Stable rank of down-up algebras

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Abstract

We investigate the behavior of finitely generated projective modules over a down-up algebra. Specifically, we show that every noetherian down-up algebra $A(\alpha, \beta, \gamma)$ has a non-free, stably free right ideal. Further, we compute the stable rank of these algebras using Stafford's Stable Range Theorem and Kmax dimension.

Keywords: Down-up algebras, stably free modules, projective modules, stable rank, Krull dimension, Kmax dimension.

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1 Introduction

The study of finitely projective modules over an arbitrary ring is a classical task in homological algebra. Investigating whether these modules are free, or at least stably free, has also great interest in geometry, topology and K-theory. One of the most well known results in this context is the Quillen-Suslin theorem about Serre's problem for the commutative polynomial ring $\Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is a field. In this particular situation, Quillen and Suslin proved independently that the finitely generated projective modules are free, see [11] for a detailed and very clear exposition about this subject. However, for noncommutative rings of polynomial type it is easy to present examples where the Quillen-Suslin Theorem fails. For instance, if T is a division ring and $S := T[x, y]$, there is an S-module M such that $M \bigoplus S \cong S^2$, but M is not free, see [16]. Moreover, Stafford developed conditions in [16, Theorem 1.2] under which the skew polynomial ring $S = R[x; \sigma, \delta]$, with R a noetherian domain, σ an automorphism of R and δ a σ-derivation, has a non-trivial stably free right ideal. These ideas have been used in [1] in order to obtain non-trivial stably free modules over the enveloping algebras of the RIT (relativistic internal time) Lie algebras. Using similar methods, Iyudu and Wisbauer gave a sufficient condition in [8] for the existence of projective non-free modules over the class of crossed products of noetherian domains with universal enveloping algebras of Lie algebras. In the current paper, we will show that there exist non-free projective modules over down-up algebras too. This fact will allow us to obtain bounds of the stable rank of these algebras.

Down-up algebras have been introduced by Benkart and Roby in [4] motivated by the study of posets. Given a field $\Bbbk$ and constants α, β, γ in $\Bbbk$, the *down-up algebra* $A = A(\alpha, \beta, \gamma)$ is the associative algebra generated over $\Bbbk$ by U and D, subject to the defining relations:

$$\begin{aligned} DU^2 &= \alpha UDU + \beta U^2D + \gamma U \\ D^2U &= \alpha DUD + \beta UD^2 + \gamma D. \end{aligned}$$

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As known examples of down-up algebras, we can mention $A(2,-1,0)$ that turns out to be isomorphic to the enveloping algebra of the Heisenberg Lie algebra of dimension 3; for the case where $\gamma \neq 0$, the algebra $A(2,-1,\gamma)$ is isomorphic to the enveloping algebra of $sl_2(\mathbb{k})$. For another interesting example, consider the quantized enveloping algebra $U_q(sl_3(\mathbb{k}))$ with generators E_i, F_i, $K^{\pm 1}$, $i = 1, 2$ and a non-zero scalar q in $\mathbb{k}$; the subalgebra of $U_q(sl_3(\mathbb{k}))$ generated by E_1, E_2 is the down-up algebra $A([2]_q, -1, 0)$, where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. For $\gamma \neq 0$, the down-up algebra $A(0,1,\gamma)$ is isomorphic to the enveloping algebra of the Lie superalgebra osp(1,2).

Kirkman, Musson and Passman proved in [10] that $A(\alpha,\beta,\gamma)$ is a noetherian algebra if and only if, the parameter β is non-zero; the latter is equivalent to saying that $A(\alpha,\beta,\gamma)$ is a domain. Furthermore, for down-up algebras, the Krull, Gelfand-Kirillov, and global dimensions have already been computed, see [3], [4] and [10]. Additionally, their representation theory, Hochschild homology and cohomology, as well as several homological and ring theoretical properties have also been studied (e.g., [5], [6], [7], [9], [20]).

In [4] the task of investigating indecomposable and projective modules for down-up algebras was proposed. We give a partial answer to this subject proving in Theorem 2.1 that all finitely generated projective modules over a noetherian down-up algebra are stably free. Moreover, we show that the class of noetherian down-up algebras does not satisfy a noncommutative version of Quillen-Suslin Theorem in the sense that there exist non-trivial stably free modules over these algebras, see Corollary 3.6 and Proposition 3.8. In view of the above, we obtain a lower bound of the stable rank of a down-up algebra and, using the Stafford's Stable Range Theorem, we achieve in Theorem 4.1 upper bounds of this value. Finally, under certain conditions, the exact value of stable rank is obtained in Theorem 4.7.

The article is organized as follows: in Section 2 we prove that every finitely generated projective module over a noetherian down-up algebra is stably free.
Section 3 is devoted to showing that the algebra $A = A(\alpha,\beta,\gamma)$, with $\beta \neq 0$, always has a non-trivial stably free right ideal. For this task, we split the problem in two cases: $\gamma \neq 0$ and $\gamma = 0$ and we use some techniques from [16] to achieve our goal.
In Section 4, bounds of the stable rank of a down-up algebra are established. Under some conditions over the roots of the polynomial $t^2 - \alpha t - \beta$, such bounds are improved. The main tool at this point is the Kmax dimension of an arbitrary ring.

2 Stability of projective modules

A ring S is called a *PSF* ring if every finitely generated projective S-module is stably free. In this section we will show that, for $\beta \neq 0$, the algebra $A = A(\alpha,\beta,\gamma)$ is a *PSF* ring. It is important to note that, as β is non-zero, A is a right (left) noetherian ring [10, Corollary 2.2] and, therefore, the rank of free A-modules and the rank of stably free A-modules are well defined.

Theorem 2.1. *Let $A = A(\alpha,\beta,\gamma)$ be a down-up algebra. If $\beta \neq 0$, then A is a* PSF *ring.*

Proof. In [10, Section 3.1] it is proved that the collection $\{V_n\}_{n\geq 0}$ given by $V_0 := \mathbb{k}$, $V_1 := \mathbb{k} + \mathbb{k}u + \mathbb{k}d$ and $V_n := (V_1)^n$, for $n \geq 2$, is a filtration of A, and that $Gr(A)$, the associated graded ring, is isomorphic to the down-up algebra $A(\alpha,\beta,0)$. Hence, $Gr(A)$ is a right (left) noetherian ring. It is also known that if A is noetherian, then gldim$(A) = 3$ [10, Theorem 4.1]; thus $Gr(A)$ is a right regular ring. Since $Gr(A)$ is a free V_0-module, it follows from [13, Theorem 12.3.2] that A is a *PSF* ring. □

Remark 2.2. Given $R = \bigoplus_{i\geq 0} R_i$ a graded ring, we know that if P is a finitely generated graded projective R-module, then P is extended from R_0; more precisely, there is a graded R-module isomorphism $R \otimes_{R_0} P_0 \cong P$, where P_0 is a graded projective R_0-module, see [11, Theorem II. 4.6]. Thus, if $A = A(\alpha, \beta, \gamma)$ is a noetherian down-up algebra with $\gamma = 0$, every finitely generated graded projective A-module P is extended from $\Bbbk$. Hence, P turns out to be a free A-module.

3 Non-trivial stably free ideals

It is well known that there exist stably free modules which are non-free over $\mathcal{U}(sl_2(\Bbbk))$ and $\mathcal{U}(\mathfrak{h})$, where $\mathfrak{h}$ denotes the Heisenberg Lie algebra of dimension 3 [16]. These algebras are examples of down-up algebras, so this raises the question whether every down-up algebra has a non-trivial stably free module or not. The goal of this section is to exhibit examples of such modules. To achieve such objective, we will distinguish two cases: $\gamma \neq 0$ and $\gamma = 0$. In the following, we assume that $\beta \neq 0$, and that $\Bbbk$ is a field of characteristic zero that contains both roots of the polynomial $t^2 - \alpha t - \beta$.

