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HIGHLIGHTS (for review)

- Motivated by the quantum harmonic oscillator states, the Hermite-Gaussian model is proposed as a generalization of the standard Gaussian one.
- Mixture and real (or imaginary) superpositions of eigenstates have a diagonal Fisher metric.
- Hermite-Gaussian model can be used for geometrical characterizations of unknown parameters in scenarios that employ quantum harmonic oscillators.
- Fisher metric of a general state of the quantum harmonic oscillator only depends on the variance.

Hermite–Gaussian model for quantum states

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Abstract

In order to characterize quantum states within the context of information geometry, we propose a generalization of the Gaussian model, which we called the *Hermite–Gaussian model*. We obtain the Fisher–Rao metric and the scalar curvature for this model, and we show its relation with the one-dimensional quantum harmonic oscillator. Using this model we characterize some families of states of the quantum harmonic oscillator. We find that for eigenstates of the Hamiltonian, mixtures of eigenstates and even or odd superpositions of eigenstates, the associated Fisher–Rao metrics —which are relevant in the context of quantum parameter estimation theory— are diagonal. Finally, we consider the action of the amplitude damping channel and discuss the relationship between the quantum decay and the different geometric indicators.

Keywords: Fisher–Rao metric, statistical models, Gaussian model, Hermite–Gaussian model

1. Introduction

The information geometry approach [1–10] studies the differential geometric structure of statistical models. A statistical model consists of a family of probability distribution functions (PDFs) parameterized by continuous variables. In order to endow these models with a geometric structure, it is necessary to define the Fisher–Rao metric [4], which in turn, is linked with the concepts of entropy and Fisher information. Once we have a statistical manifold, the main goal of the information geometry approach is to characterize the family of PDFs using geometric quantities, like the geodesic equations, the Riemann curvature tensor, the Ricci tensor or the scalar curvature.

The geometrization of thermodynamics and statistical mechanics are some of the most important achievements in this field, expressed mainly by the foundational works of Gibbs [11], Hermann [12], Weinhold [6], Mrugała [13], Ruppeiner [14], and Carathéodory [15]. These investigations lead to the Weinhold and Ruppeiner geometries, where a Riemann metric tensor in the space of thermodynamic parameters is provided and a notion of distance between macroscopic states is obtained. However, the utility of information geometry is not only limited to those areas. For instance, it has been applied in quantum mechanics leading to a quantum generalization of the Fisher–Rao metric [16], and also in nuclear plasmas [17, 18]. Moreover, generalized extensions of the information geometry approach to the non-extensive formulation of statistical mechanics [19] have been also considered [20–23]. Applications of information geometry to chaos can also be performed by considering complexity on curved manifolds [24–28], leading to a criterion for characterizing global chaos on statistical manifolds, from which some consequences concerning dynamical systems have been explored [29]. More generally, the curvature has been proved to be a quantifier which measures interactions in thermodynamical systems, where the positive or negative sign corresponds to repulsive or attractive correlations, respectively [7].

Motivated by previous works of some of us [29, 30], we propose a generalization of the Gaussian model which we call the *Hermite–Gaussian model*, and we show its relation with the one-dimensional quantum harmonic oscillator.

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The use of information geometry techniques in the description of quantum harmonic oscillators can be useful, for example, in the study of the translational modes in a quantum ion trap. These modes can be described as quantum harmonic oscillators (see discussion in [31]), that need to be characterized and controlled in order to avoid coherence losses. Given the close connection between the Fisher metric and the Cramer–Rao inequality, our contribution may serve as a tool for the characterization of unknown parameters in those scenarios. The present work can be also useful for characterizing global dynamics on a new family of curved statistical manifolds [24–28].

The paper is organized as follows. In Section II, we review the main features of the information geometry approach. In Section III, we present the Hermite–Gaussian model, we obtain the Fisher–Rao metric and the scalar curvature for this model, and we show its relation with the one-dimensional quantum harmonic oscillator. We employ the Hermite-Gaussian model to characterize some families of states of the quantum harmonic oscillator. We focus on three different families: Hamiltonian eigenstates, mixtures of eigenstates and superposition of eigenstates. Then, in Section IV, we illustrate with the exampled of the amplitude damping channel. We show that the geometrical effect of the channel is expressed in a decrease of the scalar curvature towards an asymptotical value associated to the decohered state. Finally, in Section V, we present the conclusions and some future research directions.

2. Information geometry

The information geometry approach studies the differential geometric structure possessed by families of probability distribution functions (PDFs). In this section we introduce the general features of this approach, which will be used in the next sections. The presentation is based on the book of S. Amari and H. Nagaoka [9].

2.1. Statistical models

Information geometry applies techniques of differential geometry to study properties of families of probability distribution functions parameterized by continuous variables. These families are called *statistical models*. More specifically, a statistical model is defined as follows. We consider the probability distribution functions defined on $X \subseteq \mathbb{R}^n$, i.e., the functions $p : X \to \mathbb{R}$ which satisfy

$$p(x) \geq 0, \quad \text{and} \quad \int_X p(x)dx = 1. \tag{1}$$

When X is a discrete set the integral must be replaced by a sum. A statistical model is a family S of probability distribution function over X, whose elements can be parameterized by appealing to a set of m real variables, i.e.,

$$S = \left\{ p_\theta(x) = p(x|\theta) \,\middle|\, \theta = (\theta^1, \ldots, \theta^m) \in \Theta \subseteq \mathbb{R}^m \right\}, \tag{2}$$

with $\theta \mapsto p_\theta$ an injective mapping. The dimension of the statistical model is given by the number of real variables used to parameterized the family S.

When statistical models are applied to physical systems, the interpretation of X and Θ is the following. X represents the microscopic variables of the system, which are typically difficult to determine, for instance the positions of the particles of a gas. Θ represents the macroscopic variables of the system, which can be easily measured. The set X is called the *microspace* and the variables $x \in X$ are the microvariables. The set Θ is called the *macrospace* and the variables $\theta^1, \ldots, \theta^m$ are the macrovariables.

