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$\mathcal{J}_H$-SINGULARITY AND $\mathcal{J}_H$-REGULARITY OF MULTIVARIATE STATIONARY PROCESSES OVER LCA GROUPS*

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Abstract. Let G be an LCA group, Γ its dual group, and H a closed subgroup of G such that its annihilator Λ is countable. Let M denote a regular positive semidefinite matrix-valued Borel measure on Γ and $L^2(M)$ the corresponding Hilbert space of matrix-valued functions square-integrable with respect to M. For $g \in G$, let $\mathbf{Z}_g$ be the closure in $L^2(M)$ of all matrix-valued trigonometric polynomials with frequencies from $g+H$. We describe those measures M for which $\mathbf{Z}_g = L^2(M)$ as well as those for which $\bigcap_{g\in G} \mathbf{Z}_g = \{0\}$. Interpreting M as a spectral measure of a multivariate wide sense stationary process on G and denoting by $\mathcal{J}_H$ the family of H-cosets, we obtain conditions for $\mathcal{J}_H$-singularity and $\mathcal{J}_H$-regularity.

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1. INTRODUCTION

The celebrated Whittaker–Shannon–Kotel'nikov theorem claims that if the non-stochastic spectral measure μ of a mean square continuous wide sense stationary process $\mathbf{X} := \{X(t) \colon t \in \mathbb{R}\}$ is concentrated on the interval $(-\pi, \pi)$, then $\mathbf{X}$ admits a series representation

$$X(t) = \sum_{k\in\mathbb{Z}} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}, \tag{1.1}$$

which converges in the quadratic mean. In particular, (1.1) yields

$$\overline{\mathrm{sp}}\{X(k) \colon k \in \mathbb{Z}\} = \overline{\mathrm{sp}}\{X(t) \colon t \in \mathbb{R}\}, \tag{1.2}$$

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where $\overline{\mathrm{sp}}$ stands for closed span. In a sense, formula (1.2) implies that all information on the process $\mathbf{X}$ can be extracted from observing it on the integers. Although (1.2) is weaker than (1.1) (see [11]), and is not so useful from the practical point of view, several authors were concerned with the problem of describing those processes for which (1.2) is true. The solution was given in terms of the spectral measure μ, i.e. the "isomorphic" problem in the spectral domain $L^2(\mu)$ of $\mathbf{X}$ was studied. The main idea is to replace the measure μ by a suitable periodic measure.

Lloyd [11] introduced a measure ν by setting $\nu(B) := \sum_{k=-\infty}^{\infty} \mu(B+2k\pi)$ for any Borel subset B of $\mathbb{R}$. With its aid he computed the orthoprojection in $L^2(\mu)$ onto $\overline{\mathrm{sp}}\{\mathrm{e}^{\mathrm{i}k\,\cdot}\colon k \in \mathbb{Z}\}$ and derived from this necessary and sufficient conditions for (1.2). Lloyd's approach was generalized to multivariate processes in [14] and applied to a problem of multichannel sampling in [8]. Another approach first taken by Yaglom [21] in a more or less explicit form consists in restricting the measure μ to each interval $[2\pi k, 2\pi(k+1))$, $k \in \mathbb{Z}$, shifting these restrictions to $[0, 2\pi)$ and then adding them to obtain a measure $\tilde{\mu}$. This method has the advantage that the resulting measure $\tilde{\mu}$ is finite, whereas ν is only σ-finite in general. Yaglom's approach was used in [7] and generalized to multivariate processes in [17] and [6] as well as to harmonizable symmetric α-stable processes in [9].

The present paper deals with multivariate processes on LCA groups, i.e. locally compact abelian groups with Hausdorff topology. On the basis of abstract harmonic analysis we gain a unified approach to stationary processes on various groups. Note that in [6] processes on $\mathbb{R}$ and $\mathbb{Z}$ were discussed separately. Since our results can be applied, for example, to a homogeneous field on a torus or cylinder, they are also of practical interest.

The problem reads as follows. Let G be an LCA group and H its closed subgroup. Describe all multivariate weakly stationary processes $\mathbf{X} := \{X(g)\colon g \in G\}$ on G with

$$\overline{\mathrm{sp}}\{X(h)\colon h \in H\} = \overline{\mathrm{sp}}\{X(g)\colon g \in G\}, \tag{1.3}$$

where $\overline{\mathrm{sp}}$ stands for closed matrix-linear span. If we introduce the family $\mathcal{J}_H := \{g + H\colon g \in G\}$ of H-cosets, in accordance with [18, Definition 2.10] a process $\mathbf{X}$ satisfying (1.3) can be called *$\mathcal{J}_H$-singular*.

In Section 2 we discuss $\mathcal{J}_H$-singular processes in terms of their spectral measure M, i.e. a positive semidefinite matrix-valued measure on the dual group Γ of G. Unfortunately, we have not been able to solve the problem in its full generality since our method heavily leans on the requirement that the annihilator Λ of the group H is countable, i.e. finite or countably infinite. Thus, throughout this paper it is assumed that Λ is countable. Then Theorem 2.2 gives a complete characterization of the spectral measures of $\mathcal{J}_H$-singular processes. The special case $G = \mathbb{R}$ and $H = \mathbb{Z}$ was solved in [6]. However, the formulation of the result as well as the proof given there were somewhat imprecise or even misleading and Theorem 2.2 removes this inaccuracy. Moreover, [6] was written ignoring Pourahmadi's paper [14]. Here we point out the interrelations between Pourahmadi's results and ours.

Section 3 is devoted to some consequences of Theorem 2.2.

In Section 4 we discuss a problem which arises in multichannel sampling and was solved in [8] using Lloyd's approach. It seems to us that from Theorem 2.2 one can deduce a simpler and more lucid solution of that problem.

In a sense, processes $\mathbf{X}$ with

$$\bigcap_{g\in G} \overline{\operatorname{sp}}\{X(g+h)\colon h\in H\} = \{0\}$$

form the opposite class to the class of $\mathcal{J}_H$-singular processes. In accordance with [18, Definition 2.10] they are called *$\mathcal{J}_H$-regular*. Section 5 deals with the description of the spectral measure of a $\mathcal{J}_H$-regular process. Among other things, we again generalize and improve a result of [6].

Already the early papers on stationary time series dealt with the Wold decomposition of a stationary process into its $\mathcal{J}$-regular and $\mathcal{J}$-singular parts: see [20], [10], [22] for processes on $\mathbb{Z}$ or $\mathbb{R}$ and the family $\mathcal{J} := \{(-\infty, x]\cap\mathbb{Z}\colon x\leqslant 0\}$ or $\mathcal{J} := \{(-\infty, x]\colon x\leqslant 0\}$; Salehi and Scheidt [18, Theorem 2.13] proved the existence and uniqueness of a Wold decomposition for processes on an LCA group G and a family $\mathcal{J}$ of subsets of G invariant under translation. The problem arises to describe the spectral measures of the $\mathcal{J}$-regular and $\mathcal{J}$-singular parts. Here we solve this problem for the family $\mathcal{J}_H$ generalizing [6, Theorem 5.2], where the special case that $\mathcal{J}_H$ has only two elements was discussed.

In [9], $\mathcal{J}_H$-regularity and $\mathcal{J}_H$-singularity of a univariate process on G were described without assuming that Λ is countable. It would be of theoretical as well as of practical interest to extend these results to multivariate processes since, for example, the rather natural case that $\mathbf{X}$ is a process on the two-dimensional lattice $\mathbb{Z}^2$ and $H = \mathbb{Z}\times\{0\}$ is not covered by our results.

2. $\mathcal{J}_H$-SINGULARITY

For $p, q\in\mathbb{N}$ denote by $\mathcal{M}_{p,q}$ the linear space of all $p\times q$ matrices with complex entries. Set $\mathcal{M}_q := \mathcal{M}_{q,q}$ and denote by $\mathcal{M}_q^{\geqslant}$ the cone of positive semidefinite $q\times q$ matrices. To simplify the notation, dependence on the dimension is occasionally not indicated. For example, we shall write $\mathcal{M}$ instead of $\mathcal{M}_{p,q}$ and $\mathcal{M}^{\geqslant}$ instead of $\mathcal{M}_q^{\geqslant}$. The adjoint, Moore–Penrose inverse, range, null space, and rank of a matrix X are denoted by X^*, X^+, $\mathcal{R}(X)$, $\mathcal{N}(X)$, and $\operatorname{rk} X$, resp. If X is quadratic, then $\operatorname{tr} X$ denotes its trace. The unit matrix (of arbitrary size) is denoted by I. If L is a subspace of $\mathbb{C}^q$, then $L^\perp$ stands for its orthogonal complement and P_L for the orthoprojection onto L.

For a topological space Ω, let $\mathcal{B}(\Omega)$ be the σ-algebra of its Borel subsets. If μ is a non-negative measure and ν a complex-valued measure on $\mathcal{B}(\Omega)$ such that ν is absolutely continuous with respect to μ, we shall write $\nu \ll \mu$. A function $\Phi\colon\Omega\to\Omega'$ from Ω into a topological space Ω' is called *measurable* if it is $(\mathcal{B}(\Omega),\mathcal{B}(\Omega'))$-measurable. According to Azoff's general result concerning measurability of processes in linear algebra (cf. [1]), all functions we shall deal with here can be assumed to be measurable.

Let G be an LCA group, whose group operation is written additively. Denote by Γ its dual group and by $\langle \gamma, g\rangle$ the value of $\gamma \in \Gamma$ at $g \in G$. Let M be a regular $\mathcal{M}_q^{\geqslant}$-valued measure on $\mathcal{B}(\Gamma)$. We recall the definition of the left Hilbert-$\mathcal{M}_p$-module of $\mathcal{M}_{p,q}$-valued functions on Γ, which are square-integrable with respect to M as given by Rosenberg [15]. We mention that a slightly less general definition was given about 15 years earlier by Kats [4] (see also Rozanov's book [16]). Let μ be a σ-finite non-negative measure on $\mathcal{B}(\Gamma)$ such that $M \ll \mu$. Denote by $\frac{\mathrm{d}M}{\mathrm{d}\mu}$ the Radon–Nikodym derivative. We emphasize that $\frac{\mathrm{d}M}{\mathrm{d}\mu}$ is a measurable function (and not a μ-equivalence class of functions). Two measurable $\mathcal{M}_{p,q}$-valued functions Φ and Ψ are called *M-equivalent* if $\Phi \frac{\mathrm{d}M}{\mathrm{d}\mu} = \Psi \frac{\mathrm{d}M}{\mathrm{d}\mu}$ μ-a.e. The set of all (M-equivalence classes of) functions Φ such that $\int_\Gamma \Phi \frac{\mathrm{d}M}{\mathrm{d}\mu} \Phi^* \,\mathrm{d}\mu$ exists (or, equivalently, $\int_\Gamma \operatorname{tr}\bigl(\Phi \frac{\mathrm{d}M}{\mathrm{d}\mu} \Phi^*\bigr)\,\mathrm{d}\mu < \infty$) form a left Hilbert-$\mathcal{M}_p$-module with inner product $\int_\Gamma \operatorname{tr}\bigl(\Phi \frac{\mathrm{d}M}{\mathrm{d}\mu} \Psi^*\bigr)\,\mathrm{d}\mu = \operatorname{tr} \int_\Gamma \Phi \frac{\mathrm{d}M}{\mathrm{d}\mu} \Psi^* \,\mathrm{d}\mu$, which is denoted by $L^2_{p,q}(M)$ or simply $L^2(M)$. A routine application of the chain rule for Radon–Nikodym derivatives reveals that the definition of $L^2(M)$ does not depend on the choice of μ. It is common to choose $\mu = \tau := \operatorname{tr} M$.

