# On the solvability of the Periodically Forced Relativistic Pendulum Equation on Time Scales

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**Abstract**

We study some properties of the range of the relativistic pendulum operator $\mathcal{P}$, that is, the set of possible continuous $T$-periodic forcing terms $p$ for which the equation $\mathcal{P}x = p$ admits a $T$-periodic solution over a $T$-periodic time scale $\mathbb{T}$. Writing $p(t) = p_0(t) + \overline{p}$, we prove the existence of a compact interval $\mathcal{I}(p_0)$ such that the problem has a solution if and only if $\overline{p} \in \mathcal{I}(p_0)$ and at least two different solutions when $\overline{p}$ is an interior point. Furthermore, we give sufficient conditions for nondegeneracy; specifically, we prove that if $T$ is small then $\mathcal{I}(p_0)$ is a neighbourhood of 0 for arbitrary $p_0$. Well known results for the continuous case are generalized to the time scales context.

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*Keywords*: Relativistic pendulum; Periodic solutions; Time scales; Degenerate equations; Mawhin's continuation theorem

## 1 Introduction

The $T$-periodic problem for the forced relativistic pendulum equation on time scales reads

$$\mathcal{P}x(t) := (\varphi(x^{\Delta}(t)))^{\Delta} + ax^{\Delta}(t) + b\sin x(t) = p_0(t) + s, \quad t \in \mathbb{T}, \tag{1}$$

where $a, b > 0$ and $s$ are real numbers, $\mathbb{T}$ is an arbitrary $T$-periodic nonempty closed subset of $\mathbb{R}$ for some $T > 0$, $\varphi : (-c, c) \to \mathbb{R}$ is the relativistic operator

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$\varphi(x) := \dfrac{x}{\sqrt{1-\frac{x^2}{c^2}}}$ with $c > 0$ and $p_0$ is continuous and $T$-periodic in $\mathbb{T}$, with zero average, that is, $\overline{p} := \dfrac{1}{T}\displaystyle\int_0^T p(t)\Delta t = 0$. In this work, we are concerned with the set of all possible values of $s$ such that (1) admits a $T$-periodic solution.

The time scales theory was introduced in 1988, in the PhD thesis of Stefan Hilger [11], as an attempt to unify discrete and continuous calculus. The time scale $\mathbb{R}$ corresponds to the continuous case and, hence, yields results for ordinary differential equations. If the time scale is $\mathbb{Z}$, then the results apply to difference equations. However, the generality of the set $\mathbb{T}$ produces many different situations in which the time scales formalism is useful in several applications. For example, in the study of hybrid discrete-continuous dynamical systems, see [6].

In the past decades, periodic problems involving the relativistic forced pendulum differential equation for the continuous case $\mathbb{T} = \mathbb{R}$ were studied by many authors, see [3, 4, 8, 13, 15, 16]. In particular, the works [3, 16] are concerned with the so-called *solvability set*, that is, the set $\mathcal{I}(p_0)$ of values of $s$ for which (1) has at least one $T$-periodic solution. We remark that problem (1) is $2\pi$-periodic and, consequently, if $x$ is a $T$-periodic solution then $x + 2k\pi$ is also a $T$-periodic solution for all $k \in \mathbb{Z}$. For this reason, the multiplicity results for (1) usually refer to the existence of *geometrically distinct* $T$-periodic solutions, i.e. solutions not differing in a multiple of $2\pi$.

For the standard pendulum equation with $a = 0$, the solvability set was analyzed in the pioneering work [9], where it is proved that $\mathcal{I}(p_0) \subset [-b, b]$ is a nonempty compact interval containing 0. This result was extended to the relativistic case in [8]; however, the method of proof in both works is variational and, consequently, cannot be applied to the case $a > 0$. This latter situation was studied in [10] for the standard pendulum and in [16] for the relativistic case. An interesting question, stated already in [9] is whether or not the equation may be degenerate, namely: is there any $p_0$ such that $\mathcal{I}(p_0)$ reduces to a single point? Many works are devoted to this problem and, in the standard case, nondegeneracy has been proved for an open and dense subset of $\tilde{C}_T$, the space of zero-average $T$-periodic continuous functions. However, the question for arbitrary $p_0$ remains unsolved.

