Causal Mediation Analysis for Difference-in-Difference Design and Panel Data

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Appendix A

$Y_{gt}(a) \perp A_{gt} | Y_{g0} \ \forall a \ \leftrightarrow \ Y_{gt}(a) - Y_{g0}(a) \perp A_{gt} \ \forall a$

Lemma 1. Random variables X, Y , Z satisfies $X \perp Y | Z \ \leftrightarrow \ E[X|Y, Z] = E[X|Z]$.

Proof:

$P\{X \le x, Y = y, Z = z\} = P\{X \le x, Y = y | Z = z\} P\{Z = z\}$

$= P\{X \le x | Z = z\} \times P\{Y = y | Z = z\} \times P\{Z = z\}$

$= P\{X \le x | Z = z\} \times P\{Y = y, Z = z\}$

Then we have

$P\{X \le x | Y = y, Z = z\} = \frac{P\{X \le x, Y = y, Z = z\}}{P\{Y = y, Z = z\}}$

$= \frac{P\{X \le x | Z = z\} \times P\{Y = y, Z = z\}}{P\{Y = y, Z = z\}}$

$= P\{X \le x | Z = z\}$

This proves lemma 1.

By iterated expectation and Lemma 1.

$$E\left[Y_{gt}(a) - Y_{g0}(a) \middle| A_{gt} = \mathrm{a}\right]$$
$$= \mathrm{E}[E[Y_{gt}(a) - Y_{g0}(a) | A_{gt} = \mathrm{a}, Y_{g0}]]$$
$$= \mathrm{E}[E[Y_{gt}(a) - Y_{g0}(a) | A_{gt} = \mathrm{a}]]$$
$$= \mathrm{E}[Y_{gt}(a) - Y_{g0}(a)]$$
$$\leftrightarrow \; Y_{gt}(a) - Y_{g0}(a) \perp A_{gt} \; \forall a$$

Appendix B

Assumptions

NEPT

(A1) $Y_{g0} = Y_{g0}(a)$, for all a

Exchangeability

(E1) $Y_{gt}(a) - Y_{g0}(a) \perp A_{gt} \; \forall a$

Detailed Process

$TE = E[Y_{gt}(1)] - E[Y_{gt}(0)]$

$= \{E[Y_{gt}(1)] - E[Y_{g0}]\} - \{E[Y_{gt}(0)] - E[Y_{g0}]\}$ (by (A1))

With assumptions (A1) and (E1), it could be identified as follows.

Let $\phi_{\mathrm{TE}}(\mathrm{a})$ be a causal parameter for TE.

$\phi_{\mathrm{TE}}(\mathrm{a}) := E[Y_{gt}(a)] - E[Y_{g0}]$

$= E[Y_{gt}(a)] - E[Y_{g0}(a)]$ (by (A1))

$= E[Y_{gt}(a) - Y_{g0}(a)]$

$= E[Y_{gt}(a) - Y_{g0}(a) | A_{gt} = a]$ (by (E1))

$= E[Y_{gt} - Y_{g0} | A_{gt} = a]$ (by consistency)

$= E[Y_{gt} | A_{gt} = a] - E[Y_{g0} | A_{gt} = a]$

$TE = \phi_{TE}(\mathrm{a}_1) - \phi_{TE}(\mathrm{a}_0)$

$= E[Y_{gt} - Y_{g0} | A_{gt} = \mathrm{a}_1] - E[Y_{gt} - Y_{g0} | A_{gt} = \mathrm{a}_0]$

$= \{E[Y_{gt} | A_{gt} = \mathrm{a}_1] - E[Y_{g0} | A_{gt} = \mathrm{a}_1]\} - \{E[Y_{gt} | A_{gt} = \mathrm{a}_0] - E[Y_{g0} | A_{gt} = \mathrm{a}_0]\}$

Appendix C

Assumptions

NEPT

(A2) $Y_{g0} = Y_{g0}(a, m)$, for all a and m

Exchangeability

(E1) $Y_{gt}(a) - Y_{g0}(a) \perp A_{gt} \; \forall a$

(E2) $Y_{gt}(a, m) - Y_{g0}(a, m) \perp M_{gt} \mid A_{gt} \; \forall a, m$

Detailed Process

$CDE(\mathrm{m}) = E\left[Y_{gt}(\mathrm{a}_1, m)\right] - E\left[Y_{gt}(\mathrm{a}_0, m)\right]$

$= \left\{E\left[Y_{gt}(\mathrm{a}_1, m)\right] - E[Y_{g0}]\right\} - \left\{E\left[Y_{gt}(\mathrm{a}_0, m)\right] - E[Y_{g0}]\right\}$ (by (A2))

With assumptions (A2), (E1) and (E2), it could be identified as follows.