Let H be a closed subgroup of G and $\Lambda := \{\lambda \in \Gamma\colon \langle \lambda, g\rangle = 1 \text{ for all } g \in H\}$ its annihilator. Recall that Λ is a closed subgroup of Γ. For $g \in G$, denote by $\tilde{g}$ its H-coset $g + H$. The set $\overline{\operatorname{sp}}\{\langle \cdot, g+h\rangle I\colon h \in H\}$, i.e. the closure in $L^2_{p,q}(M)$ of the $\mathcal{M}_{p,q}$-linear hull of all functions $\langle \cdot, g+h\rangle I$, $h \in H$, is denoted by $\mathbf{Z}_{\tilde{g}}^{(p,q)}(M)$ or often by $\mathbf{Z}_{\tilde{g}}$ for short. Set $\mathbf{Z} := \mathbf{Z}_{\tilde{0}}$.

Since the operator of multiplication by the function $\langle \cdot, g\rangle$ is obviously a unitary operator in $L^2(M)$, the equality $\mathbf{Z}_{\tilde{g}} = L^2(M)$ is satisfied for all $g \in G$ if and only if it is satisfied for some $g \in G$.

LEMMA 2.1. *For arbitrary $p \in \mathbb{N}$, the space $\mathbf{Z}^{(p,q)}$ coincides with $L^2_{p,q}(M)$ if and only if $\mathbf{Z}^{(1,q)} = L^2_{1,q}(M)$.*

Proof. From the Cauchy inequality it follows easily that $\Phi \in L^2_{p,q}(M)$ if and only if all rows of Φ belong to $L^2_{1,q}(M)$. Note that $\mathbf{Z}^{(p,q)} = L^2_{p,q}(M)$ if and only if for any $\Phi \in L^2_{p,q}(M)$ the relations

$$\int_\Gamma \Phi(\gamma) \frac{\mathrm{d}M}{\mathrm{d}\tau}(\gamma) \langle \gamma, h\rangle X^* \,\tau(\mathrm{d}\gamma) = 0, \qquad h \in H,\ X \in \mathcal{M}_{p,q}, \tag{2.1}$$

yield $\Phi = 0$ in $L^2_{p,q}(M)$. Since the condition (2.1) is satisfied if and only if $\int_\Gamma \phi(\gamma) \frac{\mathrm{d}M}{\mathrm{d}\tau}(\gamma) \langle \gamma, h\rangle \tau(\mathrm{d}\gamma) = 0$ for all $h \in H$ and all rows ϕ of Φ, it is not hard to derive the assertion. ∎

Taking into account the preceding lemma we can give the following definition.

DEFINITION 2.1. A regular $\mathcal{M}_q^{\geqslant}$-valued measure M on $\mathcal{B}(\Gamma)$ is called $\mathcal{J}_H$-*singular* if $\mathbf{Z} = L^2(M)$.

Considering M as a spectral measure of a q-variate stationary process $\mathbf{X}$ on G, we can conclude from Kolmogorov's isomorphism theorem that M is $\mathcal{J}_H$-singular if the process $\mathbf{X}$ is. The goal of the present section is to describe all $\mathcal{J}_H$-singular measures. To do this we first introduce some measures related to M.

A subset T of Γ is called a *transversal* (*with respect to* Λ) if it meets each Λ-coset just once, i.e. $T \cap (\lambda + T) = \emptyset$, $\lambda \in \Lambda \setminus \{0\}$, and $\bigcup(\lambda + T) = \Gamma$, where, by convention, the index of summation or of a union of sets is λ and runs through Λ if not indicated. A transversal may be intuitively treated as representing the factor group Γ/Λ. According to [2, Theorem 1] we can and will assume that $T \in \mathcal{B}(\Gamma)$. For $\lambda \in \Lambda$, let M_λ be the restriction of M to $\mathcal{B}(\lambda + T)$, $\tau_\lambda := \operatorname{tr} M_\lambda$, $\tilde{M}_\lambda(\tilde{B}) := M_\lambda(\lambda + \tilde{B})$, $\tilde{B} \in \mathcal{B}(T)$, $\tilde{\tau}_\lambda := \operatorname{tr} \tilde{M}_\lambda$. Set $\tilde{M} := \sum \tilde{M}_\lambda$, $\tilde{\tau} := \operatorname{tr} \tilde{M} = \sum \tilde{\tau}_\lambda$ and $\sigma(B) := \sum \tilde{\tau}((B \cap (\lambda + T)) - \lambda)$, $B \in \mathcal{B}(\Gamma)$. All measures just defined are regular and the measures designated by a tilde are measures on $\mathcal{B}(T)$. Note that $\tau \ll \sigma$, that $\tilde{\tau}$ is the restriction of σ to $\mathcal{B}(T)$, and that the measure σ is periodic, i.e. $\sigma(\lambda + B) = \sigma(B)$, $B \in \mathcal{B}(\Gamma)$, $\lambda \in \Lambda$. Setting

$$\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma) := \frac{\mathrm{d}M}{\mathrm{d}\sigma}(\lambda + \gamma), \qquad \gamma \in T,\ \lambda \in \Lambda, \tag{2.2}$$

can establish a 1-1 correspondence between the set of Radon–Nikodym derivatives $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ and the set of families of Radon–Nikodym derivatives $\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}$, $\lambda \in \Lambda$.

LEMMA 2.2. *For $g \in G$, the set $\mathbf{Z}_{\tilde{g}}(\tilde{M})$ coincides with $L^2(\tilde{M})$.*

Proof. If $\tilde{\Phi} \in L^2(\tilde{M})$ satisfies $\int_T \tilde{\Phi}(\gamma) \frac{\mathrm{d}\tilde{M}}{\mathrm{d}\tilde{\tau}}(\gamma)\langle \gamma, h\rangle\, \tilde{\tau}(\mathrm{d}\gamma) = 0$ for all $h \in H$, from [12, Lemma 3.1] it follows $\tilde{\Phi}\frac{\mathrm{d}\tilde{M}}{\mathrm{d}\tilde{\tau}} = 0$ $\tilde{\tau}$-a.e., which implies that $\tilde{\Phi} = 0$ in $L^2(\tilde{M})$. Thus $\mathbf{Z}(\tilde{M}) = L^2(\tilde{M})$, and hence $\mathbf{Z}_{\tilde{g}}(\tilde{M}) = L^2(\tilde{M})$ for all $g \in G$. ∎

Let $g \in G$. Since for $\lambda \in \Lambda$, the function $\langle \lambda, \cdot \rangle$ is constant on each H-coset, we can set $\langle \lambda, \tilde{g} \rangle := \langle \lambda, g \rangle$ and define an operator $V_{\tilde{g}}$ on $L^2(\tilde{M})$ by

$$(V_{\tilde{g}}\tilde{\Phi})(\gamma) := \langle \lambda, \tilde{g}\rangle \tilde{\Phi}(\gamma - \lambda), \qquad \gamma \in \lambda + T,\ \lambda \in \Lambda,\ \tilde{\Phi} \in L^2(\tilde{M}).$$

Let $V := V_{\tilde{0}}$. Despite its simplicity the following lemma is crucial for our considerations.

LEMMA 2.3. *For any $g \in G$, the operator $V_{\tilde{g}}$ establishes an isometric isomorphism between $L^2_{p,q}(\tilde{M})$ and $\mathbf{Z}_{\tilde{g}}^{(p,q)}(M)$. In particular, V maps $L^2_{p,q}(\tilde{M})$ onto $\mathbf{Z}^{(p,q)}(M)$ isometrically.*

Proof. It is obvious that $V_{\tilde{g}}$ is an $\mathcal{M}_p$-linear map. From the periodicity of σ, formula (2.2), and the monotone convergence theorem we obtain

$$\begin{aligned}\int_\Gamma (V_{\tilde{g}}\tilde{\Phi})\frac{\mathrm{d}M}{\mathrm{d}\sigma}(V_{\tilde{g}}\tilde{\Phi})^*\,\mathrm{d}\sigma &= \sum_{\lambda\in\Lambda}\int_{\lambda+T}(V_{\tilde{g}}\tilde{\Phi})\frac{\mathrm{d}M}{\mathrm{d}\sigma}(V_{\tilde{g}}\tilde{\Phi})^*\,\mathrm{d}\sigma\\ &= \sum_{\lambda\in\Lambda}\int_T (V_{\tilde{g}}\tilde{\Phi})(\lambda+\gamma)\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\lambda+\gamma)(V_{\tilde{g}}\tilde{\Phi})(\lambda+\gamma)^*\,\sigma(\mathrm{d}\gamma)\\ &= \sum_{\lambda\in\Lambda}\int_T \tilde{\Phi}\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}\tilde{\Phi}^*\,\mathrm{d}\tilde{\tau} = \int_T \tilde{\Phi}\frac{\mathrm{d}\tilde{M}}{\mathrm{d}\tilde{\tau}}\tilde{\Phi}^*\,\mathrm{d}\tilde{\tau}\end{aligned}$$

for $\tilde{\Phi}\in L^2_{p,q}(\tilde{M})$, which shows that $V_{\tilde{g}}$ is an isometry. To show that the image of $V_{\tilde{g}}$ is equal to $\mathbf{Z}^{(p,q)}_{\tilde{g}}(M)$ note that the function $\langle\cdot, g+h\rangle X$, $h\in H$, $X\in\mathcal{M}_{p,q}$, is the image of its restriction to T and apply Lemma 2.2. ■

An M-equivalence class is called *periodic* if it contains a function Φ such that $\Phi(\gamma+\lambda)=\Phi(\gamma)$ for all $\gamma\in\Gamma$ and $\lambda\in\Lambda$. The following description of $\mathbf{Z}(M)$ was obtained by Pourahmadi [14, Lemma 2.4] under the assumption that there exists a Radon–Nikodym derivative $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ with periodic range function. We mention that in the case of $\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)\big)=\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma)\big)$, $\gamma\in\Gamma$, $\lambda\in\Lambda$, the measures τ and σ are equivalent. To see this note first that $\tau(B)=0$ yields $\frac{\mathrm{d}M}{\mathrm{d}\sigma}=0$ σ-a.e. on B, hence, $\frac{\mathrm{d}M}{\mathrm{d}\sigma}=0$ σ-a.e. on $\kappa+B$ and $\tau(\kappa+B)=0$ for all $\kappa\in\Lambda$. It follows that

$$\sigma(B)=\sum_{\lambda\in\Lambda}\tilde{\tau}((B\cap(\lambda+T))-\lambda)=\sum_{\lambda\in\Lambda}\sum_{\kappa\in\Lambda}\tau((B\cap(\lambda+T))-\lambda+\kappa)=0$$

for $B\in\mathcal{B}(\Gamma)$.