Given a physical system, we can define many statistical models. First, we have to choose the microvariables to be considered, and then we have to choose the macrovariables which parameterized the PDFs defined on the microspace. All statistical models are equally valid, but no all of them are equally useful. In general, the choice of the statistical model would be based on pragmatic considerations.

2.2. Geometric structure of statistical models

In order to apply differential geometry to statistical models, it is necessary to endow them with a metric structure. This is accomplished by means of the Fisher–Rao metric

$$\mathbf{I} = I_{ij} = \int_X dx\, p(x|\theta) \frac{\partial \log p(x|\theta)}{\partial \theta^i} \frac{\partial \log p(x|\theta)}{\partial \theta^j}, \qquad i, j = 1, \ldots, m. \tag{3}$$

The metric tensor $\mathbf{I}$ gives to the macrospace a geometrical structure. Therefore, the family S is a statistical manifold, i.e., a differential manifold whose elements are probability distribution functions.

From the Fisher–Rao metric, we can obtain the line element between two nearby PDFs with parameters $\theta^i + d\theta^i$ and θ^i

$$ds = \sqrt{I_{ij} d\theta^i d\theta^j}, \qquad i, j = 1, \ldots, m.$$

Using the metric tensor we can obtain the geodesic equations for the macrovariables θ_i along with relevant geometrical quantities, like the Riemann curvature tensor, the Ricci tensor or the scalar curvature.

$$\textit{Geodesic equations:} \qquad \frac{d^2\theta^k}{d^2\tau} + \Gamma^k_{ij} \frac{d\theta^i}{d\tau} \frac{d\theta^j}{d\tau} = 0, \tag{4}$$

$$\textit{Christoffel symbols:} \qquad \Gamma^k_{ij} = \frac{1}{2} I^{im} \left(I_{mk,l} + I_{ml,k} - I_{kl,m} \right), \tag{5}$$

$$\textit{Riemman curvature tensor:} \qquad R_{iklm} = \frac{1}{2} \left(I_{im,kl} + I_{kl,im} - I_{il,km} - I_{km,il} \right) + I_{np} \left(\Gamma^n_{kl} \Gamma^p_{im} - \Gamma^n_{km} \Gamma^p_{il} \right), \tag{6}$$

$$\textit{Ricci tensor:} \qquad R_{ik} = I^{lm} R_{limk}, \tag{7}$$

$$\textit{Scalar curvature:} \qquad R = I^{ik} R_{ik}. \tag{8}$$

The comma in the subindex denotes the partial derivative operation (of first and second orders), I^{kl} is the inverse of I_{ij}, and τ is a parameter that characterizes the geodesic curves.

Moreover, the Fisher–Rao metric gives information about the estimators of the macrovariables. Given an unbiased estimator $\mathbf{T} = (T_1, ..., T_m)$ of the parameters $(\theta_1, ..., \theta_m)$, i.e., $E(\mathbf{T}) = (\theta_1, ..., \theta_m)$, the Cramér–Rao bound gives a lower bound for the covariance matrix of $\mathbf{T}$,

$$\mathrm{cov}(\mathbf{T}) \geq \mathbf{I}^{-1}, \tag{9}$$

where the matrix inequality $A \geq B$ means that the matrix $A - B$ is positive semi-definite. In particular, this relation gives bounds for the variance of the unbiased estimators T_i,

$$\mathrm{var}(T_i) \geq \{\mathbf{I}^{-1}\}_{ii}, \tag{10}$$

This bound is important when looking for optimal estimators. In what follows, we present an important statistical model used in the information geometry approach, the Gaussian model.

2.3. Gaussian model

One of the most relevant statistical models used in the information geometry approach is the *Gaussian model*. This model is useful due to its versatility for describing multiple phenomena: linear diffusion in Brownian motion, error statistical distribution in experiments, Central Limit Theorem in probability theory, wave-packet function modelling a free particle, Gaussian noise in master equations, among others. The Gaussian model is obtained by choosing the family S as the set of multivariate Gaussian distributions. For instance, if $(x_1, \ldots, x_n) \in \mathbb{R}^n$ are the microvariables and there are no correlations between them, then $(\mu_1, \ldots, \mu_n, \sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n \times \mathbb{R}^n_+$ are the set of macrovariables, where μ_i and σ_i^2 correspond to the mean value and the variance of the microvariable x_i.

If we consider only one microvariable x, the Gaussian model is given by the following probability distribution function

$$p(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \tag{11}$$

which is parameterized by the mean value μ and the standard deviation σ. From equations (3) to (8), one can obtain the Fisher–Rao metric and the scalar curvature of this model,

$$I_{\alpha\beta} = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix} \quad \text{with} \quad \alpha,\beta = \mu,\sigma, \tag{12}$$

$$R = -1. \tag{13}$$

The Gaussian model is a curved manifold with constant curvature. In some contexts, the sign of the curvature is interpreted as modeling interactions, like in the 3D Bose gas, the 3D Fermi gas and the ideal gas, where their respective curvatures are negative, positive and zero [8].

In the next section, we introduce a generalization of the Gaussian model, based on the eigenstates of the quantum harmonic oscillator Hamiltonian.

