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Abstract: In this paper we introduce a new approach to representing both TU-games and NTU-games as special economic structures. Instead of representing a game as a market – an exchange economy with concave utility functions – as in the extant literature, we represent an arbitrary game as a coalition production economy with a public good. The economy provides an indirect description of the game. Our model uses the idea of a social planner or arbitrager who, through arbitrage on productive activities, seeks to minimize the net cost of providing the public good subject to the constraint that the society satisfies its reservation welfare level. Note that in constrast to the prior literature on representing games as economic structures, we require neither that the game be balanced nor that there are large numbers of players. Our approach to representing a game as an economic structure uses techniques from consumer demand theory and, in particular, the notion of a compensated demand correspondence. The main results of this paper exhibit a relationship between cooperative game theory and consumer demand theory.

1 Introduction

In this paper we introduce a new approach to representing games with and without side payments, respectively, TU and NTU games, as special economic structures. These special economic structures are *coalition production economies with a public good*. A coalition production economy consists of a set of productive sectors that can produce either a single good or a number of differentiated goods. Associated to this economy there is a public good that affects the productivity of the sectors. (Imagine, for example, that this public good is aggregate employment.) We define a social welfare function for the society. A social planner plays the role of an arbitrager. The social planner organizes the production/consumption of the sectors according to a given vector of prices and costs and seeks to minimize the net production cost of the public good, provided that the society satisfies its reservation utility level. This minimization problem is called the *social planner's problem*.

From the perspective of cooperative game theory, the interest of our economic structures comes from the links that will be established between coalition production economies and cooperative games in characteristic form. Different examples of the construction of particular coalition production economies from any given TU or NTU game are studied in this paper. Our approach to representing a game as a coalition economic structure uses techniques from consumer demand theory. In particular the notion of a *compensated demand correspondence* and of an *expenditure function* are used in this paper for the reinterpretation of a game from a new economic point of view.[1] The main result of this paper exhibits a link between compensated demand correspondences that solve social planner's problems with best coalitions correspondences associated to the

[1] We refer the reader to [10] for an excellent introduction to these concepts.

indirect functions of games.

From cooperative game theory, *indirect functions* and *best coalitions correspondences* of TU-games and NTU-games, introduced in [8] and [11], respectively, are the basic tools to be used through all the paper. These functions and their associated best coalitions correspondences provide dual representations of games; they contain the same information as the corresponding characteristic functions.

The reader will observe that although we follow a different approach to those of Shapley and Shubik [12], Billera [1] and Billera and Bixby [2], we are building on ideas in those papers: economic structures induce games and games induce special economic structures. To obtain an equivalence between games and markets, the prior papers require totally balancedness; this is not the case in our paper since games and coalition production economies that are derived from games contain essentially the same information. We also note that we do not require that there be large numbers of players, as in [15] and [16], for example. Our approach appears to be quite distinct from any in the prior literature.

The remainder of this work is organized as follows. The definition of the special economic structure, the coalition production economy with a public good, and that of the associated social planner's problem on this economy are given in Section 2. The next section introduces some general definitions on both TU and NTU games. Particular examples of coalition production economies derived from games are studied in this section and the relationship between both structures is shown; in particular, compensated demand correspondences of social planner's problems are related to best coalitions correspondences. Different numerical examples are sketched through all this section.

2 Games with or without Transferable Utility

Let N be a finite set, called the set of *players,* with cardinality greater than or equal to 2. A subset S of N is called a *coalition.* A game with transferable utility, *a TU-game*, is a pair $(N; v)$ where $v : 2^N \to \mathbf{R}$ is a real valued function, called the *characteristic function* of the game, defined on the power set of N. Typically, it is required that $v(\emptyset) = 0$. A TU-game $(N; v)$ is *monotone* if $\forall S, T \subset N$ such that $S \subset T$, we have $v(S) \le v(T)$.

An *NTU-game*, or a game with non transferable utility, is a pair $(N; V)$ where V is a correspondence, called the *characteristic correspondence*, assigning a subset $V(S) \subset \mathbf{R}^S$ to each coalition $S \subset N$, such that

(a) $V(S)$ is nonempty closed and convex for all S,

(b) $\{x \mid x \in V(S) \text{ and } x_i \geq y,\ \forall i \in S,\ \forall y \in V(\{i\})\}$ is bounded for all S.

We can consider $V(S)$ as a subset of $\mathbf{R}^N$ by putting zeros in the coordinates that refer to players that are in $N \setminus S$; thus, for every $S \subset N$, we denote by $V'(S)$ the following set:

$$V'(S) = \left\{x \in \mathbf{R}^N : x_S \in V(S),\ x_{N \setminus S} = 0\right\}.$$

An NTU-game $(N; V)$ is *comprehensive* if for each coalition S and each vector x in $V(S)$, if $z \leq x$ then $z \in V(S)$. A game $(N; V)$ is *compactly generated* if there exists some compact subset of $\mathbf{R}^N$ such that, for each coalition S and for each allocation x in $V(S)$, there exists some allocation y in such that $(y_i)_{i \in S} \in V(S)$ and $x_i \leq y_i$ for every player i in S. In this case we say that (compactly) generates $(N; V)$. An NTU-game $(N; V)$ is *monotone* if $\forall S, T \subset N$ such that $S \subset T$, we have that $\forall x \in V(S)$ the allocation y such that $y_S = x$ and $y_{T \setminus S} = 0$ belongs to $V(T)$.