THEOREM 2.1. *The space* $\mathbf{Z}$ *is exactly the space of periodic* M*-equivalence classes of* $L^2(M)$.

Proof. From the definition of V it is clear that all elements of $\mathbf{Z}$ are periodic. If $\Psi\in L^2(M)$ is periodic and orthogonal to $\mathbf{Z}$, then, similarly to the proof of Lemma 2.2,

$$0=\int_\Gamma \Psi(\gamma)\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma)\langle\gamma,h\rangle\,\sigma(\mathrm{d}\gamma)=\int_T \Psi(\gamma)\frac{\mathrm{d}\tilde{M}}{\mathrm{d}\tilde{\tau}}(\gamma)\langle\gamma,h\rangle\,\tilde{\tau}(\mathrm{d}\gamma)$$

for all $h\in H$. Therefore, $\Psi=0$ in $L^2(M)$ by [12, Lemma 3.1]. ■

A finite set $\{L_j\}$ of subspaces of $\mathbb{C}^q$ is called *direct* if their sum $\sum_j L_j$ is direct, i.e. for every $v\in\sum_j L_j$ there exist unique $u_j\in L_j$ such that $v=\sum_j u_j$. We mention some facts from linear algebra, whose elementary proofs are omitted.

LEMMA 2.4. *The following assertions are equivalent:*

(i) *the set* $\{L_j\}$ *is direct,*
(ii) *if* $\sum_j u_j = 0$, $u_j \in L_j$, *then* $u_j = 0$ *for all* j,
(iii) *for any index* k, *the intersection* $L_k \cap \sum_{j\neq k} L_j$ *is the null space.*

An infinite set $\{L_j : j \in J\}$ of subspaces of $\mathbb{C}^q$ is called *direct* if there exists a finite subset J' of J such that $\{L_j : j \in J'\}$ is direct, $\sum_{j\in J'} L_j = \sum_{j\in J} L_j$, and $L_j = \{0\}$ for $j \in J \setminus J'$. It is not hard to see that under the additional requirement that $L_j \neq \{0\}$ for $j \in J'$, the set J' is unique and contains at most q elements.

Now we are ready to describe the set of $\mathcal{J}_H$-singular measures.

THEOREM 2.2. *Let* Λ *be countable. The following assertions are equivalent:*

(i) *the measure* M *is* $\mathcal{J}_H$*-singular,*
(ii) *for all families* $\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}$, $\lambda \in \Lambda$, *there exists* $\tilde{B} \in \mathcal{B}(T)$ *such that*

$$\tilde{\tau}(T \setminus \tilde{B}) = 0 \quad \textit{and} \quad \Big\{\mathcal{R}\Big(\tfrac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma)\Big) : \lambda \in \Lambda\Big\} \textit{ is direct for all } \gamma \in \tilde{B}, \tag{2.3}$$

(iii) *there exist a family* $\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}$, $\lambda \in \Lambda$, *and* $\tilde{B} \in \mathcal{B}(T)$ *satisfying* (2.3),
(iv) *there exists a family* $\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}$, $\lambda \in \Lambda$, *such that* $\big\{\mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma)\big) : \lambda \in \Lambda\big\}$ *is direct for all* $\gamma \in T$,
(v) *for each version of* $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ *there exists* $B \in \mathcal{B}(\Gamma)$ *such that*

$$\sigma(\Gamma \setminus B) = 0 \quad \textit{and} \quad \Big\{\mathcal{R}\Big(\tfrac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma + \lambda)\Big) : \lambda \in \Lambda\Big\} \textit{ is direct for all } \gamma \in B, \tag{2.4}$$

(vi) *there exist* $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ *and* $B \in \mathcal{B}(\Gamma)$ *satisfying* (2.4),
(vii) *there exists* $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ *such that* $\big\{\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)\big) : \lambda \in \Lambda\big\}$ *is direct for all* $\gamma \in \Gamma$,
(viii) *there exist* $\frac{\mathrm{d}M}{\mathrm{d}\tau}$ *and* $B \in \mathcal{B}(\Gamma)$ *such that* $\tau(\Gamma \setminus B) = 0$ *and* $\big\{\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\tau}(\gamma+\lambda)\big) : \lambda \in \Lambda\big\}$ *is direct for all* $\gamma \in B$.

Proof. Lemma 2.1 implies that it is enough to handle the case $p = q$.

(i)⇒(ii). Let $\kappa \in \Lambda$ and $1_{\kappa+T}$ be the indicator function of $\kappa+T$. If $\mathbf{Z} = L^2(M)$, Lemma 2.3 gives the existence of $\tilde{\Phi} \in L^2(\tilde{M})$ satisfying $V\tilde{\Phi} = 1_{\kappa+T}I$, which yields $\mathcal{N}(\tilde{\Phi}) \cap \mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}_\kappa}\big) = \{0\}$ $\tilde{\tau}_\kappa$-a.e. and $\mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}_\lambda}\big) \subseteq \mathcal{N}(\tilde{\Phi})$ $\tilde{\tau}_\lambda$-a.e. $\lambda \in \Lambda \setminus \{\kappa\}$. An application of the chain rule gives $\mathcal{N}(\tilde{\Phi}) \cap \mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}\big) = \{0\}$ and $\mathcal{R}(\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}) \subseteq \mathcal{N}(\tilde{\Phi})$ for $\lambda \in \Lambda \setminus \{\kappa\}$, $\tilde{\tau}$-a.e., hence

$$\mathcal{R}\bigg(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}\bigg) \cap \sum_{\lambda\neq\kappa} \mathcal{R}\bigg(\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}\bigg) = \{0\} \qquad \tilde{\tau}\text{-a.e.}$$

Since Λ is countable, there exists $\tilde{B} \in \mathcal{B}(T)$ satisfying $\tilde{\tau}(T \setminus \tilde{B}) = 0$ and

$$\mathcal{R}\bigg(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}(\gamma)\bigg) \cap \sum_{\lambda\neq\kappa} \mathcal{R}\bigg(\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma)\bigg) = \{0\}$$

for all $\gamma \in \tilde{B}$ and all $\kappa \in \Lambda$. Now apply Lemma 2.4.

(ii)⇒(iii) is clear and (iii)⇒(iv) can be obtained by setting $\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}} = 0$ on $\tilde{B}$ for $\lambda \in \Lambda$. The equivalences (ii)⇔(v), (ii)⇔(vi), and (iv)⇔(vii) are immediate consequences of (2.2).

(vii)⇒(viii). Choose a function $\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}$ and set $B := \big\{\gamma \in \Gamma\colon \frac{\mathrm{d}\tau}{\mathrm{d}\sigma}(\gamma) > 0\big\}$. If $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ satisfies (vii), define

$$\frac{\mathrm{d}M}{\mathrm{d}\tau} := \frac{\mathrm{d}M}{\mathrm{d}\sigma}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}\right)^{-1} \text{ on } B \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}\tau} = 0 \text{ on } \Gamma \setminus B.$$

Let $\gamma \in B$ and $\lambda \in \Lambda$. If $\gamma + \lambda \in B$, then $\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)\big) = \mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\tau}(\gamma+\lambda)\big)$. If $\gamma+\lambda \in \Gamma\setminus B$, then $\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\tau}(\gamma+\lambda)\big) = 0$. Since a family of subspaces of $\mathbb{C}^q$ remains direct if some of them are replaced by the null space, the assertion is proved.

(viii)⇒(i). Assume that $\frac{\mathrm{d}M}{\mathrm{d}\tau}$ and $B \in \mathcal{B}(\Gamma)$ satisfy (viii). Choose $\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}$ such that $B = \{\gamma \in \Gamma\colon \frac{\mathrm{d}\tau}{\mathrm{d}\sigma}(\gamma) > 0\}$, set $\tilde{B} := \bigcup_{\lambda\in\Lambda}((B \cap (\lambda + T)) - \lambda)$ and note that

$$\begin{aligned}
\tilde{\tau}(T \setminus \tilde{B}) &= \sum_{\lambda\in\Lambda} \tau((\lambda + (T \setminus \tilde{B})) \cap (\lambda + T)) \\
&\leqslant \sum_{\lambda\in\Lambda} \tau((((\lambda + T) \setminus (B \cap (\lambda + T))) \cap (\lambda + T)) \\
&= \sum_{\lambda\in\Lambda} \tau((\lambda + T) \setminus (B \cap (\lambda + T))) = \tau(\Gamma \setminus B) = 0.
\end{aligned}$$

Let $\Phi \in L^2(M)$. If $\gamma \in \tilde{B}$, define $\tilde{\Phi}(\gamma)$ in such a way that for all $\lambda \in \Lambda$, the restriction of the operator $\tilde{\Phi}(\gamma)$ to $\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\tau}(\gamma+\lambda)\big)$ coincides with the corresponding restriction of $\Phi(\gamma+\lambda)$. If $\gamma \in T\setminus\tilde{B}$, set $\tilde{\Phi}(\gamma) = 0$. Since $\tilde{\Phi}(\gamma) = \sum \Phi(\gamma+\lambda)P_\lambda(\gamma)$, where $P_\lambda(\gamma)$ denotes the orthoprojection onto $\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\tau}(\gamma+\lambda)\big)$, the function $\tilde{\Phi}$ is measurable. Moreover,

$$\int_T \tilde{\Phi}\frac{\mathrm{d}\tilde{M}}{\mathrm{d}\tilde{\tau}}\tilde{\Phi}^*\,\mathrm{d}\tilde{\tau} = \sum_{\lambda\in\Lambda}\int_T \tilde{\Phi}\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}\tilde{\Phi}^*\,\mathrm{d}\tilde{\tau} = \sum_{\lambda\in\Lambda}\int_{\lambda+T} \Phi\frac{\mathrm{d}M}{\mathrm{d}\sigma}\Phi^*\,\mathrm{d}\sigma = \int_\Gamma \Phi\frac{\mathrm{d}M}{\mathrm{d}\sigma}\Phi^*\,\mathrm{d}\sigma$$

by the monotone convergence theorem and the periodicity of σ. Therefore, $\tilde{\Phi} \in L^2(\tilde{M})$ and $V\tilde{\Phi} = \Phi$, which yields $\mathbf{Z} = L^2(M)$ by Lemma 2.3. ∎

If $q = 1$, i.e. if $M = \mu$ is a regular finite non-negative measure, then $\tau = \mu$ and the condition of $\mathcal{J}_H$-singularity can be given another form (cf. [11, Theorem 1] for the special case that H is a closed subgroup of $\mathbb{R}$ and [12, Theorem 4.6] for the general case).