3. Hermite–Gaussian model

We propose a generalization of the Gaussian model, called the *Hermite-Gaussian model*, which is motivated by the quantum harmonic oscillator. Given the microspace $X = \mathbb{R}$ and the macrospace $\Theta = \{(\mu,\sigma)\}$, we define for each n the Hermite–Gaussian model as the family of probability distribution functions given by (see Appendix B)

$$p_n(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} a_n^2 H_n^2\left(\frac{x-\mu}{\sqrt{2}\sigma}\right), \qquad a_n = \frac{1}{\sqrt{2^n n!}}. \tag{14}$$

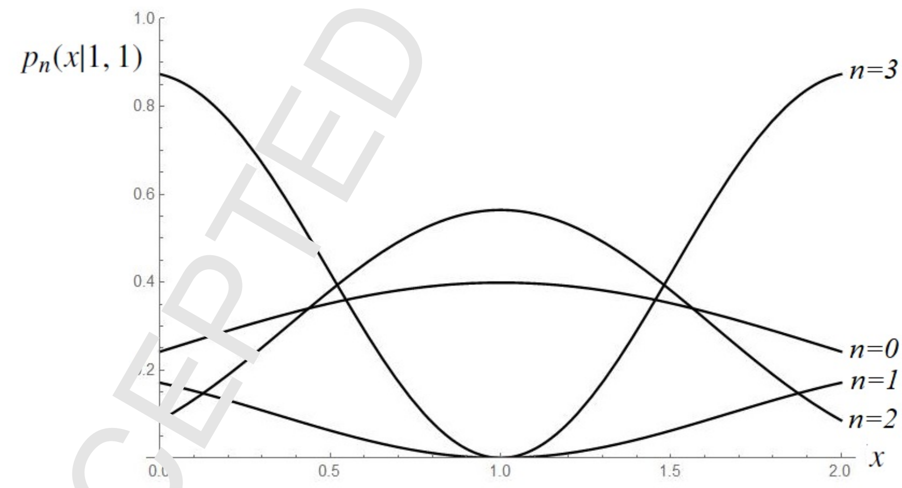

Figure 1: Some plots of the PDFs of the Hermite-Gaussian model for $n = 0, 1, 2, 3$. The curves correspond to the expression of $p_n(x|\mu,\sigma)$ of Eq. (14) for $\mu = \sigma = 1$.

In particular, if $n = 0$, the Gaussian model is recovered. The Fisher–Rao metric of the Hermite–Gaussian model takes the form

$$I_{\alpha\beta}^{(n)} = \int_X \frac{1}{p(x|\mu,\sigma)} \partial_\alpha p(x|\mu,\sigma) \partial_\beta p(x|\mu,\sigma) dx, \qquad \alpha,\beta = \mu,\sigma. \tag{15}$$

and its explicit formula is the following (see Appendix B)

$$I_{\alpha\beta}^{(n)} = \begin{pmatrix} \frac{2n+1}{\sigma^2} & 0 \\ 0 & \frac{2(n^2+n+1)}{\sigma^2} \end{pmatrix}. \tag{16}$$

Taking into account that the scalar curvature is given by

$$R^{(n)} = -\frac{1}{n^2+n+1}, \tag{17}$$

we can express the Fisher–Rao metric in terms of $R^{(n)}$

$$I_{\alpha\beta}^{(n)} = \begin{pmatrix} \frac{2n+1}{\sigma^2} & 0 \\ 0 & -\frac{2}{\sigma^2 R^{(n)}} \end{pmatrix}. \tag{18}$$

From the Fisher–Rao metric, we can compute the Cramér–Rao bound for unbiased estimators of the parameters μ and σ. This bound is of fundamental importance for the theory of parameter estimation. The lower covariance matrix of any pair of unbiased estimators T_1, T_2 of the parameters μ, σ, is given by

$$\text{cov}\,(T_1, T_2) \geq \begin{pmatrix} \frac{\sigma^2}{2n+1} & 0 \\ 0 & -\frac{\sigma^2 R^{(n)}}{2} \end{pmatrix}. \tag{19}$$

For the covariance of the estimators we obtain

$$\text{var}\,(T_1) \geq \frac{\sigma^2}{2n+1}, \tag{20}$$

$$\text{var}\,(T_2) \geq \frac{\sigma^2}{2(n^2+n+1)} = -\frac{\sigma^2 R^{(n)}}{2}. \tag{21}$$

In what follows, we show the connection between the Hermite–Gaussian model and the quantum harmonic oscillator. We use these model to characterize the PDFs given by quantum states of the harmonic oscillator. We focus on Hamiltonian eigenstates, mixtures of eigenstates and superposition of eigenstates.

3.1. Hamiltonian Eigenstates

The relation between the Hermite–Gaussian model and the quantum harmonic oscillator is straightforward. We start considering the Hamiltonian of the harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_0^2 (\hat{x} - x_0)^2, \tag{22}$$

where m is the mass, ω_0 is the frequency, x_0 is the equilibrium position of the oscillator, and $\hat{x}$ and $\hat{p}$ are the position and momentum operators. This model is relevant for the study of quantum ion traps [31]. Its eigenstates $|n\rangle$ satisfy the time-independent Schrödinger equation, $\hat{H}|n\rangle = E_n|n\rangle$, with $E_n = \hbar\omega_0(n + \frac{1}{2})$. Moreover, the eigenstates satisfy orthogonality and completeness relations

$$\langle n|m\rangle = \delta_{nm} \qquad \text{(orthogonality)}$$

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I} \qquad \text{(completeness)}$$

where $\hat{I}$ is the identity operator.

The wave function of the eigenstate $|n\rangle$, in the coordinate representation, is given by

$$\varphi_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{\sqrt{2\pi}\sigma}} e^{-\frac{(x-\mu)^2}{4\sigma^2}} a_n H_n\left(\frac{x-\mu}{\sqrt{2}\sigma}\right), \tag{23}$$

with $\mu = x_0$, $\sigma^2 = \frac{\hbar}{2m\omega_0}$, and $a_n = \frac{1}{\sqrt{2^n n!}}$. Then, the PDF of the position operator for the eigenstate $|n\rangle$ is $P_n(x) = |\varphi_n(x)|^2$.

Therefore, if we consider the eigenstate $|n\rangle$ of an harmonic oscillator with parameters μ and σ, the PDF of the position operator $P_n(x)$ is equal to the probability distribution function $p_n(x|\mu, \sigma)$ of the Hermite–Gaussian model, given in equation (14). Moreover, the Fisher–Rao metric and the scalar curvature associated with the probability distribution function $P_n(x)$ are given in equations (16) and (17), respectively.