The *indirect function* of a TU-game $(N; v)$ is the function $\pi : \mathbf{R}^N \to \mathbf{R}$ defined by

$$\pi(w) = \max_{S \subset N} \left\{ v(S) - \sum_{i \in S} w_i \right\},\ \forall w \in \mathbf{R}^N. \tag{1}$$

A coalition $T \subset N$ is said to be a *best coalition* for $w \in \mathbf{R}^N$ if it solves the preceding maximization problem, i.e., if it satisfies

$$\pi(w) = v(T) - \sum_{i \in T} w_i.$$

In [8] it is shown that the indirect function provides a dual representation of a TU-game: each TU-game $(N; v)$ can be recovered from its indirect function π by

$$v(S) = \min_{w \in \mathbf{R}^N} \left\{ \pi(w) + \sum_{i \in S} w_i \right\},\ \forall S \subset N.$$

The *best coalitions correspondence* $\Phi : \mathbf{R}^N \to\to 2^N$ is defined to be the mapping assigning to each $w \in \mathbf{R}^N$ the set of all best coalitions for w:

$$\Phi(w) = \left\{ T \subset N : \pi(w) = v(T) - \sum_{i \in T} w_i \right\},$$

$$\forall w \in \mathbf{R}^N.$$

Note that this set is always nonempty because of the definition of the indirect function in (1).

On the other hand, the concept of *indirect function* for compactly generated and comprehensive NTU-games was introduced in [11] for its dual representation. The *indirect function* $\pi : \mathbf{R}^N_{++} \times \mathbf{R}^N \longrightarrow \mathbf{R}$ of a compactly generated and comprehensive NTU-game $(N; V)$ is defined by

$$\pi(\lambda, w) = \max_{S \subset N,\ x \in V(S)} \left\{ \sum_{i \in S} (\lambda_i x_i - w_i) \right\},\ \forall (\lambda, w) \in \mathbf{R}^N_{++} \times \mathbf{R}^N.$$

It is shown in [11] that the indirect function π of a compactly generated and comprehensive NTU-game $(N; V)$ contains all the information on the game itself. For any NTU-game the characteristic correspondence can be recovered by its indirect function π by

$$V(S) = \left\{ y \in \mathbf{R}^S :\ \sum_{i \in S} (\lambda_i y_i - w_i) \leq \pi(\lambda, w),\ \forall (\lambda, w) \in \mathbf{R}^N_{++} \times \mathbf{R}^N \right\},\ \forall S \subset N.$$

Similarly to the TU case, a coalition $T \subset N$ is said to be a *best coalition* for $(\lambda, w) \in \mathbf{R}^N_{++} \times \mathbf{R}^N$ if there exists some $x \in V(T)$, called a *best production*, for $T \subset N$ such that

$$\pi(\lambda, w) = \sum_{i \in T} (\lambda_i x_i - w_i).$$

The *best coalitions correspondence* $\Phi : \mathbf{R}^N_{++} \times \mathbf{R}^N \to\!\!\to \mathbf{R}^N \times 2^N$ associated to the NTU-game $(N; V)$ is the mapping assigning to each $(\lambda, w) \in \mathbf{R}^N_{++} \times \mathbf{R}^N$ the set of all pairs of best coalitions with their associated best productions:

$$\Phi(\lambda, w) = \left\{ (T, x') \in 2^N \times \mathbf{R}^N : x' \in V'(T),\ \pi(\lambda, w) = \sum_{i \in T} (\lambda_i x'_i - w_i) \right\},\ \forall (\lambda, w) \in \mathbf{R}^N_{++} \times \mathbf{R}^N.$$

Note that this correspondence takes nonempty values; the reason for this nonemptiness follows from the definition of the indirect function and from the fact that we are considering compactly generated and comprehensive NTU-games.

For the case of monotone games it is shown in [8] and [11] that we only need to know the restriction to $\mathbf{R}^N_+$ of the indirect function for representing the initial game. We can recover a monotone TU-game $(N; v)$ from its indirect function by

$$v(S) = \min_{w \in \mathbf{R}^N_+} \left\{ \pi(w) + \sum_{i \in S} w_i \right\},\ \forall S \subset N; \tag{2}$$

for a monotone, compactly generated and comprehensive NTU-game $(N;V)$, one has

$$V(S)=\left\{y\in\mathbf{R}^S:\ \sum_{i\in S}(\lambda_i y_i-w_i)\leq\pi(\lambda,w),\ \forall(\lambda,w)\in\mathbf{R}^N_{++}\times\mathbf{R}^N_+\right\},\ \forall S\subset N;$$

in the latter case, since the indirect function is homogeneous of degree 1 we can write

$$V(S)=\left\{y\in\mathbf{R}^S:\ \sum_{i\in S}(\lambda_i y_i-w_i)\leq\pi(\lambda,w),\ \forall(\lambda,w)\in\Delta\right\},\ \forall S\subset N,$$

where

$$\Delta=\left\{(\lambda,w)\in\mathbf{R}^N_{++}\times\mathbf{R}^N_+\ :\ \sum_{i=1}^n(\lambda_i+w_i)=1\right\}. \tag{3}$$

