Weighted Fano varieties and infinitesimal Torelli problem

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A B S T R A C T

We solve the infinitesimal Torelli problem for 3-dimensional quasi-smooth $\mathbb{Q}$-Fano hypersurfaces with at worst terminal singularities. We also find infinite chains of double coverings of increasing dimension which alternatively distribute themselves in examples and counterexamples for the infinitesimal Torelli claim and which share the analogue, and in some cases the same, Hodge-diagram properties as the length 3 Gushel–Mukai chain of prime smooth Fanos of coindex 3 and degree 10.

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1. Introduction

Minimal model program suggests us to formulate inside the category of varieties with terminal singularities many questions which were initially asked for smooth varieties. The construction of the period map, and the related Torelli type problems are definitely among these problems.

1.1. Infinitesimal Torelli

A $\mathbb{Q}$-Fano variety is a projective variety X such that X has at worst $\mathbb{Q}$-factorial terminal singularities, $-K_X$ is ample and $\mathrm{Pic}(X)$ has rank 1; cf. [9]. In this paper we give a full answer to the infinitesimal Torelli problem in the case of quasi-smooth $\mathbb{Q}$-Fano hypersurfaces of dimension 3 with terminal singularities and with Picard number 1.

In Section 2.1 of this paper the reader can find a basic dictionary and up to date references needed to understand the statement of the infinitesimal Torelli problem and where to find the meaning of infinitesimal variation of Hodge structures (IVHS; to short). Here we stress only that the method à la Griffiths based on the extension of the Macaulay's theorem in algebra to the weighted case, cf. [25], almost never works in our case.

There is a complete list of quasi-smooth Fano hypersurfaces with $\rho = 1$. We have the 'famous 95', that is a list of 95 families of quasi-smooth, anti-canonically polarized Fano threefold of index 1. The list was first considered in [18], and later on showed to be complete by [19, Corollary 2.5] see also cf. [7]. The 'famous 95' families have been prominent in the context of explicit birational geometry: for a modern survey we refer to [5], while for an account of those in term of Hodge theory we refer to [1]. We prove:

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Theorem [A]. *Let $\mathcal{M}$ be the space of quasi-smooth hypersurface of degree d with only terminal singularities in weighted projective space $w\mathbb{P} = w\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum_{i=1}^{4} a_i$, modulo automorphisms of $w\mathbb{P}$. Then there is an open dense subset of $\mathcal{M}$ on which infinitesimal Torelli holds.*

See Theorem 3.7.

The situation changes drastically when we go to the higher index case. There are 35 families of Fano threefolds of higher index, that we can find listed for example in [20], see our Table 1. Out of the 35 , we have 8 out of them that are rigid and with $h^{2,1} = 0$. These are, in particular, the families no. 105, 113, 119, 120, 123, 126, 129, 130, and we will call them *Hodge-rigid*. We are left with 27 families, and they split with respect to the infinitesimal Torelli property. We prove:

Theorem [B]. *Let $\mathcal{M}$ be the space of quasi-smooth weighted hypersurfaces of degree d in weighted projective 4-space and of index > 1 modulo automorphisms. Then*

1. *for families no. 115, 121, 122 and 127 the infinitesimal Torelli theorem does not hold;*
2. *the families no. 105, 113, 119, 120, 123, 126, 129, 130 are Hodge-rigid;*
3. *for the remaining 23 families, there is an open dense subset of $\mathcal{M}$ on which the infinitesimal Torelli holds.*

See Theorem 3.14.

The behaviour with respect to the infinitesimal Torelli problem is summarized in Table 1

We point out that for the four (AT)-cases of Table 1, that is in the cases of failure for the Torelli problem, we are also able to construct explicitly the deformations along which infinitesimal Torelli fails; we call them *anti-Torelli* deformations. Indeed we get explicit polynomial basis for the vector subspace of 'anti-Torelli' deformations; see Section 3.5.

1.2. GM-type infinite towers

We consider as one of the main result of this work the explicit construction of infinite towers of quasi-smooth varieties which exhibit a failure of infinitesimal Torelli and simultaneously a behaviour analogous to the Gushel–Mukai varieties deeply studied in a list of recent papers, for example [10–13].

Recall first that a Gushel–Mukai variety (in the following, shortened as GM) is the datum of the intersection of the cone over the Grassmannian $G = \mathrm{Gr}(2,5)$ with an appropriate linear space, and a general global section of $\mathcal{O}_G(2)$. One curious feature of the GM variety is that the Hodge numbers satisfy a periodic behaviour: the even dimensional varieties of GM-type will satisfy (with one further dimension to remove in the K3 case)

$$H_{\mathrm{prim}}^{\dim(X)}(X^{\mathrm{even}}, \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C}^{20} \oplus \mathbb{C}$$

while the odd one

$$H^{\dim(X)}(X^{\mathrm{odd}}, \mathbb{C}) \cong \mathbb{C}^{10} \oplus \mathbb{C}^{10}.$$

The fact that the Hodge structure on $H^4(X, \mathbb{C})$ is of K3 type and that the study of the period map for X shares many similarities to the case of cubic fourfold add importance to the study of GM-varieties.

In Section 4 we perform a geometric construction that yields examples that shares common similarities with the above varieties. Although our ambient space is not a Grassmannian variety, we decided to adopt the name of Gushel–Mukai like since we always find Hodge similarities like above, and because the geometrical construction is similar to the GM case.

In particular, starting from one of the hereby considered Fano threefold $X_d \subset w\mathbb{P}$ (or, better, a surface lying in it as quasi-linear section) we realize a double cover $\varphi : Y \to w\mathbb{P}$ branched over X_d. Iterating this process we get an infinite chain (or 'tower') of varieties, satisfying the periodic equality (with the dichotomy odd/even) of the Hodge groups like in the GM case. Moreover, since (unlike in the GM case) there is no Grassmannian to 'bound' the dimension, our towers can go up and produce examples in any dimension. If the 'even' Hodge structure comes from the Hodge structure of a (weighted) K3 surface, we say that our tower is of *K3-type*.

Let us call an *even member* of the tower an X_d^{2k} obtained by doing an even number of step in the construction above, similarly we will define the *odd members*. In Theorem 4.4 we show:

Theorem [C]. *Let $X_d^0 = V(f_0) \subset \mathbb{P}(a_0, \ldots, a_n)$. Suppose that $d \equiv 0 \pmod 2$. Let X_d^{2k} be any even member of the tower, of dimension $n + 2k$. There is an isomorphism of IVHS*

$$\phi : H^n(X_d^0, \mathbb{C}) \xrightarrow{\sim} H^{n+2k}(X_d^{2k}, \mathbb{C})[-2k].$$

In particular the central Hodge numbers of X_d^0 are the same of the Hodge numbers of X_d^{2k} up to a degree k shift, that is

$$\left(h_{X_d^{2k}}^{n+2k,0}, h_{X_d^{2k}}^{n+2k-1,1}, \ldots, h_{X_d^{2k}}^{1,n+2k-1}, h_{X_d^{2k}}^{0,n+2k} \right) = \left(0, \ldots, 0, h_{X_d^0}^{n,0}, \ldots, h_{X_d^0}^{0,n}, 0, \ldots, 0 \right),$$

with $2k$ zeros on the last vector. The same holds for odd members, with an equality between the Hodge theory of X_d^1 and X_d^{2k+1}, for any k.

One of the most interesting case is the construction of an infinite tower whose initial odd element is a general quartic double solid 4.4.2; there is a strong renew of interest about the (ir)rationality problem on quartic double solid, see [6,21], after Voisin's result that for any integer $k = 0, \ldots, 7$, a very general nodal quartic double solid with k nodes is not stably rational; see: [27]. Actually we show that up to dimension 4 onwards the members of our tower are rational; see Proposition 4.7.

Nevertheless, for us, the most beautiful case is the following one which, in a further analogy with the GM case, holds an infinite series of examples and counterexamples to the Torelli problem according to the parity of the dimension.

Theorem [D]. *Any quasi-smooth odd dimensional member $X_{14}^{2k+1} \subset \mathbb{P}(2, 3, 4, 5, 7^{2k+1})$ is of Anti-Torelli type, while any even dimensional quasi-smooth member $X_{14}^{2k} \subset \mathbb{P}(2, 3, 4, 5, 7^{2k})$ is of Torelli type. In particular we have an infinite chain of GM-type varieties which are examples or counterexamples for the Torelli problem, with alternate dimensions.*

See Theorem 4.8.

We start now with a recap of known results on weighted projective varieties and Torelli problem.

2. Preliminaries

2.1. Weighted projective hypersurfaces and Jacobian ring

We denote with $\mathbb{P}(a_0, \ldots, a_n)$ or $w\mathbb{P}$ the weighted projective space with variables $x_0, \ldots, x_n$ with wt $(x_i) = a_i$. For the standard theory of weighted projective spaces we refer to [22] and [15].