The purpose of this work is to extend the results in [3] and [16] to the context of time scales. To this end, we prove in the first place that the set $\mathcal{I}(p_0)$ is a nonempty compact interval. The method of proof is inspired in a simple idea introduced in [10] for the standard pendulum equation, which basically employs the Schauder Theorem and the method of upper and lower solutions. Moreover, by a Leray-Schauder degree argument it shall be proved that if $s$ is an interior point of $\mathcal{I}(p_0)$, then the problem admits at least two geometrically distinct periodic solutions.

Furthermore, sufficient conditions shall be given in order to guarantee that $0 \in \mathcal{I}(p_0)$. We recall that, when $a \neq 0$, this is not trivial even for the continuous case. For the standard pendulum equation, there exist well known examples with $0 \notin \mathcal{I}(p_0)$ for arbitrary values of $T$; for the relativistic case, it was proved in [3] that, if $cT \leq \sqrt{3}\pi$, then $0 \in \mathcal{I}(p_0)$ and it is an interior point when the inequality is strict. It is worth noticing that, however, the problem is open for larger values of $T$. As we shall see, a slight improvement of this bound can be deduced from the results in the present paper. We remark that the computation is independent on $p_0$: in other words, we shall prove that if $T$ is sufficiently small, then the range of the operator $\mathcal{P}$ contains a set of the form $\tilde{C}_T + [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$.

We highlight that our results are concerned with equations on time scales that involve $\varphi-$laplacian of relativistic type, for which the literature is scarce.

The paper is organized as follows. In Section 2, we establish the notation, terminology and preliminary results which will be used throughout this paper. In Section 3 we prove that the set $\mathcal{I}(p_0)$ is a nonempty compact interval and that two geometrically distinct $T$-periodic solutions exist when $s$ is an interior point. Finally, Section 4 is devoted to find sufficient conditions such that $0 \in \mathcal{I}(p_0)$ and improve the condition obtained in [3] for the continuous case.

## 2 Notation and preliminaries

Fix $T > 0$ and assume that $\mathbb{T}$ is $T$-periodic, i.e. $\mathbb{T} + T = \mathbb{T}$. Let $C_T = C_T(\mathbb{T}, \mathbb{R})$ be the Banach space of all continuous $T$-periodic functions on $\mathbb{T}$ endowed with the uniform norm

$$\|x\|_\infty = \sup_{\mathbb{T}} |x(t)| = \sup_{[0,T]_{\mathbb{T}}} |x(t)|$$

and let $C_T^1 = C_T^1(\mathbb{T}, \mathbb{R})$ denote the Banach space of all continuous $T$-periodic functions on $\mathbb{T}$ that are $\Delta$-differentiable functions with continuous $\Delta$-derivatives, endowed with the usual norm

$$\|x\|_1 = \sup_{[0,T]_{\mathbb{T}}} |x(t)| + \sup_{[0,T]_{\mathbb{T}}} |x^\Delta(t)|.$$

Equation (1) can be written as

$$(\varphi(x^\Delta(t)))^\Delta = f(t, x(t), x^\Delta(t)) \quad t \in \mathbb{T}, \tag{2}$$

where $f : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the continuous function given by $f(t,u,v) := p_0(t) + s - au - b\sin(u)$. A function $x \in C_T^1$ is said to be a solution of (2) if $\varphi(x^\Delta) \in C_T^1$ and verifies $(\varphi(x^\Delta(t)))^\Delta = f(t, x(t), x^\Delta(t))$ for all $t \in \mathbb{T}$. We remark that necessarily $\|x\|_\infty < c$.

For $x \in C_T$, the average, the maximum value and the minimum value of $x$ shall be denoted respectively by $\overline{x}$, $x_{\max}$ and $x_{\min}$.

### 2.1 Upper and lower solutions and degree

Let us define $T$-periodic lower and upper solutions for problem (2) as follows.