3.1 Case $\gamma \neq 0$

For this case, we will proceed as in [1], [8] and [16]: first, we consider a subalgebra $\widetilde{A}$ of A for which there exists a right (left) non-trivial stably free ideal K. Afterwards, we extend such ideal K to the whole algebra A using results from [16].
Let λ and μ be the roots of $t^2 - \alpha t - \beta$, so that $\alpha = \lambda + \mu$ and $\beta = -\lambda\mu$. Since β is non-zero, it follows that λ and μ are both non-zero. For $\gamma \neq 0$, there is an isomorphism $A(\alpha, \beta, \gamma) \cong A(\alpha, \beta, 1)$, see [5, Lemma 4.1 (ii)], so we assume $\gamma = 1$ without loss of generality. Under these conditions, the multiplication rules in A are given by:

$$[D, [D, U]_\lambda]_\mu = D \qquad\qquad [[D, U]_\lambda, U]_\mu = U,$$

where $[a, b]_\eta$ denotes the expression $ab - \eta ba$. Let $\omega := [d, u]_\lambda = du - \lambda ud$ and consider the algebra $\widetilde{A} := \Bbbk[u][\omega; \sigma, \delta]$ with σ an automorphism of $\Bbbk[u]$ such that $\sigma(u) := \mu^{-1}u$, and δ the σ-derivation defined by $\delta(u) := -\mu^{-1}u$. Let us see that $\widetilde{A}$ is a subalgebra of A: indeed, let $\phi : \widetilde{A} \to A$ be determined by $u \mapsto U$ and $\omega \mapsto [D, U]_\lambda$; then:

$$\begin{aligned} U[D, U]_\lambda &= U(DU - \lambda UD) = UDU - \lambda U^2 D \\ &= \mu^{-1}(\mu UDU - \lambda\mu U^2 D - U) = \mu^{-1}(DU^2 - \lambda UDU - U) \\ &= \mu^{-1}[D, U]_\lambda U - \mu^{-1}U, \end{aligned}$$

and therefore, ϕ turns out to be an algebra homomorphism. The set $\mathcal{B}_1 := \{u^i\omega^j \mid i, j \in \mathbb{N}\}$ is a $\Bbbk$-basis for $\widetilde{A}$ and $\phi(\mathcal{B}_1) = \{U^j(DU - \lambda UD)^j \mid i, j \in \mathbb{N}\}$. Since $\mathcal{B} := \{U^i(DU + aUD + b)^j D^k \mid i, j, k \in \mathbb{N}\}$ is a $\Bbbk$-basis of A for any $a, b \in \Bbbk$ [20, Lemma 2.2], then $\phi(\mathcal{B}_1)$ is linearly independent in A and $\widetilde{A}$ is a subalgebra of A.

We will strongly use the following remarkable result from [16]:

Lemma 3.1. *[16, Corollary 1.6] Let R be a noetherian domain and let $S = R[x; \sigma, \delta]$ be an Ore extension. Suppose that there exists a non-unit $r \in R$ such that $\sum_{i \geq 0} \delta^i(r)R = R$. Then $K = rS \cap xS$ is a non-trivial, stably free right ideal of S.*

This latter lemma will allow us to carry out the first step to achieve our goal.

Lemma 3.2. *The subalgebra $\widetilde{A}$ has a stably free right ideal that is non-free.*

Proof. In view of the fact that $\Bbbk[u]$ is a domain and σ is an automorphism, we have that $\widetilde{A}$ is also a domain. The element $r = 1 + u$ is non-invertible in $\Bbbk[u]$, with the property that

$$r + \delta(r)\mu = 1 + u + (-\mu^{-1}u)\mu = 1 + u - u = 1.$$

Thus, $\sum_{i\geq 0} \delta^i(r)\Bbbk[u] = \Bbbk[u]$ and Lemma 3.1 asserts that $\widetilde{A}$ has a right stably free ideal K which is non-free. The ideal K is defined by $\{f \in \widetilde{A} \mid rf \in \omega\widetilde{A}\}$ and is isomorphic to $r\widetilde{A}\cap\omega\widetilde{A}$. Moreover, in the proof of Lemma 3.1 it is proved that $\widetilde{A} = r\widetilde{A}+\omega\widetilde{A}$; specifically, we have $1 = r(1+\mu\omega)+\omega(-\mu\sigma(r))$. This equality allows us to obtain generators for the right ideal K: in effect, we claim that $a = (\omega + \mu^{-1})(1 + u) - \mu^{-1}$ and $b = \omega^2 + \mu^{-1}\omega$ are polynomials such that $K = a\widetilde{A} + b\widetilde{A}$. To show this, we first note that:

$$\begin{aligned} ra &= (u+1)(\omega+\mu^{-1})(u+1) - \mu^{-1}(u+1) = u\omega(u+1) + \omega(u+1) + \mu^{-1}(u+1)^2 - \mu^{-1}(u+1) \\ &= \omega(u+1)(\mu^{-1}u+1) - \mu^{-1}((u+1)^2 - u(u+1) - (u+1)) = \omega(u+1)(\mu^{-1}u+1) \in \omega\widetilde{A}, \end{aligned}$$

and,

$$\begin{aligned} rb &= \omega(\omega+\mu^{-1}) + u\omega^2 + \mu^{-1}u\omega = \omega(\omega+\mu^{-1}) + \mu^{-1}\omega u\omega - \mu^{-1}u\omega + \mu^{-1}u\omega \\ &= \omega(\omega + \mu^{-1}u\omega + \mu^{-1}) \in \omega\widetilde{A}. \end{aligned}$$

Suppose that $\mathcal{B}_1$ is ordered by the deglex order $\prec$ with $u \prec \omega$, and let f be a non-zero element in K. We claim that if $lm(f) = u^{\delta_1}\omega^{\delta_2}$ is the leading monomial of f, then $\delta_1 + \delta_2 \geq 2$ and $\delta_2 \geq 1$: indeed, it is clear that either δ_1 or δ_2 is non-zero. If $\delta_1 + \delta_2 = 1$, we have that $\delta_1 = 0$ or $\delta_2 = 0$. In the first case, $f = c_1\omega + c_2u + c_3$ with $c_i \in \Bbbk$ for $i = 1, 2, 3$ and c_1 not zero. Then,

$$\begin{aligned} rf =&(u+1)(c_1\omega + c_2u + c_3) = c_1u\omega + c_2u^2 + c_3u + c_1\omega + c_2u + c_3 \\ =&c_1(\mu^{-1}\omega u - \mu^{-1}u) + c_2u^2 + (c_3 + c_2)u + c_1\omega + c_3 \\ =&\omega(c_1\mu^{-1}u + c_1) + c_2u^2 + (c_2 + c_3 - c_1\mu^{-1})u + c_3 \in \omega\widetilde{A}. \end{aligned}$$

