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BOUNDARY ASYMPTOTICS OF THE ERGODIC FUNCTIONS ASSOCIATED WITH FULLY NONLINEAR OPERATORS THROUGH A LIOUVILLE TYPE THEOREM

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Abstract. We prove gradient boundary blow up rates for ergodic functions in bounded domains related to fully nonlinear degenerate/singular elliptic operators. As a consequence, we deduce the uniqueness, up to constants, of the ergodic functions. The results are obtained by means of a Liouville type classification theorem in half-spaces for infinite boundary value problems related to fully nonlinear, uniformly elliptic operators.

1. **Introduction.** A classical result from harmonic function theory, see e.g. [1], states that if v is a positive harmonic function in the upper half-space $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N \ : \ x_N > 0\}$, vanishing on $\partial\mathbb{R}^N_+$, then, necessarily, $v(x) = c\, x_N$ for some $c > 0$.

By performing the change of unknown

$$v(x) = e^{-u(x)}\,,$$

the same result may be read in terms of u as

$$\left\{ \begin{array}{c} -\Delta u + |\nabla u|^2 = 0 \ \text{ in } \mathbb{R}^N_+ \\ u = +\infty \ \text{ on } \partial\mathbb{R}^N_+ \end{array} \right. \iff u(x) = -\log x_N + c\,,\ c \in \mathbb{R}\,.$$

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We are interested here in extending the above result to more general equations. Namely, we focus on the second order, fully nonlinear infinite boundary value problem

$$\begin{cases} -F(D^2u) + |\nabla u|^\beta = 0 & \text{in } \mathbb{R}^N_+ \\ u(x',0) = +\infty & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{1}$$

where $\beta > 1$ and $F : \mathcal{S}_N \to \mathbb{R}$ is a positively homogeneous uniformly elliptic operator, that is a continuous function defined on the space of $N \times N$-square symmetric matrices $\mathcal{S}_N$, positively homogeneous of degree 1, and satisfying

$$a\,\mathrm{tr}(P) \le F(X+P) - F(X) \le A\,\mathrm{tr}(P)\,, \tag{2}$$

for all $X, P \in \mathcal{S}_N$, with $P \ge O$, for some positive constants $A \ge a > 0$. In (1), D^2u stands for the hessian matrix of the unknown function u, and ∇u for its gradient.

We assume that $u \in C(\mathbb{R}^N_+)$ is a viscosity solution of (1), satisfying further

(H1) the boundary condition is uniformly satisfied, namely

$$\forall\, M > 0\,,\ \exists\, \epsilon_M > 0 \ \text{ such that } u(x', x_N) \ge M,\ \forall\, x' \in \mathbb{R}^{N-1} \ \text{ and } x_N \le \epsilon_M\,,$$

(H2) u is bounded for x_N bounded away from zero and from infinity, that is

$$\forall\, \delta \in (0,1)\,,\ \exists\, C_\delta > 0 \ \text{ such that } |u(x', x_N)| \le C_\delta,\ \forall\, x' \in \mathbb{R}^{N-1} \ \text{ and } x_N \in \left[\delta, \frac{1}{\delta}\right]$$

Our first result is the following Liouville type classification theorem.

Theorem 1.1. *Let $1 < \beta \le 2$ and assume that F satisfies (2). If $u \in C(\mathbb{R}^N_+)$ is a viscosity solution of (1) satisfying assumptions* (H1) *and* (H2)*, then, for some $c \in \mathbb{R}$,*

$$u(x', x_N) \equiv u(x_N) = \begin{cases} \dfrac{1}{2-\beta}\left(\dfrac{F(e_N \otimes e_N)^{\frac{1}{2-\beta}}}{(\beta-1)\,x_N}\right)^{\frac{2-\beta}{\beta-1}} + c & \text{if } \beta < 2\,, \\ -F(e_N \otimes e_N)\log x_N + c & \text{if } \beta = 2\,. \end{cases}$$

We observe that the logarithmic change of variable useful in the case of Laplace operator cannot be used for a fully nonlinear operator F, even when $\beta = 2$, since the new variable v will be merely a subsolution and not a solution of the homogeneous equation. Theorem 1.1 will be instead established by showing first that any viscosity solution u actually is a monotone decreasing function of the only variable x_N, and then integrating the resulting ODE. The one dimensional symmetry of u will be obtained in turn by applying the well known so called sliding method, introduced in [4]. Indeed, after showing that the comparison principle holds true for bounded viscosity sub- and supersolutions in horizontal strips, any solution u will be proved to satisfy the inequality

$$u \le u_t\,, \tag{3}$$

where $u_t(x) = u(x + t\nu)$ stands for the translated function with respect to any direction $\nu = (\nu', \nu_N)$ with $\nu_N \le 0$ and $t > 0$. We emphasize that the proof for Theorem 1.1 is even more simple than the original proof given in [4], where inequality (3) is first established for t large, and then for every $t > 0$ by a contradiction argument. In our case, since the comparison principle for solutions in horizontal strips holds true independently of the size of the solutions to be compared, inequality (3) is established simultaneously for all $t > 0$.