It is important to remark that the Fisher–Rao metric is diagonal, and the scalar curvature is always negative and decreases with the quantum number n, tending to zero in the limit of high quantum numbers. Moreover, from the Cramér–Rao bound we obtain that the minimal variance of the estimators of the parameter μ grows with σ^2 and decreases with the eigenstate number, and the minimal variance of estimators of the parameter σ also grows with σ^2 but decreases with the square of the eigenstate number. Equivalently, the minimal variance of the estimators of σ is proportional to the scalar curvature.

3.2. General states

We are going to consider the PDF of the position operator obtained from general states of the harmonic oscillator. Let us consider the basis of the Hamiltonian eigenstates $\{|n\rangle\}_{n\in\mathbb{N}}$ and a state represented by a density matrix $\hat{\rho}$ of the form

$$\hat{\rho} = \sum_{n,m} \lambda_{nm} |n\rangle\langle m|. \tag{24}$$

The probability distribution function of the position operator is given by

$$P(x) = \langle x|\hat{\rho}|x\rangle = \sum_{n,m} \lambda_{nm}\varphi_n(x)\varphi_m(x) = \sum_{n,m} \frac{\lambda_{nm} a_n a_m}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} H_n\left(\frac{x-\mu}{\sqrt{2}\sigma}\right) H_m\left(\frac{x-\mu}{\sqrt{2}\sigma}\right), \tag{25}$$

where $\varphi_n(x)$ is the wave function of the eigenstate $|n\rangle$, given in equation (23).

For practical reasons, we define the function $f(y)$,

$$f(y) = \sum_{n,m} \frac{\lambda_{nm} a_n a_m}{\sqrt{2\pi}} e^{-y^2} H_n(y) H_m(y). \tag{26}$$

Then, we have $P(x) = \frac{f(y(x))}{\sigma}$, with $y(x) = \frac{x-\mu}{\sqrt{2}\sigma}$.

In order to calculate the Fisher–Rao metric associated with $P(x)$, we need the partial derivatives $\partial_\mu P(x)$ and $\partial_\sigma P(x)$, which are given by

$$\partial_\mu P(x) = \partial_\mu \left(\frac{f(y(x))}{\sigma}\right) = \frac{-f'(y(x))}{\sqrt{2}\sigma^2}, \tag{27}$$

$$\partial_\sigma P(x) = \partial_\sigma \left(\frac{f(y(x))}{\sigma}\right) = \frac{-f(y(x))}{\sigma^2} + \frac{-y(x) f'(y(x))}{\sigma^2}, \tag{28}$$

with $f'(y) = \frac{d}{dy} f(y)$.

Replacing the PDF (25) and the partial derivatives (27) and (28) in the integral of equation (15), and making the change of variable $y = y(x)$, we obtain the Fisher–Rao metric

$$I_{\mu\sigma} = I_{\sigma\mu} = \int_{-\infty}^{+\infty} \frac{\partial_\mu P(x) \partial_\sigma P(x)}{P(x)} dx = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \left(f'(y) + \frac{y\,(f'(y))^2}{f(y)} \right) dy = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{y\,(f'(y))^2}{f(y)} dy,$$

$$I_{\mu\mu} = \int_{-\infty}^{+\infty} \frac{\left(\partial_\mu P(x)\right)^2}{P(x)} dx = \frac{1}{\sqrt{2}\sigma^2} \int_{-\infty}^{+\infty} \frac{(f'(y))^2}{f(y)} dy,$$

$$I_{\sigma\sigma} = \int_{-\infty}^{+\infty} \frac{(\partial_\sigma P(x))^2}{P(x)} dx = \frac{\sqrt{2}}{\sigma^2} \int_{-\infty}^{+\infty} \frac{(f(y) + y f'(y))^2}{f(y)} dy = \frac{\sqrt{2}}{\sigma^2} \int_{-\infty}^{+\infty} \left(-f(y) + 2\,(y f(y))' + \frac{y^2 (f'(y))^2}{f(y)} \right) dy =$$
$$= \frac{\sqrt{2}}{\sigma^2} \int_{-\infty}^{+\infty} \frac{y^2 (f'(y))^2}{f(y)} dy - \frac{1}{\sigma^2},$$

where in the first equation we used that $\int_{-\infty}^{+\infty} f'(y) dy = 0$, and in the last equation we used that $\int_{-\infty}^{+\infty} f(y) dy = \frac{1}{\sqrt{2}}$ and $\int_{-\infty}^{+\infty} (y f(y))' dy = 0$.

Therefore, we can write the Fisher–Rao metric as follows:

$$I_{\alpha\beta} = \frac{1}{\sigma^2} \begin{pmatrix} \tilde{I}_{\mu\mu} & \tilde{I}_{\mu\sigma} \\ \tilde{I}_{\mu\sigma} & \tilde{I}_{\sigma\sigma} \end{pmatrix}, \tag{29}$$

where $\tilde{I}_{\mu\sigma}$, $\tilde{I}_{\mu\sigma}$ and $\tilde{I}_{\mu\sigma}$ are independent of μ and σ, and they are given by

$$\tilde{I}_{\mu\sigma} = \int_{-\infty}^{+\infty} \frac{y\,(f'(y))^2}{f(y)} dy,$$
$$\tilde{I}_{\mu\mu} = \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \frac{(f'(y))^2}{f(y)} dy,$$
$$\tilde{I}_{\sigma\sigma} = \sqrt{2} \int_{-\infty}^{+\infty} \frac{y^2 (f'(y))^2}{f(y)} dy - 1.$$

From the Fisher–Rao metric and using equations (5) to (8), we can obtain the scalar curvature

$$R = \frac{2\tilde{I}_{\mu\mu}}{\tilde{I}_{\mu\sigma}^2 - \tilde{I}_{\mu\mu}\tilde{I}_{\sigma\sigma}}. \tag{30}$$

The Cramér–Rao bound gives the lower covariance matrix of any pair of unbiased estimators T_1, T_2 of the parameters μ, σ,

$$\operatorname{cov}(T_1, T_2) \geq \frac{\sigma^2}{\tilde{I}_{\mu\mu}\tilde{I}_{\sigma\sigma} - \tilde{I}_{\mu\sigma}^2} \begin{pmatrix} \tilde{I}_{\sigma\sigma} & -\tilde{I}_{\mu\sigma} \\ -\tilde{I}_{\mu\sigma} & \tilde{I}_{\mu\mu} \end{pmatrix}. \tag{31}$$

Finally, we can express the variance of T_2 in terms of the scalar curvature,

$$\operatorname{var}(T_2) \geq -\frac{\sigma^2 R}{2}. \tag{32}$$

Corollary 1: The Fisher–Rao metric for a general state of the harmonic oscillator is independent of the parameter μ and it only depends on the parameter σ by a general factor $1/\sigma^2$.