**Definition 2.1** *A lower solution $\alpha$ (resp. upper solution $\beta$) of (2) is a function $\alpha \in C_T^1$ with $\left\|\alpha^\Delta\right\|_\infty < c$ such that $\varphi(\alpha^\Delta)$ is continuously $\Delta$-differentiable and*

$$\left(\varphi\left(\alpha^\Delta(t)\right)\right)^\Delta \geq f(t, \alpha(t), \alpha^\Delta(t)) \quad (resp.\ \left(\varphi\left(\beta^\Delta(t)\right)\right)^\Delta \leq f(t, \beta(t), \beta^\Delta(t))) \tag{3}$$

*for all $t \in \mathbb{T}$. Such lower (upper) solution is called* strict *if the inequality (3) is strict for all $t \in \mathbb{T}$.*

Following the ideas in [1], the problem of finding $T$-periodic solutions of (2) over the closure of the set

$$\Omega_{\alpha,\beta} := \{x \in C_T^1 : \alpha(t) \leq x(t) \leq \beta(t) \text{ for all } t\}$$

can be reduced to a fixed point equation $x = M_f(x)$, where $M_f : \overline{\Omega}_{\alpha,\beta} \to C_T^1$ is a compact operator defined from the nonlinear version of the continuation method (see e.g. [14]), namely

$$M_f(x) := \overline{x} + \overline{N_f x} + K(N_f x - \overline{N_f x}),$$

where $N_f$ is the Nemitskii operator associated to $f$ and $K : \tilde{C}_T \to \tilde{C}_T$ is the (nonlinear) compact operator given by $K\xi = x$, with $x \in C_T^1$ the unique solution of the problem $(\varphi(x^\Delta(t)))^\Delta = \xi(t)$ with zero average. For the purposes of the present paper, we shall only need the following result, which is an adaptation of Theorem 3.7 in [1]:

**Theorem 2.2** *Suppose that (2) has a lower solution $\alpha$ and an upper solution $\beta$ such that $\alpha(t) \leq \beta(t)$ for all $t \in \mathbb{T}$. Then problem (1) has at least one $T$-periodic solution. If furthermore $\alpha$ and $\beta$ are strict, then $\deg_{LS}(I - M_f, \Omega_{\alpha,\beta}(0), 0) = 1$, where $\deg_{LS}$ stands for the Leray-Schauder degree.*

# 3 The solvability set $\mathcal{I}(p_0)$

In this section, we shall prove that the solution set $\mathcal{I}(p_0)$ is a nonempty compact set; furthermore, employing the method of upper and lower solutions it shall be proved that $\mathcal{I}(p_0)$ is an interval. Finally, the excision property of the degree will allow to verify that if $s$ is an interior point of $\mathcal{I}(p_0)$, then the problem has at least 2 geometrically different $T$-periodic solutions.

**Theorem 3.1** *Assume that $p_0 \in C_T$ has zero average. Then, there exist numbers $d(p_0)$ and $D(p_0)$, with $-b \leq d(p_0) \leq D(p_0) \leq b$, such that (1) has at least one $T$-periodic solution if and only if $s \in [d(p_0), D(p_0)]$.*

<u>*Proof*</u>: The proof is similar to the continuous case; therefore, we shall only sketch it in the following steps.

<u>Step 1</u> (An associated integro-differential problem). Observe that if $x \in C_T^1$ is a solution of (1), then, integrating equation (1) from 0 to $T$, it is seen that

$s = \frac{b}{T}\int_0^T \sin(x(t))\Delta t$. Therefore, it proves convenient to consider the integro-differential Dirichlet problem

$$\begin{cases} (\varphi(x^{\Delta}(t)))^{\Delta} + ax^{\Delta}(t) + b\sin x(t) = p_0(t) + s(x), \quad t \in (0,T)_{\mathbb{T}} \\ x(0) = x(T), \end{cases} \tag{4}$$

with $s(x) := \frac{b}{T}\int_0^T \sin(x(t))\Delta t$. By Schauder's fixed point theorem, it is straightforward to prove that for each $r \in \mathbb{R}$ there exists at least one solution $x \in C([0,T]_{\mathbb{T}})$ of (4) such that $x(0) = x(T) = r$.

