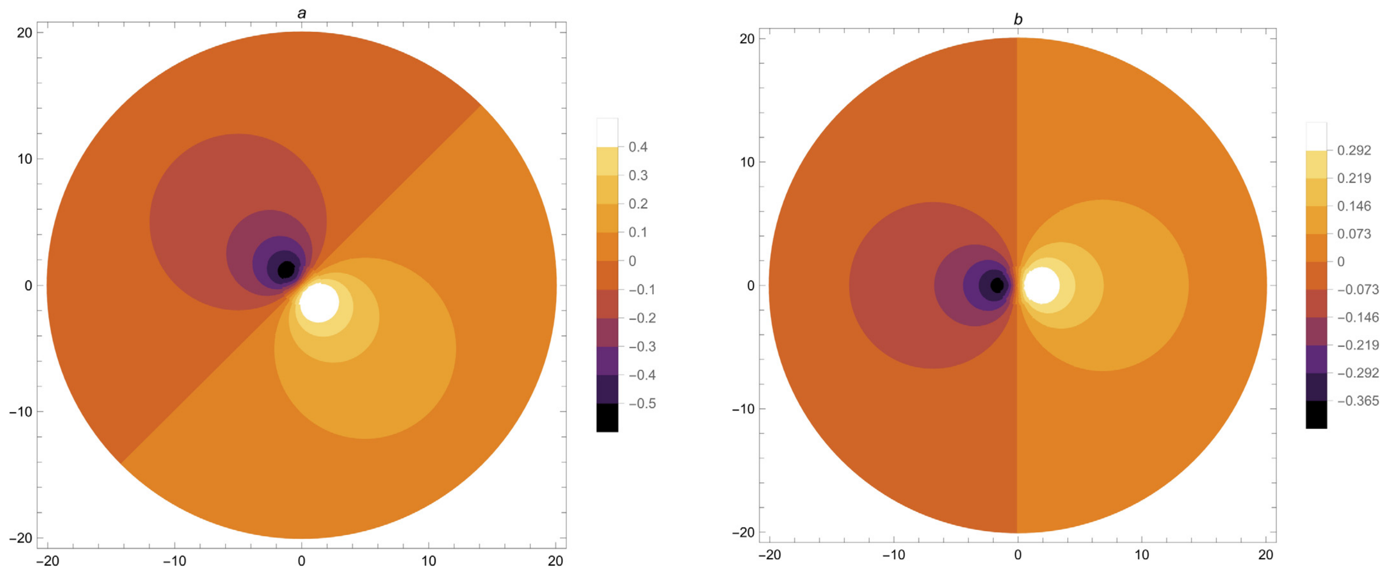
$$J^m_{0j} = \sum_l \epsilon_{jl} B^{(l)}. \tag{35}$$

In our case j_0^m reduces to

$$j_0^m = x B^{(1)} + y B^{(2)}. \tag{36}$$

We show in Fig. 4 a plot of this density j_0^m for the case in which $(m^{(1)}, m^{(2)}) = (1, -1)$. Similar plots can be obtained for the different sectors.

The Lagrangian (1) proposed in [1], when coupled to appropriate currents leads to Gauss law of a symmetric tensor gauge theory coupled to an external *electric* charge which encodes conservation



(a) sector (1,-1) (b) sector (1,0)

Fig. 2. Contour plots for the magnetic field associated to the field $\tilde{B}[a_i]$ for different winding numbers $(m^{(1)}, m^{(2)})$. Parameters have been chosen so that $g^a = 1$ and $\eta^a = 1$. Panel (*a*) corresponds to contour plots for sector $(1, -1)$ while panel (*b*) to sector $(1, 0)$.

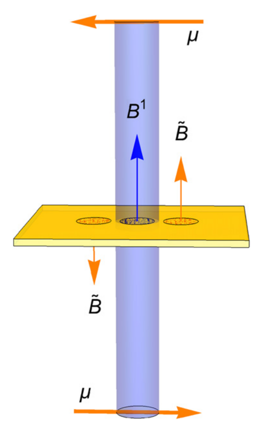


Fig. 3. We represent in three spatial dimensions the magnetic field associated to a $B^{(1)}$ vortex with $m^{(1)} = 1$, (light blue) and the $\tilde{B}$ field associated to a_i (orange). Orange arrows μ represent the two effective magnetic dipoles, in this case in the x direction.