COROLLARY 2.1. *The equality $\mathbf{Z} = L^2(\mu)$ is true if and only if there exists $B \in \mathcal{B}(\Gamma)$ such that $\mu(\Gamma \setminus B) = 0$ and $B \cap (\lambda + B) = \emptyset$ for all $\lambda \in \Lambda \setminus \{0\}$.*

Proof. If $B \in \mathcal{B}(\Gamma)$ is a set as in the assertion, set $\frac{\mathrm{d}\mu}{\mathrm{d}\sigma} = 1_B$ to obtain Theorem 2.2(viii). If Theorem 2.2(vii) is true, choose $B := \{\gamma \in \Gamma\colon \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(\gamma) > 0\}$. ∎

Another consequence of Theorem 2.2 is the following multivariate extension of Corollary 2.1, which was obtained by Pourahmadi [14, Theorem 4.1] if $G = \mathbb{R}$ and H is a closed subgroup of $\mathbb{R}$.

COROLLARY 2.2. *Let the range of* $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ *be periodic. Then* M *is* $\mathcal{J}_H$*-singular if and only if there exists* $B \in \mathcal{B}(\Gamma)$ *such that* $\sigma(\Gamma \setminus B) = 0$ (*or equivalently* $\tau(\Gamma \setminus B) = 0$) *and* $B \cap (\lambda + B) = \emptyset$ *for* $\lambda \in \Lambda \setminus \{0\}$.

3. FURTHER CONSEQUENCES OF THEOREM 2.2

To derive further consequences of Theorem 2.2 we recall some elementary facts from linear algebra.

LEMMA 3.1. *Let* $r, q, n \in \mathbb{N}$.

(i) *Let* $X \in \mathcal{M}_{r,q}$ *and* $\operatorname{rk} X = q$. *A set* $\{L_j\}$ *of subspaces of* $\mathbb{C}^q$ *is direct if and only if* $\{XL_j\}$ *is direct.*
(ii) *If* $X \in \mathcal{M}_{r,q}$, $\operatorname{rk} X = q$, *and* $Y_j \in \mathcal{M}_q^{\geqslant}$ *for* $j \in J$, *then the set* $\{\mathcal{R}(Y_j)\colon j \in J\}$ *is direct if and only if* $\{\mathcal{R}(XY_jX^*)\colon j \in J\}$ *is direct.*
(iii) *If* $X \in \mathcal{M}_q$, $Y \in \mathcal{M}_q^{\geqslant}$ *and* $Z \in \mathcal{M}_{q,r}$ *are such that* $\mathcal{R}(X^*) \subseteq \mathcal{R}(Z)$, *then* $\mathcal{R}(XYX^*) = \mathcal{R}(XYZ)$.
(iv) *If* $X_k \in \mathcal{M}_q$ *for* $k \in \{1, \dots, n\}$, $Y_j \in \mathcal{M}_q^{\geqslant}$ *for* $j \in J$, *and* $\{\mathcal{R}(X_kY_jX_k^*)\colon j \in J\}$ *is direct for each* k, *then* $\{\mathcal{R}((X_1^*, \dots, X_n^*)^*Y_j(X_1^*, \dots, X_n^*))\colon j \in J\}$ *is direct.*

REMARK. For a brief account of operator (or matrix) matrices see e.g. [3].

Proof of Lemma 3.1. (i) Since $L_j = \{0\}$ if and only if $XL_j = \{0\}$, we can assume that $\{L_j\}$ is a finite set. Note that $\sum_j u_j = 0$ for $u_j \in L_j$ if and only if $\sum_j Xu_j = 0$, and apply Lemma 2.4.
(ii) Let $L_j := \mathcal{R}(Y_j) = \mathcal{R}(Y_jX^*)$ and apply (i).
(iii) From $\mathcal{N}(XYX^*) = \mathcal{N}(Y^{1/2}X^*) = \mathcal{N}(YX^*)$ it follows that $\mathcal{R}(XYX^*) = \mathcal{R}(XY)$; now use $\mathcal{R}(XYX^*) \subseteq \mathcal{R}(XYZ) \subseteq \mathcal{R}(XY)$.
(iv) Set $Z := (X_1^*, \dots, X_n^*)$ and assume that $\sum_j Z^*Y_jZu_j = 0$ for a finite set of vectors $u_j \in \mathbb{C}^q$. It follows that $\sum_j X_kY_jZu_j = 0$ for $k \in \{1, \dots, n\}$. Since $\mathcal{R}(X_kY_jX_k^*) = \mathcal{R}(X_kY_jZ)$ by (iii), the set $\{\mathcal{R}(X_kY_jZ)\colon j \in J\}$ is direct if the set $\{\mathcal{R}(X_kY_jX_k^*)\colon j \in J\}$ is. Therefore, from Lemma 2.4 we obtain $X_kY_jZu_j = 0$ for all $k \in \{1, \dots, n\}$ and all j. Thus, $\{\mathcal{R}(Z^*Y_jZ)\colon j \in J\}$ is direct. ∎

COROLLARY 3.1.

(i) *Let* $F\colon \Gamma \to \mathcal{M}_{r,q}$ *be a measurable function such that* $\operatorname{rk} F = q$ σ*-a.e. and* F *is constant on* $\gamma + \Lambda$ *for* $\tilde{\tau}$*-a.a.* $\gamma \in T$. *The measure* $F\mathrm{d}MF^*$ *is* $\mathcal{J}_H$*-singular if and only if* M *is.*

(ii) *Let* $F_k\colon \Gamma \to \mathcal{M}_q$, $k \in \{1,\dots,n\}$, *be measurable functions such that* $F := (F_1^*,\dots,F_n^*)^*$ *has rank* q σ*-a.e. and* F *is constant on* $\gamma+\Lambda$ *for* $\tilde{\tau}$*-a.a.* $\gamma \in T$. *If* $\mathbf{Z}(F_k \mathrm{d}M F_k^*) = L^2(F_k \mathrm{d}M F_k^*)$ *for all* k, *then* $\mathbf{Z}(M) = L^2(M)$.

Proof. Assertion (i) follows from Theorem 2.2 and Lemma 3.1(ii). To obtain (ii) note that Theorem 2.2 and Lemma 3.1(iv) imply that $\mathbf{Z}(F\mathrm{d}MF^*) = L^2(F\mathrm{d}MF^*)$ if $\mathbf{Z}(F_k \mathrm{d}M F_k^*) = L^2(F_k \mathrm{d}M F_k^*)$ for all k and then apply (i). ∎

The following result (see [14, Lemma 2.2], compare also [13]), is a special case of Corollary 3.1(ii).

COROLLARY 3.2. *If* $\mathbf{Z}(m_{kk}) = L^2(m_{kk})$ *for all scalar measures* m_{kk} *on the principal diagonal of* M, *then* $\mathbf{Z}(M) = L^2(M)$.

We recall an assertion obtained by Lloyd. Let $G = \mathbb{R}$, $H = \mathbb{Z}$, and μ be a finite non-negative measure on $\mathcal{B}(\mathbb{R})$.

THEOREM 3.1 ([11, Theorem 2]). *For any* $n \in \mathbb{N}$ *the following conditions are equivalent:*

(i) *for all* $m \in \mathbb{Z}$ *the function* $\mathrm{e}^{\mathrm{i}\frac{m}{n}\cdot}$ *belongs to* $\mathbf{Z}(\mu)$,
(ii) *for some integer* m *relatively prime to* n, *the function* $\mathrm{e}^{\mathrm{i}\frac{m}{n}\cdot}$ *belongs to* $\mathbf{Z}(\mu)$,
(iii) *the measure* μ *is concentrated on a set* $B \in \mathcal{B}(\mathbb{R})$ *disjoint from each of the translates* $B + 2\pi m$ *with* $m \in \mathbb{Z} \setminus \{kn\colon k \in \mathbb{Z}\}$.

To prove a generalization of Lloyd's theorem to $\mathcal{M}_q^{\geqslant}$-valued measures on LCA groups, for simplicity of presentation we confine ourselves to the quadratic case $p = q$. For $g \in G$, set $\Lambda_g := \{\lambda \in \Lambda\colon \langle \lambda, g\rangle = 1\}$ and $G_g := \{h \in G : \Lambda_g \subseteq \Lambda_h\}$. Let R be a set of representatives of the Λ_g-cosets (with respect to Λ). For $\gamma \in T$ and $\lambda \in R$ define $L_\lambda(\gamma) := \sum_{\kappa\in\Lambda_g} \mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda+\kappa)\big)$.

THEOREM 3.2. *Let* Λ *be countable. The following conditions are equivalent:*

(i) *for all* $h \in G_g$ *the function* $\langle\cdot, h\rangle I$ *belongs to* $\mathbf{Z}$,
(ii) *there exists* $h \in G$ *such that* $\Lambda_h = \Lambda_g$ *and* $\langle \cdot, h\rangle I \in \mathbf{Z}$,
(iii) *for* $\tilde{\tau}$*-a.a.* $\gamma \in T$ *the set* $\{L_\lambda(\gamma)\colon \lambda \in R\}$ *is direct.*

Proof. (i)⇒(ii) is clear.

(ii)⇒(iii). Let $h \in G$ be such that $\Lambda_h = \Lambda_g$ and $\langle\cdot,h\rangle I \in \mathbf{Z}$. By Lemma 2.3 there exists $\tilde{\Phi} \in L^2(\tilde{M})$ with $(V\tilde{\Phi})(\cdot) = \langle \cdot, h\rangle I$, which yields

$$\tilde{\Phi}(\gamma)\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda+\kappa) = \langle \gamma+\lambda, h\rangle \frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda+\kappa)$$

for $\tilde{\tau}$-a.a. $\gamma \in T$ and all $\lambda \in R$, $\kappa \in \Lambda_g$. Thus, if $u \in \mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda+\kappa)\big)$, then $\tilde{\Phi}(\gamma)u = \langle\gamma,h\rangle\langle\lambda,h\rangle u$.

We prove by induction that for $\tilde{\tau}$-a.a. $\gamma \in T$, all finite subsets of $\{L_\lambda(\gamma) : \lambda \in R\}$ are direct. Clearly, all singletons are direct. Assume that all subsets of $\{L_\lambda(\gamma)\colon \lambda \in R\}$ with exactly n elements are direct. Let $\lambda_j \in R$, $u_j \in L_{\lambda_j}(\gamma)$, $j \in \{1,\dots,n+1\}$, and $\sum_{j=1}^{n+1} u_j = 0$. It follows $\sum_{j=1}^{n+1} \tilde{\Phi}(\gamma)u_j = \langle\gamma,h\rangle \sum_{j=1}^{n+1}\langle\lambda_j,h\rangle u_j = 0$ as well as $\langle\gamma,h\rangle\langle\lambda_1,h\rangle \sum_{j=1}^{n+1} u_j = 0$, hence $\sum_{j=2}^{n+1}(\langle\lambda_j,h\rangle - \langle\lambda_1,h\rangle)u_j = 0$ and we obtain $(\langle\lambda_j,h\rangle - \langle\lambda_1,h\rangle)u_j = 0$ for $j \in \{2,\dots,n+1\}$ from the induction assumption. Since $\Lambda_h = \Lambda_g$ and λ_j and λ_1 are from different Λ_g-cosets if $j \neq 1$, we conclude that $u_j = 0$ for $j \in \{1,\dots,n\}$. Now apply Lemma 2.4.