Let $\phi_{\mathrm{CDE}}(\mathrm{a})$ be a causal parameter for CDE.

$\phi_{CDE}(a) := E\left[Y_{gt}(a, m)\right] - E[Y_{g0}]$

$= E\left[Y_{gt}(a, m)\right] - E\left[Y_{g0}(a, m)\right]$ (by (A2))

$= E\left[Y_{gt}(a, m) - Y_{g0}(a, m)\right]$

$= E\left[Y_{gt}(a, m) - Y_{g0}(a, m) | A_{gt} = a\right]$ (by (E1))

$= E\left[Y_{gt}(a, m) - Y_{g0}(a, m) | A_{gt} = a, M_{gt} = m\right]$ (by (E2))

$= E\left[Y_{gt} - Y_{g0} \middle| A_{gt} = a, M_{gt} = m\right]$ (by consistency)

$CDE(\mathrm{m}) = \phi_{CDE}(\mathrm{a}_1) - \phi_{CDE}(\mathrm{a}_0)$

$= E\left[Y_{gt} - Y_{g0} \middle| A_{gt} = \mathrm{a}_1, M_{gt} = m\right] - E\left[Y_{gt} - Y_{g0} \middle| A_{gt} = \mathrm{a}_0, M_{gt} = m\right]$

$= \left\{E\left[Y_{gt} \middle| A_{gt} = \mathrm{a}_1, M_{gt} = m\right] - E\left[Y_{g0} \middle| A_{gt} = \mathrm{a}_1, M_{gt} = m\right]\right\} -$

$\left\{E\left[Y_{gt} \middle| A_{gt} = \mathrm{a}_0, M_{gt} = m\right] - E\left[Y_{g0} \middle| A_{gt} = \mathrm{a}_0, M_{gt} = m\right]\right\}$

Appendix D

Assumptions

NEPT

(A1) $Y_{g0} = Y_{g0}(a)$, for all a

(A2) $Y_{g0} = Y_{g0}(a, m)$, for all a and m

(A3) $Y_{g0} = Y_{g0}\left(a, M_{gt}(a^*)\right)$, for all a and a^*

(A4) $M_{g0} = M_{g0}(a)$, for all a

Exchangeability

(E1) $Y_{gt}(a) - Y_{g0}(a) \perp A_{gt} \ \forall a$

(E2) $Y_{gt}(a, m) - Y_{g0}(a, m) \perp M_{gt} \mid A_{gt} \ \forall a, m$

(E3) $M_{gt}(a) - M_{g0}(a) \perp A_{gt} \ \forall a$

(E4) $Y_{gt}(a, m) - Y_{g0}(a, m) \perp M_{gt}(a^*) | A_{gt} \ \forall a, a^*, m$

Detailed Process

With assumptions (A1)-(A4) and (E1)-(E4).

Let $\phi_{\mathrm{NE,Y}}(a, a^*)$ be a causal parameter for the effect of M_{gt} on Y_{gt}.

$\phi_{\mathrm{NE,Y}}(a, a^*) := E\left[Y_{gt}\left(a, M_{gt}(a^*)\right)\right] - E[Y_{g0}]$

$= E\left[Y_{gt}\left(a, M_{gt}(a^*)\right)\right] - E\left[Y_{g0}\left(a, M_{gt}(a^*)\right)\right]$ (by (A3))

$= \int_m E\left[Y_{gt}(a, m) - Y_{g0}(a, m) | M_{gt}(a^*) = m\right] f_{M_{gt}(a^*)}(m) dm$

$= \int_m E\left[Y_{gt}(a, m) - Y_{g0}(a, m)\right] f_{M_{gt}(a^*)}(m) dm$ (by (E4))

$= \int_m E\left[Y_{gt}(a, m) - Y_{g0}(a, m) | A_{gt} = a\right] f_{M_{gt}(a^*)}(m) dm$ (by (E1))

