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GLOBAL BIFURCATION OF CAPILLARY-GRAVITY STRATIFIED WATER WAVES

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Abstract. We study steady periodic water waves with variable vorticity and density, where we allow for stagnation points in the flow, and where we admit the capillarity effects of surface tension. Using global bifurcation theory, we extend a local curve of nontrivial solutions of the governing equations to a global continuum of solutions. Furthermore, we obtain a description of the behaviour of the solutions along this continuum.

1. Introduction

In this paper we consider a model for stratified water waves which experience the capillarity effects of surface tension. Stratified water waves are flows where the fluid density varies as a function of the streamlines. Physically, the density of a fluid may vary significantly when certain factors— such as the salinity, temperature, pressure, topography, or oxygenation—of the fluid body fluctuates dramatically, cf. the discussion in [19,29]. Surface tension plays a key role for small to medium amplitude water waves, and in particular for waves which are wind generated. For such wind generated waves there is an appreciable layer of vorticity adjacent to the surface, cf. [8,17]. All of these factors are encompassed in the model which is considered in this paper.

The first results concerning the existence of small-amplitude waves for stratified flows were obtained by Dubreil-Jacotin [11], in 1937. More recently, rigorous results concerning the existence of small and large amplitude stratified flows were obtained via bifurcation methods in [37], building on techniques first applied to rotational homogeneous flows in [6]. The existence of small and large amplitude waves for stratified flows with the additional complication of surface tension was then addressed in [38], building on results in [35]. These papers all excluded stagnation points from their models. The presence of stagnation points in a fluid has long been a source of great mathematical and physical interest, dating back to Kelvin's work concerning *cat's eyes* and Stokes conjecture on the wave of greatest height (see [2, 19, 34] for discussions of these phenomena). Allowing for stagnation points adds massive complications to the mathematical problem (cf. [7]), but we note that there have been two recent papers [9, 36] where different approaches were successfully used, for homogeneous fluids, to prove the existence of small amplitude waves with constant vorticity where stagnation points or critical layers occur. Recently, in [12], for stratified flows which have a linear density distribution, the existence of small amplitude waves was established. The existence of large amplitude waves was with capillary effects and stagnation points traveling over a homogeneous fluids

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was proved only recently [32]. In this paper, we allow for both variable vorticity and variable density, subject to conditions (A1)-(A3), (B1)-(B3) of Section 3 holding.

Using global bifurcation techniques we construct a global continuum of steady periodic stratified water waves, which is either unbounded or contains a wave of largest admissible amplitude, and which extends the local bifurcations curves of [19]. The relevant governing equations are reformulated as a nonlinear, nonlocal operator, and, building on the results of [19], we find a local bifurcation curve of non-trivial solutions via the Crandall-Rabinowitz [1,10] local bifurcation theorem. In this paper, in Theorem 4.1, we show that this formulation of the governing equations is amenable to global bifurcation techniques, and accordingly we can extend the local bifurcation curves to a global continuum of solutions. The continuum contains waves of large amplitude in the sense that the solutions either become unbounded, or they approach a wave which has the largest admissible amplitude, as defined in (2.1). The outline of the paper is the following: after presenting in Section 2 the mathematical model for capillary-gravity stratified water waves we recall in Section 3 the local bifurcation results obtained in [19]. Section 4 is dedicated to the proof of our main result, Theorem 4.1.

2. The mathematical model

In this model we consider a two-dimensional inviscid, incompressible fluid with variable density $\rho > 0$ occupying the domain

$$\Omega_\eta := \{(x,y) \,:\, x \in \mathbb{S} \text{ and } -1 < y < \eta(t,x)\},$$

where the unknown wave surface profile η is assumed to satisfy

$$|\eta(t,x)| < 1 \qquad \text{for all } (t,x). \tag{2.1}$$

The symbol $\mathbb{S}$ denotes the unit circle, whereby functions on $\mathbb{S}$ are identified with $2\pi-$periodic functions on $\mathbb{R}$. The line $y = 0$ represents the location of the mean water level, and so for any fixed time t we have

$$\int_{\mathbb{S}} \eta(t,x)\,dx = 0. \tag{2.2}$$

We seek traveling wave solutions of the governing equations, which presupposes a functional $(x,y,t)-$dependence of the form $(x-ct,y)$:

$$\eta(t,x) = \eta(x-ct), \qquad (\rho,u,v,P)(t,x,y) = (\rho,u,v,P)(x-ct,y),$$

where (u,v) is the velocity field, P is the pressure function and $c > 0$ is the constant wave speed. This assumption is equivalent to the flow being steady in the frame of reference moving with the wave speed c, and in this frame the fluid motion is prescribed by the two-dimensional Euler equations and the continuity equation