(iii)$\Rightarrow$(i). Let $h \in G_g$. For $\gamma \in T$, define a linear operator $\tilde{\Phi}(\gamma)$ on $\mathbb{C}^q$ such that its restriction to $L_\lambda(\gamma)$ coincides with the restriction of the operator of multiplication by $\langle\lambda,h\rangle$. If $\lambda' \in \Lambda$, $\lambda' = \lambda + \kappa$, $\lambda \in R$, $\kappa \in \Lambda_g$, we obtain $(V\tilde{\Phi})(\gamma+\lambda') = \tilde{\Phi}(\gamma) = \langle\lambda,h\rangle I$ on $L_\lambda(\gamma)$, hence $(V\tilde{\Phi})(\cdot) = \langle\cdot,h\rangle I \in \mathbf{Z}$ by Lemma 2.3. ■

From Theorems 2.2 and 3.2 one can immediately derive an extension of [11, Theorem 1] (cf. [14, Theorem 4.1]).

THEOREM 3.3. *Let Λ be countable. If there exists $g \in G$ with $\Lambda_g = \{0\}$, the following conditions are equivalent:*

(i) $\mathbf{Z} = L^2(M)$,
(ii) $\langle\cdot,g\rangle I \in \mathbf{Z}$ *for some $g \in G$ such that $\Lambda_g = \{0\}$,*
(iii) *the set $\big\{\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)\big)\colon \lambda \in \Lambda\big\}$ is direct for $\tilde{\tau}$-a.a. $\gamma \in T$.*

In this theorem the condition that $\Lambda_g = \{0\}$ for some $g \in G$ cannot be omitted.

EXAMPLE 3.1. If $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times G_1$, where $\mathbb{Z}_2$ is the group of order 2 and G_1 is an arbitrary LCA group, $H := \{0\} \times \{0\} \times G_1$, and hence $\Lambda = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$, it is easy to verify that $\Lambda_g \neq \{0\}$ for all $g \in G$.

If $\mathbf{Z} \neq L^2(M)$, it is of interest to compute the orthogonal projection of an arbitrary $\Phi \in L^2(M)$ onto $\mathbf{Z}$. Note first that the series $\sum_{\lambda\in\Lambda} \Phi(\gamma+\lambda)\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)$ converges for $\tilde{\tau}$-a.a. $\gamma \in T$ and define a function $\tilde{\Phi}\colon T \to \mathcal{M}$ such that

$$\tilde{\Phi}(\gamma) = \sum_{\lambda\in\Lambda} \Phi(\gamma+\lambda)\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)\bigg(\sum_{\lambda\in\Lambda}\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)\bigg)^+$$

for $\tilde{\tau}$-a.a. $\gamma \in T$. We omit the proof of the following theorem since it is quite similar to the proofs of [6, Lemma 4.1 and Theorem 4.2].

THEOREM 3.4. *Let $\Phi \in L^2(M)$. Then:*

(i) $\tilde{\Phi} \in L^2(\tilde{M})$,
(ii) *the orthogonal projection of Φ onto $\mathbf{Z}$ equals $V\tilde{\Phi}$.*

The proof of Theorem 3.4 is based on simple Hilbert space geometry and some straightforward computations and does not make use of Theorem 2.2. Therefore, one could start with Theorem 3.4 and try to derive the results of Section 2 from it. For special classes of measures M such an approach was taken by several authors (see [11], [14], [8]). However, it does not seem to be easy to infer Theorem 2.2 from Theorem 3.4.

4. A PROBLEM OF MULTICHANNEL SAMPLING

Now we discuss a problem arising in multichannel sampling. Let $\mathbf{X}$ be a q-variate stationary process on G and M its spectral measure. Let $m \in \mathbb{N}$ and $\mathbf{Y}_j$, $j \in \{1, \dots, m\}$, be multivariate processes on G depending on $\mathbf{X}$ linearly. The problem is to find conditions under which observations of all processes $\mathbf{Y}_j$ at a subgroup H of G give full information on $\mathbf{X}$. In a slightly more general form this problem can be formulated as follows. Let $p_j \in \mathbb{N}$, $j \in \{1, \dots, m\}$, $p := \sum_{j=1}^{m} p_j$, $F_j \in L^2_{p_j,q}(M)$, $\mathbf{F} := (F_1^*, \dots, F_m^*)^*$. Denote by $\mathbf{Z}(M;\mathbf{F})$ the closure of the $\mathcal{M}_p$-linear span of all functions of the form $\langle \cdot, h_j \rangle X_j F_j$, $h_j \in H$, $X_j \in \mathcal{M}_{p,p_j}$, $j \in \{1, \dots, m\}$, in $L^2_{p,q}(M)$. Give necessary or sufficient conditions for the equality $\mathbf{Z}(M;\mathbf{F}) = L^2_{p,q}(M)$ to hold.

Since $\mathbf{F}$ is defined only up to M-equivalence, we can and will assume that $\mathcal{N}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}\big) \subseteq \mathcal{N}(\mathbf{F})$ σ-a.e. If there exists a set $B \in \mathcal{B}(\Gamma)$ satisfying $\sigma(B) > 0$ and $L(\gamma) := \mathcal{N}(\mathbf{F}(\gamma)) \cap \mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma)\big) \neq \{0\}$ for all $\gamma \in B$, then $\mathbf{Z}(M;\mathbf{F}) \neq L^2_{p,q}(M)$, because if $\Phi \in L^2_{p,q}(M)$ is such that the restriction of the operator $\Phi(\gamma)$ to $L(\gamma)$ is different from 0 for all $\gamma \in B$, then $\Phi \in L^2_{p,q}(M) \setminus \mathbf{Z}(M;\mathbf{F})$. Let us assume that

$$\mathcal{N}(\mathbf{F}) = \mathcal{N}\left(\frac{\mathrm{d}M}{\mathrm{d}\sigma}\right) \quad \sigma\text{-a.e.} \tag{4.1}$$

Let N be an $\mathcal{M}_p^{\geqslant}$-valued measure defined by $\mathrm{d}N = \mathbf{F}\mathrm{d}M\mathbf{F}^*$.

LEMMA 4.1. *Under condition* (4.1) *the map* $\Psi \mapsto \Psi\mathbf{F}$, $\Psi \in L^2_{p,q}(N)$, *establishes an isometric isomorphism between* $L^2_{p,q}(N)$ *and* $L^2_{p,q}(M)$.

Proof. The only thing requiring proof is the surjectivity of the map. If $\Phi \in L^2_{p,q}(M)$, set $\Psi := \Phi(\mathbf{F}^*\mathbf{F})^+\mathbf{F}^*$. Since $\Psi\mathbf{F} = \Phi P_{\mathcal{R}(\mathbf{F}^*)}$ and (4.1) yield $\mathcal{R}(\mathbf{F}^*) = \mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}\big)$ σ-a.e., it follows that $\Psi\mathbf{F} = \Phi$ in $L^2_{p,q}(M)$. Moreover, $\int_\Gamma \Psi \frac{\mathrm{d}N}{\mathrm{d}\sigma} \Psi^* \,\mathrm{d}\sigma = \int_\Gamma \Phi \frac{\mathrm{d}M}{\mathrm{d}\sigma} \Phi^* \,\mathrm{d}\sigma$, which shows that $\Psi \in L^2_{p,q}(N)$. ∎

THEOREM 4.1. *Assume that* $\mathbf{F}$ *satisfies* (4.1). *Then* $\mathbf{Z}(M,\mathbf{F}) = L^2_{p,q}(M)$ *if and only if there exists* $B \in \mathcal{B}(\Gamma)$ *such that* $\sigma(\Gamma \setminus B) = 0$ *and the set* $\{\mathcal{R}(\mathbf{F}(\gamma+\lambda)\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)\mathbf{F}(\gamma+\lambda)^*) \colon \lambda \in \Lambda\}$ *is direct for all* $\gamma \in B$.

Proof. Apply Lemma 4.1 and Theorem 2.2. ∎

If $M = \mu$ is a scalar measure, then $\mathcal{R}(\mathbf{F}(\gamma))$ is a one-dimensional subspace of $\mathbb{C}^p$ spanned by the vector $\mathbf{F}(\gamma)$ and the result can be given a more lucid form.

COROLLARY 4.1. *Assume that* $q = 1$ *and* (4.1) *is satisfied. Then* $\mathbf{Z}(\mu;\mathbf{F}) = L^2_{p,1}(\mu)$ *if and only if for* σ*-a.a.* $\gamma \in \Gamma$ *there exists a subset* Λ' *of* Λ *such that* $\mathbf{F}(\gamma+\lambda) = 0$ *for* $\lambda \in \Lambda \setminus \Lambda'$, *and the vectors* $\mathbf{F}(\gamma+\lambda)$, $\lambda \in \Lambda'$, *are linearly independent.*

In the case $p = q = 1$ another description of measures μ with $\mathbf{Z}(\mu; \mathbf{F}) = L^2_{1,1}(\mu)$ was obtained in [8, Theorem 3.5]. To illustrate the usefulness of Corollary 4.1 we mention that the assertions of Examples 4.4 and 4.6 as well as of Proposition 4.6 of [8] are its straightforward consequences, whereas some computations were needed to infer them from [8, Theorem 3.5].

5. $\mathcal{J}_H$-REGULARITY

In accordance with the notion of $\mathcal{J}_H$-regularity of a stationary process mentioned in the introduction we give the following definition.

DEFINITION 5.1. A regular $\mathcal{M}_p^{\geqslant}$-valued measure M on $\mathcal{B}(\Gamma)$ is called *$\mathcal{J}_H$-regular* if $\bigcap_{g\in G} \mathbf{Z}_{\tilde{g}} = L^2(M)$.

For this definition to be correct it is necessary that the notion of $\mathcal{J}_H$-regularity does not depend on the dimension p; this independence can be seen from the proof of Theorem 5.1.

To characterize the set of all $\mathcal{J}_H$-regular measures we first prove an auxiliary result from abstract harmonic analysis. For brevity we shall say that an LCA group G has *property* (P) if for all $n \in \mathbb{N}$ and pairwise different $g_j \in G$ there exist $\gamma_j \in \Gamma$, $j \in \{1, \dots, n\}$, such that the $n \times n$ matrix $(\langle \gamma_k, g_j \rangle)$ is invertible.

LEMMA 5.1. *For any n rows of an invertible matrix $X \in \mathcal{M}_m$, there exist n columns of X such that the corresponding $n \times n$ submatrix X is invertible.*

Proof. If such columns did not exist, then the associated $\binom{m}{n} \times \binom{m}{n}$ matrix, whose elements are the $n \times n$ minors of X (in lexicographical order) would have a zero row, which would contradict the invertibility of the associated matrix. ∎

LEMMA 5.2. *Any countable LCA group G has property* (P).