Theorem 1.1 belongs to the large family of symmetry results for solutions of nonlinear elliptic equations. Besides [4], we just mention the works [12, 11, 10], and we refer to the references therein, as recent symmetry results for solutions of semilinear, quasilinear and fully nonlinear equations respectively. Moreover, we refer to [3] for symmetry results in halfspaces for nonlocal operators. In all previous results, the symmetry is obtained for entire solutions or solutions vanishing on the boundary. Up to our knowledge, Theorem 1.1 is the first application of the sliding method to infinite boundary value problems.

An interesting feature of the Liouville property for problem (1) relies in its connection with the so called ergodic problem associated with fully nonlinear degenerate/singular operators. Let us recall that, given a C^2 bounded domain $\Omega \subset R^N$ and a continuous function $f \in C(\Omega)$, an ergodic constant related to Ω and f is a constant $c_\Omega \in \mathbb{R}$ such that there exist solutions $u \in C(\Omega)$ (called ergodic functions) of the infinite boundary value problem

$$\begin{cases} -|\nabla u|^\alpha F(D^2u) + |\nabla u|^\beta = f + c_\Omega & \text{in } \Omega\,, \\ u = +\infty & \text{on } \partial\Omega\,, \end{cases} \tag{4}$$

where F is a positively homogeneous operator satisfying (2) as before, and the exponents α and β satisfy respectively $\alpha > -1$ and $\alpha + 1 < \beta \leq \alpha + 2$.

Problem (4) has been originally studied in the semilinear case $\alpha = 0$ and $F(D^2u) = \Delta u$ in [16], where the terminology "ergodic" has been introduced and its connection with a state constraint optimal stochastic control problem has been showed. Further contributions have been given in [18], and in [17] for p-Laplace operator as principal part. We refer also to [2] for analogous results related to nonlocal operators.

In the fully nonlinear degenerate/singular setting, problem (4) has been recently studied in [6, 7], where it has been proved in particular that if f is bounded and Lipschitz continuous, then ergodic pairs (c_Ω, u) do exist.

By keeping the same terminology for unbounded domains, Theorem 1.1 states that the ergodic constant for $\Omega = \mathbb{R}^N_+$ and $f \equiv 0$ is $c = 0$, and the ergodic function is unique and coincides with the one variable function $u(x_N)$ given in the statement of Theorem 1.1. Indeed, as proved in [5, 14], one has

$$-|\nabla u|^\alpha F(D^2u) + |\nabla u|^\beta = 0 \iff -F(D^2u) + |\nabla u|^{\beta-\alpha} = 0$$

Note that, in the present assumptions, one has $1 < \beta - \alpha \leq 2$, and thus Theorem 1.1 applies.

For bounded domains $\Omega \subset R^N$, the uniqueness, up to additive constants, of solution for problem (4) has been recently established in [8] in the case $\beta < \alpha + 2$. As a general strategy, uniqueness of solution can be obtained as a consequence of the strong maximum principle. For the singular/degenerate operators appearing in (4), the strong maximum principle holds true in the region where the gradients of solutions do not vanish, see [5]. Hence, uniqueness of solution is reduced to first order lower bounds. These in turn can be obtained as a consequence of asymptotic boundary expansions for the gradient of ergodic functions.