3 Coalition Production Economies

3.1 Coalition Production Economies with a Public Good

We define a *coalition production economy* as an economy with a set $N=\{1,2,...,n\}$ of *productive sectors* that produce/consume collectively some output $y\in\mathbf{R}$ of a good that can be sold (or, if y is negative, bought) at unitary price. A *coalition production economy with a public good* is a coalition production economy where the productivity of coalitions depends on a certain quantity t of a public good that will be consumed by the society; it is assumed that this good is in the interval $[0,1]$. Formally, associated to a coalition production economy with a public good is a *production function*

$$\mu:[0,1]\times 2^N\to\mathbf{R}$$

such that, for each $t\in[0,1]$ and $S\in 2^N$ (2^N denotes the power set[2] of N), $\mu(t,S)$ specifies the amount of the good that can be produced by the productive sectors in S for a certain quantity t of the public good. This production function is assumed to satisfy $\mu(t,\emptyset)=0$.

[2] The *power set* of a set F is

$$2^F=\{S:S\subset F\}.$$

The cost of having a productive sector $i \in N$ at full activity is an exogenous fixed cost denoted by $w_i \in \mathbf{R}_+$. The *net cost* of the economic activity is given by a function

$$c : \mathbf{R}_+^N \times [0,1] \times 2^N \times \mathbf{R}_+ \to \mathbf{R}.$$

For each $w \in \mathbf{R}_+^N$, $t \in [0,1]$, $S \in 2^N$ and $x \in \mathbf{R}_+$, $c(w,t,S,y)$ is the difference between the cost of the activity of the sectors in S and the income received from selling an output y.

It is assumed that the society is socially homogeneous, that is, the preferences of all individuals can be represented by the same utility function. The members of the society only care about the amount of the public good they can consume. Because of social homogeneity, the utility function U of any individual can be taken as a social welfare function; this accords with the unanimity property of social choice theory.

Associated with any coalition production economy with a public good there is a social planner who hires productive sectors and organizes production/consumption to minimize net costs while providing a certain reservation level of utility u_0. Formally, for given costs $w \in \mathbf{R}_+^N$ and reservation utility u_0, the *social planner's problem* is written as follows:

$$\begin{cases} \min\ c(w,t,S,\mu(t,S)) \\ s.t. \\ \quad t \in [0,1], \\ \quad S \subset N, \\ \quad U(t) \geq u_0. \end{cases}$$

Note that for this minimization problem w and u_0 are exogenously given to the social planner.

The *minimum expenditure* or *minimum net cost* for this problem is represented by the function $e : [0,1] \times \mathbf{R}_+^N \to \mathbf{R}$ defined by

$$e(u_0, w) = \inf \{c(w,t,S,\mu(t,S)) : t \in [0,1],\ S \subset N,\ U(t) \geq u_0\},\ \forall (u_0,w) \in [0,1] \times \mathbf{R}_+^N.$$

The mapping $\sigma : [0,1] \times \mathbf{R}_+^N \to\!\!\to [0,1] \times 2^N$ assigning to each pair $(u_0, w) \in [0,1] \times \mathbf{R}_+^N$ the set of arguments that solve the social planner's problem is called the *compensated demand correspondence:*

$$\sigma(u_0, w) =$$

$$\left\{(t,S) \in [0,1] \times 2^N :\ U(t) \geq u_0 \textit{ and } e(u_0,w) = c(w,t,S,\mu(t,S))\right\}, \tag{4}$$

$$\forall (u_0, w) \in [0,1] \times \mathbf{R}_+^N.$$

3.2 The Family of Games Induced by a Coalition Production Economy

Associated to the coalition production economy with a public good is a family of games that contains the rules of the interaction across productive sectors. For a given t the production function μ induces the following *(induced) family of games* in characteristic form $\{(N; v_{\mu,t})\}_{t\in[0,1]}$, where

$$v_{\mu,t}(S) = \mu(t, S).$$

The interest of these induced games will be explained later on; it will be shown that for particular models of coalition production economies that are induced by an initially given game, there is a game in the induced family that coincides with the initial one.

3.3 Coalition Production Economies with Differentiated Production and a Public Good

By making few changes in the coalition production economy with one produced good model, it is possible to describe general situations where each productive sector in N produces a differentiated good. The following three new modifications of the initial model fully describe the differentiated production case:

(a) The production function μ is a correspondence into $\mathbf{R}^N$: $\mu(t, S)$ is now a subset of $\mathbf{R}^N$ for each $t \in [0, 1]$ and $S \subset N$.

(b) Each differentiated good can be sold according to an exogenous vector of prices $\lambda \in \mathbf{R}^N_+$. If productive sector $i \in N$ produces a certain good, each unit of this good can be sold at price $\lambda_i \in \mathbf{R}_+$.

(c) Apart from the selection of the productive sectors $S \subset N$ and the selection of the public good $t \in [0, 1]$, in this model the social planner has another argument in the minimization problem: the planner must choose the optimal vector of outputs $x' \in \mathbf{R}^N$ in $\mu(t, S)$.