$= \int_m E\left[Y_{gt}(a, m) - Y_{g0}(a, m) | A_{gt} = a, M_{gt} = m\right] f_{M_{gt}(a^*)}\left(m_{gt}\right) dm$ (by (E2))

$= \int_m E\left[Y_{gt} - Y_{g0} | A_{gt} = a, M_{gt} = m_{gt}\right] f_{M_{gt}(a^*)}(m) dm$ (by consistency)

And with (A4), (E3) and (C3), let $\phi_{\mathrm{NE,M}}\left(a_{gt}\right)$ be a causal parameter for the effect of A_{gt} on M_{gt}, we can also identify $\phi_{\mathrm{NE,M}}\left(a_{gt}\right)$:

$\phi_{\mathrm{NE,M}}\left(a_{gt}\right) := E\left[M_{gt}(a)\right] - E[M_{g0}]$

$= E\left[M_{gt}(a)\right] - E\left[M_{g0}(a)\right]$ (by (A4))

$= E\left[M_{gt}(a) - M_{g0}(a)\right]$

$= E\left[M_{gt}(a) - M_{g0}(a) \middle| A_{gt} = a\right]$ (by (E3))
$= E\left[M_{gt} - M_{g0} \middle| A_{gt} = a\right]$ (by consistency)

The rest of the identification is related to the variable type of M_{gt}, so it will be discussed in the section of estimation.

Since M_{gt} is binary, the integration of m could be reduced to a summation of $M_{gt} = 1$ and $M_{gt} = 0$.

$\phi_{\text{NE,Y}}(a, a^*)$
$= \sum_{m=0}^{1} E\left[Y_{gt} - Y_{g0} \middle| A_{gt} = a, M_{gt} = m\right] Pr\left(M_{gt} = m \middle| A_{gt} = a^*\right)$
$= Pr\left(M_{gt} = 1 \middle| A_{gt} = a^*\right)\left\{E\left[Y_{gt} - Y_{g0} \middle| A_{gt} = a, M_{gt} = 1\right] - E\left[Y_{gt} - Y_{g0} \middle| A_{gt} = a, M_{gt} = 0\right]\right\} + E\left[Y_{gt} - Y_{g0} \middle| A_{gt} = a, M_{gt} = 0\right]$

Then the process of identification is finished, it contains no more counterfactual terms. The original definition of NDE is $E\left[Y_{gt}\left(a_1, M_{gt}(a_0)\right) - Y_{gt}\left(a_0, M_{gt}(a_0)\right)\right]$ and NIE is $E\left[Y_{gt}\left(a_1, M_{gt}(a_1)\right) - Y_{gt}\left(a_1, M_{gt}(a_0)\right)\right]$, but by assumption (A3), it can be extended to $\phi_{\text{NE,Y}}(a_1, a_0) - \phi_{\text{NE,Y}}(a_0, a_0)$ and $\phi_{\text{NE,Y}}(a_1, a_1) - \phi_{\text{NE,Y}}(a_1, a_0)$. Then we can use the result of identified $\phi_{\text{NE,Y}}(a, a^*)$, and also estimate NDE and NIE by an DID estimator of Y.

$NIE = E\left[Y_{gt}\left(a_1, M_{gt}(a_1)\right) - Y_{gt}\left(a_1, M_{gt}(a_0)\right)\right]$
$= \phi_{\text{NE,Y}}(a_1, a_1) - \phi_{\text{NE,Y}}(a_1, a_0)$
$= \left\{\left[E\left[M_{gt} \middle| A_{gt} = a_1\right] - E\left[M_{g0} \middle| A_{gt} = a_1\right]\right] - \left[E\left[M_{gt} \middle| A_{gt} = 0\right] - E\left[M_{g0} \middle| A_{gt} = 0\right]\right]\right\}\left\{\left[E\left[Y_{gt} \middle| A_{gt} = a_1, M_{gt} = 1\right] - E\left[Y_{g0} \middle| A_{gt} = a_1, M_{gt} = 1\right]\right] - \left[E\left[Y_{gt} \middle| A_{gt} = a_1, M_{gt} = 0\right] - E\left[Y_{g0} \middle| A_{gt} = a_1, M_{gt} = 0\right]\right]\right\}$