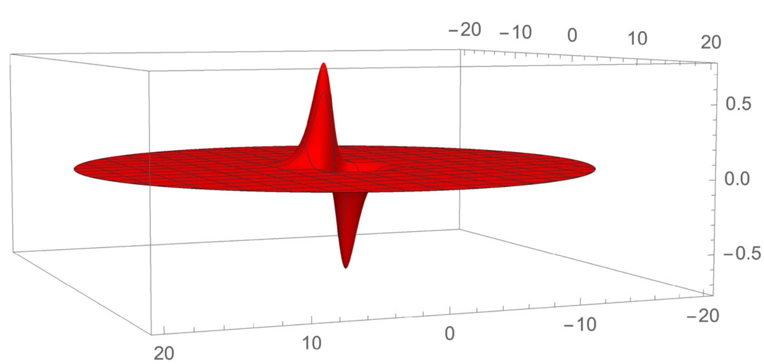


Fig. 4. Plot of the density j_0^m for the $(1, -1)$ sector. Parameters have been chosen so that $g^a = 1$ and $\eta^a = 1$.

not only of such charge but also conservation of an electric dipole moment [9]. In the present work, we have shown that the model described by Lagrangian (6), presents an interesting magnetics sector where in addition to the Nielsen-Olesen type vortex solutions typical of the standard $U(1)$ Higgs models, the coupling between the gauge fields and the vector field V_i gives rise to additional *magnetic* fields which are qualitatively similar to those produced by an effective magnetic dipole as reflected by the conserved density j_0^m.

In this letter we have analyzed the vortices at the BPS point. It is well known that for conventional superconductors, this relation between coupling constants signals the boundary between Type I and Type II superconductors. At this point, vortex-vortex interactions also change from attractive to repulsive. It was later discovered that at this point the model coincided with the bosonic sector of an N=2 SUSY version of the Abelian Higgs model [13]. A relevant question that arises is then how these features can be present in this system and what could it be the role of this kind of structures in fracton models. Even a more interesting situation could be expected if in addition to Maxwell term considered here, a Chern-Simons term is added. It is well-known that in the presence of Chern Simons term in the standard case vortices that carry both, electric and magnetic charge are present [14,15]. It is also well known that BPS equations can be found for conveniently tuned models both in the relativistic and non-relativistic cases [16–18]. We expected that the analysis presented here can be also applied to this case. We hope to report on this issue soon.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

F.A.S. is financially supported by PIP-CONICET (PIP688), and UNLP grants X910. G.S.L. is financially supported by PICT2016-1212, PIP-CONICET 11220150100653CO, and UBACyT 20020170100496BA.

References

[1] L. Radzihovsky, M. Hermele, Phys. Rev. Lett. 124 (2020) 050402.
[2] M. Shifman, Advanced Topics in Quantum Field Theory, Cambridge University Press, Cambridge, UK, 2012.
[3] E. Fradkin, Field Theories of Condensed Matter Physics, Cambridge University Press, Cambridge, UK, 2013.
[4] J. Ellis, M. Lewicki, arXiv:2009.06555 [astro-ph.CO].
[5] J. Polchinski, Introduction to cosmic F- and D-strings, arXiv:hep-th/0412244.
[6] C. Chamon, Phys. Rev. Lett. 94 (2005) 040402.
[7] M. Pretko, Phys. Rev. B 95 (2017) 115139;
M. Pretko, Phys. Rev. B 96 (2017) 125151.
[8] R.M. Nandkishore, M. Hermele, Annu. Rev. Condens. Matter Phys. 10 (2019) 295.
[9] M. Pretko, X. Chen, Y. You, Int. J. Mod. Phys. A 35 (2020) 2030003.
[10] Sagar Vijay, Jeongwan Haah, Liang Fu, Phys. Rev. B 92 (2015) 235136.
[11] E.B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976) 449, Yad. Fiz. 24 (1976) 861.
[12] H.J. de Vega, F.A. Schaposnik, Phys. Rev. D 14 (1976) 110.
[13] J.D. Edelstein, C. Nunez, F. Schaposnik, Phys. Lett. B 329 (1994) 39.
[14] S.K. Paul, A. Khare, Phys. Lett. B 174 (1986) 420, Erratum: Phys. Lett. B 177 (1986) 453.
[15] H.J. de Vega, F.A. Schaposnik, Phys. Rev. Lett. 56 (1986) 2564.
[16] J. Hong, Y. Kim, P.Y. Pac, Phys. Rev. Lett. 64 (1990) 2230.
[17] R. Jackiw, E.J. Weinberg, Phys. Rev. Lett. 64 (1990) 2234.
[18] R. Jackiw, S.Y. Pi, Phys. Rev. D 42 (1990) 3500, Erratum: Phys. Rev. D 48 (1993) 3929.