We recall that the weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ is said to be *well-formed* if

$$\mathrm{hcf}(a_0, \ldots, \hat{a}_i, \ldots, a_n) = 1 \text{ for each } i$$

We want now to define a reasonable class of varieties in a (well-formed) weighted projective space, that will play the role of the smooth one in straight case. We have the following definition

Definition 2.1. Let X be a closed subvariety of a well-formed weighted projective space, and let us denote by A_X the affine cone over X, that is the completion of $A_X^\bullet = \pi^{-1}(X)$, where $\pi : \mathbb{A}^{n+1} \setminus 0 \longrightarrow w\mathbb{P}$ is the canonical projection. We say that X is *quasi-smooth* if A_X is smooth outside its vertex $\underline{0}$.

The above definition is telling us that the only (possible) singularities of X come just from the automorphisms of the ambient space. The key idea is that a quasi-smooth subvariety $X \subset w\mathbb{P}$ is a *V-manifold*, that is isomorphic to the quotient of a complex manifold by a finite group of holomorphic automorphisms. Operative definitions of quasi-smoothness can be found, for example, in [18, Theorem 8.1].

A $\mathbb{Q}$-Fano 3-fold here means a projective 3-fold with only terminal singularities whose anticanonical divisor is ample. We assume as well (this being always satisfied in the hypersurface case) that the rank ρ of its second Betti homology is 1. The Fano index ι_X of a Fano 3-fold X is

$$\iota_X = \max\{m \in \mathbb{Z} > 0 \mid -K_X = mA \text{ for some Weil divisor A }\}$$

A Weil divisor A for which $-K_X = \iota_X A$ is called a primitive ample divisor.

Recall that if $X_d = V(f)$ is a hypersurface of degree d, we can associate to X_d a canonical deformation ring, the Jacobian ring

$$R_f := \mathbb{C}[x_0, \ldots, x_{n+1}]/J(f),$$

where $J(f)$ is its Jacobian ideal, that is, in both smooth and quasi-smooth cases, the (graded homogeneous) ideal generated by the partial derivatives of f.

This is indeed finite dimensional as vector space if and only if X is quasi-smooth. The Jacobian ring be interpreted as the ring of infinitesimal deformation of the affine cone A_X (see [14] for a more general treatment). A well-known feature of the Jacobian ring of a hypersurface is that some homogeneous slices of it can be identified with the (primitive) cohomology of the underlying X. This is the content of the *Griffiths Residue theorem* [15] or for a modern account, see [4]. In particular we have that, if $H_{\mathrm{prim}}^{p,n-p}(X)$ is the $(p, n - p)$-primitive part of $H^n(X, \mathbb{C})$, then

$$H_{\mathrm{prim}}^{p,n-p}(X) \cong R_{(n-p+1)d-s},$$

$s = \sum_{i=0}^{n+1} a_i$ is the sum of the weights and d is the degree of f.

Griffiths description of the primitive cohomology for a smooth hypersurface is a powerful tool to attack the Torelli problem. The problem was originally solved by Donagi, cf. [16]. A modern survey can be found, for example, in Claire Voisin's book [27], that here we recall briefly.

Assume that X_d is a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$. Assume that the dimension of X is at least 3 (in particular, all the deformations of X are projective). Thanks to the Lefschetz hyperplane the only interesting part of the Hodge structure

of X is located in degree n. The starting point is realizing how the differential of the period map for an Hodge structure of weight n restricted to its primitive subspaces

$$d\wp : H^1(T_X) \to \bigoplus_{p=0}^{n} \text{Hom}(H^{p,q}_{\text{prim}}(X), H^{p-1,q+1}_{\text{prim}}(X)),$$

can be rewritten thanks to the Griffiths description as

$$R_d \longrightarrow \bigoplus_{p=0}^{n} \text{Hom}(R_{(p+1)d-n-2}, R_{(p+2)d-n-2}),$$

where R denote as usual the Jacobian ring and the subscript refers to its homogeneous components.

We can rephrase this using local duality theorem (Theorem 2.2 of [25]). First recall the concept of *socle*. For a graded k-Algebra A, the socle is defined as

$$\text{Soc}(A) = \{h \in A \mid hg = 0 \text{ for all } g \in \bigoplus A_i\}.$$

In general the socle could be either empty or in different degrees, but in case of the Jacobian ring it coincides with a specific degree component R_σ, cf. Corollary A3, [25]. One has in particular $R_{\sigma+j} = 0$ for every $j > 0$. The local duality theorem is then

Theorem 2.2 (*Thm 2.2 in [25]*). *Let $f_0, \ldots, f_{n+1}$ a regular sequence of weighted homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_{n+1}]$ and let*

$$A = \mathbb{C}[x_0, \ldots, x_{n+1}]/(f_0, \ldots, f_{n+1}).$$

Suppose a_i is the weight of x_i and $d_i = \deg f_i$. Then for any $0 \le a \le \sigma$ the pairing given by multiplication

$$A_a \times A_{\sigma-a} \longrightarrow A_\sigma$$

is non-degenerate, where $\sigma = \sum d_i - a_i$ is the top degree.

If we pick $A = R$, the Jacobian ring, the theorem applies since the partial derivatives form a regular sequence.

The above theorem allows us to check, instead of the injectivity of

$$R_d \longrightarrow \text{Hom}(R_a, R_b)$$

the surjectivity of

$$R_b \times R_{\sigma-(a+b)} \to R_{\sigma-a}.$$

However, the above theorem assures the non-degeneracy of the multiplication map only when the socle is involved.

The non-degeneracy of the general multiplication map

$$R_a \times R_b \longrightarrow R_{a+b}$$

is indeed tackled by the *Macaulay's theorem*.

Theorem 2.3 (*Thm. 1 in [25]*). *Let $f_0, \ldots, f_{n+1}$ be a regular sequence of homogeneous polynomials of degree $d_0, \ldots, d_{n+1}$ in $\mathbb{C}[x_0, \ldots, x_{n+1}]$ and let*

$$R = \mathbb{C}[x_0, \ldots, x_{n+1}]/(f_0, \ldots, f_{n+1}).$$

Then R is a finite dimensional graded $\mathbb{C}$-algebra with top degree $\sigma = \sum(d_i - 1)$ and the multiplication map

$$\mu : R_a \times R_b \longrightarrow R_{a+b}$$

is nondegenerate for $a + b \le \sigma$.

In turn this is enough to guarantee the infinitesimal Torelli theorem for smooth projective hypersurfaces in $\mathbb{P}^{n+1}$. We focus now on the case of quasi-smooth hypersurfaces in weighted projective spaces.

2.2. Infinitesimal Torelli for weighted hypersurfaces

A natural extension of the Donagi work is consider the case of quasi-smooth hypersurfaces in weighted projective spaces. Let alone the generic Torelli, one fails to get a general answer even for the infinitesimal Torelli.

The main obstacle is that the weighted version of Macaulay's theorem does not hold for any weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ (more in general, for any weighted graded rings). Counterexamples can be easily given. Consider for example the case of $R = \mathbb{C}[x, y]/(x^2, y^3)$. The multiplication map $\mu : R_1 \times R_3 \to R_4$ is degenerate, since $x_0 \cdot R_3 = 0$ with $x_0 \ne 0$.

Nevertheless on some weighted graded rings some generalization of Macaulay's theorem holds. Based on a detailed analysis, Tu was able to identify in [25] several classes of quasi-smooth hypersurface for which the infinitesimal Torelli theorem actually holds. We recap briefly here his results.

For $w\mathbb{P} = \mathbb{P}(a_0, \ldots, a_{n+1})$ set

$$s = \sum a_i, \quad m = lcm(a_0, \ldots, a_{n+1})$$

and for any subset $J = (j_1, \ldots, j_n)$ of $\{0, \ldots, n+1\}$,

$$m(q|J) = lcm(q_{j_1}, \ldots, q_{j_n}).$$

We define

$$G = -s + \frac{1}{n+1} \sum_{2 \leq k \leq n+2} \binom{n}{k-2}^{-1} \sum_{|J|=k} m(q|J).$$

An estimate for G is

$$G \leq -s + m(n+1);$$

in particular we notice that for the standard projective space $\mathbb{P}^{n+1}$ we have $s = n+2$, $m = 1$, $G = -1$. What we have is

Theorem 2.4 (*Theorem 2.8 in [25]*). *Let $R = \mathbb{C}[x_0, \ldots, x_{n+1}]/J$ be the weighted ring defined by the ideal J of a regular sequence $f_0, \ldots, f_{n+1}$. Set $d_i = \deg f_i$, $a_i = $ weight x_i and $\sigma = \sum(d_i - a_i)$. The natural map*

$$R_a \to \mathrm{Hom}(R_b, R_{a+b})$$

is injective

1. *if b is a multiple of m and $\sigma - (a+b) \geq \max(G+1, 0)$, or*
2. *if $\sigma - (a+b)$ is a multiple of m and $b \geq G+1$.*

It holds:

Theorem 2.5 (*Theorem 2.10 in [25]*). *Let p an integer between 1 and n for which $gcd(m,p)$ divides s. Then there are infinitely many non-negative integers $k \geq ((n+1)p/(n+1-p)) - (s/m)$ for which $d = (s+km)/p$ is a positive integer. The infinitesimal Torelli theorem holds for quasi-smooth hypersurfaces of degree d in $\mathbb{P}(a_0, \ldots, a_{n+1})$.*

For other specific choices of the weights it is known:

Theorem 2.6 (*Theorem 4.1 in [17]*). *Let $\mathbb{P}$ a weighted projective space $\mathbb{P}(a_0, \ldots, a_{n+1})$ for which $a_0 = a_1 = 1$ and m divides s, and let $\mathcal{M}$ the space of quasi-smooth weighted hypersurfaces of degree d in $\mathbb{P}$ modulo automorphisms of $\mathbb{P}$. Assume d is a multiple of m and $d \geq \max(3s, s+m(n+1))$. Then there is an open dense subset of $\mathcal{M}$ on which the period map is defined and injective.*

Many interesting cases are still left open. In particular the answer was not known for $\mathbb{Q}$-Fano threefolds hypersurfaces, despite their importance in terms of birational geometry. In this paper we give an answer to this problem, following a careful analysis of the Jacobian rings involved.