$$\left\{\begin{array}{rcll} \rho(u-c)u_x + \rho v u_y &=& -P_x & \text{in } \Omega_\eta, \\ \rho(u-c)v_x + \rho v v_y &=& -P_y - g\rho & \text{in } \Omega_\eta, \\ u_x + v_y &=& 0 & \text{in } \Omega_\eta, \end{array}\right. \tag{2.3a}$$

where we have imposed the non-diffusive condition

$$\rho_x(u-c) + \rho_y v = 0, \tag{2.3b}$$

cf. [26, 39]. The boundary conditions are prescribed by

$$\begin{cases} v &= (u-c)\eta' & \text{on } y=\eta(x), \\ P &= P_0 - \sigma\eta''/(1+\eta'^2)^{3/2} & \text{on } y=\eta(x), \\ v &= 0 & \text{on } y=-1, \end{cases} \tag{2.3c}$$

with P_0 being the constant atmospheric pressure at the surface, σ the coefficient of surface tension and g is the gravitational constant of acceleration. The first kinematic surface condition in (2.3c) implies a non-mixing condition, namely that the wave surface consists of the same fluid particles for all times, while the last kinematic boundary condition in (2.3c) ensures that the bottom of the ocean is impermeable. The contribution of the surface tension is felt on the free surface where it exerts a force which is proportional to the curvature of the surface at any given point, the constant of proportionality being the coefficient of surface tension [2, 25]. The continuity equation (2.3a) and relation (2.3b) enable us to define the pseudo-streamfunction ψ by

$$\partial_x\psi = -\sqrt{\rho}v \quad \text{and} \quad \partial_y\psi = \sqrt{\rho}(u-c) \qquad \text{in } \overline{\Omega}_\eta. \tag{2.4}$$

The level sets of this function are the streamlines of the steady flow, and ψ is formulated for $\rho > 0$ by the expression

$$\psi(x,y) := \lambda + \int_{-1}^{y} \sqrt{\rho}(x,s)(u(x,s)-c)\,ds, \qquad (x,y)\in\overline{\Omega}_\eta.$$

Since ψ is constant on the free surface we have the freedom to choose λ such that $\psi = 0$ on $y = \eta(x)$. We then have

$$\lambda = \int_{-1}^{\eta(x)} \sqrt{\rho}(x,s)(c-u(x,s))\,ds,$$

where λ is a constant called the *mass flux*. We note that, for waves without stagnation points, without underlying currents containing strong non-uniformities, and which are not near breaking, the mass flux will be positive, $\lambda > 0$, cf. the discussion in [19]. However, we do not exclude points of stagnation in this paper, and so λ is not restricted to being strictly positive. Bernoulli's theorem states that the quantity

$$E := P + \rho\frac{(u-c)^2+v^2}{2} + g\rho y \tag{2.5}$$

is constant along streamlines, and in particular at the wave surface we obtain

$$\frac{|\nabla\psi|^2}{2} - \sigma\frac{\eta''}{(1+\eta'^2)^{3/2}} + g\rho(0)y = Q \qquad \text{on } y=\eta(x)$$

for some constant $Q \in \mathbb{R}$, where Q is known as the *hydraulic head* of the flow. The governing equations (2.3) can now be reformulated in terms of ψ, giving us the Long-Yih governing equations [26, 39] for steady stratified water waves:

$$\Delta\psi = l(y,\psi) \qquad \text{in } \Omega_\eta, \tag{2.6a}$$

$$\psi = 0 \qquad \text{on } y=\eta(x), \tag{2.6b}$$

$$\psi = \lambda \qquad \text{on } y=-1, \tag{2.6c}$$

$$\frac{|\nabla\psi|^2}{2} - \sigma\frac{\eta''}{(1+\eta'^2)^{3/2}} + g\rho(0)y = Q \qquad \text{on } y=\eta(x), \tag{2.6d}$$

where we define $l : [-1,1]\times\mathbb{R}\to\mathbb{R}$ to be the function

$$l(y,\psi) \;:=\; gy\rho'(-\psi)+\beta(-\psi), \qquad (y,\psi)\in[-1,1]\times\mathbb{R}.$$