Corollary 2: The scalar curvature for a general state of the harmonic oscillator is independent of the parameters μ and σ, and it only involves integrals of the dimensionless function $f(y)$ and its derivative $f'(y)$.

Corollary 3: The lower variance of unbiased estimators of the parameter σ is proportional to $\sigma^2 R$.

3.3. Mixtures of Hamiltonian eigenstates

We consider quantum states which are mixtures of the Hamiltonian eigenstates. Mixtures of eigenstates are particular cases of the states given in equation (24), with $\lambda_{nm} = \delta_{nm}\lambda_n$ i.e., $\hat{\rho} = \sum_n \lambda_n |n\rangle\langle n|$. Therefore, the probability distribution function of the position operator, the Fisher–Rao metric and the scalar curvature can be obtained from the general expressions (25), (29) and (30), considering $\lambda_{nm} = \delta_{nm}\lambda_n$.

In this case, the PDF of the position operator takes the form

$$P(x) = \sum_n \lambda_n |\varphi_n(x)| = \sum_n \lambda_n p_n(x|\mu, \sigma).$$

The diagonal elements of the Fisher–Rao metric are zero, and the elements $I_{\mu\sigma} = I_{\sigma\mu}$ are given in equation (29),

$$I_{\mu\sigma} = I_{\sigma\mu} = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{y\,(f'(y))^2}{f(y)} dy, \tag{33}$$

with $f(y) = \sum_n \frac{\lambda_n a_n^2}{\sqrt{2\pi}} e^{-y^2} H_n^2(y)$. Since Hermite polynomials $H_n(y)$ are even or odd functions of the variable y, $H_n^2(y)$ are even functions. Then, $f(y)$ is also an even function and its derivative $f'(y)$ is an odd function. Finally, the integrand of equation (33) is an odd function of y. Therefore, $I_{\mu\sigma} = I_{\sigma\mu} = 0$,

Finally, the scalar curvature is obtained from equation (30),

$$R = -\frac{2}{\tilde{I}_{\sigma\sigma}}.$$

As an example, we consider the mixture state $\hat{\rho}_{01} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$. The Fisher–Rao metric is given by

$$I_{\alpha\beta}^{(01)} = \frac{1}{\sigma^2} \begin{pmatrix} 2 + \sqrt{2e\pi}\left(\mathrm{Erf}\left(\frac{1}{\sqrt{2}}\right) - 1\right) & 0 \\ 0 & 2 + \sqrt{2e\pi}\left(1 - \mathrm{Erf}\left(\frac{1}{\sqrt{2}}\right)\right) \end{pmatrix}, \tag{34}$$

where $\mathrm{Erf}(x)$ is the Gauss error function, with $\mathrm{Erf}\left(\frac{1}{\sqrt{2}}\right) \approx 0.317$. The scalar curvature is approximately $R^{(01)} \approx -0.604$.

3.4. Superposition of Hamiltonian eigenstates

We consider quantum states which are superpositions of Hamiltonian eigenstates. Superpositions of eigenstates of the form $|\psi\rangle = \sum_n \alpha_n |n\rangle$ are particular cases of states given in equation (24), with $\lambda_{nm} = \alpha_n \alpha_m^*$, i.e., $\hat{\rho} = |\psi\rangle\langle\psi| = \sum_{nm} \alpha_n \alpha_m^* |n\rangle\langle m|$. Therefore, the PDF of the position operator, the Fisher–Rao metric and the scalar curvature can be obtained from the general expressions (25), (29) and (30), considering $\lambda_{nm} = \alpha_n \alpha_m^*$

3.4.1. Even or odd superpositions

In this section we focus on a family of superpositions that yield analytic expressions. If we consider a superposition of eigenstates with only even or odd eigenstates, i.e.,

$$\hat{\rho} = \sum_{\substack{n,m \\ \text{even indices}}} \alpha_n \alpha_m^* |n\rangle\langle m|, \quad \text{or} \quad \hat{\rho} = \sum_{\substack{n,m \\ \text{odd indices}}} \alpha_n \alpha_m^* |n\rangle\langle m|,$$

we obtain that the diagonal elements of the Fisher–Rao metric are zero. The proof is similar to the case of mixtures of eigenstates. The diagonal elements are given in equation (29),

$$I_{\mu\sigma} = I_{\sigma\mu} = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{y\,(f'(y))^2}{f(y)} dy, \tag{35}$$

with

$$f(y) = \sum_{\substack{\text{even or odd} \\ \text{indices}}} \frac{\alpha_n \alpha_m^* a_n a_m}{\sqrt{2\pi}} e^{-y^2} H_n(y) H_m(y).$$

If the indices n, m can only take even or odd values, then the product $H_n(y)\, H_m(y)$ is always an even function of the variable y. Then, $f(y)$ is also an even function and its derivative $f'(y)$ is an odd function. Finally, the integrand of equation (35) is an odd function of y, and the result of the integral is zero.