3. Fano hypersurfaces threefolds and Torelli problem

We first study the infinitesimal Torelli for the 'famous 95'. These are Fano threefolds of index 1 first discovered by Iano-Fletcher and Reid. They are listed online in the graded ring database [2], together with several thousands of other Fanos in higher codimension. We do not include here the complete list of 95 hypersurfaces: the interested reader can easily consult the above database.

3.1. The infinitesimal Torelli for Fano varieties of index 1

Recall from the introduction the Griffiths description of the differential of the period map for an Hodge structure of weight k

$$\mathrm{d}\wp : H^1(T_X) \to \bigoplus_{p=0}^{k} \mathrm{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)).$$

In particular, its injectivity is enough for the infinitesimal Torelli theorem to hold. Assume that X_d is a smooth hypersurface of degree d in $w\mathbb{P}^n(a_0, \ldots, a_n)$ of dimension at least 3, and that $H^2(\mathcal{O}_X) = 0$ (equivalently, all the deformations of X are

projective). Denote by $s = \sum a_i$. The Griffiths–Steenbrink description of the primitive cohomology of a quasi-smooth hypersurface reduces the problem to the injectivity of the polynomial map

$$R_d \longrightarrow \bigoplus_{p=0}^{n} \mathrm{Hom}(R_{(p+1)d-s}, R_{(p+2)d-s}),$$

where R denotes as usual the Jacobian ring and the subscript refers to its homogeneous components.

For a quasi-smooth Fano threefolds we have $H^{3,0}(X) = H^{0,3}(X) = 0$: moreover if we focus on the index 1 condition what we have to verify in order to have the infinitesimal Torelli for any of the previous 95 families is to check the injectivity of the natural map

$$R_d \to \mathrm{Hom}(R_{d-1}, R_{2d-1}).$$

If we use local duality Theorem 2.2, since the socle is located in degree $\sigma = 3d - 2\iota_X$ we can rephrase the problem as follows

Remark 3.1. Let X as above. The infinitesimal Torelli theorem holds for X if the natural map

$$\mathrm{Sym}^2(R_{d-1}) \to R_{2d-2}$$

is surjective.

We first try to apply the condition of weighted Macaulay's theorem recalled in the introduction. Consider for example the case (5) of the list above, that is $X_7 \subset \mathbb{P}(1, 1, 1, 2, 3)$. Using the notations of Theorem 2.5, the equations that need to be satisfied is $d = 7 = (8 + 6k)/p$, together with the condition $k \geq (4p/4 - p) - 4/3$, but this is clearly not possible. By similar standard computations we get

Lemma 3.2. Only the families no. 1 and no. 2 of the Fletcher-Reid list (that is, $X_4 \subset \mathbb{P}^4$ and $X_5 \subset \mathbb{P}(1^4, 2)$) satisfies the numerical conditions of the weighted Macaulay's theorem.

In the other cases, a partial answer can be obtained by looking at the surjectivity of the multiplication map in the ambient ring. More precisely

Proposition 3.3 (*Proposition 2.3 in [25]*)**.** *Let $R = \mathbb{C}[x_0, \ldots, x_n]/J$ be a weighted ring for which local duality holds, and let σ be the top degree of R. Given non-negative integers a, b satisfying $a + b \leq \sigma$ and denoting with $S = \mathbb{C}[x_0, \ldots, x_n]$ if*

$$S_b \times S_{\sigma-(a+b)} \to S_{\sigma-a}$$

is surjective, then $R_a \to \mathrm{Hom}(R_b, R_{a+b})$ is injective.

In our case, we have $a = d, b = d - 1$ and $\sigma = \sum d - 2a_i = 5d - 2s$, with $s = d + 1$. Therefore we have to check the result for $S_{d-1} \times S_{d-1} \to S_{2d-2}$, or equivalently the surjectivity of the natural map

$$\mathrm{Sym}^2(S_{d-1}) \to S_{2d-2}.$$

Lemma 3.4. *Of the 93 remaining families, only the number 5, that is $X_7 \subset \mathbb{P}(1, 1, 1, 2, 3)$ satisfies the surjectivity already at the level of polynomial ring.*

Proof. On X_7 the multiplication map $\mathrm{Sym}^2(S_6) \to S_{12}$ can be verified to be surjective. All the other cases yields counterexamples. For example consider $X_6 \subset \mathbb{P}(1^4, 3)$. $\mathrm{Sym}^2(S_5) \to S_{10}$ is not surjective as we can see by the element $x_4^3 x_0$, where x_4 is the variable of weights 3. Indeed is clear that $S_5 = \langle \mathrm{Sym}^5(x_0, \ldots, x_3), \mathrm{Sym}^2(x_0, \ldots, x_3) \cdot x_4 \rangle$. An extensive computer search using Macaulay2 confirms this statement. $\square$

Corollary 3.5. *Let X any quasi smooth member of the family $X_7 \subset \mathbb{P}(1, 1, 1, 2, 3)$. Then the differential of the Period map*

$$\mathrm{d}\wp : H^1(X, T_X) \to \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X))$$

is injective, and therefore the local Torelli theorem holds.

To prove the theorem for the other cases, we have to use the following lemma

Lemma 3.6. *Let $\pi : \mathcal{X} \to U$ a flat family of quasi-smooth hypersurfaces such that for the central fibre $X_0 = \pi^{-1}(0)$ the infinitesimal Torelli property holds. Then the same holds for the fibres $\pi^{-1}(U)$ over an open $0 \in U$.*

Proof. We recall that $\mathcal{M}_0$ is a complex manifold and the period domain is a variety. Hence the condition of the differential of the period map being of maximal rank is an open condition. $\square$

Theorem 3.7. *Let $X_d \subset \mathbb{P}(a_0, \ldots, a_4)$ one of the 92 families left (not 1,2,5). Then the Local Torelli theorem holds generically for X_d.*

Table 1
List of Fano threefolds of index 1 and their behaviour with respect to Torelli problem. The notation AT/T/R stands for (respectively) Anti-Torelli, Torelli, Rigid.

No.	$X_d \subset \mathbb{P}(a_0, \ldots, a_4)$	Ind	Torelli	No.	$X_d \subset \mathbb{P}(a_0, \ldots, a_4)$	Ind	Torelli
96	$X_3 \subset \mathbb{P}(1, 1, 1, 1, 1)$	2	T	113	$X_4 \subset \mathbb{P}(1, 1, 2, 2, 3)$	5	R
97	$X_4 \subset \mathbb{P}(1, 1, 1, 1, 2)$	2	T	114	$X_6 \subset \mathbb{P}(1, 2, 3, 4)$	5	T
98	$X_6 \subset \mathbb{P}(1, 1, 1, 2, 3)$	2	T	115	$X_6 \subset \mathbb{P}(1, 2, 2, 3, 3)$	5	AT
99	$X_{10} \subset \mathbb{P}(1, 1, 2, 3, 5)$	2	T	116	$X_{10} \subset \mathbb{P}(1, 2, 3, 4, 5)$	5	T
100	$X_{18} \subset \mathbb{P}(1, 2, 3, 5, 9)$	2	T	117	$X_{15} \subset \mathbb{P}(1, 3, 4, 5, 7)$	5	T
101	$X_{22} \subset \mathbb{P}(1, 2, 3, 7, 11)$	2	T	118	$X_6 \subset \mathbb{P}(1, 1, 2, 3, 5)$	6	T
102	$X_{26} \subset \mathbb{P}(1, 2, 5, 7, 13)$	2	T	119	$X_6 \subset \mathbb{P}(1, 2, 2, 3, 5)$	7	R
103	$X_{38} \subset \mathbb{P}(2, 3, 5, 11, 19)$	2	T	120	$X_6 \subset \mathbb{P}(1, 2, 3, 3, 4)$	7	R
104	$X_2 \subset \mathbb{P}(1, 1, 1, 1, 1)$	3	R	121	$X_8 \subset \mathbb{P}(1, 2, 3, 4, 5)$	7	AT
105	$X_3 \subset \mathbb{P}(1, 1, 1, 1, 2)$	3	T	122	$X_{14} \subset \mathbb{P}(2, 3, 4, 5, 7)$	7	AT
106	$X_4 \subset \mathbb{P}(1, 1, 1, 2, 2)$	3	T	123	$X_6 \subset \mathbb{P}(1, 2, 3, 3, 5)$	8	R
107	$X_6 \subset \mathbb{P}(1, 1, 2, 2, 3)$	3	T	124	$X_{10} \subset \mathbb{P}(1, 2, 3, 5, 7)$	8	T
108	$X_{12} \subset \mathbb{P}(1, 2, 3, 4, 5)$	3	T	125	$X_{12} \subset \mathbb{P}(1, 3, 4, 5, 7)$	8	T
109	$X_{15} \subset \mathbb{P}(1, 2, 3, 5, 7)$	3	T	126	$X_6 \subset \mathbb{P}(1, 2, 3, 4, 5)$	9	R
110	$X_{21} \subset \mathbb{P}(1, 3, 5, 7, 8)$	3	T	127	$X_{12} \subset \mathbb{P}(2, 3, 4, 5, 7)$	9	AT
111	$X_4 \subset \mathbb{P}(1, 1, 1, 2, 3)$	4	T	128	$X_{12} \subset \mathbb{P}(1, 4, 5, 6, 7)$	11	T
112	$X_6 \subset \mathbb{P}(1, 1, 2, 3, 3)$	4	T	129	$X_{10} \subset \mathbb{P}(2, 3, 4, 5, 7)$	11	R
				130	$X_{12} \subset \mathbb{P}(3, 4, 5, 6, 7)$	13	R