The fact that each produced good can be sold at possibly different prices makes the net cost function dependent on the exogenous price vector $\lambda \in \mathbf{R}^N_+$. The *net cost* of the economic activity in the differentiated case is given by the function

$$c : \mathbf{R}^N_+ \times \mathbf{R}^N_+ \times [0, 1] \times 2^N \times \mathbf{R}^N \to \mathbf{R}$$

where, for each $\lambda \in \mathbf{R}^N_+$, $w \in \mathbf{R}^N_+$, $t \in [0, 1]$, $S \in 2^N$ and $x' \in \mu(t, S)$, $c(\lambda, w, t, S, x')$ is the difference between the cost of the activity of the sectors in S and the income received from the collective production of x'.

Since no other changes are introduced in the model, the social planner's problem for given prices $\lambda \in \mathbf{R}^N_+$, costs $w \in \mathbf{R}^N_+$ and reservation utility u_0 is written as follows:

$$\begin{cases} \min \ c(\lambda, w, t, S, x') \\ s.t. \\ \ \ t \in [0,1], \\ \ \ S \subset N, \\ \ \ x' \in \mu(t,S), \\ \ \ U(t) \geq u_0. \end{cases}$$

Note that for this minimization problem λ, w and u_0 are exogenously given to the social planner while t, S and x' are endogenous.

The same considerations can be made when describing the family of games that is implicit in this economic structure. For a given t, the production function μ induces the following *family of NTU-games* in characteristic form:

$$\{(N; V_{\mu,t})\}_{t\in[0,1]},$$

where

$$V_{\mu,t}(S) = \mu(t,S) \subset \mathbf{R}^S.$$

In the next section we will study the relationship between these games and the games that induce some special coalition production economies.

4 Coalition Production Economies Induced by Games

4.1 Coalition Production Economies Induced by TU-Games

Typically, for an initially given monotone TU-game $(N; v)$, we define a coalition production economy with a public good where the set N represents a set of *industrial sectors* in a society and the *level of employment* of the workers in this sectors is the public good associated to the economy (if $t = 0$ nobody in the society is working and if $t = 1$ all society is fully employed).

If $v(S)$ represents the amount of the good that can be jointly produced by the industries in coalition $S \subset N$ when all workers are employed in these industries then, for some other fixed level of employment $t \in [0,1]$, the production of the good by S is assumed to be proportional to the level of employment; the production function of the coalition production economy is given by

$$\mu(t,S) = tv(S).$$

Consequently, if w_i denotes the cost of production for industry $i \in N$ then when the level of employment is t the cost of production is tw_i.

The workers are paid for their work in the industrial sectors with a fixed salary. If α, which is assumed to be a real value greater than $v(N)$, denotes the total amount of money that the industrial planner pays to the workers when they are all employed then, for some other level of employment t, the cost of all salaries is equal to αt. For simplicity, it is assumed that the social welfare function U on the public good is the identity:

$$U(t) = t, \ \forall t \in [0, 1].$$

In this particular model the reservation utility level u_0 can be interpreted as the minimum level of employment that has been fixed by the trade unions of the industrial sectors.

The net cost in this particular model for having the industries in a coalition S producing with a certain level of employment t, for given costs $w \in \mathbf{R}_+^N$ and output level x (measured in money units), is

$$c(w, t, S, x) = -x + \sum_{i \in S} tw_i + \alpha t. \tag{5}$$

The net cost is, then, the difference between costs (given by $\sum_{i \in S} tw_i + \alpha t$) and the income (given by x). For given costs $w \in \mathbf{R}_+^N$ and reservation utility $u_0 \in [0, 1]$, the industrial planner's problem can be written as follows:

$$\begin{cases} \min -tv(S) + \sum_{i \in S} tw_i + \alpha t \\ s.t. \\ \quad S \subset N, \\ \quad u_0 \leq t \leq 1. \end{cases}$$

We can relate its compensated demand correspondence with the best coalitions correspondence associated to the game. Consider first the following lemma, which states the link between the expenditure function $e(u_0, w)$ and the indirect function $\pi(w)$.

Lemma 1 *Consider a monotone TU-game $(N; v)$ with indirect function denoted by π. The minimum expenditure, e, for the industrial planner's problem on the induced coalition production economy satisfies the following two conditions for all $(u_0, w) \in [0, 1] \times \mathbf{R}_+^N$:*
(a) $e(u_0, w) = u_0(\alpha - \pi(w))$,
(b) $e(u_0, w) \geq 0$.

Proof. First of all, let us solve the industrial planner's problem. Since $w \in \mathbf{R}_+^N$, $\alpha > v(N)$ and the game is monotone, for any $S \subset N$ we have

$$-v(S) + \sum_{i \in S} w_i + \alpha > 0, \tag{6}$$

which implies that the minimum is attained at $t = u_0$ and part (b). Now, the industrial planner's problem, for given costs w, can be reduced to the computation of

$$e(u_0, w) = u_0(\alpha - \max_{S \subset N} \left\{ v(S) - \sum_{i \in S} w_i \right\}).$$

By taking into account the definition of the indirect function in (1), we conclude that

$$e(u_0, w) = u_0(\alpha - \pi(w)),$$

which proves part (a), and hence the lemma.■

Next, the main result of this section is proved; it provides a new economic interpretation to the best coalitions correspondence of a monotone TU-game:

Proposition 2 . *Consider a monotone TU-game $(N; v)$ with best coalitions correspondence denoted by Φ. The compensated demand function, σ, for the industrial planner's problem on the induced coalition production economy satisfies the following condition:*

$$\sigma(u_0, w) = \{u_0\} \times \Phi(w), \ \forall (u_0, w) \in (0, 1] \times \mathbf{R}_+^N.$$