Again, we obtain that the scalar curvature, given in equation (30), is

$$R = -\frac{2}{\tilde{I}_{\sigma\sigma}}.$$

3.4.2. Real or imaginary superpositions

Analytic expressions can also be obtained for superpositions of eigenstates that involve only real coefficients, i.e., $\hat{\rho} = \sum_{n,m} \alpha_n \alpha_m |n\rangle\langle m|$. In order to compute the Fisher–Rao metric, we need the function $f(y)$, given in (26), and it derivative $f'(y)$,

$$\begin{aligned} f(y) &= \frac{e^{-y^2}}{\sqrt{2\pi}} \left(\sum_n \alpha_n a_n H_n(y) \right)^2, \\ f'(y) &= \frac{2e^{-y^2}}{\sqrt{2\pi}} \left(\sum_n \alpha_n a_n H_n(y) \right) \left[\sum_n \alpha_n a_n \left(nH_{n-1}(y) - \frac{H_{n+1}(y)}{2} \right) \right], \end{aligned} \tag{36}$$

where in the last equation we have used the recurrence relation of the Hermite polynomials (A.2). Replacing expressions (36) in the Fisher–Rao metric (29), and taking into account relations (A.1) and (A.2), we obtain

$$\begin{aligned} I_{\mu\sigma} = I_{\sigma\mu} &= \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{4ye^{-y^2}}{\sqrt{2\pi}} \left[\sum_n \alpha_n a_n \left(nH_{n-1}(y) - \frac{H_{n+1}(y)}{2} \right) \right]^2 dy = \\ &= \frac{1}{\sigma^2} \sum_n \alpha_n \left(\alpha_{n-3} \sqrt{n(n-1)(n-2)} + \alpha_{n-1}\, n\sqrt{n} + \alpha_{n+1}(n+1)\sqrt{n+1} + \alpha_{n+3} \sqrt{(n+3)(n+2)(n+1)} \right), \end{aligned}$$

$$\begin{aligned} I_{\mu\mu} &= \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{2e^{-y^2}}{\sqrt{\pi}} \left[\sum_n \alpha_n a_n \left(nH_{n-1}(y) - \frac{H_{n+1}(y)}{2} \right) \right]^2 dy = \\ &= \frac{1}{\sigma^2} \sum_n \alpha_n \left(-\alpha_{n-2} \sqrt{n(n-1)} + \alpha_n(2n+1) - \alpha_{n+2} \sqrt{(n+2)(n+1)} \right), \end{aligned}$$

$$\begin{aligned} I_{\sigma\sigma} &= \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} \frac{4y^2e^{-y^2}}{\sqrt{\pi}} \left[\sum_n \alpha_n a_n \left(nH_{n-1}(y) - \frac{H_{n+1}(y)}{2} \right) \right]^2 dy - \frac{1}{\sigma^2} = \\ &= \frac{1}{\sigma^2} \sum_n \alpha_n \left(-\alpha_{n-4} \sqrt{n(n-1)(n-2)(n-3)} + \alpha_n(2n^2+2n+3) - \alpha_{n+4} \sqrt{(n+4)(n+3)(n+2)(n+1)} \right) - \frac{1}{\sigma^2}. \end{aligned}$$

If we consider a superposition of eigenstates with only imaginary coefficients, we obtain a similar result, but replacing the coefficients α_n by its imaginary part, i.e., by $\mathrm{Im}(\alpha_n)$.

4. Example: amplitude damping channel

In this section we illustrate how the scalar curvature changes in connection with the dynamics of a physical processes. In particular, we consider a dynamical evolution of a two-level system given by the amplitude damping channel. This channel has several applications in the context of quantum information processing for modeling the effects of quantum noise. It describes in a simplified way the spontaneous decay process of a two-level quantum system due to the effect of the quantum noise of an environment.

We consider an initial state $\hat{\rho}$ in a superposition of the ground state and the first excited state of the harmonic oscillator. In order to use the results of the subsection 3.4.2 we consider a superposition $|\psi\rangle = a|0\rangle + b|1\rangle$ with real coefficients $(a, b \in \mathbb{R})$. Its density matrix is given by

$$|\psi\rangle\langle\psi| = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}. \tag{37}$$

Using the results of the section 3.4.2, we obtain the elements $I_{\mu\mu}$, $I_{\mu\sigma}$ and $I_{\sigma\sigma}$ of the Fisher matrix of the state $|\psi\rangle\langle\psi|$

$$I_{\alpha\beta}(\psi) = \frac{1}{\sigma^2}\begin{pmatrix} a^2 + 3b^2 & ab \\ ab & 3a^2 + 7b^2 - 1 \end{pmatrix} \tag{38}$$

and its scalar curvature

$$R_\psi = \frac{2(a^2 + 3b^2)}{(2ab)^2 - (a^2 + 3b^2)(3a^2 + 7b^2 - 1)} \tag{39}$$

Since $a^2 + b^2 = 1$, the scalar curvature can be rewritten as

$$R_\psi = \frac{-3 + 2a^2}{9 - 14a^2 + 6a^4} \quad , \quad -1 \leq a \leq 1. \tag{40}$$

Now, we consider the time evolution given by the amplitude damping channel. It is important to remark that, if the system is in the ground state there is no emission, and it continues in the ground state. But, if the system is in the excited state, after an interval of time τ , there is a probability p that the state has decayed to the ground state due to spontaneous emission. In terms of Kraus operators, the amplitude damping channel can be expressed as

$$\hat{\mathcal{E}}_\tau(\hat{\rho}) = \hat{A}_0\hat{\rho}\hat{A}_0^\dagger + \hat{A}_1\hat{\rho}\hat{A}_1^\dagger \tag{41}$$

with the Krauss operators given by

$$\hat{A}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \quad , \quad \hat{A}_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \tag{42}$$

where p is the probability of decay during the time interval τ.

We restrict the initial state to the subspace generated by the ground state and the first excited state. An arbitrary state of the two-level system is of the form $\hat{\rho} = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|$. If we apply the amplitude damping channel n times [32], we obtain the state $\hat{\mathcal{E}}_{n\tau}(\hat{\rho})$

$$\hat{\mathcal{E}}_{n\tau}(\hat{\rho}) = \begin{pmatrix} \rho_{00} + \rho_{11}(1 - (1-p)^n) & (\sqrt{1-p})^n\rho_{01} \\ (\sqrt{1-p})^n\rho_{10} & (1-p)^n\rho_{11} \end{pmatrix}, \tag{43}$$

which is the state of the system at time $n\tau$. For long times, when $n \to \infty$, the limit state $\hat{\mathcal{E}}_\infty(\hat{\rho})$ becomes

$$\hat{\rho}_\infty = \hat{\mathcal{E}}_\infty(\hat{\rho}) = |0\rangle\langle 0|. \tag{44}$$

Therefore, when time goes to infinite, all initial states decay to the ground state as a consequence of the quantum noise of the enviroment.