Proof. The idea of the proof is simple. We check the surjectivity of the dual of the infinitesimal period map on a quasi-smooth element X_0. A genericity condition can be found for example in [18]. Once this is confirmed, by Lemma 3.6 the result will hold in a Zariski open set $0 \in U$, where we can think of X_0 as central fibre of a flat family $\pi : \mathcal{X} \to U$.

The check on the central fibre can be done with the help of the following Macaulay2 code

```
def=(d, weights)->(
R=QQ[x,y,z,t,w, Degrees=>weights];
 f=random(d,R); J=Jacobian matrix f;
B=(R/ideal J);L1=flatten entries (symmetricPower(2, basis(d-1,B)));
L2=flatten entries basis(2*d-2,B);
APP=select(L2, a-> not member(a, L1));
IL=ideal(L1);  print APP, print LINCOMB,
for i in APP do if ((i % IL)!=0) then print i else print 0
)
```

Actually all the computations could be solved by hand in principle. To have a concrete grasp of how this work we refer to Example 3.13, where we deal with the (most) interesting case of index > 1. An extensive computer search with Macaulay2 confirms that this holds for any of the 92 remaining families. □

Remark 3.8. It is because of Lemma 3.6 that some authors adopt the somewhat confusing expression "local Torelli" instead of "infinitesimal Torelli".

3.2. The infinitesimal Torelli for Fano varieties of index > 1

We now investigate the case of higher index Fano threefolds. Unlike the index 1 case, here the situation is much more various and complicated: in particular we will have (other than the Hodge-trivial examples) some families for which the infinitesimal Torelli fails, and some families for which it holds. We recall in Table 1 the list of Fano threefold of higher index, from Okada (see [20]), ordered according to the index.

What do we have to check? Let us call the index of any X ι_X, that is $\omega_X \cong \mathcal{O}_X(-\iota_X)$. Keeping notations as in the introduction we have

Lemma 3.9. *Let $X_d \subset \mathbb{P}(a_0, \ldots, a_4)$ a quasi-smooth Fano threefold hypersurface of index ι_X. Then the infinitesimal Torelli theorem holds if the natural map*

$$R_{d-\iota_X} \times R_{d-\iota_X} \to R_{2d-2\iota_X}$$

is surjective.

Proof. The success of infinitesimal Torelli corresponds to the injectivity of the map

$$R_d \to \mathrm{Hom}(R_{d-\iota_X}, R_{2d-\iota_X}),$$

where $R_{d-\iota_X} \cong H^{2,1}(X)$ and $R_{2d-\iota_X} \cong H^{1,2}(X) \cong (H^{2,1}(X))^\vee$ by Serre duality. Equivalently, using Local duality theorem (Theorem 2.2 of [25]) what we need to verify is that the surjectivity of natural multiplication map

$$R_{d-\iota_X} \times R_{\sigma-(2d-\iota_X)} \to R_{\sigma-d}.$$

On the other hand, since $s = \sum a_i = d + \iota_X$, we have $\sigma = 5d - 2s = 3d - 2\iota_X$. $\square$

We will now analyse separately the three interesting cases, that is Hodge-rigid families, the anti-Torelli and the Torelli.

3.3. Hodge-rigid families

Recall that we call an $X_d \subset w\mathbb{P}$ *Hodge-rigid* if both $H^1(X, T_X) = 0$ and $H^{2,1}(X) = 0$. Now, for these threefolds we do not have any Torelli-type question to ask: therefore we want to classify and remove those cases from our list. Now, we recall that from Griffiths–Steenbrink theory we have $R_d \cong H^1(X, T_X)$, and by Serre duality

$$H^1(X, \Omega_X^2) \cong H^1(T_X \otimes \omega_X) = H^1(X, T_X(-\iota_X)) = R_{d-\iota_X}.$$

From this it clearly follows that if $d < \iota_X$, then $H^{2,1}(X) = 0$.

Proposition 3.10. *The following families satisfies* $H^{2,1}(X) = H^1(X, T_X) = 0$. *Therefore they are Hodge-rigid.*

- no. 104 $X_2 \subset \mathbb{P}^4$;
- no. 113 $X_4 \subset \mathbb{P}(1, 1, 2, 2, 3)$;
- no. 119 $X_6 \subset \mathbb{P}(1, 2, 2, 3, 5)$;
- no. 120 $X_6 \subset \mathbb{P}(1, 2, 3, 3, 4)$;
- no. 123 $X_6 \subset \mathbb{P}(1, 2, 3, 3, 5)$;
- no. 126 $X_6 \subset \mathbb{P}(1, 2, 3, 4, 5)$;
- no. 129 $X_{10} \subset \mathbb{P}(2, 3, 4, 5, 7)$;
- no. 130 $X_{12} \subset \mathbb{P}(3, 4, 5, 6, 7)$.

Proof. The vanishing of $H^{2,1}(X)$ is assured by the condition $d < \iota_X$ above. To verify the vanishing of $H^1(X, T_X)$, simply notice that for every member of the list above one has $d > \sigma$. Therefore, by definition of socle, $R_d = 0$. $\square$

We go on a bit further with the analysis of the Hodge-rigid families. First, from the classification of Okada [20] we see that each member of the Hodge rigid families has the general member rational, and moreover we just computed that they do not have any deformation. We want now to define a notion of *strong rigidity*. Recall from [3], if A_X denotes the affine cone over X, then $(T_{A_X}^1)_k$, for $k < 0$ parametrizes the extension of X as projective variety (and as well contains the smoothings of A_X). If $X = V(f)$ is a hypersurface we have $(T_{A_X}^1)_k \cong (R_f)_{d-k}$ and moreover $(R_f)_{d-k} \cong H^1(X, T_X(d - k))$. We say that X is *strongly rigid* if $(R_f)_j = 0$, $j \neq 0$. A strongly rigid hypersurface therefore will not have any extension (rather than the trivial one). We note that X_d will satisfy $(T_{A_X}^1)_{-1} \cong (R_f)_{d-1} = 0$. In particular we will have to add non-trivial weights in order to extend (when possible) the Hodge-rigid members.

Corollary 3.11. *Amongst the Hodge-Rigid families, the* no. 104, 126, 129 *are strongly rigid.*

Proof. The Jacobian rings of the examples above have Hilbert–Poincarè series equal to 1. Therefore they have $R_0 = 1$ and $R_k = 0$, $k \neq 0$. $\square$

3.4. Torelli and Anti-Torelli families

We now investigate first the families that does not satisfy the infinitesimal Torelli property.

Theorem 3.12. *Let X_d a quasi-smooth member of one of the four families no. 115, 121, 122, 127. Then the infinitesimal Torelli does not hold for X_d.*

Proof. We will analyse separately the four different cases.

no. 115. Pick any general quasi smooth member of the family of index $\iota_X = 5$, $X_6 \subset \mathbb{P}(1, 2, 2, 3, 3)$, where we name coordinates x, y_0, y_1, z_0, z_1. Since $d - \iota_X = 1$ we have to check the surjectivity of the map: $R_1 \times R_1 \to R_2$. Now, since the partial derivatives form a regular sequence, if we compute the Hilbert–Poincaré series of the Jacobian ring we have

$$\text{HP}(R) = \prod \frac{(1 - t^{d-a_i})}{(1 - t^{a_i})} = 1 + t + 3t^2 + 3t^3 + 4t^4 + 3t^5 + 3t^6 + t^7 + t^8.$$

Therefore, since there is only one generator of degree one, we have $\text{Sym}^2(R_1) \cong\, <x^2>$, whereas R_2 contains, for example, y_0, y_1, and the natural map cannot be surjective. Dually speaking we have the standard map from R_d given by

$$\mathbb{C}^3 \cong R_6 \to \text{Hom}(R_1, R_7) \cong \mathbb{C},$$

and this cannot be injective.