Here β is the so-called Bernoulli function, which corresponds to vorticity in the setting of homogeneous fluid, and we note that it can be shown that ρ and β are indeed constant along streamlines, cf. [19]. The velocity formulation (2.3) is equivalent to (2.6) if we assume that $u<c$ in $\overline{\Omega}_\eta$. On the other hand, even if the latter condition is not satisfied, any solution of (2.6) defines a solution of the velocity formulation (2.3). We also note, from the divergence structure of the curvature operator

$$\kappa(\eta) := \frac{\eta''}{(1+\eta'^2)^{3/2}} = \left(\frac{\eta'}{(1+\eta'^2)^{1/2}}\right)' \qquad \text{for } \eta\in\mathcal{V},$$

and from (2.2), that the head Q can be determined from (2.6d), namely

$$Q = \int_{\mathbb{S}} \frac{|\nabla\psi|^2}{2}(x,\eta(x))\,dx, \tag{2.7}$$

if we normalize the integral such that $\int_{\mathbb{S}} 1\,dx = 1$. Here, we define

$$\mathcal{V} := \{\eta\in \widehat{C}^{2+\alpha}_{e,k}(\mathbb{S}) \,:\, |\eta|<1\}.$$

where the subspace $\widehat{C}^{m+\alpha}_{e,k}(\mathbb{S})$, $m\in\mathbb{N}$, of the Hölder space $C^{m+\alpha}(\mathbb{S})$ consists of even functions which are $2\pi/k-$periodic and have integral mean zero, and we define $C^{m+\alpha}_{e,k}(\overline{\Omega})$, as the subspace of $C^{m+\alpha}_{per}(\overline{\Omega})$ containing only even and $2\pi/k-$periodic functions in the x variable. The evenness condition imposes a symmetry on the free-surface (and on the underlying flow), which is natural for homogeneous rotational water waves [3,4]. Although we seek solutions of the problem (2.6) with the property that $\eta\in\widehat{C}^{m+\alpha}_{e,k}(\mathbb{S})$, the wave surface has in fact more regularity. Indeed, in the absence of stagnation points, it was shown in [18] that if the Bernoulli function is Hölder continuous and the variable density function has a first derivative which is Hölder continuous, then the free-surface profile is the graph of a smooth function. Moreover, if the Bernoulli function and the streamline density function are both real analytic functions then all of the streamlines, including the wave profile, are real analytic for all three physical regimes. These results concur with the recent literature determining the *a priori* regularity of water waves with vorticity [5, 9, 15, 16, 27, 28, 30, 31].

3. Local bifurcation

In this section we present the main local bifurcation result of [19], while outlining details which are most pertinent to reformulating (2.6) in a form which renders it amenable to the global bifurcation analysis of the next Section. The stratified capillary-gravity water wave problem (2.6) is an over-determined semilinear Dirichlet system, with (2.7) an additional boundary condition to be satisfied on the wave surface. In [19] it was proven that, for any fixed λ, the semilinear system (2.6a)–(2.6c) is well-posed when the following conditions hold:

(A1) $\rho\in C^{4-}(\mathbb{R},(0,\infty)), \beta\in C^{3-}(\mathbb{R}),$ and $\beta,\rho'\in BC^2(\mathbb{R})$;

(A2) $\partial_\psi l(y,\psi)\geq 0$ for all $(y,\psi)\in[-1,1]\times\mathbb{R}$;

(A3) $2l(y,\psi)+g(1+y)\rho'(-\psi)\leq 0$ for all $(y,\psi)\in[-1,0]\times(-\infty,0]$.

Here, $BC^2(\mathbb{R})$ is the subspace of $C^2(\mathbb{R})$ which contains only functions with bounded derivatives up to order 2, while, given $m \in \mathbb{N}$ with $m \geq 1$,

$$C^{m-}(\mathbb{R}) := \{h \in C^{m-1}(\mathbb{R}) \,:\, h^{(m-1)} \text{ is Lipschitz continuous}\}.$$

A consequence of this result is the existence of laminar-flow solutions

$$\psi := \psi_\lambda(y) \tag{3.1}$$

to (2.6). Laminar flows have a flat surface, $\eta(x) \equiv 0$, with non-trivial underlying sheared fluid motion. Further to this, the existence of non-laminar flow solutions to the entire system (2.6), which correspond to waves with non-trivial undulating free-surfaces, was proven in [19] by recasting the water wave problem (2.6) in terms of a non-local, nonlinear operator reformulation, which we will present in (3.2) below. By calculating the relevant Fréchet derivatives of this operator, and employing certain restrictions on the density and Bernoulli function, namely