Proof. The proof is just a direct consequence of (4) and the previous lemma. One has

$$\sigma(u_0, w) =$$

$$\left\{ (t, S) \in [0, 1] \times 2^N : t \geq u_0, \ u_0 \left(\alpha - \pi(w)\right) = t \left(-v(S) + \sum_{i \in S} w_i + \alpha \right) \right\}.$$

By (6), we can write

$$\sigma(u_0, w) = \left\{ (u_0, S) \in [0, 1] \times 2^N : u_0 \left(\alpha - \pi(w)\right) = u_0 \left(-v(S) + \sum_{i \in S} w_i + \alpha \right) \right\},$$

that is, we have

$$\sigma(u_0, w) = \{u_0\} \times \{S \in 2^N : \pi(w) = v(S) - \sum_{i \in S} w_i\},$$

implying that

$$\sigma(u_0, w) = \{u_0\} \times \Phi(w),$$

which closes the proof of the proposition.■

Note that from the nonemptiness of the best coalitions correspondence Φ at every w in $\mathbf{R}^N$ we obtain that the compensated demand function takes nonempty values everywhere. We can interpret any best coalitions correspondence as a compensated demand function by making the Cartesian product with the reservation utility level u_0.

It is obvious that from this model we can recover the initial game $(N; v)$ from the associated coalition production economy. Among all the games induced by the model, $\{(N; v_{\mu,t})\}_{t\in[0,1]}$, there is a particular one that coincides with the initial game: One can easily check that $(N; v_{\mu,1})$ is exactly the initial game; indeed, by construction one has

$$v_{\mu,1}(S) = \mu(1, S) = v(S),\ \forall S \subset N.$$

On the other hand, for the case of monotone TU-games $(N; v)$ the characteristic function v can be reinterpreted from a different point of view. Using the net cost function c and the expenditure function e of the minimization problem of the industrial planner, the initial game is recovered. The following result shows this fact:

Corollary 3 *Consider a monotone TU-game $(N; v)$ and the expenditure function e of its associated industrial planner's problem. Then, for any u_0 in $(0, 1]$ and $S \subset N$, it holds that $v(S)$ is the maximum real value δ for which*

$$u_0(-\delta + \sum_{i\in S} w_i + \alpha) \geq e(u_0, w),\ \forall w \in \mathbf{R}^N_+.$$

Proof. From part (a) in the previous Lemma 1, we have

$$\pi(w) = \alpha - \frac{e(u_0, w)}{u_0},\ \forall u_0 \in (0, 1],\ \forall w \in \mathbf{R}^N_+,$$

whence, from (2), we derive

$$v(S) = \min_{w\in\mathbf{R}^N_+} \left\{ \alpha - \frac{e(u_0, w)}{u_0} + \sum_{i\in S} w_i \right\},$$

implying that $v(S)$ is the maximum real value δ that satisfies

$$\delta \leq \alpha - \frac{e(u_0, w)}{u_0} + \sum_{i\in S} w_i,\ \forall w \in \mathbf{R}^N_+.$$

This closes the proof of this corollary.■

Concerning the economic interpretation of this result, we can view the characteristic function of the game at any coalition S, $v(S)$, as the maximum output that coalition S can obtain with a higher net cost for the industrial planner than the minimal one, given by e, for any given vector of costs $w \in \mathbf{R}_+^N$ and for any given reservation utility level u_0 in $(0,1]$.

An example:

Consider the following monotone TU-game $(N;v)$ with two productive sectors:

$$\begin{array}{l} v(\{1,2\}) = 1, \\ v(\{1\}) = \frac{1}{3}, \\ v(\{2\}) = \frac{1}{4}, \\ v(\emptyset) = 0. \end{array}$$

The induced production correspondence μ, for every $t \in [0,1]$, is given by

$$\begin{array}{l} \mu(t,\{1,2\}) = t, \\ \mu(t,\{1\}) = \frac{t}{3}, \\ \mu(t,\{2\}) = \frac{t}{4}, \\ \mu(t,\emptyset) = 0. \end{array}$$

For given α, $u_0 > 0$ and costs $w = (\frac{1}{6}, 1) \in \mathbf{R}_+^2$, the industrial planner's problem is written as follows:

$$\left\{ \begin{array}{l} \min \ -\mu(t,S) + \sum\limits_{i \in S} t w_i + \alpha t \\ s.t. \\ \quad S \subset \{1,2\}, \\ \quad u_0 \leq t \leq 1. \end{array} \right.$$

The solution to this problem is given by the expression

$$\sigma\left(u_0, \left(\frac{1}{6}, 1\right)\right) = \{u_0, \{1\}\},$$

reflecting that, for these prices and costs, 1 is selected by the industrial planner to be producing with an employment level equal to u_0. In this example, the minimum expenditure is given by

$$e\left(u_0, \left(\frac{1}{6}, 1\right)\right) = u_0 \left(\alpha - \frac{1}{6}\right) \geq 0.$$

Finally, if we compute the indirect function π at $w = \left(\frac{1}{6}, 1\right)$, we obtain that

$$\pi\left(\frac{1}{6}, 1\right) = \frac{1}{6},$$

and that the best coalitions correspondence Φ satisfies

$$\Phi\left(\frac{1}{6}, 1\right) = \{1\}.$$

One can easily check Lemma 1 and Proposition 1 for this particular selection of costs.