No. 121. The same phenomenon occurs for $X_8 \subset \mathbb{P}(1, 2, 3, 4, 5)$, of index 7, with coordinates x, y, z, v, w. We have indeed that the Hilbert–Poincaré series is

$$HP(R) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 3t^6 + 2t^7 + 2t^8 + t^9 + t^{10}$$

and for $\mathbb{C}^2 \cong R_8 \to \text{Hom}(R_1, R_9) \cong \mathbb{C}$ injectivity clearly fails.

No. 122. This is $X_{14} \subset \mathbb{P}(2, 3, 4, 5, 7)$, of index $\iota_X = 7$. Here we have as Hilbert–Poincaré series

$$\begin{aligned}
HP(R) = {} & 1 + t^2 + t^3 + 2t^4 + 2t^5 + 3t^6 + 3t^7 + 5t^8 + 4t^9 + 6t^{10} \\
& + 5t^{11} + 7t^{12} + 6t^{13} + 7t^{14} + 6t^{15} + 7t^{16} + 5t^{17} \\
& + 6t^{18} + 4t^{19} + 5t^{20} + 3t^{21} + 3t^{22} + 2t^{23} + 2t^{24} + t^{25} + t^{26} + t^{28}.
\end{aligned}$$

Here we have $R_{14} \cong \mathbb{C}^7$ and $R_7 \cong \mathbb{C}^3$, so we cannot conclude immediately injectivity. Nevertheless, $\text{Sym}^2(R_7)$ has dimension 6, so the surjectivity is excluded.

No. 127. This is $X_{12} \subset \mathbb{P}(2, 3, 4, 5, 7)$. Here the Hilbert–Poincaré series is

$$\begin{aligned}
HP(R) = {} & 1 + t^2 + t^3 + 2t^4 + t^5 + 3t^6 + 2t^7 + 3t^8 + 2t^9 + 3t^{10} + 2t^{11} + 3t^{12} + t^{13} \\
& + 2t^{14} + t^{15} + t^{16} + t^{18}.
\end{aligned}$$

Thus $R_{12} \cong \mathbb{C}^3$, while $\text{Hom}(R_3, R_{15}) \cong \mathbb{C}$. $\quad\square$

3.4.1. The Torelli families of index > 1

The other 24 families behave in the opposite way. We were able to check the surjectivity statement already at the level of the ring S, that is $\text{Sym}^2 S_{d-\iota_X} \to S_{2d-2\iota_X}$, for the families no. 98, 111, 118, 128. This is enough to guarantee the surjectivity at the level of the Jacobian ring since $S_{2d-2\iota_X} \to R_{2d-2\iota_X}$ is the quotient map. No. 111 and No. 118 exhibits a curious behaviour. They are respectively $X_4 \subset \mathbb{P}(1, 1, 1, 2, 3)$ and $X_6 \subset \mathbb{P}(1, 1, 2, 3, 5)$. They both verify $d = \iota_X$ and therefore $d = \sigma$. In particular, one has to check either the surjectivity of the natural map $R_0 \times R_0 \to R_0$ or the injectivity of $R_\sigma \to \text{Hom}(R_0, R_\sigma)$, and this trivially holds. For all the other families we proceed by the means of an extensive computer search, as in the index 1 case. We recall here an example.

Example 3.13. Consider $X_6 \subset \mathbb{P}(1, 1, 2, 3, 4)$ in the class no. 114 given by the equation $x_0^6 + x_1^6 + x_2^3 + x_3^2 + x_2 x_4$. Note that X_6 is quasi-smooth. The Hilbert Series of its Jacobian ring is

$$HP(R) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 4t^5 + 3t^6 + 2t^7 + t^8$$

We have to check the surjectivity $\text{Sym}^2 R_1 \to R_2$. The space R_1 is generated by x_0, x_1, while R_2 is generated by x_0^2, $x_0 x_1$, x_1^2. Indeed the variable x_2 of weight 2 is in the Jacobian ideal. $\text{Sym}^2 R_1$ is 3-dimensional, and equal to R_2. In particular the multiplication map is surjective. Therefore infinitesimal Torelli holds.

The same genericity argument of Lemma 3.6 yields

Theorem 3.14. *Let $\mathcal{M}$ the space of quasi-smooth weighted hypersurfaces of degree d in $\mathbb{P}$ modulo automorphisms of $\mathbb{P}$, for any of the non Hodge-trivial families of quasi-smooth Fano Threefolds of index $i_X > 1$. Then*

- *for the families no. 115, 121, 122 and 127 the infinitesimal Torelli theorem does not hold;*
- *for the remaining 24 families, there is an open dense subset of $\mathcal{M}$ on which the infinitesimal Torelli theorem holds.*

3.5. Explicit Anti-Torelli deformations

Here we list again the four families of anti-Torelli Fano threefold hypersurfaces

No	$X_d \subset \mathbb{P}(a_0, \ldots, a_4)$	Ind.	$h^1(T_X)$	$h^{2,1}$
115	$X_6 \subset \mathbb{P}(1, 2, 2, 3, 3)$	5	3	1
121	$X_8 \subset \mathbb{P}(1, 2, 3, 4, 5)$	7	2	1
122	$X_{14} \subset \mathbb{P}(2, 3, 4, 5, 7)$	7	7	3
127	$X_{12} \subset \mathbb{P}(2, 3, 4, 5, 7)$	9	3	1

We now analyse carefully these four examples in terms of Torelli and Anti-Torelli deformations.

Table 2
Torelli and Anti-Torelli for family no. 115.

R_1	$\langle x_0 \rangle$
R_7	$\langle x_0^3 x_1 x_2 \rangle$
R_6	$\langle x_0^4 x_1, x_0^4 x_2, x_0^2 x_1 x_2 \rangle$
K_f	$\langle x_0^4 x_1, x_0^4 x_2 \rangle$

Table 3
Torelli and Anti-Torelli for family no. 121.

R_1	$\langle x_0 \rangle$
R_9	$\langle x_0^5 x_1^2 \rangle$
R_8	$\langle x_0^6 x_1, x_0^4 x_1^2 \rangle$
K_f	$\langle x_0^6 x_1 \rangle$

Table 4
Torelli and Anti-Torelli for family no. 127.

R_3	$\langle x_1 \rangle$
R_{15}	$\langle x_0^4 x_1^2 \rangle$
R_{12}	$\langle x_0^3 x_1^2, x_0 x_1^2 x_2, x_0^4 x_2 \rangle$
K_f	$\langle x_0^3 x_1^2, x_0 x_1^2 x_2 \rangle$

No. 115. Let us start analysing the first of the example. To get explicit examples of Anti-Torelli deformations we use the following member of the family defined by the polynomial

$$f = x_0^6 + x_1^3 + x_2^3 + x_3^2 + x_4$$

where $V(f) \subset \mathbb{P}(1, 2, 2, 3, 3)$. $V(f)$ is clearly quasi-smooth since f is of Fermat-type. Recall by the previous section that

$$\mathrm{d}\wp : R_d \to \mathrm{Hom}(R_{d-\iota_X}, R_{2d-\iota_X})$$

cannot be injective since $R_6 \cong \mathbb{C}^3$ and $R_1 \cong R_7 \cong \mathbb{C}$. Denote thus by K_f the kernel of the period map associated with this specific central member of the family giving the period map. Any element in K_f will give rise to a deformation of *Anti-Torelli* type. In Table 2 we list the generators of the interesting graded component of the Jacobian ring R_f and the anti-Torelli deformations associated with f.

To compute K_f note that both $(x_0^4 x_1) x_0$ and $(x_0^4 x_2) x_0$ belong to J_f while $(x_0^2 x_1 x_2) x_0 = x_0^3 x_1 x_2$ is the generator we already found for R_7. Therefore a family of Anti-Torelli deformation with central fibre the fixed $X_0 = V(f)$ is given by $V(f + \lambda x_0^4 x_1 + \mu x_0^4 x_2)$.

No. 121. Here we use as the central fibre the following member of the family defined by the polynomial

$$f = x_0^8 + x_1^4 + x_2 x_4 + x_3^2$$

where $V(f) \subset \mathbb{P}(1, 2, 3, 4, 5)$. This is not Fermat-type but the Jacobian ideal is generated by $(x_0^7, x_1^3, x_4, x_3, x_2)$. Moreover the tangent monomials at the singular points induced by the ambient space are linear. Therefore the above variety is quasi-smooth. Even here

$$\mathrm{d}\wp : R_d \to \mathrm{Hom}(R_{d-\iota_X}, R_{2d-\iota_X})$$

cannot be injective since $R_8 \cong \mathbb{C}^2$ and $R_1 \cong R_9 \cong \mathbb{C}$. In Table 3 we list the generators of the interesting graded component of the Jacobian ring R_f and the anti-Torelli deformations associated with f.