Step 2 ($\mathcal{I}(p_0)$ is is nonempty and bounded). Let $x$ be a solution of (4) such that $x(0) = x(T) = r$, then integration over $[0,T]_{\mathbb{T}}$ yields

$$\varphi(x^{\Delta}(T)) - \varphi(x^{\Delta}(0)) + b\int_0^T \sin x(t)\Delta t = Ts(x),$$

and hence $\varphi(x^{\Delta}(T)) = \varphi(x^{\Delta}(0))$. It follows that $x$ may be extended $T$-periodically to a solution of (1) with $s = s(x)$. In other words,

$$\mathcal{I}(p_0) = \{s(x) : x \text{ is a solution of (4) for some } r \in [0,2\pi]\} \neq \emptyset.$$

Moreover, it is clear from definition that $|s(x)| \leq b$, so $\mathcal{I}(p_0) \subset [-b,b]$.

Step 3 ($\mathcal{I}(p_0)$ is connected). Assume that $s_1, s_2 \in \mathcal{I}(p_0)$ are such that $s_1 < s_2$, and let $x_1$ and $x_2$ be $T$-periodic solutions of (1) for $s_1$ and $s_2$, respectively. Then for any $s \in (s_1,s_2)$ it is verified that $x_1$ and $x_2$ are strict upper and a lower solutions of (1), respectively. Replacing $x_1$ by $x_1 + 2k\pi$, with $k$ the first integer such that $x_2 < x_1 + 2k\pi$ and applying Theorem 2.2 with $\alpha = x_2$ and $\beta = x_1 + 2k\pi$, we conclude that problem (1) has at least one $T$-periodic solution, whence $s \in \mathcal{I}(p_0)$.

Step 4 ($\mathcal{I}(p_0)$ is closed). Let $\{s_n\} \subset \mathcal{I}(p_0)$ converge to some $s$, and let $x_n \in C_T^1$ be a solution of (1) for $s_n$. Without loss of generality, we may assume that $x_n(0) \in [0,2\pi]$. Because $\left\|x_n^{\Delta}\right\|_{\infty} < c$, by Arzelà-Ascoli theorem there exists a subsequence (still denoted $\{x_n\}$) that converges uniformly to some $x$. Furthermore, from (1) we deduce the existence of a constant $C$ independent of $n$ such that $|(\varphi(x_n^{\Delta}(t)))^{\Delta}| \leq C$ for all $t$. We claim that $\varphi(x_n^{\Delta})$ is also uniformly bounded, that is, $\|x_n^{\Delta}\|_{\infty}$ is bounded away from $c$. Indeed, otherwise passing to a subsequence we may suppose for example that $\varphi(x_n^{\Delta})_{\max} \to +\infty$. Because $\varphi(x_n^{\Delta}(t_1)) - \varphi(x_n^{\Delta}(t_0)) \leq C(t_1 - t_0)$ for all $t_1 > t_0$, we deduce from periodicity that $\varphi(x_n^{\Delta})_{\max} - \varphi(x_n^{\Delta})_{\min} \leq CT$ and, consequently, $\varphi(x_n^{\Delta})_{\min} \to +\infty$. This implies that $(x_n^{\Delta})_{\min} \to c$, which contradicts the fact that $x_n^{\Delta}$ has zero average. Using Arzelà-Ascoli again, we may assume that $\varphi(x_n^{\Delta})$ converges uniformly to some function $v$ and, from the identity $x_n(t) = x_n(0) + \int_0^t x_n^{\Delta}(\xi)\Delta\xi$ we deduce that $x \in C_T^1$ and $x^{\Delta} = \varphi^{-1}(v)$. Now integrate the equation for each $n$ and take limit for $n \to \infty$ to obtain

$$\varphi(x^{\Delta}(t)) = \varphi(x^{\Delta}(0)) + \int_0^t [s + p_0(\xi) - b\sin(x(\xi))]\Delta\xi - a[x(t) - x(0)].$$

In turn, this implies that $x$ is a solution of (4) with $s(x) = s$; hence, $\mathcal{I}(p_0)$ is closed and the proof is complete. $\square$

The following theorem establishes the existence of at least two geometrically different $T$-periodic solutions to problem (1).