Therefore, $c_1 = c_2 = c_3 = 0$ and $f = 0$, which is a contradiction. A similar result is obtained if we assume $\delta_1 = 1$ and $\delta_2 = 0$; hence $\delta_1 + \delta_2 \geq 2$. Now, suppose $\delta_2 = 0$; in such case $f = c_\delta u^\delta + f_1$, where $lm(f_1) \prec lm(f)$, $\delta \geq 2$ and $c_\delta \neq 0$. So,

$$rf =(u+1)(c_\delta u^\delta + f_1) = c_\delta u^{\delta+1} + uf_1 + f \in \omega A;$$

in order that $rf \in \omega A$ necessarily $c_\delta = 0$, but this contradicts our choice of f. Consequently, $f = c_\delta u^{\delta_1}\omega^{\delta_2} + f_1$ where $\delta_1 + \delta_2 \geq 2$, $\delta_2 \geq 1$, $lm(f_1) \prec lm(f)$ and c_δ is a non-zero scalar. In these conditions, $lm(f)$ is divisible by $lm(a) = u\omega$ or $lm(b) = \omega^2$. Applying a right division algorithm, we get $f = aq_1 + bq_2 + h$, where h is reduced with respect to a and b. If $h \neq 0$, we have that $lm(h)$ is not divisible neither by $lm(a)$ nor $lm(b)$; i.e., if $lm(h) = u^{\epsilon_1}\omega^{\epsilon_2}$, then $\epsilon_2 = 0$ or $\epsilon_1 + \epsilon_2 \leq 1$. But $h = f - aq_1 - bq_2 \in K$ and we obtain a contradiction. Whence $h = 0$, $f = aq_1 + bq_2$ and $K = a\widetilde{A} + b\widetilde{A}$. □

Remark 3.3. Lemma 3.1 is a corollary of a more general result by Stafford [16, Theorem 1.2]: given a noetherian domain R and $S = R[x; \sigma, \delta]$, with σ an automorphism of R and δ a σ-derivation, if there exists a non-unit r in R and some $s \in R$ such that $S = rS + (x + s)S$, then S has a non-trivial stably free right ideal. This assertion also has a version for Laurent skew polynomial rings and it was used as a unified way for producing non-trivial stably free right ideals over Weyl algebras, rings of polynomials with coefficients in a division ring in at least two variables, group rings of poly (infinite cyclic) groups and enveloping algebras of non-abelian finite dimensional Lie algebras. However, these modules do not always exist: for example, if R is a division ring, any projective module over S is free (see [13, Proposition 11.5.3]). Moreover, given $S = R[x; \delta]$ with R a commutative local ring with maximal ideal Q and δ a non-zero derivation of R, it is proved in [16, Corollary 4.6] that every stably free right ideal of S is free if and only if $\delta(Q) \subseteq Q$ and $\mathrm{Kdim}(R) = 1$.

The second step is extending this right ideal K to A in such a way that we obtain a non-trivial, stably free right ideal of A. To achieve this, we will use the following fact.

Proposition 3.4. *[16, Proposition 2.3] Let A and B be domains such that $A \subset B$ and B is faithfully flat as left A-module, satisfying the following property:*

$$\text{If } a \text{ and } b \text{ are non-zero elements of } B \text{ such that } ab \in A, \text{ then } a = a_1 c \\ \text{and } b = c^{-1} b_1 \text{ for some unit } c \text{ in } B \text{ and elements } a_1, b_1 \in A. \qquad (\clubsuit)$$

Under these conditions, if P is a projective right ideal of A that is not cyclic, then $PB \cong P \otimes_A B$ is a projective right ideal of B that is also non-cyclic. Further, if P is stably free then so is PB.

Proposition 3.5. *The rings $\widetilde{A}$ and A are domains such that $\widetilde{A} \subset A$ and they satisfy the hypotheses of Theorem 3.4.*

Proof. Since β is non-zero, both A and $\widetilde{A}$ are domains. Inasmuch as $\mathcal{B} = \{u^i(du - \lambda du)^j d^k \mid i, j, k \in \mathbb{N}\}$ is a $\Bbbk$-basis of A, it follows that $\mathcal{B}_2 := \{d^k \mid k \in \mathbb{N}\}$ is an $\widetilde{A}$-basis for A as a left $\widetilde{A}$-module: indeed, it is obvious that $\mathcal{B}_2$ generates ${}_{\widetilde{A}}A$. Taking into account that A is a domain, in order to prove linear independence, it is enough to show that if $\sum_{l=1}^m a_l d^l = 0$, with $a_l \in \widetilde{A}$, $1 \le l \le m$, then $a_l = 0$ for each l. However, $a_l = \sum_{k=1}^{s_l} c_k^{(l)} u^{i_k^l} \omega^{j_k^l}$ for certain elements $c_k^{(l)} \in \Bbbk \setminus \{0\}$, thus

$$0 = \sum_{l=1}^{m} a_l d^l = \sum_{l=1}^{m} \sum_{k=1}^{s_l} c_k^{(l)} u^{i_k^l} \omega^{j_k^l} d^l.$$

Since $u^{i_k^l} \omega^{j_k^l} d^l \in \mathcal{B}$, we get $\sum_{l=1}^m \sum_{k=1}^{s_l} c_k^{(l)} u^{i_k^l} \omega^{j_k^l} d^l = \sum_t d_t x^{\alpha_t}$, with $x^\alpha \in \mathcal{B}$ and $d_t := \sum_{x^{\alpha_t} = u^{i_k^l} \omega^{j_k^l} d^l} c_k^{(l)}$. As a consequence, $d_t = 0$ for all t. Note that, given l and t, there exists just one $u^{i_k^l} \omega^{j_k^l} \in \mathcal{B}_1$ such that $x^{\alpha_t} = u^{i_k^l} \omega^{j_k^l} d^l$, whence the set $\{d_t\}$ coincides with $\{c_k^{(l)}\}$. Therefore, all $c_k^{(l)} = 0$ and $a_l = 0$ for all l. So, A is $\widetilde{A}$-free and, in particular, A turns out to be a faithfully flat left $\widetilde{A}$-module. Finally, to prove that condition ($\clubsuit$) is satisfied, we define the following subsets of A: set $F_0 := \widetilde{A}$ and $F_n := F_0 U_n$ for $n \ge 1$, where $U_n := {}_{\Bbbk}\langle d^k \mid k \le n \rangle$. It is clear that $A = \bigcup_{n \in \mathbb{N}} F_n$ and $F_p \subseteq F_q$ for $p < q$. Using multiplication rules in A we obtain $F_p F_q \subseteq F_{p+q}$, and it follows that $\{F_n\}_{n \in \mathbb{N}}$ is a filtration of the algebra A. Let $f, g \in A$ be non-zero elements such that $fg \in F_0$. Since $A = \bigcup_{n \in \mathbb{N}} F_n$, there exist p and $q \in \mathbb{N}$ with the property that $f \in F_p \setminus F_{p-1}$ and $g \in F_q \setminus F_{q-1}$. In this way, $f = \sum_\delta f_\delta d_\delta \in F_p$ and $g = \sum_\epsilon g_\epsilon d_\epsilon \in F_q$, where $f_\delta, g_\epsilon \in F_0$, $d_\delta \in U_p$ and $d_\epsilon \in U_q$. Hence, $f = a_1 x^{\overline{\delta}_1} d^{\delta_1} + \cdots + a_r x^{\overline{\delta}_s} d^{\delta_s}$ with $x^{\overline{\delta}_i} \in \mathcal{B}_1$ and $\delta_i \le p$ for each i; analogously, $g = b_1 x^{\overline{\epsilon}_1} d^{\epsilon_1} + \cdots + b_t x^{\overline{\epsilon}_t} d^{\epsilon_t}$, with $x^{\overline{\epsilon}_j} \in \mathcal{B}_1$, and $\epsilon_j \le q$. We can suppose $d^{\delta_1} = d^p$ and $d^{\epsilon_1} = d^q$, so $fg = a_1 b_1 x^{\overline{\delta}_1 + \overline{\epsilon}_1} d^{p+q} + f_0 g_0$, with $f_0 g_0$ a polynomial in d, both of degree less or equal to $p + q$. But $fg \in F_0$, so $p + q = 0$ and $\delta_i = \epsilon_j = 0$ for all i and j, i.e., $f, g \in F_0$. This finishes the proof. $\square$

Corollary 3.6. *If $A = A(\alpha, \beta, \gamma)$ is a down-up algebra with $\gamma \neq 0$, then A has a non-trivial, stably free right ideal.*