- (B1) $\partial_{\psi\psi} l \geq 0$ on $[-1, 0] \times (-\infty, 0]$;
- (B2) $2\partial_\psi l(y, \psi) - g(1 + y)\rho''(-\psi) \geq 0$ for all $(y, \psi) \in [-1, 0] \times (-\infty, 0]$;
- (B3) $\partial_\psi l(y, 0) \leq 2$ for all $y \in [-1, 0]$,

it was then shown that there exists a sequence of values of the bifurcation parameter λ for which the Crandall-Rabinowitz local bifurcation theorem applies. Regarding the additional assumptions (B1)-(B3), we remark that for the particular setting of homogeneous flows (ρ constant), we can see that (B1) requires that the vorticity function $\gamma''(-\psi) \geq 0$. Also, in the case of homogeneous flows, (B2) is simply equivalent to (A2), and (B3) requires that on the surface $\gamma'(0) \geq -2$. In particular, for homogeneous flows, our analysis covers the constant vorticity, and in particular the irrotational case, when ρ=const. and β =const.. We note that, as we will outline below, our analysis places a restriction on the wavenumber k, which greatly simplifies the local bifurcation problem. The local bifurcation problem for capillary-gravity waves, which does not make such restrictions on the wave-number, was first rigorously studied in [20–22] in the context of homogeneous, irrotational waves. It is immediately evident that in this physical setting the local bifurcation phenomena is very involved, and so the restrictions that we place on the wavenumber k are vital for our local and global bifurcation analysis to apply.

3.1. **The operator reformulation.** Assuming that (A1)-(A3) are satisfied, the Long-Yih governing equations (2.6) are equivalent to a nonlinear and nonlocal equation for a compact perturbation of the identity. This will be seen below when we reformulate the problem by exploiting the invertibility of the curvature operator, allowing us to implement the global bifurcation results from [24] in the analysis of our reformulated problem. To proceed, we define, for each $\eta \in \mathcal{V}$, the mapping $\Phi_\eta : \Omega = \mathbb{S} \times (-1, 0) \to \Omega_\eta$ by the relation

$$\Phi_\eta(x, y) := (x, (y + 1)\eta(x) + y), \qquad (x, y) \in \Omega.$$

This operator Φ_η is a diffeomorphism for all $\eta \in \mathcal{V}$, and it transforms the unknown domain problem (2.6) onto the rectangle Ω. Corresponding to the semilinear elliptic operator in (2.6a) we introduce the transformed elliptic operator $\mathcal{A} : \mathcal{V} \times C^{2+\alpha}_{e,k}(\overline{\Omega}) \to C^{\alpha}_{e,k}(\overline{\Omega})$ with

$$\mathcal{A}(\eta, \tilde{\psi}) := \Delta(\tilde{\psi} \circ \Phi_\eta^{-1}) \circ \Phi_\eta - l(y, \tilde{\psi} \circ \Phi_\eta^{-1}) \circ \Phi_\eta, \qquad (\eta, \tilde{\psi}) \in \mathcal{V} \times C^{2+\alpha}_{e,k}(\overline{\Omega}).$$

Furthermore, corresponding to (2.6d), we define the boundary operator $\mathcal{B} : \mathcal{V} \times C^{2+\alpha}_{e,k}(\overline{\Omega}) \to C^{\alpha}_{e,k}(\mathbb{S})$ by the relation

$$\mathcal{B}(\eta, \tilde{\psi}) := \frac{\operatorname{tr} |\nabla(\tilde{\psi} \circ \Phi_\eta^{-1})|^2 \circ \Phi_\eta}{2},$$

with tr denoting the trace operator with respect to $\mathbb{S} = \mathbb{S} \times \{0\}$. The problem (2.6) can now be reformulated as the following non-local and non-linear equation

$$\Psi(\sigma, \lambda, \eta) = \kappa(\eta) \tag{3.2}$$

where $\Psi : (0, \infty) \times \mathbb{R} \times \mathcal{V} \to \widehat{C}^{\alpha}_{e,k}(\mathbb{S})$ is given by

$$\Psi(\sigma, \lambda, \eta) := \frac{1}{\sigma} \left[\mathcal{B}(\eta, \mathcal{T}(\lambda, \eta)) + g\rho(0)\eta - \int_{\mathbb{S}} \mathcal{B}(\eta, \mathcal{T}(\lambda, \eta))\, dx \right], \tag{3.3}$$

with $\mathcal{T} : \mathbb{R} \times \mathcal{V} \to C^{2+\alpha}_{e,k}(\overline{\Omega})$ being the solution operator to the semi-linear Dirichlet problem

$$\begin{cases} \mathcal{A}(\eta, \tilde{\psi}) &= 0 \quad \text{in} \quad \Omega, \\ \tilde{\psi} &= 0 \quad \text{on} \quad y = 0, \\ \tilde{\psi} &= \lambda \quad \text{on} \quad y = -1. \end{cases} \tag{3.4}$$