4.2 Coalition Production Economies Induced by NTU-Games

The main difference between this model and the previous one is that, in this case, the coalition production economy is constructed from an initial NTU-game $(N;V)$ instead of from a TU-game. For that NTU-game, we define a coalition production economy with *differentiated production* and a public good where the set N represents a set of *industrial sectors* in a society and the *level of employment* of the workers in this sectors is the public good associated to the economy. The following three points make the main differences between the two models:

(a) The induced production function of the TU-case is replaced by a correspondence in the NTU-case: if $V(S) \subset \mathbf{R}^S$ represents the set of vectors consisting of different possible outcomes of each differentiated good that can be produced by the industries in coalition $S \subset N$ when all workers are employed in these industries then, for some other fixed level of employment $t \in [0,1]$, the production of the good by S is assumed to be proportional to the level of employment; the production correspondence of the coalition production economy is given by

$$\mu(t,S) = t \cdot V'(S) \subset \mathbf{R}^N.$$

(b) Each differentiated good can be sold according to an exogenous vector of prices $\lambda \in \mathbf{R}^N_+$: if sector $i \in N$ produces a certain good then each unit of the good can be sold at price $\lambda_i \in \mathbf{R}_+$.

(c) For any coalition S, the cost function has as one of its arguments the output vector x that can be selected from the set $V(S) \subset \mathbf{R}^S$ (x will be denoted by x' if it is embedded in $\mathbf{R}^N$ by including zeros conveniently)

The net cost of the economic activity described in this model is given by

$$c(\lambda, w, t, S, x') = -\sum_{i \in S} t\lambda_i x'_i + \sum_{i \in S} tw_i + \alpha t;$$

we assume that $V(N) \subset \alpha 1_N - \mathbf{R}^N_+$. The net cost is, then, the difference between costs (given by $\sum\limits_{i \in S} tw_i + \alpha t$) and income (given by $\sum\limits_{i \in S} t\lambda_i x'_i$).

Since no other changes are introduced in the model, the industrial planner's problem for given prices $\lambda \in \mathbf{R}^N_+$, costs $w \in \mathbf{R}^N_+$ and reservation utility u_0 is written as follows:

$$\begin{cases} \min -\sum\limits_{i \in S} t\lambda_i x'_i + \sum\limits_{i \in S} tw_i + \alpha t \\ s.t. \\ \quad t \in [0,1], \\ \quad S \subset N, \\ \quad x' \in V'(S), \\ \quad U(t) \geq u_0. \end{cases}$$

Note that, for this minimization problem, λ, w and u_0 are exogenously given to the social planner, while t, S and x' are endogenous. For simplicity, as in the model of the preceding subsection it is also assumed that the social welfare function is the identity.

Similarly to the TU-case, if we consider the industrial planner's problem induced by a monotone game, we obtain that the compensated demand correspondence can be related to the best coalitions correspondence associated to the game[3]. Consider first the following preliminary result:

Lemma 4 *Consider a compactly generated, comprehensive and monotone NTU-game $(N;V)$ with indirect function denoted by π. The minimum expenditure, e, for the industrial planner's problem satisfies the following conditions $\forall u_0 \in [0,1]$ and $\forall(\lambda, w) \in \Delta$:*
(a) $e(u_0,(\lambda,w)) = u_0(\alpha - \pi(\lambda,w))$,
(b) $e(u_0,(\lambda,w)) > 0$.

Proof. First of all, let us solve the industrial planner's problem

$$\begin{cases} \min\ t\left(-\sum\limits_{i\in S}\lambda_i x_i' + \sum\limits_{i\in S} w_i + \alpha\right) \\ s.t. \\ \quad S \subset N, \\ \quad x' \in V'(S), \\ \quad u_0 \leq t \leq 1. \end{cases}$$

Since prices and costs are in the set Δ (see (3)), $V(N) \subset \alpha 1_N - \mathbf{R}_+^N$ and the game is monotone, we have

$$-\sum_{i\in S}\lambda_i x_i' + \sum_{i\in S} w_i + \alpha > 0, \tag{7}$$

which implies that the minimum is attained at $t = u_0$ and part (b). Now, the industrial planner's problem reduces to the computation of

$$e(u_0,(\lambda,w)) = u_0\left(\alpha - \max_{S\subset N,\ x'\in V'(S)} \sum_{i\in S}(\lambda_i x_i' - w_i)\right).$$

By taking into account the definition of the indirect function and the link between V and V', we conclude that

$$e(u_0,(\lambda,w)) = u_0(\alpha - \pi(\lambda,w)), \tag{8}$$

which proves part (a), and hence the lemma.■

The main result of this section is stated next; it provides a new economic interpretation to best coalitions correspondences of NTU-games.

[3] This link is proved for costs and prices in the set Δ (see (3)).