Proof. By Lemma 3.2, the algebra $\widetilde{A}$ has a stably free right ideal K that is not free. Since $\widetilde{A}$ and A satisfy the hypotheses of Theorem 3.4, the right ideal $KA \cong K \otimes_{\widetilde{A}} A$ is a non-trivial stably free right ideal of A. In the proof of Lemma 3.2, we showed that $K = a\widetilde{A} + b\widetilde{A}$ with $a = (\omega + \mu^{-1})(1 + u) - \mu^{-1}$ and $b = \omega^2 + \mu^{-1}\omega$; thus $KA = (a\widetilde{A} + b\widetilde{A})A$. We shall prove that $KA = aA + bA$: given a non-zero polynomial $f \in aA + bA$, there exist $f_1, f_2 \in A$ such that $f = af_1 + bf_2$. Writing f_1 and f_2 in terms of $\mathcal{B}$, we have $f = \sum c_i a x^{\delta_i} + \sum e_j b x^{\epsilon_j}$. Note that each term in the last expression can be written as $(\lambda_1 a x^{\delta'} + \lambda_2 b x^{\delta'}) x^\delta$, where $x^{\delta'} \in \mathcal{B}_1$, $x^\delta \in \mathcal{B}_2$ and λ_1, λ_2 are not both zero. Hence, f can be expressed as a sum of elements in $(a\widetilde{A} + b\widetilde{A})A$ which implies that $f \in (a\widetilde{A} + b\widetilde{A})A$. Since $KA \subseteq aA + bA$, the equality holds. $\square$

3.2 Case $\gamma = 0$

It is known that if $\gamma = 0$, the isomorphism $A(\alpha, \beta, 0) \cong \Bbbk[u][\omega; \theta][d; \sigma, \delta]$ holds, for certain automorphisms θ, σ and a σ-derivation δ depending on α, β, see [10, Theorem 3.3]. We will present an alternative proof of the existence of this isomorphism using the universal property of skew polynomial rings that will be more suitable for our purpose.

Lemma 3.7. *For the down-up algebra $A = A(\alpha, \beta, 0)$, there exist ω and automorphisms θ and σ, together with a σ-derivation δ, such that A is isomorphic to an iterated skew polynomial ring of the form $\Bbbk[u][\omega; \theta][d; \sigma, \delta]$.*

Proof. As usual we denote by U and D the obvious generators of A. Let $R_1 := \Bbbk[u]$ and $R := \Bbbk[u][\omega; \theta]$, where θ is the automorphism of R_1 given by $\theta(u) = \mu^{-1}u$. If $\phi_0 : R_1 \to A$ is defined by $\phi_0(u) = U$, then ϕ_0 can be extended to a ring homomorphism with the property that, for $a \in R_1$ the following holds:

$$\begin{aligned}\phi_0(u)[D,U]_\lambda =& UDU - \lambda U^2 D = \mu^{-1}(\mu UDU - \lambda\mu U^2 D)\\ =&\mu^{-1}(DU - \lambda UD)U = [D,U]_\lambda \phi_0(\theta(u)).\end{aligned}$$

Taking $y := [D,U]_\lambda$, in [13, §1.2.5] asserts that there exists a unique ring homomorphism $\phi_1 : R_1[\omega; \theta] \to A$ such that $\phi_1 \circ \iota = \phi_0$, with $\iota : R_1 \to R_1[\omega; \theta]$ the natural inclusion. Now, consider the ring $R[d; \sigma, \delta] = \Bbbk[u][\omega; \theta][d; \sigma, \delta]$, where $\sigma : R \to R$ is the automorphism $\sigma(u) = \lambda^{-1}u$, $\sigma(\omega) = \mu^{-1}\omega$, and δ the σ-derivation on R determined by $\delta(u) = -\lambda^{-1}\omega$ and $\delta(\omega) = 0$. For the aforementioned homomorphism ϕ_1, note that

$$\begin{aligned}\phi_1(u)D = UD =& \lambda^{-1}DU - \lambda^{-1}[D,U]_\lambda\\ =& D\phi_1(\sigma(u)) + \phi_1(\delta(u));\\ \phi_1(\omega)D =& DUD - \lambda D^2 = \mu^{-1}(D^2U - \lambda DUD) = \mu^{-1}D[D,U]_\lambda\\ =& D\phi_1(\sigma(\omega));\end{aligned}$$

if we set now $y := D$, again from [13, §1.2.5], we obtain a unique ring homomorphism $\phi_2 : R[d; \sigma, \delta] \to A$ such that $\phi_2 \circ \iota' = \phi_1$, with $\iota' : R \to R[d; \sigma, \delta]$ the inclusion. In particular, $\phi_2(u) = U$, $\phi_2(\omega) = [D,U]_\lambda$ and $\phi_2(d) = D$; thus, ϕ_2 is surjective. Since $\mathcal{B} = \{u^i\omega^j d^k \mid i,j,k \in \mathbb{N}\}$ is a $\Bbbk$-basis of $R[d; \sigma, \delta]$ and $\phi_1(\mathcal{B}) = \{U^i(DU - \lambda UD)^j D^k \mid i,j,k \in \mathbb{N}\}$, we know that ϕ_2 is an isomorphism and we have proved the statement. ☐

Proposition 3.8. *The down-up algebra $A(\alpha, \beta, 0)$ has a non-trivial, stably free right ideal.*

Proof. By Lemma 3.7, the isomorphism $A \cong \Bbbk[u][\omega; \theta][d; \sigma, \delta]$ holds, then it is enough to show that the latter ring satisfies the statement. Taking $r = 1 + u\omega$, we have that

$$\begin{aligned}r(1 - u\omega) - \delta(r)(\mu^{-2}\lambda u^2) =& (1 + u\omega)(1 - u\omega) - (-\lambda^{-1}\mu^{-1}\omega^2)(\mu^{-2}\lambda u^2)\\ =& 1 - u\omega u\omega + \mu^{-3}\omega^2 u^2\\ =& 1 - \mu^{-3}\omega^2 u^2 + \mu^{-3}\omega^2 u^2 = 1;\end{aligned}$$

since A is a domain and σ is an automorphism, by Lemma 3.1 the algebra A has a stably free right ideal K which is non-free. The ideal K is given by $\{f \in A \mid rf \in dA\}$ and turns out to be isomorphic to $rA \cap dA$. We assert that K is generated by the polynomials $a = d^2$, and $b = du\omega + \lambda^{-1}\mu\omega^2 + \mu^2 d$: indeed, we start noting that

$$\begin{aligned}ra =& (u\omega + 1)d^2 = uwd^2 + d^2\\ =& \mu^{-2}\lambda^{-2}d^2 u\omega - (\lambda^{-2}\mu^{-2} + \lambda^{-1}\mu^{-3})d\omega^2 \in dA,\end{aligned}$$

and

$$
\begin{aligned}
rb =&(u\omega+1)(du\omega+\lambda^{-1}\mu\omega^2+\mu^2 d)\\
=&(d\sigma(u\omega)+\delta(u\omega))u\omega+\lambda^{-1}\mu u\omega^3+\mu^2(d\sigma(u\omega)+\delta(u\omega))+du\omega+\lambda^{-1}\mu\omega^2+\mu^2 d\\
=&(\lambda^{-1}\mu^{-1}du\omega-\lambda^{-1}\mu^{-1}\omega^2)u\omega+\lambda^{-1}\mu u\omega^3+\mu^2(\lambda^{-1}\mu^{-1}du\omega-\lambda^{-1}\mu^{-1}\omega^2)+du\omega+\lambda^{-1}\mu\omega^2+\mu^2 d\\
=&d(\lambda^{-1}u^2\omega^2+(\lambda^{-1}\mu+1)u\omega+\mu^2)-\lambda^{-1}\mu^{-1}\omega^2 u\omega+\lambda^{-1}\mu u\omega^3-\lambda^{-1}\mu\omega^2+\lambda^{-1}\mu\omega^2\\
=&d(\lambda^{-1}u^2\omega^2+(\lambda^{-1}\mu+1)u\omega+\mu^2)\in dA.
\end{aligned}
$$