In [19] it was shown, using ideas from [13], that the assumptions (A1)-(A3) ensure that $\mathcal{T}$ is well-defined between these spaces, that the problems (2.6) and (3.2) are equivalent, and furthermore:

- $\mathcal{T} \in C^2(\mathbb{R} \times \mathcal{V}, C^{2+\alpha}_{e,k}(\overline{\Omega}))$;
- Since $\mathcal{B}$ depends analytically on $(\eta, \tilde{\psi})$, we have

$$\Psi \in C^2((0, \infty) \times \mathbb{R} \times \mathcal{V}, \widehat{C}^{\alpha}_{e,k}(\mathbb{S})). \tag{3.5}$$

The following Lemma enables us to reformulate the problem (3.2) in a form which is most appropriate for the global bifurcation methods of the next Section, by expressing it as a compact perturbation of the identity.

Lemma 3.1. *Let $\mathcal{R}$ denote the open subset of $\widehat{C}^{\alpha}_{e,k}(\mathbb{S})$ defined by*

$$\mathcal{R} := \left\{ \xi \in \widehat{C}^{\alpha}_{e,k}(\mathbb{S}) \ : \ \sup_{[0,2\pi]} \left| \int_0^x \xi(x)\, dx \right| < 1 \right\}.$$

Then, the operator $\kappa : \mathcal{V} \to \mathcal{R}$ is a diffeomorphism.

Proof. The fact that curvature operator is bijective follows from its definition. Let now $\xi \in \mathcal{R}$ be given. If $\kappa(\eta) = \xi$, then we obtain by integration that

$$\frac{\eta'(x)}{(1 + \eta'^2(x))^{1/2}} = \zeta(x) := \int_0^x \xi(t)\, dt \qquad \text{for all } x \in \mathbb{R}.$$

Therefore η' and ζ have the same sign and

$$\eta' = \frac{\zeta}{(1 - \zeta^2)^{1/2}}.$$

Integrating once more with respect to time and using the fact that η has integral mean zero, we conclude that curvature operator $\kappa : \mathcal{V} \to \mathbb{R}$ is invertible and its inverse is given by

$$\kappa^{-1}(\xi) := \int_0^x \frac{\zeta(t)}{(1-\zeta^2(t))^{1/2}}\,dt - \frac{1}{2\pi}\int_0^{2\pi}\int_0^x \frac{\zeta(t)}{(1-\zeta^2(t))^{1/2}}\,dtdx \tag{3.6}$$

for $\xi \in \mathcal{R}$, whereby ζ is the odd anti-derivative of ξ. □

In virtue of Lemma 3.1, we conclude that if (σ,λ,η) is a solution of problem (3.2), then $(\sigma,\lambda,\eta) \in \mathcal{U} := \Psi^{-1}(\mathcal{R})$. With this notation, the problem (3.2) can be reformulated as

$$F(\sigma,\lambda,\eta) := \eta + f(\sigma,\lambda,\eta) = 0 \qquad \text{in } \widehat{C}^{2+\alpha}_{e,k}(\mathbb{S}), \tag{3.7}$$

where $f \in C^\omega(\mathcal{U}, \widehat{C}^{2+\alpha}_{e,k}(\mathbb{S}))$ is the mapping defined by

$$f(\sigma,\lambda,\eta) := -\kappa^{-1}(\Psi(\sigma,\lambda,\eta)) \qquad \text{for } (\sigma,\lambda,\eta) \in \mathcal{U}. \tag{3.8}$$

3.2. **Local bifurcation results.** We note that $(\lambda,\sigma,0) \in \mathcal{U}$ for all $\lambda \in \mathbb{R}$ and that

$$F(\sigma,\lambda,0) = 0. \tag{3.9}$$

. The local existence Theorem 3.2 follows upon showing that, for certain values of λ, the operator $F_\eta(\lambda,\sigma,0)$ satisfies the relevant criteria of the Crandall-Rabinowitz theorem [1, 10], namely: it is a Fredholm operator of index zero, with a one-dimensional kernel, and it satisfies the transversality condition. The Fréchet deriviative $F_\eta(\lambda,\sigma,0)$ takes the form of a Fourier multiplier, with weights determined as follows. Let the Fourier series expansions of η be

$$\eta = \sum_{m=1}^{\infty} a_m \cos(mkx),$$

then, as in [19], we obtain that

$$F_\eta(\lambda,0)\sum_{m=1}^{\infty} a_m \cos(kmx) = \sum_{m=1}^{\infty} \mu_m(\sigma,\lambda) a_m \cos(kmx),$$