Proof. Let $[g_1, \dots, g_n]$ be the subgroup of G generated by n pairwise different elements $g_j \in G$. Note that any countable LCA group is discrete, and therefore it is enough to show that $[g_1, \dots, g_n]$ has property (P). By the fundamental structure theorem for finitely generated abelian groups, $[g_1, \dots, g_n]$ is a finite direct product of cyclic groups.

Let us first show that any cyclic group has property (P). Let $\mathbb{Z}_m := \{0, 1, \dots, m-1\}$ be the finite cyclic group of order m, whose group operation is addition modulo m. Its characters can be identified with all maps of the form $j \mapsto \exp\{2\pi \mathrm{i} jk/m\}$, $j \in \mathbb{Z}_m$. Since the $m \times m$ matrix $(\exp\{2\pi \mathrm{i} jk/m\})$ is an invertible Vandermonde matrix, it follows from Lemma 5.1 that $\mathbb{Z}_m$ has property (P). If $\mathbb{Z}$ is the infinite cyclic group of integers, its characters can be identified with all maps of the form $j \mapsto \mathrm{e}^{\mathrm{i} j\alpha}$, $\alpha \in [0, 2\pi)$. Shifting a set $\{j_1, \dots, j_n\} \subseteq \mathbb{Z}$ by $-\min\{j_1, \dots, j_n\}$, we can assume that all elements j_k are non-negative. Set $\ell := \max\{j_1, \dots, j_n\}$, choose numbers $\alpha_k \in [0, 2\pi)$, $k \in \{0, \dots, \ell\}$, such that the

Vandermonde matrix $(e^{ij\alpha_k})_{j,k=0}^{\ell}$ is invertible and apply Lemma 5.1 to conclude that $\mathbb{Z}$ has property (P).

To complete the proof let us show that if two LCA groups G_1 and G_2 have property (P), then so does $G_1 \times G_2$. Let $\{(g'_{11}, g'_{21}), \dots, (g'_{1n}, g'_{2n})\}$ be a subset of $G_1 \times G_2$. Let $g_{11}, \dots, g_{1s}$ be those elements of G_1 which appear at least once as the first component of these pairs. Define $g_{21}, \dots, g_{2t}$ analogously for the second component. By assumption there exist $\gamma_{11}, \dots, \gamma_{1s} \in \Gamma_1$ and $\gamma_{21}, \dots, \gamma_{2t} \in \Gamma_2$ such that the $s \times s$ matrix $(\langle \gamma_{1k}, g_{1j} \rangle)$ and the $t \times t$ matrix $(\langle \gamma_{2m}, g_{2\ell} \rangle)$ are invertible. Since their tensor product $(\langle \gamma_{1k}, g_{1j} \rangle \langle \gamma_{2m}, g_{2\ell} \rangle)$ is invertible, the fact that $G_1 \times G_2$ has property (P) again follows from Lemma 5.1. ∎

Now we give a description of $\mathcal{J}_H$-regular $\mathcal{M}_p^{\geqslant}$-valued measures (cf. [7, Theorem 2.2] for the case $q = 1$).

THEOREM 5.1. *If Λ is countable, the following assertions are equivalent:*

(i) *the measure M is $\mathcal{J}_H$-regular,*

(ii) *for all families $\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}$, $\lambda \in \Lambda$, there exists $\tilde{B} \in \mathcal{B}(\Gamma)$ such that*

$$\tilde{\tau}(T \setminus \tilde{B}) = 0 \quad \textit{and} \quad \mathcal{R}\bigg(\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma)\bigg) \subseteq \sum_{\kappa \neq \lambda} \mathcal{R}\bigg(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}(\gamma)\bigg) \tag{5.1}$$

for all $\gamma \in \tilde{B}$ and $\lambda \in \Lambda$,

(iii) *there exist a family $\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}$, $\lambda \in \Lambda$, and $\tilde{B} \in \mathcal{B}(\Gamma)$ satisfying* (5.1).

Proof. (i)⇒(ii). Assume that (ii) is not true. Let a family of Radon–Nikodym derivatives $\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}$, $\lambda \in \Lambda$, and $\tilde{C} \in \mathcal{B}(T)$ be such that $\tilde{\tau}(\tilde{C}) > 0$ and the inclusion $\mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma)\big) \subseteq \sum_{\kappa \neq \lambda} \mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}(\gamma)\big)$ is not true for some $\lambda \in \Lambda$ and all $\gamma \in \tilde{C}$. Choose a bounded measurable function $\tilde{\Phi}\colon T \to \mathcal{M}_{p,q}$ satisfying $\mathcal{N}(\tilde{\Phi}) = \sum_{\kappa \neq \lambda} \mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}\big)$ on $\tilde{C}$ and $\tilde{\Phi} = 0$ on $T \setminus \tilde{C}$. For $g \in G$, the function $\langle -\lambda, \tilde{g} \rangle \tilde{\Phi}$ is not the zero element of $L^2(\tilde{M})$ and its image under the isometry $V_{\tilde{g}}$ equals $\tilde{\Phi}(\gamma - \lambda)$ for $\gamma \in \lambda + T$ and $\langle \kappa - \lambda, \tilde{g} \rangle \tilde{\Phi}(\gamma - \kappa)$ for $\gamma \in \kappa + T$, $\kappa \in \Lambda \setminus \{\lambda\}$. Therefore, as an element of $L^2(M)$, the function $V_{\tilde{g}}(\langle -\lambda, \tilde{g} \rangle \tilde{\Phi})$ equals $\tilde{\Phi}(\gamma - \lambda)$ for $\gamma \in \lambda + T$, and 0 outside $\lambda + T$. In particular, it does not depend on g, which means by Lemma 2.3 that it belongs to $\bigcap_{g \in G} \mathbf{Z}_{\tilde{g}}$. Thus M is not $\mathcal{J}_H$-regular.

(ii)⇒(iii) is trivial.

(iii)⇒(i). Let (iii) be satisfied. We shall show that $\Phi \in \bigcap_{g \in G} \mathbf{Z}_{\tilde{g}}$ yields $\Phi = 0$ in $L^2(M)$. By Lemma 2.3 for any H-coset $\tilde{g}$, the function Φ is the image of some $\tilde{\Phi}_{\tilde{g}} \in L^2(\tilde{M})$ under the isometry $V_{\tilde{g}}$. Since Λ is countable, there exists $\tilde{C}_{\tilde{g}} \in \mathcal{B}(T)$ such that $\tilde{\tau}(\tilde{C}_{\tilde{g}}) = 0$ and

$$\langle \lambda, \tilde{g} \rangle \tilde{\Phi}_{\tilde{g}}(\gamma) \frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma) = \Phi(\gamma + \lambda) \frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma), \qquad \gamma \in T \setminus \tilde{C}_{\tilde{g}},\ \lambda \in \Lambda. \tag{5.2}$$

For $\lambda_0 \in \Lambda$ and a subset $S := \{\lambda_1, \ldots, \lambda_q\}$ of $\Lambda \setminus \{\lambda_0\}$ set

$$\tilde{B}_S := \left\{ \gamma \in T \colon \mathcal{R}\left(\frac{\mathrm{d}\tilde{M}_{\lambda_0}}{\mathrm{d}\tilde{\tau}}(\gamma)\right) \subseteq \sum_{j=1}^{q} \mathcal{R}\left(\frac{\mathrm{d}\tilde{M}_{\lambda_j}}{\mathrm{d}\tilde{\tau}}(\gamma)\right) \right\}.$$

According to Lemma 5.1 we can choose $\tilde{g}_1, \ldots, \tilde{g}_q \in G/H$ such that the matrix $(\langle -\lambda_k, \tilde{g}_j \rangle)_{j,k=0}^{q}$ is invertible. Define $\tilde{C}_S := \bigcup_{j=0}^{q} \tilde{C}_{\tilde{g}_j}$ and $\tilde{D} := \bigcup_S (\tilde{B}_S \setminus \tilde{C}_S)$, where S runs through all subsets of $\Lambda \setminus \{\lambda_0\}$ with exactly q elements. Note that $\tilde{B} \subseteq \bigcup_S \tilde{B}_S$ and $\tilde{\tau}(T \setminus \tilde{D}) = 0$. Let $\gamma \in \tilde{D}$. Choose a subset $S = \{\lambda_1, \ldots, \lambda_q\}$ of $\Lambda \setminus \{\lambda_0\}$ such that $\gamma \in \tilde{B}_S \setminus \tilde{C}_S$. If $u_0 \in \mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_{\lambda_0}}{\mathrm{d}\tilde{\tau}}(\gamma)\big)$, there exist $u_k \in \mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_{\lambda_k}}{\mathrm{d}\tilde{\tau}}(\gamma)\big)$ satisfying $\sum_{k=0}^{q} u_k = 0$. Setting $\tilde{g} = \tilde{g}_j$ in (5.2), we obtain the homogeneous linear system

$$0 = \tilde{\Phi}_j\Big(\sum_{k=0}^{q} u_k\Big) = \sum_{k=0}^{q} \langle -\lambda_k, \tilde{g}_j \rangle \Phi(\gamma + \lambda_k) u_k, \quad j \in \{0, \ldots, q\}.$$

Since its coefficient matrix is invertible, it follows that $\Phi(\gamma + \lambda_0)u_0 = 0$, which yields $\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma + \lambda_0)\big) \subseteq \mathcal{N}(\Phi(\gamma + \lambda_0))$. Since $\lambda_0 \in \Lambda$ was arbitrary, we obtain $\Phi = 0$ in $L^2(M)$. ∎

From Theorems 5.1 and 2.2 it follows that an $\mathcal{M}_3^{\geqslant}$-valued measure on $\mathcal{B}(\mathbb{R})$ defined by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \delta_0 + \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \delta_{2\pi} + \begin{pmatrix} 9 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix} \delta_{4\pi},$$

where δ_x denotes the Dirac measure at $x \in \mathbb{R}$, is $\mathcal{J}_{\mathbb{Z}}$-singular and all measures on its principal diagonal are $\mathcal{J}_{\mathbb{Z}}$-regular measures. Therefore, in general $\mathcal{J}_H$-singularity of M does not tell us anything about the $\mathcal{J}_H$-regularity or $\mathcal{J}_H$-singularity of the measures on its principal diagonal. The following corollary of Theorem 5.1 shows that $\mathcal{J}_H$-regularity of M yields $\mathcal{J}_H$-regularity of the diagonal measures.