Let us recall that the analysis performed in [6] yields that the ergodic constant c_Ω is unique under the assumption that

$$F(\nabla d(x) \otimes \nabla d(x)) \quad \text{is of class } C^2 \text{ for } x \text{ in a neighborhood of } \partial\Omega. \tag{5}$$

Moreover, any ergodic function u is proved to satisfy, for any $\epsilon > 0$,

$$\begin{aligned} &\frac{C(x)-\epsilon}{d(x)^\chi} - D_\epsilon \leq u(x) \leq \frac{C(x)+\epsilon}{d(x)^\chi} + D_\epsilon \quad \text{if } \chi > 0\,, \\ &|\log d(x)|\,(C(x)-\epsilon) - D_\epsilon \leq u(x) \leq |\log d(x)|\,(C(x)+\epsilon) + D_\epsilon \quad \text{if } \chi = 0\,, \end{aligned} \tag{6}$$

where $d(x)$ denotes the distance function from $\partial\Omega$, $\chi = \frac{2+\alpha-\beta}{\beta-1-\alpha}$, $D_\epsilon > 0$ is a constant depending on the data and on ϵ and $C(x)$ is any $\mathcal{C}^2(\Omega)$ function satisfying, for x in a neighborhood of $\partial\Omega$,

$$\begin{aligned} C(x) &= \left((\chi+1)F(\nabla d(x)\otimes\nabla d(x))\right)^{\frac{1}{\beta-\alpha-1}}\chi^{-1} \quad &\text{if} \quad \chi > 0, \\ C(x) &= F(\nabla d(x)\otimes\nabla d(x)) \quad &\text{if} \quad \chi = 0. \end{aligned} \tag{7}$$

Estimates (6), which clearly imply the boundary asymptotic identities

$$\begin{aligned} &\lim_{d(x)\to 0} \frac{u(x)\,d(x)^\chi}{C(x)} = 1 \text{ if } \chi > 0, \\ &\lim_{d(x)\to 0} \frac{u(x)}{|\log d(x)|\,C(x)} = 1 \text{ if } \chi = 0\,, \end{aligned} \tag{8}$$

suggest the right scaling to be applied to the function u for an asymptotic boundary analysis. In the case $\chi > 0$, it is natural to consider, for fixed $x_0 \in \partial\Omega$ and $\delta > 0$, the rescaled function

$$u_\delta(\zeta) = \delta^\chi u(x_0 + \delta\,\zeta)\,,$$

defined for $\zeta \in \frac{1}{\delta}(\Omega - x_0)$. By the first identity in (8), it immediately follows that

$$u_\delta(\zeta) \to v(\zeta) := \frac{C(x_0)}{(\nabla d(x_0)\cdot\zeta)^\chi} \quad \text{as } \delta \to 0$$

locally uniformly in the halfspace $H = \{\zeta \,:\, \zeta\cdot\nabla d(x_0) > 0\}$. Moreover, the regularity results and a priori estimates proved in [8] yield that the sequence $\{u_\delta\}$ is bounded in $C^{1,\gamma}_{loc}(H)$ for some $\gamma > 0$. This implies that, up to a subsequence, $\nabla u_\delta(\zeta) \to \nabla v(\zeta)$ locally uniformly in H, and this amounts to the gradient boundary asymptotics

$$\lim_{d(x)\to 0} \frac{d(x)^{\chi+1}\nabla u(x)\cdot\nabla d(x)}{C(x)} = -\chi\,. \tag{9}$$

As a consequence, the uniqueness of the ergodic function is proved in the case $\chi > 0$, see Theorem 1.2 of [8].

In the case $\chi = 0$, i.e. $\beta = \alpha + 2$, an analogous argument leads to consider the scaled-translated function

$$u_\delta(\zeta) = u(x_0 + \delta\,\zeta) + C(x_0)\,\log\delta\,.$$

By estimates (6) and again by the results of [8], one has that the sequence $\{u_\delta\}$ is converging, up to a subsequence, in $C^1_{loc}(H)$. However, in the present case we cannot immediately detect from (8) the limit function v. We need to use the equations satisfied by u_δ, in order to derive the equation satisfied by v and to apply Theorem 1.1. This yields the following result, which establishes the analogous of (9) in the case $\chi = 0$.

Theorem 1.2. *Assume that F satisfies (2) and (5), let $f \in C(\Omega)$ be bounded and let $u \in C(\Omega)$ be a solution of problem (4) with $\beta = \alpha + 2$. Then, one has*

$$\lim_{d(x)\to 0} \frac{d(x)\nabla u(x)\cdot\nabla d(x)}{C(x)} = -1\,. \tag{10}$$

As a final remark, let us compare formulas (9) and (10): the inconsistency of the limit values for $\chi = 0$, due to the fact that ergodic functions for $\beta < \alpha + 2$ converge to ergodic functions relative to $\beta = \alpha + 2$ only locally in Ω, reflects once more the gap between the cases $\beta < \alpha + 2$ and $\beta = \alpha + 2$.