**Theorem 3.2** *Assume that $p_0 \in C_T$ has zero average. If $s \in (d(p_0), D(p_0))$, then the problem (1) has at least two geometrically different $T$-periodic solutions.*

*Proof*: For $s \in (d(p_0), D(p_0))$, let $s_1 = d(p_0) < s < D(p_0) = s_2$, $x_1$ and $x_2$ be as in Step 3 of the previous proof, then $x_1$ and $x_2$ are strict upper and lower solutions for $s$, respectively. Due to the $2\pi$-periodicity of (1), we may assume that $x_2 < x_1$ and $x_2 + 2\pi \not\leq x_1$. From Theorem 2.2 and the excision property of the Leray-Schauder degree, we deduce the existence of three different solutions $y_1, y_2, y_3 \in C_T^1$ such that

$$\begin{gathered} x_2(t) < y_1(t) < x_1(t), \\ x_2(t) + 2\pi < y_2(t) < x_1(t) + 2\pi \\ x_2(t) < y_3(t) < x_1(t) + 2\pi \end{gathered}$$

for all $t \in \mathbb{T}$. If $y_2 = y_1 + 2\pi$, then $y_3 \neq y_1, y_1 + 2\pi$ and the conclusion follows. $\square$

# 4 Sufficient conditions for $0 \in \mathcal{I}(p_0)$

In this section, we shall obtain conditions guaranteeing that 0 belongs to the solvability set. Even in the continuous case, this is not clear when $a \neq 0$ since, as it is well known, counter-examples exist for the standard pendulum equation for arbitrary periods. In the relativistic case, however, it is proved that $0 \in \mathcal{I}(p_0)$ when $T$ is sufficiently small and counter-examples for large values of $T$ are not yet known. Here, we shall improve the bounds for $T$ obtained in previous works, as mentioned in the introduction. The results shall be expressed in terms of the optimal constant of the inequality

$$\|x - \overline{x}\|_\infty \leq k\|x^\Delta\|_\infty, \qquad x \in C_T^1$$

which shall be denoted by $k(\mathbb{T})$. For instance, for arbitrary $\mathbb{T}$ it is readily seen that $k(\mathbb{T}) \leq \frac{T}{2}$, because $x^\Delta$ has zero average and hence, due to periodicity,

$$x_{\max} - x_{\min} \leq \int_{t_{\min}}^{t_{\max}} [x^\Delta(t)]^+ \Delta t \leq \int_0^T [x^\Delta(t)]^+ \Delta t = \frac{1}{2}\int_0^T |x^\Delta(t)|\Delta t.$$

In the continuous case, the Sobolev inequality $\|x - \overline{x}\|_\infty \leq \sqrt{\frac{T}{12}}\|x'\|_2$ yields $k(\mathbb{R}) \leq \frac{T}{2\sqrt{3}}$.

Recall that, from Theorem 4.1 and Example 5.3 in [2], in order to prove the existence of $T$-periodic solutions for $s=0$ it suffices to verify that the equation

$$\left(\frac{x^{\Delta}(t)}{\sqrt{1-\frac{x^{\Delta}(t)^2}{c^2}}}\right)^{\Delta} = \lambda[p_0(t) - ax^{\Delta}(t) - b\sin x(t)] \tag{5}$$

has no $T$-periodic solutions with average $\pm\frac{\pi}{2}$.

Let $x \in C^1_T$ be a solution of (5) and suppose for example that $\overline{x} = \frac{\pi}{2}$, then

$$\left|x(t) - \frac{\pi}{2}\right| \leq ck(\mathbb{T}).$$

In particular, if $ck(\mathbb{T}) \leq \frac{\pi}{2}$, then $x(t) \in [0,\pi]$ for all $t \in \mathbb{T}$ and upon integration of equation (5) we get the following contradiction:

$$0 = b\int_0^T \sin(x(t))\Delta t > 0.$$

For example, the condition $cT \leq \pi$ is sufficient for arbitrary $\mathbb{T}$ and, in the continuous case, the condition $cT \leq \sqrt{3}\pi$ is retrieved.