As an example, we consider the amplitude damping channel with decay probability $p = 0.1$ during the time interval τ, and an initial state $|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$. The evolved state at time $n\tau$ is given by $\hat{\rho}_n = \hat{\mathcal{E}}_{n\tau}(\hat{\rho})$. In Figure 2 shows the values of the associated scalar curvature for each state $\hat{\rho}_n$. We can see that the amplitude damping channel transforms the scalar curvature to the asymptotic value $R_\infty = -1$, which corresponds to the decay state $\hat{\rho}_\infty = |0\rangle\langle 0|$.

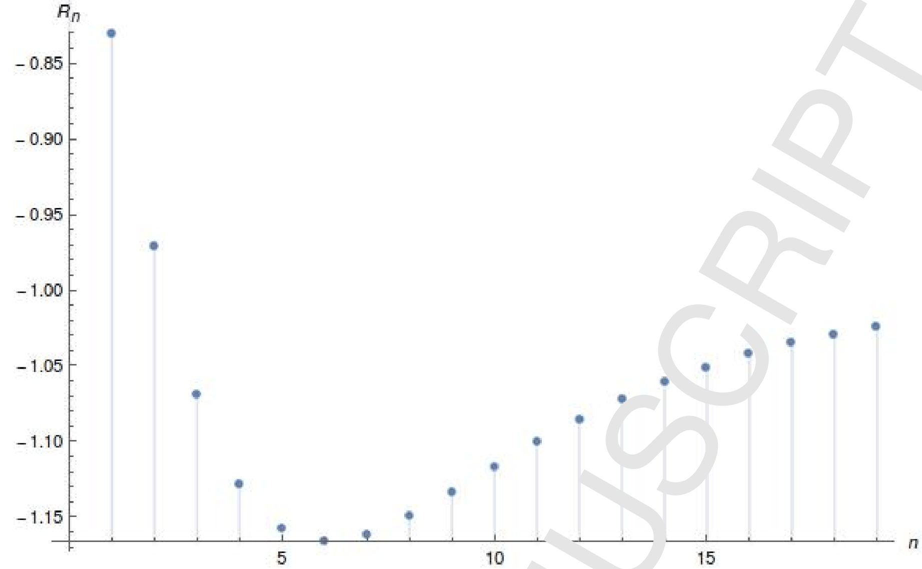

Figure 2: Plot of the scalar curvature of the evolved states $\hat{\rho}_n = \hat{\mathcal{E}}_{n\tau}(\hat{\rho})$, with an initial state $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and decay probability $p = 0.1$.

5. Conclusions

In this work we have proposed a generalization of the Gaussian model -namely, the *Hermite-Gaussian model*- and we have studied its properties from the point of view of the information geometry approach. We have shown its relation with the probabilities associated to the one-dimensional quantum harmonic oscillator model and analytic expressions for some particular classes of states were provided. Specifically, we found that for finite mixtures of eigenstates and finite superpositions of (even or odd) eigenstates the Fisher metric is always diagonal. Real and imaginary superpositions of eigenstates do not imply a diagonal Fisher metric and the matrix elements are given in terms of a series sum. The computation of the Fisher metric results fundamental in the derivation of the Cramer-Rao inequality, which plays a key role in parameter estimation theory. Our contribution could be useful for characterizing the different parameters associated to a quantum harmonic oscillator.

An analytic expression for the scalar curvature was obtained for the case of diagonal Fisher, being negative and inversely proportional to the $\sigma\sigma$ element. We have illustrated the dynamics of the model using the amplitude dumping channel. We have showed that the geometrical effect of the channel is to decrease the initial value of the scalar curvature of the Hermite-Gaussian model towards its asymptotic and minimum value $R = -1$ which corresponds to the ground state.

Appendix A. Hermite polynomials

The Hermite polynomials H_n are given by the expression

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2},$$

and their orthogonality relation is

$$\int_{-\infty}^{+\infty} e^{-y^2} H_n(y) H_m(y) dy = \sqrt{\pi}\, 2^n\, n!\, \delta_{n,m}. \tag{A.1}$$

An important feature of these polynomials is that if n is even, $H_n(y)$ is an even function; and if n is odd, $H_n(y)$ is an odd function.

Some relevant recurrence relations are the following:

$$H'_n(y) = 2nH_{n-1}(y), \qquad H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}. \tag{A.2}$$

Appendix B. Hermite–Gaussian model

For parameters μ and σ, the probability distribution of the n–Hermite–Gaussian model is

$$p_n(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2} a_n^2 H_n^2(y), \quad \text{with} \quad a_n = \frac{1}{\sqrt{2^n n!}}, \quad y = \frac{x-\mu}{\sqrt{2}\sigma}. \tag{B.1}$$

In order to obtain the elements of the metric tensor, we need to calculate the partial derivatives of the probability distribution. It easy to show that

$$\partial_\mu p_n(x) = -\frac{p'_n(y)}{\sqrt{2}\,\sigma}, \tag{B.2}$$

$$\partial_\sigma p_n(x) = -\frac{p_n(y) + y p'_n(y)}{\sigma}, \tag{B.3}$$

with

$$p'_n(y) = \frac{dp_n}{dy}(y) = \frac{2a_n^2}{\sqrt{2\pi}\sigma} e^{-y^2} H_n(y) \left(nH_{n-1}(y) - \frac{1}{2} H_{n+1}(y) \right), \tag{B.4}$$

where we have used the recurrence relations (A.2). It should be noted that $p_n(y)$ is even an function of y, thus $p'_n(y)$ is an odd function of y.