The paper is organized as follows: in Section 2 we give the proofs of the comparison principle in strips and of Theorem 1.1; in Section 3 we prove Theorem 1.2 and deduce the uniqueness of the ergodic function in the case $\chi = 0$, referring to [8] for the proof.

2. **Proof of Theorem 1.1.**

Theorem 1.1 will be obtained as an easy consequence of the following comparison result in strips.

Proposition 1. *Suppose that u and v are bounded and Lipschitz continuous functions on the strip $\Sigma = \mathbb{R}^{N-1} \times (b, c)$, satisfying in the viscosity sense*

$$-F(D^2u) + |\nabla u|^\beta \leq 0 \leq -F(D^2v) + |\nabla v|^\beta \qquad in\ \Sigma\,.$$

Then

$$u(x) \leq v(x) + \sup_{\partial\Sigma}(u - v)\,, \qquad \forall\, x \in \Sigma\,.$$

Proof. Suppose by contradiction that

$$S := \sup_{\Sigma}(u - v) > \sup_{\partial\Sigma}(u - v)\,,$$

and let us set $w(x) = v(x) + S$. Then, one has $u(x) \leq w(x)$ for all $x \in \Sigma$ and there exists a sequence $\left\{x_j = \left(x'_j, (x_N)_j\right)\right\} \subset \Sigma$ such that $(u - w)(x_j) \to 0$. Up to a (not relabeled) subsequence, we have $(x_N)_j \to \bar{x}_N \in [b, c]$.

We define $u_j(x', x_N) = u(x' + x'_j, x_N)$ and $w_j(x', x_N) = w(x' + x'_j, x_N)$. By Ascoli-Arzelà Theorem, again up to a subsequence, u_j and w_j converge locally uniformly to some $\bar{u}$ and $\bar{w}$, which are sub and supersolution respectively. Furthermore,

$$\bar{u}(0, \bar{x}_N) = \lim_{j\to\infty} u\left(x'_j, (x_N)_j\right) = \lim_{j\to\infty} w\left(x'_j, (x_N)_j\right) = \bar{w}(0, \bar{x}_N)\,.$$

Since $\bar{u} \leq \bar{w}$ in Σ, $\bar{u} \leq \bar{w} + \sup_{\partial\Sigma}(u - v) - S < \bar{w}$ on $\partial\Sigma$ and $\bar{u}(0, \bar{x}_N) = \bar{w}(0, \bar{x}_N)$, we deduce that $\bar{x}_N \in (b, c)$ and, by the strong comparison principle (see [13]), that $\bar{u} \equiv \bar{w}$ in $\overline{\Sigma}$, which contradicts the fact that $\bar{u} < \bar{w}$ on $\partial\Sigma$. $\square$

Proof of Theorem 1.1.

The conclusion will be obtained by showing that u is a monotone decreasing function depending on the only variable x_N.

Let us first observe that any solution u of the equation in (1) is locally $C^{1,\gamma}$ in $\mathbb{R}^N_+$ for some $\gamma \in (0, 1)$ depending on N, a, A and β, see e.g. [8]. Moreover, as proved e.g. in [9] and [6], u satisfies the local Lipschitz estimate

$$|Du(x)| \leq \frac{C}{x_N^{\frac{1}{\beta-1}}}\,, \tag{11}$$

for a positive constant C depending only on N, a, A and $\beta > 1$.

Next, for a fixed unitary vector $\nu = (\nu', \nu_N) \in \mathbb{R}^N$ with $\nu_N < 0$, and for $t > 0$, let us consider the function

$$u_t(x) := u(x + t\,\nu)\,,$$

defined in $\mathbb{R}^N_t := \{(x', x_N) \in \mathbb{R}^N \ :\ x_N > -t\,\nu_N\}$. Clearly, u_t is a solution of the infinite boundary value problem (1) in $\mathbb{R}^N_t$.

We claim that $u \leq u_t$ in $\mathbb{R}^N_t$.