However, the previous bound can be improved. To this end, suppose that $\frac{\pi}{2} < ck(\mathbb{T}) < \pi$ and $\overline{x} = \frac{\pi}{2}$, then

$$x(t) \in \left[\frac{\pi}{2} - ck(\mathbb{T}), \frac{\pi}{2} + ck(\mathbb{T})\right] \subset \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

for all $t \in \mathbb{T}$ and hence

$$\sin x(t) \geq -\sin(A) > -1, \qquad \text{where } A = ck(\mathbb{T}) - \frac{\pi}{2}$$

Fix $\delta \in (0, \frac{\pi}{2})$ and consider the set

$$C_\delta = \left\{t \in [0,T]_{\mathbb{T}} : |x(t) - \frac{\pi}{2}| \leq \delta\right\},$$

then

$$\begin{array}{rcl} 0 & = & \int_0^T \sin(x(t))\Delta t \geq \int_{C_\delta}(\sin(x(t)) + \sin(A))\Delta t - T\sin(A) \\ & \geq & \left[\sin(\frac{\pi}{2} - \delta) + \sin(A)\right]\mathfrak{m}(C_\delta) - T\sin(A) \\ & = & \cos(\delta)\mathfrak{m}(C_\delta) - [T - \mathfrak{m}(C_\delta)]\sin A \end{array} \tag{6}$$

where $\mathfrak{m}(C_\delta)$ is the measure of the set $C_\delta$ associated to the $\Delta$-integral, namely $\mathfrak{m}(C_\delta) = \int_{C_\delta}\Delta t$. Clearly, a contradiction is obtained when the latter term of (6) is positive.

On the other hand, notice that if $x(t_0) \leq \frac{\pi}{2}$ and $t_1 > t_0$ is such that $x(t_1) \geq \frac{\pi}{2} + \delta$, then

$$\delta \leq x(t_1) - x(t_0) = \int_{t_0}^{t_1} x^{\Delta}(s)\Delta s < c(t_1 - t_0).$$

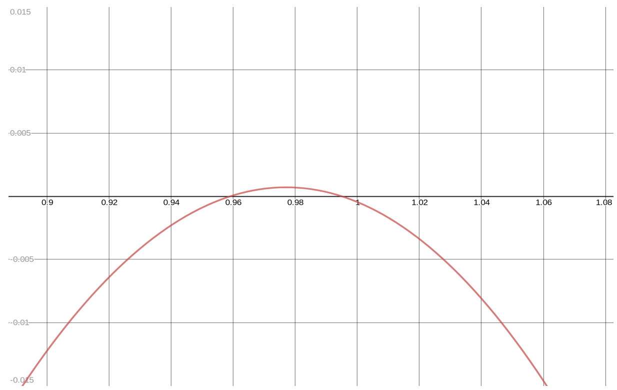


Figure 1: Graph of $\varphi$ for $\mathbb{T} = \mathbb{R}$ with $cT = 6.318$

Thus, by periodicity, it is seen that $\mathfrak{m}(C_\delta) > \frac{2\delta}{c}$. Hence, a sufficient condition for the existence of at least one $T$-periodic solution is that, for some $\delta \in (0, \frac{\pi}{2})$,

$$\cos(\delta)\frac{2\delta}{c} \geq \left(T - \frac{2\delta}{c}\right) \sin A.$$

Note, furthermore, that if the previous inequality is strict, then a contradiction is still obtained as in (6) when we add a small parameter $s$ to the function $p_0$ in (5). Thus we have proved:

**Theorem 4.1** *Assume that $ck(\mathbb{T}) < \pi$ and define the function*

$$\varphi(\delta) := 2\delta\cos(\delta) + (cT - 2\delta)\cos\left(ck(\mathbb{T})\right).$$

*If $\varphi(\delta) \geq 0$ for some $\delta \in (0, \frac{\pi}{2})$, then $0 \in \mathcal{I}(p_0)$. Furthermore, if the previous inequality is strict, then $0 \in \mathcal{I}(p_0)^\circ$.*