Here again $(x_0^6 x_1) x_0 \in J_f$, while $x_0^5 x_1^2$ generates R_9. Therefore a family of Anti-Torelli deformation with central fibre the fixed X_f will be given by $V(f + \lambda x_0^6 x_1)$.

No. 127. Here we use the following member of the family defined by the polynomial

$$f = x_0^6 + x_1^4 + x_2^3 + x_3 x_4$$

where $V(f) \subset \mathbb{P}(2, 3, 4, 5, 7)$. Here same Jacobian criteria of the previous example apply. Even in this case $\mathrm{d}\wp : R_d \to \mathrm{Hom}(R_{d-\iota_X}, R_{2d-\iota_X})$ cannot be injective since $R_{12} \cong \mathbb{C}^3$ and $R_3 \cong R_{15} \cong \mathbb{C}$. In Table 4 we list the generators of the interesting graded component of the Jacobian ring R_f and the anti-Torelli deformations associated with f.

The Kernel verifications follow as in the previous cases. Therefore a family of Anti-Torelli deformation with central fibre the fixed X_f will be given by $V(f + \lambda x_0^3 x_1^2 + \mu x_0 x_1^2 x_2)$.

Table 5
Torelli and Anti-Torelli for family no. 127.

R_7	$\langle x_0^2 x_1, x_0 x_3, x_1 x_2 \rangle$
R_{21}	$\langle x_0^4 x_1 x_3^2, x_0^3 x_3^3, x_2^4 x_3 \rangle$
R_{14}	$\langle x_0 x_1 x_2 x_3, x_0^4 x_1^2, x_0^5 x_2, x_0^3 x_1 x_3, x_0^2 x_1 x_2, x_0^2 x_3^2, x_1^2 x_2^2 \rangle$
K_f	$\langle x_0 x_1 x_2 x_3 \rangle$

No. 122. Here we use the following member of the family defined by the polynomial

$$f = x_0^7 + x_0 x_2^3 + x_1^3 x_3 + x_2 x_3^2 + x_4^2$$

where $V(f) \subset \mathbb{P}(2, 3, 4, 5, 7)$. Looking at the equation the Jacobian ideal has no zeros.

Unlike the previous examples, here the non-injectivity cannot be deduced a priori, but follows from a careful examination of the multiplication map. In Table 5 we list the generators of the interesting graded component of the Jacobian ring R_f and the anti-Torelli deformations associated with f.

To check the kernel K_f we proceed as follows. First we check using the previous tools that the element of R_{2d-2t} in the cokernel of multiplication map is $x_0^5 x_2$. Then we dualize via $(R_k)^\vee \cong R_{\sigma-k}$. Here $R_d = R_{14} = R_{\sigma-d}$. The (non-canonical) isomorphism between R_{14} and itself yielding duality pairs up $x_0^5 x_2$ with $\frac{1}{32} x_0 x_1 x_2 x_3$ with respect to the socle basis $x_0^5 x_1 x_3^3$. One can in fact verifies that the above multiplication gives exactly the socle generator, whereas any other multiplication of $x_0^5 x_2$ with any other basis element of R_{14} lies in J_f. Indeed one can identify $\frac{1}{32} x_0 x_1 x_2 x_3$ as dual element of $x_0^5 x_2$. Therefore a family of Anti-Torelli deformation with central fibre the fixed X_f will be given by $V(f + \lambda x_0 x_1 x_2 x_3)$.

We therefore raise the following question:

Question 3.15. *For any of the above examples, consider any deformation of X_d induced by one of the $\xi \in K_f$, that is of Anti-Torelli-type. Is there any geometrical reason for the failure of the Torelli theorem in these specific directions?*

4. Gushel–Mukai type infinite towers

We start our analysis of an interesting geometrical phenomenon which links the geometry of some weighted Fano hypersurfaces to the Gushel–Mukai geometry recalled in the introduction. We first will explain the necessary algebraic background in order to understand this strange phenomenon, and later on we will analyse in great details some example of this construction.

4.1. Hyperplane section principle for graded rings

The key phenomenon here is the *hyperplane section principle* for graded rings. This is a standard result in commutative algebra. We will follow closely the treatment of [23]

Let R be a graded ring, and $x_0 \in R$ a graded element of degree a_0. Suppose that x_0 is a regular element of R, that is, a non-zerodivisor. Then multiplication by x_0 is an inclusion $R \to R$ with image the principal ideal (x_0), and one arrives at the exact sequence

$$0 \to (x_0) \to R \to \bar{R} \to 0,$$

where $\bar{R} = R/(x_0)$. Geometrically, Proj $(\bar{R})$ is the hyperplane section of Proj (R) given by $x_0 = 0$. The hyperplane section principle says that under these assumptions, we can deduce a lot of the structure of $\bar{R}$ from R and vice-versa.

Theorem 4.1 (*Hyperplane Section Principle, [23]*)**.**

1. *Let $\overline{x_1}, \dots, \overline{x_k}$ be homogeneous element that generate $\bar{R}$, and $x_1, \dots, x_k \in R$ any homogeneous elements that map to $\overline{x_1}, \dots, \overline{x_k} \in \bar{R}$. Then R is generated by $x_0, x_1, \dots, x_k$.*
2. *Under the assumption of (1), let $\bar{f}_1, \dots, \bar{f}_n$ be homogeneous generators of the ideal of relations holding between $\overline{x_1}, \dots, \overline{x_k}$. Then there exist homogeneous relations $f_1, \dots, f_n$ holding between $x_0, x_1, \dots, x_n$ in R such that the f_i reduces to $\bar{f}_i$ modulo x_0 and $f_1, \dots, f_n$ generate the relation between $x_0, x_1, \dots, x_n$.*
3. *Similar for the syzygies.*

4.2. Construction of towers

Let us start from a weighted hypersurface $X_d^0 = V(f_0) \subset \mathbb{P}(a_0, \dots, a_n)$. Suppose that $d \equiv 0 \ (mod\ 2)$ and call $d = 2t$. Consider now

$$X_d^1 \xrightarrow{2:1} \mathbb{P}(a_0, \dots, a_n)$$

a double cover of the ambient weighted projective space, branched over X_d^0. Since d is even, X_d^1 has a model as

$$X_d^1 = V(f_0 + y_1^2) \subset \mathbb{P}(a_0, \ldots, a_n, t).$$

Let us call $f_1 := f_0 + y_1^2$. Suppose now that $\omega_{X_d^0} \cong \mathcal{O}_{X_d^0}(m_0)$: by adjunction one has
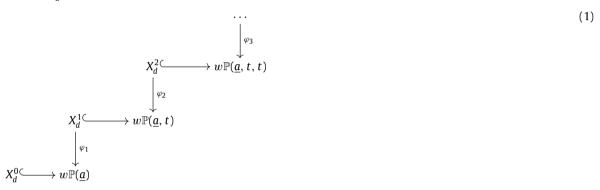
$$\omega_{X_d^1} \cong \mathcal{O}_{X_d^1}(-\sum a_i - t + d) \cong \mathcal{O}_{X_d^1}(m_0 - t) =: \mathcal{O}_{X_d^1}(m_1).$$

We have the following immediate result

Lemma 4.2. X_d^0 is quasi-smooth if and only if X_d^1 is.

Proof. Simply notice that $J_{f_1} = (J_{f_0}, y_1)$. Therefore the only possible new singularities of X_d^1 comes just from the new ambient space. □

Set $R_{f_0} = S/(J_{f_0})$ and similarly R_{f_1} for X_d^1. It is clear that $R_{f_1} \cong S[y_1]/(J_{f_0}, y_1)$, therefore by the hyperplane section principle one has $R_{f_0} \cong R_{f_1}$. We pick any general member $X_d^1 = V(g) \subset \mathbb{P}(a_0, \ldots, a_n, t)$. Since the partial derivatives form a regular sequence, one clearly has $\dim(R_{f_1})_k = \dim(R_g)_k$, for all k and we have an isomorphism of $\mathbb{C}$-vector spaces between the two Jacobian rings. Now we see that by completing the square the "double cover type" does not form a proper subfamily inside the space of all $X_d^1 \subset \mathbb{P}(a_0, \ldots, a_n, t)$. Clearly the process can go on to give an infinite chain of double covers

$$\cdots \tag{1}$$

$$\downarrow \varphi_3$$

$$X_d^2 \lhook\joinrel\longrightarrow w\mathbb{P}(\underline{a}, t, t)$$

$$\downarrow \varphi_2$$

$$X_d^1 \lhook\joinrel\longrightarrow w\mathbb{P}(\underline{a}, t)$$

$$\downarrow \varphi_1$$

$$X_d^0 \lhook\joinrel\longrightarrow w\mathbb{P}(\underline{a})$$

where any X_d^j is a double cover of the projective space $\mathbb{P}(a_0, \ldots, a_n, t^{j-1})$ branched on X_d^{j-1}. By hyperplane section principle

Proposition 4.3. *For any X_d^j obtained with tower construction one has $R_{f_j} \cong R_{f_0}$.*

We point out that we can reverse the chain, and considering X_d^1 as an extension of X_d^0. Recall that, in general, $Y \subset \mathbb{P}^{n+1}$ is said to be an *extension* of $X \subset \mathbb{P}^n$ if we have $\dim(X) = n$, $\dim(Y) = n + 1$ and there exists an immersion $i : \mathbb{P}^n \to \mathbb{P}^{n+1}$ such that $X = Y \cap i(\mathbb{P}^n)$. In the weighted case of $X \subset w\mathbb{P}(a_0, \ldots, a_n)$ we thus have to look of the several graded components $(T_{A_X}^1)_{-a_i}$. Moreover, for the case of a hypersurface of degree d we have

$$T_{A_X}^1 \cong R_f(d).$$

To sum up: given X_d^0 we also have X_d^1 as

$$X_d^1 = V(f_0 + h_t \cdot y_1) \subset \mathbb{P}(a_0, \ldots, a_n, t),$$

where y_1 is a new variable of weight t and more important

$$h_t \in (T_{A_{X_d^0}}^1)_{-t} \cong (R_{f_0})_{d-t} \cong (R_{f_0})_t.$$

It is trivial to see that there is no difference between this and a double cover model.