In order to prove the claim, we suppose that $\mathcal{B}$ is ordered by the deglex order $\prec$ with $u\prec\omega\prec d$. Let f be a non-zero polynomial in A. We shall show that if $f\in K$, then $lm(f)$ is divisible by $lm(a)=d^2$ or $lm(b)=u\omega d$. Let $lm(f)=u^{\delta_1}\omega^{\delta_2}d^{\delta_3}$ be the leading monomial of f. A straightforward reasoning allows to derive that if $f\in K$, then necessarily $\delta_3\geq 1$. We consider the following possibilities:

- $\delta_1=\delta_2=0$: in such case $\delta_3\geq 2$, since otherwise $f=c_1d+c_2\omega+c_3u+c_4$ with $c_i\in\Bbbk$, $i=1,2,3,4$ and c_4 non-zero. So,

  $$(u\omega+1)f=d(c_1\lambda^{-1}\mu^{-1}u\omega+c_1)+c_2u\omega^2+c_3\mu u^2\omega-c_1\lambda^{-1}\mu^{-1}\omega^2+c_4u\omega+c_2\omega+c_3u+c_4\in dA$$

  implies that $c_1=c_2=c_3=c_4=0$; i.e., $f=0$ which is contrary to our choice of f. Thus, $\delta_3\geq 2$.

- $\delta_3=1$: in this situation we must have $\delta_1,\delta_2\geq 1$. By the above, we get that either δ_1 or δ_2 is not zero. Suppose $\delta_1\neq 0$ and $\delta_2=0$. Thus $f=cu^{\delta_1}d+f_1$ with $lm(f_1)\prec lm(f)$ and $c\neq 0$; then

  $$
  \begin{aligned}
  (u\omega+1)f =&c\mu^{\delta_1}(d\sigma(u^{\delta_1+1}\omega)+\delta(u^{\delta+1}\omega))+c(d\sigma(u^{\delta_1})+\delta(u^{\delta_1}))+u\omega f_1+f_1\\
  =&c\mu^{\delta_1}(\lambda^{-(\delta_1+1)}\mu^{-1}du^{\delta_1+1}\omega-\lambda^{-1}\mu^{-1}p_{\delta_1+1}(\lambda^{-1},\mu)u^{\delta_1}\omega^2)+c\lambda^{-\delta_1}du^{\delta_1}\\
  &-c\lambda^{-1}p_{\delta_1}(\lambda^{-1},\mu)u^{\delta_1-1}\omega+u\omega f_1+f_1\\
  =&d(c\lambda^{-(\delta_1+1)}u^{\delta_1+1}\omega+c\lambda^{-\delta_1}u^{\delta_1})-c\lambda^{-1}\mu^{\delta_1-1}p_{\delta_1+1}(\lambda^{-1},\mu)u^{\delta_1}\omega^2\\
  &-c\lambda^{-1}p_{\delta_1}(\lambda^{-1},\mu)u^{\delta_1-1}\omega+u\omega f_1+f_1\in dA,
  \end{aligned}
  $$

  where $p_t(\lambda^{-1},\mu)=\lambda^{-(t-1)}\mu^{t-1}+\lambda^{-(t-2)}\mu^{t-2}+\cdots+\lambda^{-1}\mu+1$ is the expression that appears in the calculation of $\delta(u^t)$ and $\delta(u^t\omega)$. Specifically, using induction over $t\geq 1$, it can be shown that $\delta(u^t)=-\lambda^{-1}p_t(\lambda^{-1},\mu)u^{t-1}\omega$, and $\delta(u^t\omega)=-\lambda^{-1}\mu^{-1}p_t(\lambda^{-1},\mu)u^{t-1}\omega^2$ for $t\geq 1$. Let $\epsilon_1=c\lambda^{-1}\mu^{\delta_1-1}p_{\delta_1+1}(\lambda^{-1},\mu)$, $\epsilon_2=c\lambda^{-1}p_{\delta_1}(\lambda^{-1},\mu)$ and $c'\in\Bbbk$ the coefficient of $u^{\delta_1-1}\omega$ in f_1. Thus $\mu^{\delta_1-1}c'-\epsilon_1=0$ and $c'-\epsilon_2=0$. Rewriting these equations, we obtain that $cp_{\delta_1+1}(\lambda^{-1},\mu)=cp_{\delta_1}(\lambda^{-1},\mu)$. Hence $c(p_{\delta_1+1}(\lambda^{-1},\mu)-p_{\delta_1}(\lambda^{-1},\mu))=0$; but $p_{\delta_1+1}(\lambda^{-1},\mu)-p_{\delta_1}(\lambda^{-1},\mu)=\lambda^{-\delta_1}\mu^{\delta_1}$ and, since λ and μ are non-zero, it follows that $c=0$. This is a contradiction, therefore $\delta_2\geq 1$.

- $\delta_3=1$ and $\delta_2\neq 0$. If $\delta_1=0$, the polynomial f is written as $f=c\omega^{\delta_2}d+f_1$ with $lm(f_1)\prec lm(f)$ and $c\neq 0$. In this case

  $$
  \begin{aligned}
  (u\omega+1)f =&c(d\sigma(u\omega^{\delta_2+1})+\delta(u\omega^{\delta_2+1}))+cd\sigma(\omega^{\delta_2})+u\omega f_1+f_1\\
  =&c\lambda^{-(\delta_2+1)}\mu^{-1}du\omega^{\delta_2+1}-c\lambda^{-1}\mu^{-(\delta_2+1)}\omega^{\delta_2+2}+c\mu^{-\delta_2}d\omega^{\delta_2}+u\omega f_1+f_1\\
  =&d(c\lambda^{-(\delta_2+1)}\mu^{-1}u\omega^{\delta_2+1}+c\mu^{-\delta_2}\omega^{\delta_2})-c\lambda^{-1}\mu^{-(\delta_2+1)}\omega^{\delta_2+2}+u\omega f_1+f_1\in dA.
  \end{aligned}
  $$

  Since each term in $u\omega f_1$ is multiplied by u and $\deg(f_1)\leq\delta_2+1$, it is necessary that $c\lambda^{-1}\mu^{-(\delta_2+1)}=0$. Thus $c=0$, which contradicts our choice of f. Consequently, $\delta_1\geq 1$.

Therefore, given a non-zero polynomial $f\in K$ and applying a right division algorithm, there exist $q_1,q_2,h\in A$ such that $f=aq_1+bq_2+h$, with h reduced with respect to a and b. If $h\neq 0$, then h is not divisible neither by $lm(a)$ nor by $lm(b)$. But $h=f-aq_1-bq_2\in K$ and we get a contradiction. In consequence $h=0$, $f=aq_1+bq_2$ and $K=aA+bA$. $\square$

Remark 3.9. It is proved in [9] that a down-up algebra is isomorphic to an ambiskew ring. Specifically, in that paper it is showed that $A(\alpha, \beta, \gamma) \cong \Bbbk[\omega][u, \sigma][d; \sigma^{-1}, \delta]$, where $\omega = du - \lambda ud$, σ is the automorphism over $\Bbbk[\omega]$ given by $\sigma(\omega) = \mu\omega + \gamma$ extended to $\Bbbk[\omega][u; \sigma]$ by setting $\sigma(u) = \lambda u$, and δ the σ^{-1}-derivation with $\delta(\Bbbk[\omega]) = 0$ and $\delta(u) = -\lambda^{-1}\omega$. We could have tried to apply directly the results from [16] to this ring in order to obtain a non-free, stably free right ideal. Nevertheless, despite this isomorphism, the element r in Lemma 3.1 cannot be attained in a natural way using this approach; for this reason we decided to consider the cases $\gamma \neq 0$ and $\gamma = 0$ independently.