It is worth noticing that the bound thus obtained improves the previous one and, in particular, it guarantees that 0 is in fact an interior point of $\mathcal{I}(p_0)$ for $ck(\mathbb{T}) \leq \frac{\pi}{2}$. In the continuous case, an easy numerical computation gives the sufficient condition $cT \leq 6.318$, slightly better than the bound $cT < \sqrt{3}\pi$ obtained in [4] (see Figure 1). For arbitrary $\mathbb{T}$, if we consider the 'non-sharp' value $k(\mathbb{T}) = \frac{T}{2}$, then a sufficient condition when $cT \in (\pi, 2\pi)$ is the existence of $\delta \in (0, \frac{\pi}{2})$ such that

$$2\delta\cos(\delta) + (cT - 2\delta)\cos\left(\frac{cT}{2}\right) \geq 0.$$

Again, numerical experiments show $0 \in \mathcal{I}(p_0)^\circ$ for $cT \leq 4.19$, as shown in Figure 2.

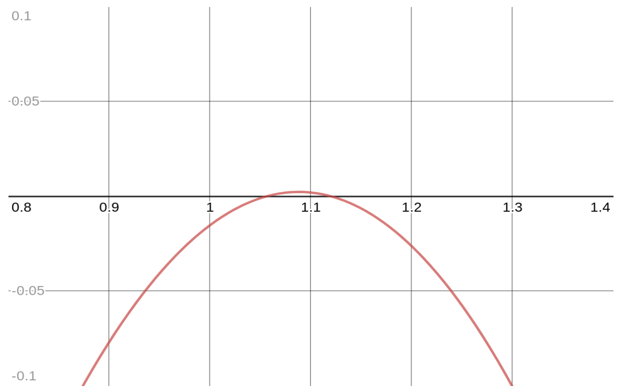


Figure 2: Graph of $\varphi$ for $k(\mathbb{T}) = \frac{T}{2}$ and $cT = 4.19$

**Remark 4.2** *A sharper analysis can be made by noticing that $\varphi$ reaches its maximum in the unique $\delta^* \in (0, \frac{\pi}{2})$ such that*

$$\cos(\delta^*) - \delta^* \sin(\delta^*) = \cos\left(ck(\mathbb{T})\right). \tag{7}$$

*Thus, replacing (7) in $\varphi$, a somewhat explicit condition on $T$ reads:*

$$2(\delta^*)^2 \sin(\delta^*) + cT \cos\left(ck(\mathbb{T})\right) \geq 0.$$

**Remark 4.3** *An estimation of the constant $k(\mathbb{T})$ could be obtained analogously to the continuous case as shown for example in [12]. Let $\{e_n\}_{n\in\mathbb{Z}} \subset C_T$ be an orthonormal basis of $L^2(0,T)_{\mathbb{T}}$ with $e_0 \equiv \frac{1}{\sqrt{T}}$ and $E_n$ be a primitive of $e_n$ such that $\overline{E}_n = 0$. Writing $x' = \sum_{n\neq 0} a_n e_n$, it follows that*

$$\|x - \overline{x}\|_\infty = \left|\sum_{n\neq 0} a_n E_n\right| \leq \|x'\|_{L^2} \sqrt{\sum_{n\neq 0} \|E_n\|_\infty^2} \leq \|x'\|_\infty \sqrt{T \sum_{n\neq 0} \|E_n\|_\infty^2}.$$

*When $\mathbb{T} = \mathbb{R}$, taking the usual Fourier basis one has that $\|E_n\|_\infty = \frac{\sqrt{T}}{2\pi n}$ and the value $k(\mathbb{R}) \leq \frac{T}{2\sqrt{3}}$ is obtained from the well known equality $\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$.*

**Remark 4.4** *As mentioned in the introduction, the previous result gives an inferior bound for the length of the solvability interval which does not depend on $p_0$, provided that $T$ is small enough. In some obvious cases, inferior bounds are obtained for arbitrary $T$: for example, if $\|p_0\|_\infty < b$ then $[-\varepsilon, \varepsilon] \subset \mathcal{I}(p_0)$ for $\varepsilon = b - \|p_0\|_\infty$. This is readily verified taking $\alpha = \frac{\pi}{2}$ and $\beta = \frac{3\pi}{2}$ as lower and upper solutions.*

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