Proposition 5 *Consider a compactly generated, comprehensive and monotone NTU-game $(N; V)$ with best coalitions correspondence denoted by Φ. The compensated demand function, σ, of the industrial planner's problem satisfies the following condition:*

$$\sigma(u_0, (\lambda, w)) = \{u_0\} \times \Phi(\lambda, w),\ \forall u_0 \in (0, 1],\ \forall (\lambda, w) \in \Delta.$$

Proof. The proof is just a direct consequence of Lemma 2. One has

$$\sigma(u_0, (\lambda, w)) =$$

$$\{(t, S, x') : \ S \subset N,\ x' \in V'(S),\ u_0 \leq t \leq 1 \ and$$

$$u_0 \left(\alpha - \pi(\lambda, w)\right) = t \left(-\sum_{i \in S} \lambda_i x_i' + \sum_{i \in S} w_i + \alpha \right) \Bigg\}.$$

Since $u_0 \neq 0$, by (7) we can write

$$\sigma(u_0, (\lambda, w)) =$$

$$\{(u_0, S, x') : \ S \subset N,\ x' \in V'(S) \ and$$

$$u_0 \left(\alpha - \pi(\lambda, w)\right) = u_0 \left(-\sum_{i \in S} \lambda_i x_i' + \sum_{i \in S} w_i + \alpha \right) \Bigg\},$$

that is, we have

$$\sigma(u_0, (\lambda, w)) =$$

$$\left\{ (u_0, S, x') : \ S \subset N,\ x' \in V'(S) \ and \ \pi(\lambda, w) = \sum_{i \in S} \lambda_i x_i' - \sum_{i \in S} w_i \right\},$$

which implies that

$$\sigma(u_0, (\lambda, w)) = \{u_0\} \times \Phi(\lambda, w)$$

and then the proof of the proposition.■

Note that the nonemptiness of the best coalitions correspondence Φ at every (λ, w) in Δ implies the nonemptiness of the compensated demand correspondence. We can interpret any best coalitions correspondence as a compensated demand function by doing a little change, namely, by making the Cartesian product with the reservation utility level u_0.

Again, it is obvious that from this coalition production economy we can obtain the initial game $(N; V)$ since both structures contain the same information. Among all the games induced by the model, $\{(N; V_{\mu,t})\}_{t \in [0,1]}$, there is a particular one which coincides with the initial one. One can easily check that the game $(N; V_{\mu,1})$ is exactly the initial one, since by construction one has

$$V_{\mu,1}(S) = \mu(1, S) = V(S),\ \forall S \subset N.$$

On the other hand, for the case of monotone, compactly generated and comprehensive NTU-games $(N;V)$ the characteristic mapping V can be reinterpreted from the point of view of the coalition production economy. Using the net cost function and the expenditure function of the minimization problem of the industrial planner, the initial game can be recovered. The following result shows this fact:

Corollary 6 *Consider a monotone, compactly generated and comprehensive NTU-game $(N;V)$ and its associated industrial planner's problem. Then, for any u_0 in $(0,1]$, one has*

$$V(S)=\left\{x\in\mathbf{R}^S : c(\lambda,w,u_0,S,x')\geq e(u_0,(\lambda,w)),\ \forall\,(\lambda,w)\in\Delta\right\},\ \forall S\subset N.$$

Proof. From the duality between indirect functions and monotone, compactly generated and comprehensive NTU-games, one has

$$V(S)=\left\{y\in\mathbf{R}^S:\ \sum_{i\in S}(\lambda_i y_i-w_i)\leq\pi(\lambda,w),\ \forall(\lambda,w)\in\Delta\right\},\ \forall S\subset N;$$

after some simple arrangements, one can write

$$V(S)=\left\{y\in\mathbf{R}^S : u_0\left(-\sum_{i\in S}(\lambda_i y_i-w_i)+\alpha\right)\geq u_0\left(\alpha-\pi(\lambda,w)\right),\ \forall\,(\lambda,w)\in\Delta\right\}.$$

Part (a) of Lemma 2 implies, jointly with the definition of c, that

$$V(S)=\left\{x\in\mathbf{R}^S : c(\lambda,w,u_0,S,x')\geq e(u_0,(\lambda,w)),\ \forall\,(\lambda,w)\in\Delta\right\},$$

which closes the proof of this corollary.■

Concerning the economic interpretation that can be derived from this result, we can view the characteristic mapping of a game at a coalition S, $V(S)$, as the set of output vectors each industry in S can obtain with a higher net cost for the production planner than the minimum expenditure e, for any vector of prices λ and costs w in the set Δ and for any given fixed reservation utility level u_0 in $(0,1]$.

An example:

Consider the following compactly generated, comprehensive and monotone NTU-game $(N;V)$ with two industrial sectors:

$$\begin{aligned}
&V(\{1,2\})=\left\{(y_1,y_2)\in\mathbf{R}^2\ :\ y_1+y_2\leq 1 \textit{ and } y_1,y_2\leq 1\right\},\\
&V(\{1\})=\left\{y\in\mathbf{R}:\ y\leq\tfrac{1}{3}\right\},\\
&V(\{2\})=\left\{y\in\mathbf{R}:\ y\leq\tfrac{1}{3}\right\},\\
&V(\emptyset)=\{y\in\mathbf{R}:\ y\leq 0\}.
\end{aligned}$$