Instead what is not clear a priori is what happens to the Hodge groups when we run up the tower.

4.3. Periodic patterns in Hodge theory

Let us call an *even member* of the tower an X_d^{2k} obtained by doing an even number of step in the construction above. Similarly we will define the *odd members*.

Theorem 4.4.

- Let X_d^{2k} any even member of the tower, of dimension $n + 2k$. We have that

$$H^{n+2k}(X_d^{2k}, \mathbb{C}) \cong H^n(X_d^0, \mathbb{C}).$$

The isomorphism is compatible with the Hodge decomposition: in particular the central Hodge numbers of X_d^0 are the same of the Hodge numbers of X_d^{2k} up to a degree k shift, that is

$$\left(h_{X_d^{2k}}^{n+2k,0}, h_{X_d^{2k}}^{n+2k-1,1}, \ldots, h_{X_d^{2k}}^{1,n+2k-1}, h_{X_d^{2k}}^{0,n+2k} \right) = \left(0, \ldots, 0, h_{X_d^0}^{n,0}, \ldots, h_{X_d^0}^{0,n}, 0, \ldots, 0 \right),$$

with $2k$ zeros on the last vector;
- the same holds for odd members, with an equality between the Hodge numbers of X_d^1 and X_d^{2k+1}, for any k.

Proof. This is just a careful analysis of the involved components of the Jacobian ring. Let us start from the base of our chain, $X_d^0 \subset \mathbb{P}(a_0, \ldots, a_n)$. We will denote as before $d = 2t$ and $s = \sum a_i$ Since by Proposition 4.3 the Jacobian ring will be the same in any step of our construction, we will drop the subscript referring to the equation and denote it simply with R. By Griffiths–Steenbrink one has

$$H_{\mathrm{prim}}^{n-p,p}(X_d^0) \cong R_{(p+1)d-s}$$

Now extend to $X_1^d \subset \mathbb{P}(a_0, \ldots, a_n, t)$. By the same argument we will have

$$H_{\mathrm{prim}}^{(n+1)-p,p}(X_d^1) \cong R_{(p+1)d-s-t}.$$

If we extend another time we have

$$H_{\mathrm{prim}}^{(n+2)-p,p}(X_d^2) \cong R_{(p+1)d-s-2t} \cong R_{pd-s} \cong H_{\mathrm{prim}}^{n-p+1,p-1}(X_d^0),$$

where we set $H_{\mathrm{prim}}^{r,s}(X) = 0$ if $r < 0$ or $s < 0$. On the other end the same yields for X_d^1 with

$$H_{\mathrm{prim}}^{(n+3)-p,p}(X_d^3) \cong R_{(p+1)d-s-3t} \cong R_{pd-s-t} \cong H_{\mathrm{prim}}^{(n+1)-p+1,p-1}(X_d^1).$$

Of course every time we perform a double extension we end up in the same graded components of the Jacobian ring, just with shift in the Hodge theory, exactly as concluded above. □

Remark 4.5. Note in particular that, since the degree of any member of the tower is constant, we will always have $H^1(T_{X_d^k})_{\mathrm{proj}} \cong R_d$. We recall that the distinction between $H^1(T_{X_d^k})_{\mathrm{proj}}$ and $H^1(T_{X_d^k})$ holds only in dimension ≤ 2.

Remark 4.6. As showed for example in [24], the Jacobian ring of a variety actually determines its (IVHS). Therefore what we have is indeed an isomorphism of IVHS

$$\phi : H^n(X_d^0) \xrightarrow{\sim} H^{n+2k}(X_d^{2k})[-2k].$$

4.4. Gushel–Mukai-type geometry

Let us now focus on our case, that is, Fano threefolds of index > 1, and pick any $X_d \subset \mathbb{P}(a_0, \ldots, a_n)$. To be in a tower one of the weights a_i has to be $a_i = d/2$. If not, one can start running the game directly from $X_d = X_d^0$, adjoining a variable of half the weight of the degree. Of course if $d \equiv 1 \pmod 2$ there is no hope of building any tower.

Looking at the table of Okada, it turns out that 30 families of Fano threefolds of index > 1 lies in a tower. At the end of the section we include a table with all the towers, and the relevant Hodge groups in both even and odd case. Note that to be Fano of K3 type, they have to satisfy $t = \iota_X$, where t is the covering variable as above. Among all, 4 of them are of K3-type, and we will focus our attention on these in particular. They corresponds to the families

1. No. 97, $X_4 \subset \mathbb{P}(1, 1, 1, 1, 2)$;
2. No. 107, $X_6 \subset \mathbb{P}(1, 1, 2, 2, 3)$;
3. No. 116, $X_{10} \subset \mathbb{P}(1, 2, 3, 4, 5)$;
4. No. 122, $X_{14} \subset \mathbb{P}(2, 3, 4, 5, 7)$.

4.4.1. A non K3-type example
Before analysing the K3-type examples, let us do a non K3-one. Pick the family no. 115, already considered before for being a counterexample to the infinitesimal Torelli problem. Again, this is the family of index $\iota_X = 5$, $X_6 \subset \mathbb{P}(1, 2, 2, 3, 3)$. The starting point of this tower is therefore the curve $X_6^0 \subset \mathbb{P}(1, 2, 2)$, with $\omega_{X_6^0} \cong \mathcal{O}_{X_6^0}(1)$. The Hilbert–Poincaré series of the Jacobian ring of the tower is again

$$\mathrm{HP}(R) = \prod \frac{(1 - t^{d-a_i})}{(1 - t^{a_i})} = 1 + t + 3t^2 + 3t^3 + 4t^4 + 3t^5 + 3t^6 + t^7 + t^8.$$

In particular any odd member will have an Hodge structure (coming from the one of a curve) concentrated in degree 1 and 7 (note that $\dim(R_1) = \dim(R_7) = 1$), while any even member will have the Hodge structure in degree -2, 4, 6, with

$R_2 = R_6 = 0$ and $\dim(R_4) = 4$. So, for example, the Fano threefold will have as Hodge diamond

$$
\begin{array}{ccccccc}
0 & & 1 & & 1 & & 0 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

and the Fano fourfold will have

$$
\begin{array}{ccccccccc}
0 & & 0 & & 5 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

We start now discussing the geometry of the GM-like varieties of the K3 type.

4.4.2. Tower on quartic double solid: a GM-like Fano of K3 type

The first example of GM-like Fano that we are going to consider is $X_4 \subset \mathbb{P}(1^4, 2)$ with coordinates $x_0, \ldots, x_3, y_1$, already famous in literature as the quartic double solid. It is a double cover of $\mathbb{P}^3$ ramified on a quartic (K3) surface. Note that the threefold itself is smooth (and not only quasi-smooth): in fact, the generic member of the family will have $y_1^2 + \cdots$, and therefore will avoid the coordinate point $P_4 = [0, 0, 0, 0, 1]$.

An infinitesimal Torelli theorem for the quartic double solid was already established by Clemens [8]. Moreover, since it shares numerical coincidences (in particular, the dimension of the intermediate Jacobian) with the Gushel–Mukai Fano threefold of index 10, Y_{10}, Tyurin conjectured the existence of a birational isomorphism between Y_{10} and X_4. This was just recently disproved by Debarre, Iliev, Manivel in [11].