4 Stable rank

In this last section we assume additionally that $\Bbbk$ is an algebraically closed field. Recall that a unimodular row $\boldsymbol{u} = (u_1, u_2, \ldots, u_s)$ with entries in a ring S is said to be *stable* if there exist $r_1, \ldots, r_{s-1}$ such that $\boldsymbol{u}' = (u_1 + u_s r_1, \ldots, u_{s-1} + u_s r_{s-1})$ is also a unimodular row. The *stable rank* of S is defined as the least non-negative integer t with the property that every unimodular row of length $t + 1$ is stable, see [12], [13, Chapter 11] and references therein for features and interesting examples of stable rank. Furthermore, if sr(S) denotes the stable rank of S, the Stafford's Stable Range Theorem states that if S is right noetherian and rKdim(S) $= d < \infty$, where rKdim(S) denotes the right Krull dimension of S in the sense of Rentschler and Gabriel, then sr(S) $\leq$ rKdim(S)+1, see [15].

In [3, Theorem 4.1] Bavula and Lenagan showed that if $A = A(\alpha, \beta, \gamma)$ is a down-up algebra with $\beta \neq 0$, then the right Krull dimension of A is equal to 2 if and only if char($\Bbbk$)$= 0$, $\alpha+\beta = 1$ and $\gamma \neq 0$; otherwise, the right Krull dimension of A is 3. Since $A \cong A^{op}$ via the map $D \mapsto U^\circ$ and $U \mapsto D^\circ$ [10, §1], we have that rKdim(A) = lKdim(A) thus we will simply refer to Kdim(A). These values of Kdim(A), combined with results of the previous section, will allow us to establish bounds of sr(A). Before doing this, recall that if $\Bbbk$ is a field, $\Bbbk_0$ its prime subfield and t the transcendence degree of $\Bbbk$ over $\Bbbk_0$, then the *Kronecker dimension* of $\Bbbk$ is defined to be t if char($\Bbbk$) > 0, and $t + 1$ if char($\Bbbk$) $= 0$.

Theorem 4.1. *Let $A = A(\alpha, \beta, \gamma)$ a noetherian down-up algebra. We have the following bounds of sr(A):*

(i) If $\alpha + \beta = 1$ and $\gamma \neq 0$ then $2 \leq$ sr(A) ≤ 3.

(ii) If $\alpha + \beta = 1$ and $\gamma = 0$ then $3 \leq$ sr(A) ≤ 4.

(iii) Otherwise, $2 \leq$ sr(A) ≤ 4.

Proof. Given an arbitrary ring S, it is well known that if M is a stably free S-module and rank(M) $\geq$ sr(S), then M is free with dimension equal to rank(M), see [13, Theorem 11.3.7 (i)]. In consequence, by Corollary 3.6 and Proposition 3.8 we get that sr(A) ≥ 2 for any noetherian down-up algebra A. Because Kdim(A) $= 2$ for the case (i), the Stable Range Theorem asserts that sr(A) ≤ 3 and the inequality is obtained.
For (ii), we have that $2 \leq$ sr(A) ≤ 4; however, in [5, Proposition 4.2] the authors proved that $\Bbbk[x, y] \cong A/I$ for some two-sided ideal I of A when $\alpha + \beta = 1$ and $\gamma = 0$. Since the stable rank of a quotient is not bigger than the stable rank of the ring, it follows that sr($\Bbbk[x, y]$) $\leq$ sr(A). Furthermore, Suslin proved in [18, Theorem 10] that if l is the Kronecker dimension of $\Bbbk$ and $n \leq l$, then sr($\Bbbk[x_1, \ldots, x_n]$) $= n + 1$. In particular, sr($\Bbbk[x, y]$) $= 3$ and $3 \leq$ sr(A) ≤ 4. □

Computing the exact value of the stable rank of an arbitrary ring is a very difficult task. However, as a significant example, the stable rank of commutative polynomial rings over fields was determined by Suslin [18]. In the noncommutative setting, it is well known that the stable rank of the n-th Weyl algebra $A_n(\Bbbk)$ is 2 when char($\Bbbk$) $= 0$ [14]. Tintera showed in [19] that if $\mathfrak{h}$ is the Heisenberg Lie algebra of dimension n over a field $\Bbbk$ "large enough", then sr($\mathcal{U}(\mathfrak{h})$) $= n$. The Kmax dimension of a ring was the main tool used by him in order to obtain this equality. We will use an analogous argument for computing the exact value of sr(A), when A is a down-up algebra with $\alpha+\beta=1$ and $\gamma=0$.

In order to recall the definition of the Kmax dimension, we begin by considering the deviation of a poset [13, Section 6.1]: let $\mathcal{P}$ be a poset, $a, b \in \mathcal{P}$ and $a \geq b$. The factor of a and b is the subposet of $\mathcal{P}$ defined as $\mathcal{P}_{a,b} = \{x \in \mathcal{P} \mid a \geq x \geq b\}$. To define the deviation of $\mathcal{P}$, or dev($\mathcal{P}$) for short, we say that dev($\mathcal{P}$) $= -\infty$ if $\mathcal{P}$ is trivial. If $\mathcal{P}$ is non-trivial but satisfies the d.c.c., then dev ($\mathcal{P}$) $= 0$. For a general ordinal α, we define dev($\mathcal{P}$) $= \alpha$ provided:

(i) dev($\mathcal{P}$) $\neq \beta < \alpha$;

(ii) in any descending chain of elements of $\mathcal{P}$, all but finitely many factors have deviation less than α.

Now, we recall the definition of the Kmax dimension of a ring S.

Definition 4.2. *Let S be an associative ring with identity and denote by $\mathcal{M}_S$ the set of maximal right ideals of S.*

(i) A right ideal I of S is called a ***Jacobson right ideal*** *if $I = J(I)$, where $J(I) := \bigcap\{M \in \mathcal{M}_S \mid I \subseteq M\}$.*

(ii) Let $\mathcal{JL}(S)$ be the set of Jacobson right ideals of S partially ordered by inclusion. The ***Kmax dimension*** *of S is defined to be the deviation of the poset $(\mathcal{JL}(S), \subseteq)$ and we denote it by Kmax(S).*

Remark 4.3. (i) Let $(\mathcal{L}(S), \subseteq)$ be the poset of right ideals of S. Note that $(\mathcal{JL}(S), \subseteq)$ is a subposet of $(\mathcal{L}(S), \subseteq)$. So, if rKdim($S$) exists, we have that Kmax(S) $\leq$ rKdim(S). In particular if S is a right noetherian ring, this inequality always holds.
(ii) There exist rings for which the inequality in (i) is strict: if $S = \mathbb{C}[x]_{(x)}[y]$, with $\mathbb{C}[x]_{(x)}$ denoting the localization of $\mathbb{C}[x]$ at powers of x, then Kmax(S) = 1 < Kdim(S) = 2 (see [17], remark to Proposition 1.6). Furthermore, for $\mathfrak{h}$ a non-abelian nilpotent Lie algebra of dimension n, Tintera proved in [19, Lemma 3] that Kmax($\mathcal{U}(\mathfrak{h})$) < Kdim($\mathcal{U}(\mathfrak{h})$) = n.
(iii) It follows from [15] or [17, Theorem B] that if S is a right noetherian ring, a Kmax version of the Stable Range Theorem holds; i.e., if Kmax(S) exists and is finite, then sr(S) $\leq$ Kmax(S) +1. The latter explains the reason leading us to introduce the Kmax dimension.

Below we summarize some important properties satisfied by the Kmax dimension.