The induced production correspondence μ is given by

$$\begin{array}{l} \mu(t,\{1,2\}) = t \cdot V'(\{1,2\}), \\ \mu(t,\{1\}) = t \cdot V'(\{1\}), \\ \mu(t,\{2\}) = t \cdot V'(\{2\}), \\ \mu(t,\emptyset) = t \cdot V'(\emptyset), \end{array}$$

where $V'(S)$ denotes the usual representation of $V(S)$ as a subset of $\mathbf{R}^N$. For given α and u_0, and for $(\lambda, w) = (\frac{2}{3}, \frac{1}{12}, \frac{1}{6}, \frac{1}{12}) \in \Delta$, the industrial planner's problem is given by

$$\left\{ \begin{array}{l} \min \ t \left(-(\frac{2}{3}x'_1 + \frac{1}{12}x'_2) + \sum\limits_{i \in S} w_i + \alpha \right) \\ s.t. \\ \quad S \subset \{1,2\}, \\ \quad x' \in V'(S), \\ \quad u_0 \leq t \leq 1. \end{array} \right.$$

Given $u_0 \in [0,1]$, the solution to this problem is given by

$$\sigma \left(u_0, \left(\frac{2}{3}, \frac{1}{12}, \frac{1}{6}, \frac{1}{12} \right) \right) = \{ (u_0, \{1,2\}, (1,0)) \},$$

reflecting that, for these prices and costs, both industries are selected together by the industrial planner. In this example, the minimum expenditure is given by

$$e \left(u_0, \left(\frac{2}{3}, \frac{1}{12}, \frac{1}{6}, \frac{1}{12} \right) \right) = u_0 \left(\alpha - \frac{5}{12} \right) \geq 0.$$

Finally, if we compute the indirect function π at $\lambda = (\frac{1}{3}, \frac{1}{3})$ and $w = (\frac{1}{6}, \frac{1}{6})$, we obtain that

$$\pi \left(\frac{2}{3}, \frac{1}{12}, \frac{1}{6}, \frac{1}{12} \right) = \frac{5}{12},$$

and that the best coalitions correspondence Φ at the same point satisfies

$$\Phi \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6} \right) = \{ (\{1,2\}, (1,0)) \}.$$

Now, one can easily check Lemma 2 and Proposition 2 for this particular selection of prices and costs.

5 Conclusions and relationships to the literature

This paper, through its use of indirect and direct representations of games and the concepts of compensated demand functions and expenditure functions from consumer demand theory has provided a new approach to representing games as economies. This approach suggests other related questions and also other approaches. First, it appears that when a game is balanced or totally balanced we may be able to obtain further relationships between the economies and games, more precisely, between equilibrium outcomes of economies and cores of games; research on this question is in progress. Another approach to the representation of games as economic structures that appears very promising is the representation of the games as economies with local public goods, or, in other words, as economies with clubs (cf. [3], [4] or [6]. For games with many players, another approach based on revealed preference and integrability theory is suggested in (**??**).

References

[1] Billera, L.J. (1974) "On games without side payments arising from a general class of markets," *Journal of Mathematical Economics* 1(2), 129-139.

[2] Billera, L.J. and R.E. Bixby (1974) "Market representations of n-person games," *Bulletin of the American Mathematical Society* 80(3), 522-526.

[3] Conley, J. and M.H. Wooders (1997) "Equivalence of the core and competitive equilibrium in a Tiebout economy with crowding types," *Journal of Urban Economics* 41(3), 421-440.

[4] Conley, J. and M.H. Wooders (2001) "Tiebout economics with differential genetic types and endogenously chosen crowding characteristics," *Journal of Economic Theory* 98(2), 261-294.

[5] Debreu, G. and H. Scarf (1963) "A limit theorem on the core of an economy," *International Economic Review* 4, 235-246.

[6] Kovalenkov, A. and M. H. Wooders (1999a) "Approximate cores of games and economies with clubs," Department of Economics, University of Warwick Working Paper No. 535 revised, *Journal of Economic Theory* (to appear).

[7] Kovalenkov, A. and M. H. Wooders (1999b) "A law of scarcity," Department of Economics, University of Warwick Working Paper No. 546, revised,

[8] Mart ínez-Legaz, J.-E. (1996) "Dual representation of cooperative games based on Fenchel-Moreau conjugation," *Optimization* 36(4), 291-319.

[9] Mas-Colell, A. (1975) "A further result on the representation of games by markets," *Journal of Economic Theory* 10(1), 117-122.

[10] Mas-Colell, A., M. Whinston and J. Green (1995) *Microeconomic Theory*, Oxford University Press, New York/Oxford.

[11] Meseguer, A. (1997) "Dual representation of games and its applications," Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, PhD Thesis.

[12] Shapley, L.S. and M. Shubik (1969) "On market games," *Journal of Economic Theory* 1(1), 9-25.

[13] Wooders, M.H. (1978) "Equilibria, the core, and jurisdiction structures in economies with a local public good," *Journal of Economic Theory* 18(2), 328-348.

[14] Wooders, M.H. (1979) "A characterization of approximate equilibria and cores in a class of coalition economies" (a revision of Stony Brook Department of Economics Working Paper No. 184), on line at http://www.warwick.ac.uk/fac/soc/Economics/wooders/.

[15] Wooders, M.H. (1983) "The epsilon core of a large replica game," *Journal of Mathematical Economics* 11(3), 277-300.

[16] Wooders, M.H. (1994) "Equivalence of games and markets," *Econometrica* 62(5), 1141-1160.