with

$$\mu_m(\sigma,\lambda) := 1 + \frac{1}{\sigma(km)^2}\left(-\psi_\lambda'^2(0) + \psi_\lambda'(0)w_m'(0) + g\rho(0)\right), \tag{3.10}$$

where w_m is the unique solution of the boundary value problem

$$\begin{cases} w_m'' - ((mk)^2 + \partial_\psi l(y,\psi_\lambda))w_m &= b_m, \ -1 < y < 0, \\ w_m(0) = w_m(-1) &= 0, \end{cases} \tag{3.11}$$

and

$$b_m(y) := 2l(y,\psi_\lambda(y)) + g(1+y)\rho'(-\psi_\lambda(y)) - (mk)^2(1+y)\psi_\lambda'(y).$$

The following local existence results derives from a careful analysis of the Fourier weights $\mu_m(\sigma,\lambda)$, and subsequent application of the Crandall-Rabinowitz local bifurcation theorem.

Theorem 3.2 ([19, Theorem 4.6]). *Let ρ and β be given such that* (A1)-(A3) *and* (B1)-(B3) *are satisfied and let $\sigma > 0$ be fixed. There exists a positive integer $K \in \mathbb{N}$ and for all $k \geq K$ a sequence $(\lambda_m)_{m\geq 1} \subset \mathbb{R}$ with $\lambda_m \to -\infty$ and:*

(i) *Given* $m \geq 1$, *there exist a continuously differentiable curve* $(\lambda_m, \eta_m) : (-\varepsilon, \varepsilon) \to \mathbb{R} \times \mathcal{V}$ *consisting only of solutions of* (3.2), *that is* $\Psi(\sigma, \lambda_m(s), \eta_m(s)) = 0$ *for all* $|s| < \varepsilon$.

(ii) *We have the following asymptotic relations*

$$\lambda_m(s) = \lambda_m + O(s), \quad \eta_m(s) = -s\cos(mkx) + O(s^2) \qquad \text{for } s \to 0.$$

All the solutions of (3.2) *close to* $(\sigma, \lambda_m, 0)$ *are either laminar flows or belong to the curve* $(\sigma, \lambda_m, \eta_m)$. *Moreover, there exists a constant* $\Lambda_- \in \mathbb{R}$ *with the property that if* $(\sigma, \lambda, 0)$ *is a bifurcation point of* (3.2) *with* $\lambda \in (-\infty, \Lambda_-)$, *then* $\lambda \in \{\lambda_m : m \geq 1\}$.

The restriction on k means physically that we consider the bifurcation problem for waves with a small wavelength. We also emphasize that this restriction rules out the possibility of double local bifurcation, which might occur in general only for small values of the parameter λ, cf. [20–22]. For irrotational waves, cf. [20–22], the possibility of a double bifurcation depends strictly on the value of the coefficient of surface tension σ.

4. Global bifurcation

The aim of this section is to extend the local curve $(\sigma, \lambda_m, \eta_m)$ to a global continuum $\mathcal{C}$, consisting of solutions of (2.6), and to describe the behavior of the solutions along this continuum. Our main result is the following.

Theorem 4.1. *Let* $\sigma \in (0, \infty)$ *be fixed and let the assumptions of Theorem 3.2 be satisfied. Moreover, let* $\mathcal{C}$ *be the maximal connected component of the set*

$$\{(\sigma, \lambda, \eta) : (\sigma, \lambda, \eta) \in \mathcal{U} \text{ is a solution of } (3.9)\}$$

containing $(\sigma, \lambda_m, 0)$.. *Then, we have:*

(i) $\mathcal{C}$ *is unbounded in* $\{\sigma\} \times (-\infty, \Lambda_-) \times \widehat{C}^{2+\alpha}_{e,k}(\mathbb{S})$, *or*

(ii) $\sup\limits_{(\sigma,\lambda,\eta)\in\mathcal{C}} \max\limits_{[0,2\pi]} |\eta| = 1.$

A key component in the proof of Theorem 4.1 is the following version of the Rabinowitz global bifurcation theorem [1,33], which applies to operators which are compact perturbations of the identity:

Theorem 4.2. *Let* X *be a Banach space and* $\mathcal{O} \subset \mathbb{R} \times X$ *a bounded and open set. Assume that*

(a) *the function* $F(\lambda, x) := x + f(\lambda, x)$ *belongs to* $C^1(\mathcal{O}, X)$,

(b) $f : \overline{\mathcal{O}} \to X$ *is completely continuous,*

and let $\mathcal{S}$ *denote the set of nontrivial solutions of the equation* $F(\lambda, x) = 0$. *If* $F_x(\lambda, 0)$ *has an odd crossing number at* $\lambda = \lambda_m$, *with* $(\lambda_m, 0) \in \mathcal{O}$, *then* $(\lambda_m, 0) \in \mathcal{S}$ *and the connected component* $\mathcal{C}$ *to which* $(\lambda_m, 0)$ *belongs*

(i) *intersects the boundary of* $\partial\mathcal{O}$, *or*

(ii) *contains some* $(0, \lambda_*) \in \mathcal{O}$ *with* $\lambda_m \neq \lambda_*$.