Lemma 4.4. *[19, Lemmas 1 and 2] (i) Given a ring extension $S \subset T$ such that T_S is a faithfully flat module, we have Kmax(S) $\leq$ Kmax(T).*
(ii) Let S be a domain for which Kmax(S) is defined and let $z \in S$ be a normal element. There is an inequality

$$\text{Kmax}(S) \leq \sup\{\ \text{Kmax}(S/zS), \text{Kmax}(S_{(z)})\},$$

where $S_{(z)}$ denotes the localization of S at the set of powers of z.

For another features of Kmax, as well as for additional descriptions and remarks, we refer to [17].

Recall that given a $\mathbb{k}$-algebra R, an automorphism σ of R and a central element a of R, the generalized Weyl algebra $R(\sigma, a)$ is defined as the algebra generated by X^+ and X^- subject to the relations: $X^-X^+ = a$, $X^+X^- = \sigma(a)$, $X^-\sigma(b) = bX^-$ and $X^+b = \sigma(b)X^+$ for all $b \in R$. For these algebras, Bavula and van Oystaeyen established in [2, Theorem 1.2] the following result for computing the Krull dimension of $T = R(\sigma, a)$ when R is commutative.

Proposition 4.5. *Let R be a commutative noetherian ring with Kdim(R) $= m$ and let $T = R(\sigma, a)$ be a generalized Weyl algebra. The Krull dimension Kdim(T) is m unless there is a height m maximal ideal P of R such that one of the following conditions holds:*

(i) $\sigma^n(P) = P$*, for some* $n > 0$*;*

(ii) $a \in \sigma^n(P)$ *for infinitely many* n*.*

If there is an ideal P *as above such that (i) or (ii) holds, then Kdim(T)* $= m + 1$*.*

To prove the main result of this section, we develop reasonings inspired into those carried out by Carvalho and Musson in [6, §5.].

In [10, §2.2] it is showed that an arbitrary noetherian down-up algebra is isomorphic to a generalized Weyl algebra: in fact, taking $R = \Bbbk[x, y]$, ϕ the automorphism of R defined by $\phi(x) = y$, $\phi(y) = \alpha y + \beta x + \gamma$ and $a = x$, the algebra $A(\alpha, \beta, \gamma)$ is isomorphic to $R(\phi, x)$ under the isomorphism φ sending X^+ to D and X^- to U; in particular, x and y correspond to UD and DU, respectively. Additionally, if $\alpha + \beta = 1$, $\gamma = 0$ and the roots λ and μ of $t^2 - \alpha t - \beta$ are different, then case 2 of [5, §1.4] holds and we have that

$$\omega_1 = \beta x + y$$
$$\omega_2 = -x + y$$

are such that $\{1, \omega_1, \omega_2\}$ is a basis of the subspace of $\Bbbk[x, y]$ generated by $1, x, y$, and moreover $\phi(\omega_1) = \omega_1$ and $\phi(\omega_2) = -\beta\omega_2$. In this case ω_2 is identified with $\omega = DU - UD$ through φ.

Benkart and Roby introduced in [4] (see also [5]) the following recursive relation in order to study Verma modules of $A(\alpha, \beta, \gamma)$:

$$s_n = \alpha s_{n-1} + \beta s_{n-2} + \gamma. \tag{4.1}$$

From [5, Lemma 2.3] it follows that for all $n \in \mathbb{Z}$, the automorphism ϕ satisfies

$$\phi^{-n}\langle x - s_0, y - s_1\rangle = \langle x - s_n, y - s_{n+1}\rangle,$$

where $\langle x - s_0, y - s_1\rangle$ denotes the two-sided ideal generated by $x - s_0$ and $y - s_1$. For $\alpha^2 + 4\beta \neq 0$ (i.e., when the roots of polynomial $t^2 - \alpha t - \beta$ are different) and $\alpha + \beta = 1$, the solution to (4.1) is given by [4, Proposition 2.12(i)]:

$$s_n = c_1\lambda^n + c_2\mu^n + \frac{\gamma n}{(2 - \alpha)}, \text{ (necessarily } \alpha \neq 2) \tag{4.2}$$

for certain fixed scalars $c_1, c_2 \in \Bbbk$ which depend on the established initial conditions.

Lemma 4.6. *Let* $A(\alpha, \beta, 0)$ *be a down-up algebra with* $\alpha + \beta = 1$*, such that roots* $1, \mu$ *are different and* μ *is not a root of unity. If* Q *is a maximal ideal of* $\Bbbk[x, y]$ *such that* $x \in \phi^n(Q)$ *for infinitely many* n*, then* $\phi^n(Q) = Q$ *for some* $n \geq 1$*.*

Proof. Let $Q = \langle x - s_0, y - s_1\rangle$ for certain $s_0, s_1 \in \Bbbk$. By hypothesis, we can suppose that $x \in Q$, namely $Q = \langle x, y - s_1\rangle$ and $s_0 = 0$. Using the initial conditions $s_0 = 0$ and $s_1 = s$, the equation (4.2) can be written as

$$s_n = \frac{s}{1 - \mu} - \frac{s}{1 - \mu}\mu^n.$$

If $s_n = 0$ for $n \geq 1$, we get $\frac{s}{1-\mu}(1 - \mu^n) = 0$. Since μ is not a root of unity, necessarily $s = 0$ and $s_n = 0$ for all n. Thus, the proof is obtained. □

Theorem 4.7. *Let* $A = A(\alpha, \beta, 0)$ *be a down-up algebra with* $\alpha + \beta = 1$*. If one the following conditions holds*

(i) $\lambda = \mu = 1$*, or*

(ii) $\mu \neq 1$ *and it is not a root of unity,*

then sr(A) $= 3$.

Proof. In the first case $\alpha = 2$, $\beta = -1$ and $A(\alpha, \beta, 0) \cong U(\mathfrak{h})$, where $\mathfrak{h}$ denotes the Heisenberg algebra of dimension 3. The assertion follows from [19, Corollary 1]. Suppose $\lambda = 1$, $\mu \neq 1$ and μ is not a root of unity. Under these conditions $\omega = DU - UD$ is a normal element of A and $A/\omega A$ is a ring. Since A is a domain and ω is not zero, then ω is a regular element of A and, therefore, Kdim($A/\omega A$) < Kdim(A) = 3, see [13, Lemma 6.3.9]. On the other hand, if $\Lambda = \{\omega^i \mid i \in \mathbb{N}\}$, then Λ is an Ore set and the ring $A_{(\omega)} := A\Lambda^{-1}$ exists. Under the isomorphism $A \cong \Bbbk[x, y](\phi, x)$ described above, the element ω is sent to $\omega_2 = y - x \neq 0$. Thus, $A_{(\omega)} \cong \Bbbk[x, y]_{(\omega_2)}(\phi, x)$ is a generalized Weyl algebra. To prove that Kdim($A_{(\omega)}$) = Kdim($\Bbbk[x, y]_{(\omega_2)}$) = 2, we must show that neither (i) nor (ii) in Theorem 4.5 is satisfied. By Lemma 4.6 it is enough to demonstrate that for any maximal ideal P of $\Bbbk[x, y]_{(\omega_2)}$ and $n > 0$, we have $\phi^n(P) \neq P$. This is equivalent to prove that if Q is a maximal ideal of $\Bbbk[x, y]$ such that $\sigma^n(Q) = Q$, then $\omega_2 \in Q$. Thus, for $Q = \langle \omega_1 - a_1, \omega_2 - a_2 \rangle$ and, since μ is not a root of unity, from [5, Lemma 2.2(ii)] it follows that $a_2 = 0$ and we get the statement about Kdim($A_{(\omega)}$). Hence, by Lemma 4.4(ii), we have Kmax(A) ≤ 2 and the Stable Range Theorem asserts that sr(A) ≤ 3. The equality follows from Proposition 4.1 (ii). $\square$

Remark 4.8. In [6, Section 5] it was proved that if $A(\alpha, \beta, \gamma)$ is a down-up algebra such that the polynomial $t^2 - \alpha t - \beta$ equals $(t - \mu)^2$, with $\mu \neq 1$ and μ a primitive n-th root of unity, then $\omega = DU - \mu UD + \frac{\gamma}{\mu - 1}$ is a normal element of A and Kdim($A_{(\omega)}$) = 2. From this and Lemma 4.4(ii) we obtain the following.