Indeed, by assumptions (H1) and (H2), we have that, uniformly with respect to $x' \in \mathbb{R}^{N-1}$,

$$u(x) - u_t(x) \to -\infty \quad \text{as } x_N \to -t\,\nu_N\,.$$

Moreover, by estimate (11), we also have

$$u(x) - u_t(x) \to 0 \quad \text{as } x_N \to +\infty\,,$$

uniformly with respect to $x' \in \mathbb{R}^{N-1}$ as well. Therefore, for any $\epsilon > 0$ there exists $\delta = \delta_{\epsilon,t} > 0$ sufficiently small such that

$$u(x) - u_t(x) \leq \epsilon \quad \text{for either } x_N \geq -t\,\nu_N + \frac{1}{\delta} \quad \text{or } x_N \leq -t\,\nu_N + \delta\,.$$

On the other hand, in the strip $\Sigma = \mathbb{R}^{N-1} \times \left(-t\,\nu_N + \delta, -t\,\nu_N + \frac{1}{\delta}\right)$, u and u_t satisfy the assumptions of Proposition 1, so that we obtain

$$u(x) - u_t(x) \leq \epsilon \qquad \forall\, x \in \mathbb{R}^N_t\,.$$

By letting $\epsilon \to 0$, we deduce the claim.

It then follows that $\frac{\partial u}{\partial \nu} \geq 0$ in $\mathbb{R}^N_+$ for all vectors $\nu = (\nu', \nu_N)$ with $\nu_N < 0$. In particular $\frac{\partial u}{\partial x_N} \leq 0$. Furthermore, by the continuity of Du, $\frac{\partial u}{\partial \nu} \geq 0$ is true also for vectors ν such that $\nu_N = 0$. Hence, we obtain that $\frac{\partial u}{\partial x_i} = 0$ for all $i = 1, \ldots, N-1$ Therefore, $u(x) \equiv u(x_N)$ is a nonincreasing one variable function satisfying in the viscosity sense

$$\begin{cases} F(u'' e_N \otimes e_N) = (-u')^\beta & \text{for } x_N > 0 \\ \qquad u(0) = +\infty \end{cases}$$

The uniform ellipticity condition (2) yields that operator F is negative when evaluated on negative matrices. Hence, it follows necessarily that $u'' \geq 0$, and, by positive homogeneity, u is a solution of the ODE

$$u'' = \frac{(-u')^\beta}{F(e_N \otimes e_N)}\,.$$

Let us emphasize that, again by (2), one has $F(e_N \otimes e_N) \geq a > 0$. A direct integration of this equation and the imposition of the initial condition yield the result. □

Remark 1. In the proof of Theorem 1.1 the uniform ellipticity assumption (2) is essential, since it guarantees the validity of the strong maximum principle and it allows the direct integration of the resulting ODE for one dimensional solutions. In the case $F(X) = a\,\mathrm{tr}(X)$ is a linear operator, the conclusion could be obtained also by performing the change of unknown $v = e^{-u/a}$ and by using harmonic functions theory, see also Remark 3.2 in [8].

Let us also remark that many variants and extensions of the above results could be obtained. As an example, locally Lipschitz zero order terms $f(u)$ could be included in the equations, both of absorbing or reaction type, as e.g. for the semilinear equations considered in [4]. In view of its application in Theorem 1.2, we concentrated specifically on the homogeneous equation (1) since this is related to the homogeneous equation for singular/degenerate operators.

3. **Proof of Theorem 1.2 and uniqueness of the ergodic functions.** In this section we consider the ergodic problems in bounded domains associated with degenerate or singular fully nonlinear operators. Namely, we consider the infinite boundary value problem (4) for a smooth bounded domain $\Omega \subset \mathbb{R}^N$, and we refer to a solution (c_Ω, u) as an ergodic pair. As recalled in the introduction, by assuming that $f \in C(\Omega)$ is bounded and Lipschitz continuous, F is positively homogeneous and satisfies (2), $\alpha > -1$ and $\alpha + 1 < \beta \le \alpha + 2$, it is proved in [6] the existence of ergodic pairs (c_Ω, u). Moreover, by further assuming the smoothness condition (5), the ergodic constant c_Ω is unique and any ergodic function u satisfies estimates (6), see [6]. Furthermore, in the case $\beta < \alpha + 2$ any ergodic function is proved to satisfy also (9) and, consequently, to be unique, see [7].

Here we are concerned with the analogous result for the case $\beta = \alpha + 2$, i.e. $\chi = 0$. Let us recall that the domain Ω is assumed to be of class C^2, and $d(x)$ denotes a smooth function equaling the distance from the boundary in a neighborhood of $\partial\Omega$.

Proof of Theorem 1.2.
We apply the scaling argument used in [15, 17] in the semilinear case.
Let us fix $x_0 \in \partial\Omega$ and, for $\delta > 0$, let us consider the rescaled function

$$u_\delta(\zeta) = u(x_0 + \delta\,\zeta) + C(x_0)\,\log\delta\,,$$

defined for ζ belonging to the translated and rescaled domain $O_\delta = \frac{\Omega - x_0}{\delta}$.