Proof. The proof follows as [24, Theorem II.3.3]. □

The idea of an odd crossing number, as stated in Theorem 4.2, goes as follows. To study the bifurcation at $(\lambda_m, 0)$ we need to know how the isolated eigenvalue 0 of $F_x(\lambda, 0)$ is perturbed when λ varies in a neighborhood of λ_m. To this end, we define the 0-group of $F_x(\lambda, 0)$ as being the set consisting of the perturbed eigenvalues of $F_x(\lambda,0)$ near 0, which depend continuously on λ, cf. [23]. We define $s(\lambda)=1$ if there are no negative real eigenvalues in the 0−group of $F_x(\lambda,0)$ and $s(\lambda)=(-1)^{i_1+\ldots+i_k}$ if $\mu_1,\ldots,\mu_k$ are all negative real eigenvalues in the 0-group having algebraic multiplicities $i_1,\ldots,i_k$, respectively. If

$$\begin{aligned}&F_x(\lambda,0) \text{ is bijective for } \lambda\in(\lambda_m-\delta,\lambda_m)\cup(\lambda_m,\lambda_m+\delta)\\ &\text{and } s(\lambda) \text{ changes at } \lambda=\lambda_m,\end{aligned} \tag{4.1}$$

then $F_x(\lambda,0)$ has an odd crossing number at $\lambda=\lambda_m$, cf. [24, Definition II.3.1]. As a consequence of $F_x(\lambda,0)$ having an odd crossing number at $\lambda=\lambda_m$, the index $i(F_x(\lambda,0),0)$ jumps at $\lambda=\lambda_m$ from -1 to $+1$ or vice versa.

Proof of Theorem 4.1. The idea of the proof is as follows. We assume that neither of the alternatives $(i)-(ii)$ in Theorem 4.1 holds for $\mathcal{C}$. We show that this assumption implies that f in (3.8) is completely continuous, and hence the operator F in (3.7) is a compact perturbation of the identity. By invoking the Rabinowitz global bifurcation Theorem 4.2 we obtain a contradiction.

So, assume that $(i)-(ii)$ do not hold. We claim that $\mathcal{C}$ is then a compact subset of $\mathcal{U}$. Indeed, let $((\sigma,\lambda_p,\eta_p))_p\subset\mathcal{C}$ be a bounded sequence in $\mathcal{C}$. If ε is sufficiently small, since the assumptions (i) and (ii) of the Theorem do not hold, we are guaranteed that $-\varepsilon^{-1}\le\lambda_p\le\Lambda_--\varepsilon$ for all $p\in\mathbb{N}$. Whence, there exists a converging subsequence of $(\lambda_p)_p$, which we also denote by $(\lambda_p)_p$. On the other hand, by the invalidity of (i) and (ii), we may assume that $\varepsilon>0$ is sufficiently small to guarantee that $\max_{[0,2\pi]}|\eta_p|\le 1-\varepsilon$ and $\|\eta_p\|_{2+\alpha}\le\varepsilon^{-1}$ for all $p\in\mathbb{N}$. Pick now $\beta\in(0,\alpha)$ and a subsequence (denoted again by $(\eta_p)_p$) which converges in $\widehat{C}^{2+\beta}_{e,k}(\mathbb{S})$ towards a function η. Then, since $\mathcal{T}$ is continuous, we deduce that $(\mathcal{T}(\lambda_p,\eta_p))_p$ is a bounded sequence in $C^{2+\beta}_{e,k}(\overline{\Omega})$. Particularly, this implies together with the assumption (A1) that $(l(y,\mathcal{T}(\lambda_p,\eta_p)\circ\Phi_\eta^{-1})\circ\Phi_\eta)_p$ is a bounded sequence in $C^{\alpha}_{e,k}(\overline{\Omega})$. We also observe that $\mathcal{T}(\lambda_p,\eta_p)$ solves for each $p\in\mathbb{N}$ the Dirichlet problem (3.4) when $\lambda=\lambda_p$ and $\eta=\eta_p$, the leading part of $\mathcal{A}(\eta_p,\tilde\psi)$ being a linear and uniformly elliptic operator