Appendix E

$E[Y_{gt}|A_{gt} = a_{gt}] = \gamma_A a_{gt} + \gamma_t + \gamma_g$ (Model 1)
$E[Y_{gt}|A_{gt} = a_{gt}, M_{gt} = m_{gt}] = \theta_A a_{gt} + \theta_M m_{gt} + \theta_t + \theta_g$ (Model 2)
$E\left[M_{gt} \middle| A_{gt} = a_{gt}\right] = \beta_A a_{gt} + \beta_t + \beta_g$ (Model 3)

The equivalence of $\widehat{\gamma_A} - \widehat{\theta_A}$ and $\widehat{\theta_M}\widehat{\beta_A}$:
$E[Y_{gt}|A_{gt} = a_{gt}] = \sum_m E[Y_{gt}|A_{gt} = a_{gt}, M_{gt} = m_{gt}] \times Pr(M_{gt}|A_{gt} = a_{gt})$
$= \sum_m (\theta_A a_{gt} + \theta_M m_{gt} + \theta_t + \theta_g) \times Pr(M_{gt}|A_{gt} = a_{gt})$
$= \theta_A a_{gt} + \theta_t + \theta_g + \theta_M \sum_m m_{gt} \times Pr(M_{gt}|A_{gt} = a_{gt})$

$$= \theta_A a_{gt} + \theta_t + \theta_g + \theta_M E[M_{gt} | A_{gt} = a_{gt}]$$
$$= \theta_A a_{gt} + \theta_t + \theta_g + \theta_M (\beta_A a_{gt} + \beta_t + \beta_g)$$
$$= \theta_A a_{gt} + \theta_M \beta_A a_{gt} + (\theta_t + \theta_M \beta_t) + (\theta_g + \theta_M \beta_g)$$
$$\gamma_A a_{gt} + \gamma_t + \gamma_g = \theta_A a_{gt} + \theta_M \beta_A a_{gt} + (\theta_t + \theta_M \beta_t) + (\theta_g + \theta_M \beta_g)$$

$$\Rightarrow (\gamma_A - \theta_A) a_{gt} = \theta_M \beta_A a_{gt}$$

Appendix F

40 countries: Argentina, Australia, Austria, Bolivia, Brazil, Canada, Chile, Colombia Croatia, Denmark, Egypt, Finland, France, Germany, Indonesia, Iraq, Ireland, Israel, Italy, Japan, Malaysia, Mexico, Netherlands, New Zealand, Peru, Philippines, Poland, Portugal, Saudi Arabia, Singapore, South Korea, Spain, Sweden, Taiwan, Thailand, Turkey, United Arab Emirates, United Kingdom, United States, and Venezuela.

Appendix G

Table S1. The summary statistics for all variables across 40 countries and 12 months.

	Mean	**Minimum**	**Maximum**	**SD**
covid-19 outbreak	128	0	3137	311.5
Extent of mobility restriction	10.83	39.57	-2.133	7.363
Search volume	45.85	0	91	21.1

Appendix H

The built models for the search volumes of 'insomnia' and the extent of mobility restriction.

E [(search volume of keyword) $_{gt}$ | (the extent of national covid-19 outbreak) $_{gt}$] = γ_A × (the extent of national covid-19 outbreak) $_{gt}$ + γ_t + γ_g (**Model 4**)

E [(search volume of keyword) $_{gt}$ | (extent of mobility restriction) $_{gt}$, (the extent of national covid-19 outbreak) $_{gt}$] = θ_M × (extent of mobility restriction) $_{gt}$ + θ_A × (the extent of national covid-19 outbreak) $_{gt}$ + θ_t + θ_g (**Model 5**)

E [(extent of mobility restriction) $_{gt}$ | (the extent of national covid-19 outbreak) $_{gt}$] = β_A × (the extent of national covid-19 outbreak) $_{gt}$ + β_t + β_g (**Model 6**)

Appendix I

Table S2. Coefficient estimations in Model 2-4.

	Estimation	**SD**	**95%CI**	**P value**
γ_A	1.41	0.72	(-0.004, 2.82)	0.051
β_A	0.75	0.28	(0.19, 1.31)	0.008
θ_A	1	0.71	(-0.40, 2.39)	0.16
θ_M	0.55	0.12	(0.32, 0.77)	<0.001