COROLLARY 5.1. *Let $F \colon \Gamma \to \mathcal{M}_{p,q}$ be a measurable function which is constant on $\gamma + \Lambda$ for $\tilde{\tau}$-a.a. $\gamma \in T$. If M is $\mathcal{J}_H$-regular, then $F \mathrm{d}M F^*$ is $\mathcal{J}_H$-regular as well. In particular, all measures on the principal diagonal of M are $\mathcal{J}_H$-regular.*

Proof. Let $\mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma)\big) \subseteq \sum_{\kappa \in \Lambda \setminus \{\lambda\}} \mathcal{R}\big(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}(\gamma)\big)$, or equivalently, for $\tilde{\tau}$-a.a. $\gamma \in T$,

$$\bigcap_{\kappa \in \Lambda \setminus \{\lambda\}} \mathcal{N}\left(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}(\gamma)\right) \subseteq \mathcal{N}\left(\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma)\right), \quad \lambda \in \Lambda. \tag{5.3}$$

Since $u \in \mathcal{N}(F(\gamma)\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}(\gamma)F(\gamma)^*)$ if and only if $F(\gamma)^*u \in \mathcal{N}(\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}(\gamma))$, from (5.3) it follows that

$$\bigcap_{\kappa\in\Lambda\setminus\{\lambda\}} \mathcal{N}\left(F(\gamma)\frac{\mathrm{d}\tilde{M}_\kappa}{\mathrm{d}\tilde{\tau}}(\gamma)F(\gamma)^*\right) \subseteq \mathcal{N}\left(F(\gamma)\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}(\gamma)F(\gamma)^*\right)$$

for $\tilde{\tau}$-a.a. $\gamma \in T$ and all $\lambda \in \Lambda$. ■

Theorem 2.2 contains several assertions equivalent to $\mathcal{J}_H$-singularity, where perhaps condition (viii) formulated in terms of M and τ is the most natural one. Now we give conditions equivalent to $\mathcal{J}_H$-regularity; it turns out that a condition in terms of M and τ does not exist in general. We start with the following lemma.

LEMMA 5.3. *Let μ and ν be regular σ-finite non-negative measures on $\mathcal{B}(\Gamma)$ such that $M \ll \mu \ll \nu$. If there exist $\frac{\mathrm{d}M}{\mathrm{d}\nu}$ and $B \in \mathcal{B}(\Gamma)$ satisfying*

$$\nu(\Gamma\setminus B) = 0 \quad \textit{and} \quad \mathcal{R}\left(\frac{\mathrm{d}M}{\mathrm{d}\nu}(\gamma+\lambda)\right) \subseteq \sum_{\kappa\neq\lambda} \mathcal{R}\left(\frac{\mathrm{d}M}{\mathrm{d}\nu}(\gamma+\kappa)\right) \tag{5.4}$$

for all $\gamma \in B$ and $\lambda \in \Lambda$, then there exists $\frac{\mathrm{d}M}{\mathrm{d}\mu}$ such that

$$\mathcal{R}\left(\frac{\mathrm{d}M}{\mathrm{d}\mu}(\gamma+\lambda)\right) \subseteq \sum_{\kappa\neq\lambda} \mathcal{R}\left(\frac{\mathrm{d}M}{\mathrm{d}\mu}(\gamma+\kappa)\right) \quad \textit{for all } \gamma\in\Gamma \textit{ and } \lambda\in\Lambda. \tag{5.5}$$

Proof. Let $\frac{\mathrm{d}M}{\mathrm{d}\nu}$ and $B \in \mathcal{B}(\Gamma)$ satisfy (5.2). Choose $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$, set $C := B \cap \{\gamma \in \Gamma : \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(\gamma) > 0\}$, and define $\frac{\mathrm{d}M}{\mathrm{d}\mu} := \frac{\mathrm{d}M}{\mathrm{d}\nu}(\frac{\mathrm{d}\mu}{\mathrm{d}\nu})^{-1}$ on C, $\gamma \in \Gamma\setminus C$, set $C(\gamma) := \{\lambda \in \Lambda : \gamma+\lambda \in C\}$, $L(\gamma) := \sum_{\lambda\in C(\gamma)} \mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\nu}(\gamma+\lambda)\big)$, and $\frac{\mathrm{d}M}{\mathrm{d}\mu}(\gamma) := P_{L(\gamma)}$. Let $\lambda \in \Lambda$. If $\gamma \in \Gamma$ is such that $\gamma+\lambda' \in C$, then $\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\mu}(\gamma+\lambda')\big) = \mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\nu}(\gamma+\lambda')\big)$ for $\lambda' \in \Lambda$. It is easy to show that if $\gamma+\lambda \notin C$ or $\gamma+\kappa \notin C$ for some $\kappa \in \Lambda\setminus\{\lambda\}$, then $\mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\mu}(\gamma+\lambda)\big) \subseteq L(\gamma) \subseteq \sum_{\kappa\neq\lambda} \mathcal{R}\big(\frac{\mathrm{d}M}{\mathrm{d}\mu}(\gamma+\kappa)\big)$. Thus, $\frac{\mathrm{d}M}{\mathrm{d}\mu}$ satisfies (5.5). ■

THEOREM 5.2. *Each of the following assertions is equivalent to Theorem* 5.1(ii)*:*

(a) *for all $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ there exists $B \in \mathcal{B}(\Gamma)$ such that*

$$\sigma(\Gamma\setminus B) = 0 \qquad \textit{and} \qquad \mathcal{R}\left(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)\right) \subseteq \sum_{\kappa\neq\lambda} \mathcal{R}\left(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\kappa)\right) \tag{5.6}$$

for all $\gamma \in B$ and $\lambda \in \Lambda$,

(b) *there exist $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ and $B \in \mathcal{B}(\Gamma)$ satisfying* (5.6)*,*

(c) *there exists $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ such that $\mathcal{R}(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)) \subseteq \sum_{\kappa\neq\lambda} \mathcal{R}(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\kappa))$ for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$.*

If one of these conditions is satisfied and μ is a regular σ-finite non-negative measure such that $M \ll \mu \ll \sigma$, then there exists $\frac{\mathrm{d}M}{\mathrm{d}\mu}$ satisfying (5.5). *If ν is a regular σ-finite non-negative measure such that $\sigma \ll \nu$ and* (5.4) *is true for some $B \in \mathcal{B}(\Gamma)$, then* (a)–(c) *are satisfied.*

Proof. (Theorem 5.1(ii) $\Rightarrow$ (a)). For $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ define a family $\frac{\mathrm{d}\tilde{M}_\lambda}{\mathrm{d}\tilde{\tau}}$, $\lambda \in \Lambda$, by (2.2). Choose $\tilde{B} \in \mathcal{B}(\Gamma)$ satisfying (5.1) and define $B := \bigcup(\lambda + \tilde{B})$. Since σ is a periodic continuation of $\tilde{\tau}$, we have $\sigma(\Gamma \setminus B) = 0$. For $\gamma \in B$, there exists $\lambda_0 \in \Lambda$ with $\gamma - \lambda_0 \in \tilde{B}$. By assumption it follows that

$$\mathcal{R}\left(\frac{\mathrm{d}\tilde{M}_{\lambda+\lambda_0}}{\mathrm{d}\tilde{\tau}}(\gamma - \lambda_0)\right) \subseteq \sum_{\kappa \neq \lambda} \mathcal{R}\left(\frac{\mathrm{d}\tilde{M}_{\kappa+\lambda_0}}{\mathrm{d}\tilde{\tau}}(\gamma - \lambda_0)\right),$$

hence, (5.6).

((a) $\Rightarrow$ Theorem 5.1(ii)) is an immediate consequence of (2.2), (a)$\Rightarrow$(b) is trivial, and (b)$\Rightarrow$(c) follows from Lemma 5.3.

(b)$\Rightarrow$(c). If $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ satisfies (c) and Δ is an arbitrary Radon–Nikodym derivative of M with respect to σ, set $C := \left\{\gamma \in \Gamma\colon \frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma) = \Delta(\gamma)\right\}$, $\tilde{C} := \bigcup((C \cap (\lambda + T)) - \lambda)$, $D := \bigcup(\tilde{C} + \lambda)$, $B := \Gamma \setminus D$. Since $\sigma(\Gamma \setminus B) = 0$ by periodicity of σ and $B = \lambda + B$, $\lambda \in \Lambda$, we get (5.6) with $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ replaced by Δ.

The concluding assertions are simple consequences of Lemma 5.3. ■

Theorem 5.2 implies that the existence of $\frac{\mathrm{d}M}{\mathrm{d}\tau}$ and $B \in \mathcal{B}(\Gamma)$ such that

$$\tau(\Gamma \setminus B) = 0 \quad \text{and} \quad \mathcal{R}\left(\frac{\mathrm{d}M}{\mathrm{d}\tau}(\gamma + \lambda)\right) \subseteq \sum_{\kappa \neq \lambda} \mathcal{R}\left(\frac{\mathrm{d}M}{\mathrm{d}\tau}(\gamma + \kappa)\right), \gamma \in B, \lambda \in \Lambda, \tag{5.7}$$

is necessary for $\mathcal{J}_H$-regularity of M. However, this condition is not sufficient.

EXAMPLE 5.2. Let $G = \mathbb{R}$, $H = \mathbb{Z}$, hence $\Gamma = \mathbb{R}$, $\Lambda = 2\pi\mathbb{Z}$. Let $q = 1$ and $M = \tau$ be the restriction of the Lebesgue measure to $\mathcal{B}([0, 2\pi))$. If $\frac{\mathrm{d}M}{\mathrm{d}\tau} = 1$ on $\mathbb{R}$, then condition (5.7) is satisfied, although M is not $\mathcal{J}_H$-regular.

6. WOLD DECOMPOSITION

From [18, Theorem 2.13] and from Kolmogorov's isomorphism theorem it follows that any regular $\mathcal{M}_q^{\geqslant}$-valued measure admits a unique decomposition into a sum of a $\mathcal{J}_H$-regular and a $\mathcal{J}_H$-singular measure. We conclude our paper with a description of these two measures.

Let M be a regular $\mathcal{M}_q^{\geqslant}$-valued measure on $\mathcal{B}(\Gamma)$ and $\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ be its Radon–Nikodym derivative. For $\gamma \in \Gamma$, set

$$L_\lambda := \mathcal{R}\left(\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma + \lambda)\right), \ \lambda \in \Lambda, \quad K_\lambda := \sum_{\kappa \in \Lambda \setminus \{\lambda\}} L_\kappa, \quad L := \bigcap_{\lambda \in \Lambda} K_\lambda,$$

and $P := P_L$. Note that for simplicity of presentation we do not indicate the dependence on γ.