$$\Delta(\tilde\psi\circ\Phi_{\eta_p}^{-1})\circ\Phi_\eta=\tilde\psi_{yy}-\frac{2(1+y)\eta_p'}{1+\eta_p}\tilde\psi_{xy}+\frac{1+(1+y)^2{\eta_p'}^2}{(1+\eta_p)^2}\tilde\psi_{yy}-(1+y)\frac{(1+\eta_p)\eta_p''-2\eta_p'^2}{(1+\eta_p)^2}\tilde\psi_y$$

with coefficients bounded in $C^{\alpha}_{e,k}(\overline{\Omega})$, uniformly with respect to p. Therefore, by Schauder's estimate, cf. [14, Theorem 6.6], we find a positive constant C such that $\|\mathcal{T}(\lambda_p,\eta_p)\|_{2+\alpha}\le C$ for all $l\in\mathbb{N}$. Invoking (3.2) and (3.3), we conclude that $\sup_p\|\eta_p''\|_{1+\alpha}<\infty$, so that (η_p) is a bounded sequence in $\widehat{C}^{3+\alpha}_{e,k}(\mathbb{S})$. Since $\widehat{C}^{3+\alpha}_{e,k}(\mathbb{S})$ is compactly embedded in $\widehat{C}^{2+\alpha}_{e,k}(\mathbb{S})$ there exists a subsequence of $(\eta_p)_p$ (not relabeled) which converges in $\widehat{C}^{2+\alpha}_{e,k}(\mathbb{S})$ to η. We still need to show that the limit point belongs to $\mathcal{C}$. Clearly, we have that $\eta\in\mathcal{V}$. Moreover, relation (3.2) is satisfied along the sequence $((\sigma,\lambda_p,\eta_p))_p$, so that it is satisfied also by the limit point (σ,λ,η). This shows that $(\sigma,\lambda,\eta)\in\mathcal{U}$, and consequently $(\sigma,\lambda,\eta)\in\mathcal{C}$.

Since $\mathcal{C}$ is compact in $\mathcal{U}$, we may cover $\mathcal{C}$ by a finite number of balls B_i, $i=1...N$, having the property that $\overline{B}_i\subset\mathcal{U}$ and $\sup_{(\sigma,\lambda,\eta)\in\mathcal{O}}\max_{[0,2\pi]}|\eta|=1-\varepsilon$ for some $\varepsilon>0$, whereby we set $\mathcal{O}:=\cup_{i=1...N}B_i$. The restriction of F to $\overline{\mathcal{O}}$ satisfies all the assumptions of Theorem 4.2. First, let us note that the arguments presented above show that f is a completely continuous map. Next, we study how the eigenvalues in the 0-group perturb when λ crosses $\lambda = \lambda_m$. This reduces to studying the behavior of $\mu_l(\lambda, \sigma)$ when λ is close to λ_m. To this end, we have from the proof of [19, Theorem 4.6] that

$$\partial_\lambda \mu_p(\sigma, \lambda_m) > 0,$$

which shows that $F_\eta(\sigma, \lambda, 0)$ has an odd crossing number at $\lambda = \lambda_m$. Using Theorem 4.2, we deduce, in virtue of $\mathcal{C} \subset \mathcal{O}$ that $\mathcal{C}$ must intersect a further bifurcation point λ_p of (3.8) with $p \neq m$. We claim that $p = Nm$ whereby $N \in \mathbb{N}$ satisfies $N \geq 2$. Indeed, if we restrict the bifurcation problem (3.8) to the space of $2\pi/(km)$ periodic functions, the set $\mathcal{C} \cap \left(\{\sigma\} \times \mathbb{R} \times \widehat{C}^{2+\alpha}_{e,m}(\mathbb{S})\right)$ satisfies non of the properties (i)-(ii), and our analysis shows that it contains a further bifurcation point λ_{Nm} with $N \geq 2$. Considering now the bifurcation problem from λ_{Nm} in the space $\{\sigma\} \times \mathbb{R} \times \widehat{C}^{2+\alpha}_{e,Nm}(\mathbb{S})$, our previous arguments show that there exists a further bifurcation point $\lambda_{N_1 m}$ on $\mathcal{C}$ with N_1 being a multiple of N. In this way we find a sequence $N_p \to \infty$ such that $\lambda_{N_p m}$ belongs to $\mathcal{C}$ for all $p \in \mathbb{N}$. But since $\lambda_n \searrow_{n\to\infty} -\infty$, this contradicts our assumption that (i) and (ii) do not hold, thereby proving the theorem. □

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