Also, we will need to express $yp'_n(y)$ in terms of Hermite polynomials,

$$\begin{aligned} yp'_n(y) &= \frac{2a_n^2}{\sqrt{2\pi}\sigma} e^{-y^2} H_n(y) \left(nyH_{n-1}(y) - \frac{1}{2} yH_{n+1}(y) \right) = \\ &= \frac{2a_n^2}{\sqrt{2\pi}\sigma} e^{-y^2} H_n(y) \left(n(n-1)H_{n-2}(y) - \frac{1}{2} H_n(y) - \frac{1}{4} H_{n+2}(y) \right), \end{aligned}$$

where we have used expression (B.4) and the recurrence relations (A.2).

Off–diagonal elements

Since the metric tensor is symmetric, it is enough to calculate the element $I^{(n)}_{\mu\sigma}$, given by

$$I^{(n)}_{\mu\sigma} = \int_{-\infty}^{+\infty} \frac{1}{p_n(x)} \partial_\mu p_n(x) \partial_\sigma p_n(x) dx. \tag{B.5}$$

Replacing expressions (B.2) and (B.3) in (B.5) and doing some easy manipulations, we obtain

$$I^{(n)}_{\mu\sigma} = \int_{-\infty}^{+\infty} \frac{1}{\sigma^2} \left(p'_n(y(x)) + y(x) \frac{[p'_n(y(x))]^2}{p_n(y(x))} \right) dx = \int_{-\infty}^{+\infty} \frac{\sqrt{2}}{\sigma} \left(p'_n(y) + y \frac{[p'_n(y)]^2}{p_n(y)} \right) dy, \tag{B.6}$$

where in the last equation we changed from variable x to the variable $y = \frac{x-\mu}{\sqrt{2}\sigma}$. Since $p_n(y)$ and $p'_n(y)$ are even and odd functions of y, respectively, then the integrand of (B.6) is an odd function. Therefore, $I^{(n)}_{\mu\sigma} = 0$.

Element $I^{(n)}_{\mu\mu}$

The element $I^{(n)}_{\mu\mu}$ is given by

$$I^{(n)}_{\mu\mu} = \int_{-\infty}^{+\infty} \frac{1}{p_n(x)} \left[\partial_\mu p_n(x) \right]^2 dx. \tag{B.7}$$

Replacing expression (B.2) in (B.7), we obtain

$$I^{(n)}_{\mu\mu} = \int_{-\infty}^{+\infty} \frac{1}{2\,\sigma^2} \frac{[p'_n(y(x))]^2}{p_n(y(x))} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2}\sigma} \frac{[p'_n(y)]^2}{p_n(y)} dy. \tag{B.8}$$

In the last step, we have changed from variable x to the variable y. Then, if we replace expressions (B.1) and (B.4) in (B.8) and we rearrange the expression, we obtain

$$I_{\mu\mu}^{(n)} = \frac{2a_n^2}{\sqrt{\pi}\sigma^2}\left[n^2\int_{-\infty}^{+\infty} e^{-y^2}H_{n-1}^2(y)dy - n\int_{-\infty}^{+\infty} e^{-y^2}H_{n-1}(y)H_{n+1}(y)dy + \frac{1}{4}\int_{-\infty}^{+\infty} e^{-y^2}H_{n+1}^2(y)dy\right] =$$
$$= \frac{2}{\sqrt{\pi}\sigma^2}\frac{1}{2^n\, n!}\left(n^2\sqrt{\pi}\,2^{n-1}\,(n-1)! + \frac{1}{4}\sqrt{\pi}\,2^{n+1}\,(n+1)!\right).$$

In the last step we have used the orthogonality relation (A.1). Finally, we obtain $I_{\mu\mu}^{(n)} = \frac{2n+1}{\sigma^2}$.

Element $I_{\sigma\sigma}^{(n)}$

The element $I_{\mu\mu}^{(n)}$ is given by

$$I_{\sigma\sigma}^{(n)} = \int_{-\infty}^{+\infty}\frac{1}{p_n(x)}\left[\partial_\sigma p_n(x)\right]^2 dx \tag{B.9}$$

Replacing expression (B.3) in (B.9), we obtain

$$I_{\sigma\sigma}^{(n)} = \int_{-\infty}^{+\infty}\frac{1}{p_n(y(x))}\left(-\frac{p_n(y(x)) + y(x)p_n'(y(x))}{\sigma}\right)^2 dx = \int_{-\infty}^{+\infty}\frac{\sqrt{2}}{\sigma}\frac{\left[p_n(y) + yp_n'(y)\right]^2}{p_n(y)}dy. \tag{B.10}$$

In the last equation we have changed from variable x to the variable y. Then, if we replace expressions (B.1) and (B.5) in (B.10) and we rearrange the expression, we obtain

$$I_{\sigma\sigma}^{(n)} = \int_{-\infty}^{+\infty}\frac{a_n^2}{\sqrt{\pi}\sigma^2}\, e^{-y^2}\left(2n(n-1)H_{n-2}(y) - \frac{1}{2}H_{n+2}(y)\right)^2 dy =$$
$$= \int_{-\infty}^{+\infty}\frac{a_n^2}{\sqrt{\pi}\sigma^2}\, e^{-y^2}\left(4n^2(n-1)^2H_{n-2}^2(y) + \frac{1}{4}H_{n+2}^2(y) - 2n(n-1)H_{n-2}(y)H_{n+2}(y)\right)dy =$$
$$= \frac{1}{\sqrt{\pi}\sigma^2}\frac{1}{2^n\, n!}\left(4n^2(n-1)^2\sqrt{\pi}\,2^{n-2}\,(n-2)! + \frac{1}{4}\sqrt{\pi}\,2^{n+2}\,(n+2)!\right).$$

In the last step, we have used the orthogonality relation (A.1). Finally, we obtain $I_{\sigma\sigma}^{(n)} = \frac{2(n^2+n+1)}{\sigma^2}$.

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