If we consider the double cover X_4^2 of $\mathbb{P}(1, 1, 1, 1, 2)$ branched on the quartic double solid (we use this notation because we consider the K3 surface as the base of the tower) we will have that the resulting variety will be quasi-smooth, acquiring $2 \times \frac{1}{2}(1, 1, 1, 1)$ points on the intersection with the weighted $\mathbb{P}(2, 2)$. If we compute the Hilbert–Poincaré series of the Jacobian ring of any member of the tower, this will be

$$
HP(R) = 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 16t^5 + 10t^6 + 4t^7 + t^8.
$$

The odd Hodge structure will be concentrated in degrees 2 and 6 (with $R_2 \cong R_6$ both 10 dimensional), while the even Hodge structure will be in degrees 0, 4, 0. It follows that for example the Fano threefold will have as Hodge Diamond

$$
\begin{array}{ccccccc}
0 & & 10 & & 10 & & 0 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

and the Fano fourfold will have

$$
\begin{array}{ccccccccc}
0 & & 1 & & 20 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

and the same for every even (and odd) dimension. Note that even in higher dimension the period map will depend only by the K3 structure (see for example [10]): in particular any member of the tower will satisfy the Torelli property. A further result regards the rationality property: while it is known that a smooth quartic double solid is irrational [26], the same does not hold in higher dimensions. We prove in fact

Proposition 4.7. *Let $X_4^j \subset \mathbb{P}(1^4, 2^j)$ a quasi-smooth member of the tower of quartic double space with $\dim X_4^j \geq 4$. Then X_4^j is rational.*

Proof. Indeed any quasi-smooth $X_4^j \subset \mathbb{P}(1^4, 2^j)$ is given up to $\mathbb{P}(1^4, 2^j)$-automorphism by an equation like:

$$
X_4^j := (f(x_0, x_1, x_2, x_3, y_1, \ldots, y_{j-2}) + y_{j-1}y_j = 0)
$$

where $f(x_0, x_1, x_2, x_3, y_1, \ldots, y_{j-2})$ is a quasi-smooth polynomial of degree 4. Now let us consider the open sub-scheme $U_j \hookrightarrow \mathbb{P}(1^4, 2^j)$ given by $y_j = 1$. We set $V_j := U_j \cap X_4^j$. By definition there exists a birational morphism between the affine variety $\mathbb{A}^{j-2}_{x_0, x_1, x_2, x_3, y_1, \ldots, y_{j-2}} \dashrightarrow V_j$ given by

$$
\phi : (x_0, x_1, x_2, x_3, y_1, \ldots, y_{j-2}) \mapsto (x_0, x_1, x_2, x_3, y_1, \ldots, y_{j-2}, f(x_0, x_1, x_2, x_3, y_1, \ldots, y_{j-2})). \quad \square
$$

Table 6
Towers from Fano threefolds of index > 1.

No	$X_d \subset \mathbb{P}(a_0, \ldots, a_4)$	Odd	Even
97	$X_4 \subset \mathbb{P}(1, 1, 1, 1, 2)$	0,10,10,0	0,1,20,1,0
98	$X_6 \subset \mathbb{P}(1, 1, 1, 2, 3)$	0,21,21,0	0,3,37,3,0
99	$X_{10} \subset \mathbb{P}(1, 1, 2, 3, 5)$	0,38,38,0	0,7,63,7,0
100	$X_{18} \subset \mathbb{P}(1, 2, 3, 5, 9)$	0,49,49,0	0,10,79,10,0
101	$X_{22} \subset \mathbb{P}(1, 2, 3, 7, 11)$	0,65,65,0	0,14,103,14,0
102	$X_{26} \subset \mathbb{P}(1, 2, 5, 7, 13)$	0,66,66,0	0,14,105,14,0
103	$X_{38} \subset \mathbb{P}(2, 3, 5, 11, 19)$	0,45,45,0	0,10,71,10,0
104	$X_2 \subset \mathbb{P}(1, 1, 1, 1, 1)$	0,0,0,0	0,0,2,0,0
106	$X_4 \subset \mathbb{P}(1, 1, 1, 2, 2)$	0,3,3,0	0,0,8,0,0
107	$X_6 \subset \mathbb{P}(1, 1, 2, 2, 3)$	0,8,8,0	0,1,17,1,0
108	$X_{12} \subset \mathbb{P}(1, 2, 3, 4, 5)$	0,19,19,0	0,3,33,3,0
111	$X_4 \subset \mathbb{P}(1, 1, 1, 2, 3)$	0,1,1,0	0,0,4,0,0
112	$X_6 \subset \mathbb{P}(1, 1, 2, 3, 3)$	0,4,4,0	0,0,9,0,0
113	$X_4 \subset \mathbb{P}(1, 1, 2, 2, 3)$	0,0,0,0	0,0,2,0,0
114	$X_6 \subset \mathbb{P}(1, 1, 2, 3, 4)$	0,2,2,0	0,0,6,0,0
115	$X_6 \subset \mathbb{P}(1, 2, 2, 3, 3)$	0,1,1,0	0,0,5,0,0
116	$X_{10} \subset \mathbb{P}(1, 2, 3, 4, 5)$	0,6,6,0	0,1,13,1,0
118	$X_6 \subset \mathbb{P}(1, 1, 2, 3, 5)$	0,1,1,0	0,0,3,0,0
119	$X_6 \subset \mathbb{P}(1, 2, 2, 3, 5)$	0,0,0,0	0,0,4,0,0
120	$X_6 \subset \mathbb{P}(1, 2, 3, 3, 4)$	0,0,0,0	0,0,2,0,0
121	$X_8 \subset \mathbb{P}(1, 2, 3, 4, 5)$	0,1,1,0	0,0,4,0,0
122	$X_{14} \subset \mathbb{P}(2, 3, 4, 5, 7)$	0,3,3,0	0,1,8,1,0
123	$X_6 \subset \mathbb{P}(1, 2, 3, 3, 5)$	0,0,0,0	0,0,1,0,0
124	$X_{10} \subset \mathbb{P}(1, 2, 3, 5, 7)$	0,2,2,0	0,0,5,0,0
125	$X_{12} \subset \mathbb{P}(1, 3, 4, 5, 7)$	0,3,3,0	0,0,7,0,0
126	$X_6 \subset \mathbb{P}(1, 2, 3, 4, 5)$	0,0,0,0	0,0,2,0,0
127	$X_{12} \subset \mathbb{P}(2, 3, 4, 5, 7)$	0,1,1,0	0,0,3,0,0
128	$X_{12} \subset \mathbb{P}(1, 4, 5, 6, 7)$	0,1,1,0	0,0,3,0,0
129	$X_{10} \subset \mathbb{P}(2, 3, 4, 5, 7)$	0,0,0,0	0,0,3,0,0
130	$X_{12} \subset \mathbb{P}(3, 4, 5, 6, 7)$	0,0,0,0	0,0,1,0,0

4.4.3. A second example of GM-like Fano of K3 type

The other example of GM-like Fano of K3 type that we want to investigate is the family no. 122, $X_{14} \subset \mathbb{P}(2, 3, 4, 5, 7)$. This is particularly interesting, since is the only GM-like of K3 type that represents as well a counterexample to the infinitesimal Torelli problem. We have already investigated the Hilbert–Poincaré series of the Jacobian ring, this being

$$\begin{aligned}
HP(R) = {} & 1 + t^2 + t^3 + 2t^4 + 2t^5 + 3t^6 + 3t^7 + 5t^8 + 4t^9 + 6t^{10} \\
& + 5t^{11} + 7t^{12} + 6t^{13} + 7t^{14} + 6t^{15} + 7t^{16} + 5t^{17} \\
& + 6t^{18} + 4t^{19} + 5t^{20} + 3t^{21} + 3t^{22} + 2t^{23} + 2t^{24} + t^{25} + t^{26} + t^{28}.
\end{aligned}$$

Here we see that the Hodge theory in even dimension is concentrated in degree 1,14, while in odd dimension in degree 7,21. It follows that for example the Fano threefold will have as Hodge Diamond

```
  0     3     3     0
     0     1     0
        0     0
           1
```

and the Fano fourfold will have

```
  0     1     8     1     0
     0     0     0     0
        0     1     0
           0     0
              1
```

with 8 being exactly the Picard rank of the K3 surface that is the step 0 of the tower. A curious behaviour appears here: one has in fact

Theorem 4.8. *Any odd dimensional member $X_{14}^{2k+1} \subset \mathbb{P}(2, 3, 4, 5, 7^{2k+1})$ will be of Anti-Torelli type, while any even dimensional member $X_{14}^{2k} \subset \mathbb{P}(2, 3, 4, 5, 7^{2k})$ will be of Torelli type. In particular we have an infinite chain of examples and counterexamples for the Torelli problem, with alternate dimensions.*

Proof. The failure in any odd dimension follows from the same reasons explained in 3.12. On the other hand in even dimension one has to check the (trivial) injectivity of the map

$$d\,\wp : R_{14} \longrightarrow \mathrm{Hom}(R_0, R_{14} \oplus \mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}, R_{14} \oplus \mathbb{C}).$$

The result follows immediately then. □

This type of construction can be obviously performed for any quasi-smooth hypersurface we considered before. However, we decided to focus in full details only in these three examples, since they shared the most interesting geometric properties. In particular we plan to continue the study of these GM-like Fano varieties of K3-type in a forthcoming work. As an example of our computations, we include now the complete list of towers coming out from the Fano threefolds of index > 1 (see Table 6).

Acknowledgements

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