Corollary 4.9. *Let $A = A(\alpha, \beta, \gamma)$ a down-up algebra such that $\alpha + \beta \neq 1$. If*

(i) $t^2 - \alpha t - \beta = (t - \mu)^2$ with μ a primitive n-th root of unity, or

(ii) $\gamma = 0$, $t^2 - \alpha t - \beta = (t - \mu)(t - \lambda)$ with $\lambda \neq \mu$ and $\frac{\lambda}{\mu}$ not a root of unity,

then $2 \leq$ *sr(A)* ≤ 3.

Proof. Part (i) follows from Remark 4.8. For (ii), suppose that λ is not a root of unity and set $\omega = \frac{1}{\lambda^2 - \lambda}(\beta(\lambda - 1)UD + \lambda(\lambda - 1)DU)$. Therefore, $\omega = DU - \mu UD$ and a straightforward calculation shows that ω is a normal element of A. Thus $A_{(\omega)}$ exists and $A_{(\omega)} \cong \Bbbk[x, y]_{(\omega')}(\phi, x)$, where $\omega' = y - \mu x$. Given a maximal ideal $Q = \langle x - s_0, y - s_1 \rangle$ of $\Bbbk[x, y]$ such that $x \in \phi^n(Q)$ for infinitely many values of n, we have that $\phi^n(Q) = Q$ for some $n \geq 1$: in fact, using the initial conditions $s_0 = 0$ and $s_1 = s$, in this case the equation (4.2) can be written as $s_n = \frac{s}{\lambda - \mu}(\lambda^n - \mu^n)$; if $s_n = 0$, necessarily $s = 0$ since $\frac{\lambda}{\mu}$ is not a root of unity, and the assertion follows. A similar reasoning to the one in the proof of Proposition 4.7, along with [5, Lemma 2.2(i)], allow to obtain Kdim($A_{(\omega)}$) = 2 and from Lemma 4.4(ii) we get the inequality sr(A) ≤ 3. $\square$

Remark 4.10. The stable rank is closely related to the cancellation property for projective modules. Recall that two finitely generated projective S-modules P and P' are called *stably isomorphic* if $P \oplus S^n \cong P' \oplus S^n$ for some n. It is said that P satisfies the *cancellation property* if any P' stably isomorphic to P is in fact isomorphic to P. Hence, if P is a finitely generated projective module over a down-up algebra A with rank(P) $\geq$ sr(A), a simple reasoning proves that P has the cancellation property.

A table summarizing the stable ranks of some important examples of down-up algebras, and one related algebra, is included below.

Algebra $A = A(\alpha, \beta, \gamma)$	**Parameters**	**Conditions on parameters**	**Bounds of sr**(A)
$\mathcal{U}(sl_2(\Bbbk))$	$\alpha = 2, \beta = -1, \gamma = -2$		$2 \leq \text{sr}(A) \leq 3$
$\mathcal{U}(osp(1,2))$	$\alpha = 0, \beta = 1, \gamma = \frac{1}{2}$		$2 \leq \text{sr}(A) \leq 3$
$\mathcal{U}(\mathfrak{h})$, $\mathfrak{h}$ the Heisenberg algebra of dimension 3	$\alpha = 2, \beta = -1, \gamma = 0$		$\text{sr}(A) = 3$
Smith's algebras similar to $U(sl_2(\Bbbk))$ with $\deg(f(h)) \leq 1$	$\alpha = 2, \beta = -1$	$\gamma = 0$	$\text{sr}(A) = 3$
		$\gamma \neq 0$	$2 \leq \text{sr}(A) \leq 3$
Conformal $sl_2(\Bbbk)$ algebras with $c \neq 0, b = 0$ $xz - azx = x$, $zy - ayz = y$, $yx - cxy = bz^2 + z$	$\alpha = c^{-1}(1 + ac)$, $\beta = -ac^{-1}$, $\gamma = -c^{-1}$	$c = 1$ or $a = 1$	$2 \leq \text{sr}(A) \leq 3$
		$a = c^{-1} \neq 1$ and a a primitive root of unity	
		Otherwise	$2 \leq \text{sr}(A) \leq 4$
Quantum Heisenberg algebra $H_q \cong \mathcal{U}_q^+(sl_3(\Bbbk))$, $q \in K^*$. $zx = qxz$, $zy = q^{-1}yz$, $xy - qyx = z$	$\alpha = q + q^{-1}$, $\beta = -1$, $\gamma = 0$ and $q \neq 1$	$q = -1$	$2 \leq \text{sr}(A) \leq 3$
		q^2 not a root of unity	
		Otherwise	$2 \leq \text{sr}(A) \leq 4$
q-analog H_q' of $U(\mathfrak{h})$, $q \neq 0, 1$. $xy - qyx = z$, $xz = qzx$ and $zy = qyz$	$\alpha = 2q, \beta = -q^2, \gamma = 0$	q is a primitive root of unity	$2 \leq \text{sr}(A) \leq 3$
		Otherwise	$2 \leq \text{sr}(A) \leq 4$
$\mathfrak{m}(\xi)$ the Witten's Deformation of $\mathcal{U}(sl_2(\Bbbk))$: $xz - \xi_1 zx = \xi_2 x$, $zy - \xi_3 yz = \xi_4 y$, $yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z$, with $\xi_6 = 0, \xi_5\xi_7 \neq 0, \xi_1 = \xi_3$ and $\xi_2 = \xi_4$	$\alpha = \frac{1+\xi_1\xi_5}{\xi_5}$, $\beta = -\frac{\xi_1}{\xi_5}$, $\gamma = -\frac{\xi_2\xi_7}{\xi_5}$	$\xi_1 = \xi_5 = 1$ and $\xi_2 = 0$	$\text{sr}(A) = 3$
		$\xi_2 = 0$, $\xi_5 = 1$, ξ_1 not a root of unity	
		$\xi_1 = 1$, $\xi_2 = 0$, ξ_5 not a root of unity	
		$\xi_2 \neq 0$, $\xi_1 = 1$ or $\xi_5 = 1$	$2 \leq \text{sr}(A) \leq 3$
		$\xi_1 \neq 1$, $\xi_1 = \xi_5^{-1}$ and ξ_1 is a primitive root of unity	
		$\xi_2 = 0$, $\xi_1, \xi_5 \neq 1$ and $\xi_1\xi_5$ not a root of unity	
		Otherwise	$2 \leq \text{sr}(A) \leq 4$
Woronowicz's algebras: $xz - \zeta^4 zx = (1 + \zeta^2)x$, $zy - \zeta^4 yz = (1 + \zeta^2)y$, $xy - \zeta^2 yx = \zeta z$ with $\zeta \neq \pm 1, 0$	$\alpha = \zeta^2(1 + \zeta^2)$, $\beta = -\zeta^6$, $\gamma = \zeta(1 + \zeta^2)$		$2 \leq \text{sr}(A) \leq 4$
Universal enveloping of Lie super algebra $sl(1,1)$	$\mathcal{U}(sl(1,1)) \cong \frac{A(0,1,0)}{\langle d^2, u^2 \rangle}$		$\text{sr}(\mathcal{U}(sl(1,1)) = 1$

Table 1: Bounds of the stable rank of $A(\alpha, \beta, \gamma)$.

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