LEMMA 6.1. *If $L = K_\lambda$, then $\bigcap_{\kappa\in\Lambda\setminus\{\lambda\}} PL_\kappa^\perp = \{0\}$.*

Proof. Let $(\cdot,\cdot)$ denote the inner product of $\mathbb{C}^q$. If $u \in \bigcap_{\kappa\in\Lambda\setminus\{\lambda\}} PL_\kappa^\perp$, then $u = Pu_\kappa$ for some $u_\kappa \in L_\kappa^\perp$, hence, $(u, v_\kappa) = (Pu_\kappa, v_\kappa) = (u_\kappa, v_\kappa) = 0$ for all $v_\kappa \in L_\kappa$, since $L_\kappa \subseteq K_\lambda = L$ if $\kappa \neq \lambda$. It follows that u is orthogonal to K_λ. Since $u \in L \subseteq K_\lambda$, we obtain $u = 0$. ∎

LEMMA 6.2. *For $\lambda \in \Lambda$, the intersection $\bigcap_{\kappa\in\Lambda\setminus\{\lambda\}} PL_\kappa^\perp$ is $\{0\}$.*

Proof. Set $L'_\lambda := L_\lambda \cap L = L_\lambda \cap \sum_{\kappa\in\Lambda\setminus\{\lambda\}} L_\kappa$ and $L' := \bigcap_{\lambda\in\Lambda} \sum_{\kappa\in\Lambda\setminus\{\lambda\}} L'_\kappa$. Clearly, $L' \subseteq L$. Conversely, if $u \in L$, then for all $\lambda \in \Lambda$,

$$u = \sum_{\kappa\in\Lambda\setminus\{\lambda\}} u_{\lambda\kappa}, \tag{6.1}$$

where $u_{\lambda\kappa} \in L_\kappa$ and, of course, only finitely many vectors on the right-hand side are different from 0. Let $\lambda_1, \lambda_2 \in \Lambda$ be distinct. Then (6.1) implies that $u_{\lambda_2\lambda_1} = u_{\lambda_1\lambda_2} + \sum_{\kappa\in\Lambda\setminus\{\lambda_1,\lambda_2\}}(u_{\lambda_1\kappa} - u_{\lambda_2\kappa}) \in L_\kappa$, hence $u_{\lambda_2\lambda_1} \in L'_{\lambda_1}$. Since λ_1 and λ_2, $\lambda_1 \neq \lambda_2$, were arbitrary, we find that $u \in L'$, which yields $L = L'$. It follows that $L = \sum_{\kappa\in\Lambda\setminus\{\lambda\}} L'_\kappa$ by definition of L'_κ, and therefore $\bigcap_{\kappa\in\Lambda\setminus\{\lambda\}} PL_\kappa^\perp \subseteq \bigcap_{\kappa\in\Lambda\setminus\{\lambda\}} P{L'_\kappa}^\perp = \{0\}$ by Lemma 6.1. ∎

THEOREM 6.1. *Let $M = M_\mathrm{r} + M_\mathrm{s}$ be the Wold decomposition of M into its $\mathcal{J}_H$-regular part M_r and $\mathcal{J}_H$-singular part M_s. Let $\left(\begin{smallmatrix} X_\lambda & Y_\lambda \\ Y_\lambda^* & Z_\lambda \end{smallmatrix}\right)$ be the block representation of $\frac{\mathrm{d}M}{\mathrm{d}\sigma}(\gamma+\lambda)$ with respect to the orthogonal decomposition $\mathbb{C}^q = L \oplus L^\perp$. Then*

$$\mathrm{d}M_\mathrm{r} = \begin{pmatrix} X_\lambda - Y_\lambda Z_\lambda^+ Y_\lambda^* & 0 \\ 0 & 0 \end{pmatrix} \mathrm{d}\sigma \tag{6.2}$$

and $\mathrm{d}M_\mathrm{s} = \left(\begin{smallmatrix} Y_\lambda Z_\lambda^+ Y_\lambda^* & Y_\lambda \\ Y_\lambda^* & Z_\lambda \end{smallmatrix}\right)\mathrm{d}\sigma$.

Proof. Theorem 4.4 of [5] implies that the theorem is equivalent to the following two assertions:

(a) The measure M_r defined by (6.2) is $\mathcal{J}_H$-regular.
(b) If N is a $\mathcal{J}_H$-regular measure such that $N \leqslant M$, then $N \leqslant M_\mathrm{r}$, where "$\leqslant$" denotes the Loewner semi-ordering.

Proof of (a). By Theorem 5.1 we have to establish the inclusion $\mathcal{R}\big(\frac{\mathrm{d}M_{\mathrm{r}}}{\mathrm{d}\sigma}(\gamma+\lambda)\big) \subseteq \sum_{\kappa\in\Lambda\setminus\{\lambda\}} \mathcal{R}\big(\frac{\mathrm{d}M_{\mathrm{r}}}{\mathrm{d}\sigma}(\gamma+\kappa)\big)$, which will be proved if we show that

$$\bigcap_{\kappa\in\Lambda\setminus\{\lambda\}} \mathcal{N}\bigg(\frac{\mathrm{d}M_{\mathrm{r}}}{\mathrm{d}\sigma}(\gamma+\kappa)\bigg) = \{0\}\oplus L^{\perp}, \qquad \lambda\in\Lambda. \tag{6.3}$$

Note first that if $u\in L$ is such that $\binom{u}{0}\in\mathcal{N}\big(\frac{\mathrm{d}M_{\mathrm{r}}}{\mathrm{d}\sigma}(\gamma+\kappa)\big)$, then $\binom{u}{-Z_\kappa^{+}Y_\kappa^{*}u}\in\mathcal{N}\big(\frac{\mathrm{d}M_{\mathrm{r}}}{\mathrm{d}\sigma}(\gamma+\kappa)\big)$. It follows that $\mathcal{N}\big(\frac{\mathrm{d}M_{\mathrm{r}}}{\mathrm{d}\sigma}(\gamma+\kappa)\big)=PL_\kappa^{\perp}\oplus L^{\perp}$ and an application of Lemma 6.2 gives (6.3).

Proof of (b). Let $u\in\mathcal{R}\big(\frac{\mathrm{d}N}{\mathrm{d}\sigma}(\gamma+\lambda_0)\big)\cap L^{\perp}$ for some $\lambda_0\in\Lambda$. By Theorem 5.1, $\mathcal{J}_H$-regularity of N yields $u\in\sum_{\kappa\in\Lambda\setminus\{\lambda_0\}}\mathcal{R}\big(\frac{\mathrm{d}N}{\mathrm{d}\sigma}(\gamma+\kappa)\big)$, hence

$$u\in\bigcap_{\lambda\in\Lambda}\sum_{\kappa\in\Lambda\setminus\{\lambda\}}\mathcal{R}\bigg(\frac{\mathrm{d}N}{\mathrm{d}\sigma}(\gamma+\kappa)\bigg)\subseteq\bigcap_{\lambda\in\Lambda}\sum_{\kappa\in\Lambda\setminus\{\lambda\}}L_\lambda=L$$

for σ-a.a. $\gamma\in\Gamma$ since the inequality $N\leqslant M$ implies that $\frac{\mathrm{d}N}{\mathrm{d}\sigma}\leqslant\frac{\mathrm{d}M}{\mathrm{d}\sigma}$ σ-a.e. Therefore $u=0$, which means that $L^{\perp}\subseteq\mathcal{N}\big(\frac{\mathrm{d}N}{\mathrm{d}\sigma}(\gamma+\lambda_0)\big)$ and so $\frac{\mathrm{d}N}{\mathrm{d}\sigma}(\gamma+\lambda_0)\leqslant\frac{\mathrm{d}M_{\mathrm{r}}}{\mathrm{d}\sigma}(\gamma+\lambda_0)$ for σ-a.a. $\gamma\in\Gamma$ by [19, Corollary, p. 392]. ■

REFERENCES

[1] E. A. Azoff, *Borel measurability in linear algebra*, Proc. Amer. Math. Soc. 42 (1974), 346–350.

[2] J. Feldman and F. P. Greenleaf, *Existence of Borel transversals in groups*, Pacific J. Math. 25 (1968), 455–461.

[3] P. Halmos, *A Hilbert Space Problem Book*, Grad. Texts in Math. 19, Springer, 1982.

[4] I. S. Kats, *On Hilbert spaces generated by monotone Hermitian matrix-functions*, Khar'kov Gos. Univ. Uch. Zap. 34 = Zap. Mat. Otd. Fiz.-Mat. Fak. i Khar'kov. Mat. Obshch. (4) 22 (1950), 95–113 (in Russian).

[5] L. Klotz, *The maximal $\mathcal{J}$-regular part of a q-variate weakly stationary process*, Probab. Math. Statist. 22 (2002), 155–165.

[6] L. Klotz, *Some remarks on an interpolation problem of A. M. Yaglom*, Teor. Veroyatn. Primen. 51 (2006), 425–433.

[7] L. Klotz and J. M. Medina, *$\mathcal{J}_H$-regular Borel measures on locally compact abelian groups*, Acta Math. Hungar. 159 (2019), 42–54.

[8] L. Klotz and J. M. Medina, *Density in $L^2(\Gamma\mu)$ of certain families of functions on LCA groups related to the multi-channel sampling problem*, Numer. Funct. Anal. Optim. 41 (2020), 1642–1665.

[9] L. Klotz and M. Riedel, *Periodic observations of harmonizable symmetric stable sequences*, Probab. Math. Statist. 25 (2005), 289–306.

[10] A. N. Kolmogorov, *Stationary sequences in Hilbert's space*, Boll. Moskov. Gos. Univ. Mat. 2 (1941), no. 6, 40 pp.

[11] S. P. Lloyd, *A sampling theorem for stationary (wide sense) stochastic processes*, Trans. Amer. Math. Soc. 92 (1959), 1–12.

[12] J. M. Medina, L. P. Klotz, and M. Riedel, *Density of spaces of trigonometric polynomials with frequencies from a subgroup in L^α-spaces*, C. R. Math. Acad. Sci. Paris 356 (2018), 586–593.
[13] T. K. Pogány and P. M. Peruničić, *On the multidimensional sampling theorem*, Glas. Mat. Ser. III 36(56) (2001), 155–167.
[14] M. Pourahmadi, *A sampling theorem for multivariate stationary processes*, J. Multivariate Anal. 13 (1983), 177–186.
[15] M. Rosenberg, *The square-integrability of matrix-valued functions with respect to a non-negative Hermitian measure*, Duke Math. J. 31 (1964), 291–298.
[16] Yu. A. Rozanov, *Stationary Random Processes*, Gos. Izdat. Fiz.-Mat. Lit., Moscow, 1963 (in Russian).
[17] H. Salehi, *On interpolation of q-variate stationary stochastic processes*, Pacific J. Math. 28 (1969), 183–191.
[18] H. Salehi and J. K. Scheidt, *Interpolation of q-variate weakly stationary stochastic processes over a locally compact abelian group*, J. Multivariate Anal. 2 (1972), 307–331.
[19] Yu. L. Shmul'yan, *An operator Hellinger integral*, Mat. Sb. (N.S.) 49 1959, 381–430 (in Russian).
[20] H. Wold, *A study in the analysis of stationary time series*, Diss. Uppsala, Almqvist & Wiksell, 1938.
[21] A. M. Yaglom, *On problems about the linear interpolation of stationary random sequences and processes*, Uspekhi Mat. Nauk (N.S.) 4 (1949), no. 4, 173–178 (in Russian).
[22] V. N. Zasuhin, *On the theory of multidimensional stationary random processes*, C. R. (Dokl.) Acad. Sci. URSS (N.S.) 33 (1941), 435–437.

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