We observe that, as $\delta \to 0$, the domains O_δ approach the limiting halfspace $H := \{\zeta \in \mathbb{R}^N \,:\, \zeta \cdot \nabla d(x_0) > 0\}$ and, by estimates (6) and the Lipschitz regularity in $\overline{\Omega}$ of $C(x)$, the functions u_δ are uniformly bounded for δ small and ζ in any compact subset of H.

Moreover, a direct computation shows that the functions u_δ are solutions of the equation

$$-|\nabla u_\delta|^\alpha F(D^2 u_\delta) + |\nabla u_\delta|^{\alpha+2} = \delta^{\alpha+2}\left[f(x_0 + \delta\,\zeta) + c_\Omega\right] \qquad \text{in } O_\delta\,.$$

By Theorem 1.1 of [7], it follows that the sequence $\{u_\delta\}$ is uniformly bounded in $C^{1,\gamma}_{\text{loc}}(H)$, for some $\gamma \in (0,1)$ depending on the data. Hence, by a standard diagonal procedure, there exists a $v \in \mathcal{C}^{1,\gamma}_{\text{loc}}(H)$ satisfying in the viscosity sense

$$-|\nabla v|^\alpha F(D^2 v) + |\nabla v|^{\alpha+2} = 0 \qquad \text{in } H\,,$$

and such that

$$u_\delta(\zeta) \to v(\zeta) \quad \text{ in } \mathcal{C}^1_{\text{loc}}(H) \text{ as } \delta \to 0.$$

By the results of [14], it follows that actually v is a viscosity solution (in the standard sense) of

$$-F(D^2 v) + |\nabla v|^2 = 0 \qquad \text{in } H\,.$$

Moreover, estimates (6) imply that v satisfies the inequalities

$$-D_\epsilon - (C(x_0) - \epsilon)\,\log(\nabla d(x_0)\cdot\zeta) \le v(\zeta) \le (C(x_0) + \epsilon)\,\log(\nabla d(x_0)\cdot\zeta) + D_\epsilon \quad \forall\,\zeta \in H\,.$$

It then follows that v is a solution of the infinite boundary value problem

$$\begin{cases} -F(D^2 v) + |\nabla v|^2 = 0 & \text{in } H \\ v = +\infty & \text{on } \partial H \end{cases}$$

satisfying further assumptions (H1) and (H2). By Theorem 1.1, after a rotation, we then deduce

$$v(\zeta) = -F(\nabla d(x_0) \otimes \nabla d(x_0))\,\log(\nabla d(x_0)\cdot\zeta) + c = -C(x_0)\,\log(\nabla d(x_0)\cdot\zeta) + c\,,$$

for some $c \in \mathbb{R}$. Since $u_\delta \to v$ in $\mathcal{C}^1_{\text{loc}}(H)$, we further obtain that, locally uniformly in H, one has, as $\delta \to 0$,

$$\nabla u_\delta(\zeta) = \delta \nabla u(x_0 + \delta \zeta) \to \nabla v(\zeta) = -C(x_0)\frac{\nabla d(x_0)}{\nabla d(x_0) \cdot \zeta}.$$

This yields that

$$\lim_{\delta \to 0} \frac{d(x_0 + \delta \zeta)\nabla u(x_0 + \delta \zeta) \cdot \nabla d(x_0 + \delta \zeta)}{C(x_0 + \delta \zeta)} = -|\nabla d(x_0)|^2 = -1,$$

and since all convergences are uniform with respect to $x_0 \in \partial\Omega$, we finally obtain the conclusion.

□

Theorem 1.2 implies, in particular, that $\nabla u \neq 0$ in a neighborhood of $\partial\Omega$, for any ergodic function u. This in turn enables the use of the strong maximum principle proved in [5], which yields the uniqueness of the ergodic function. Applying exactly the same proof of Theorem 1.2 of [8], we deduce the following uniqueness result.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2 and let F satisfy (2) and (5). Assume further that $\alpha > -1$, $\alpha + 1 < \beta \leq \alpha + 2$ and that $f \in C(\Omega)$ is bounded. Then, up to additive constants, problem (4) has at most one solution, provided that, when $\alpha \neq 0$, $\sup_\Omega f < -c_\Omega$ and $\partial\Omega$ is connected.*

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