

Short time blow-up by negative mass term for semilinear wave equations with small data and scattering damping

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Abstract

In this paper we study blow-up and lifespan estimate for solutions to the Cauchy problem with small data for semilinear wave equations with scattering damping and negative mass term. We show that the negative mass term will play a dominant role when the decay of its coefficients is not so fast, thus the solutions will blow up in a finite time. What is more, we establish a lifespan estimate from above which is much shorter than the usual one.

1 Introduction

We consider the Cauchy problem with small data for the semilinear wave equations with scattering damping and negative mass term

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu_1}{(1+t)^\beta} u_t - \frac{\mu_2}{(1+t)^{\alpha+1}} u = |u|^p, & \text{in } \mathbf{R}^n \times (0, T), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbf{R}^n, \end{cases} \quad (1)$$

where $\mu_1 \geq 0$, $\mu_2 > 0$, $\alpha < 1$, $\beta > 1$, $p > 1$, $n \in \mathbf{N}$, $T > 0$ and $\varepsilon > 0$ is a “small” parameter. This problem comes from the recent interest in

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the “wave-like” or “heat-like” behaviour of semilinear wave equations with variable coefficients damping. The Cauchy problems with small data for

$$u_{tt} - \Delta u = |u|^p \quad \text{and} \quad u_t - \Delta u = |u|^p$$

admit critical powers, respectively, the so-called Strauss exponent $p_S(n)$ and the Fujita exponent $p_F(n)$ (see [13] and [3]), where for “critical power” of a problem we mean the exponent p_c such that its small data solutions blow up for $1 < p \leq p_c$ and exist globally in time for $p > p_c$. It is of recent interest the Cauchy problem with small data for

$$u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = |u|^p. \quad (2)$$

If the Cauchy problem (2) admits a critical power related to $p_S(n)$, then we say it has a “wave-like” behaviour, while if it is related to $p_F(n)$, then we say it admits a “heat-like” behaviour. Generally speaking, if the decay rate β of the damping coefficients is large enough, then the damping term seems to have no influence and then we get a “wave-like” behaviour; otherwise, we get a “heat-like” behaviour. According to the different value of β , we recover four cases (overdamping, effective, scaling invariant, scattering), based on the works by Wirth [20–22] (see also [2, 4–6, 8, 9, 16–19] and references therein).

On the other hand, people are paying more attention to the Cauchy problem for

$$u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t + \frac{\mu_2^2}{(1+t)^2} u = |u|^p,$$

which includes scale-invariant damping and mass in the mean time. In some sense, this model describes the interplay between the damping and mass. For this problem, the quantity

$$\delta := (\mu_1 - 1)^2 - 4\mu_2^2$$

plays an important role to the behaviour of the solutions. We refer the reader to [10–12] and references therein.

Naturally, we want to consider the corresponding problem with scattering damping and mass term. Very recently, the authors [7] studied the Cauchy problem (1) with fast decay rate in the coefficients of the mass term, thus, $\alpha > 1$, in which we proved that the blow-up results and the upper bound lifespan estimates are the same as that of the semilinear wave equations with scattering damping but without mass term, see [9]. This implies that the negative mass term seems to have no influence on the behaviour of the solutions. In this work, we are devoted to studying the case $\alpha < 1$. Our motivation to

study a negative mass term is simply a mathematical interest by [7], but one may refer to the introduction of Yagdjian and Galstian [23] which mentions its physical background. From our main result listed in Theorem 1 below, it seems that the negative mass term will affect the qualitative properties of the small data solutions of the Cauchy problem (1), due to two reasons: firstly, the non-existence of global energy solutions can be established for all $p > 1$ and $n \geq 1$; moreover, the upper bound of the lifespan is smaller than the usual one and it looks like a log-type with respect to ε .

2 Definitions and Main Result

First of all, let us introduce energy solutions of Cauchy problem (1).

Definition 2.1. *We say that u is an energy solution for problem (1) over $\mathbf{R}^n \times [0, T)$ if*

$$u \in C([0, T), H^1(\mathbf{R}^n)) \cap C^1([0, T), L^2(\mathbf{R}^n)) \cap L_{loc}^p(\mathbf{R}^n \times (0, T))$$

satisfies $u(x, 0) = \varepsilon f(x)$ in $H^1(\mathbf{R}^n)$, $u_t(x, 0) = \varepsilon g(x)$ in $L^2(\mathbf{R}^n)$ and

$$\begin{aligned} & \int_{\mathbf{R}^n} u_t(x, t) \phi(x, t) dx - \int_{\mathbf{R}^n} \varepsilon g(x) \phi(x, 0) dx \\ & + \int_0^t ds \int_{\mathbf{R}^n} \{-u_t(x, s) \phi_t(x, s) + \nabla u(x, s) \cdot \nabla \phi(x, s)\} dx \\ & + \int_0^t ds \int_{\mathbf{R}^n} \left\{ \frac{\mu_1}{(1+s)^\beta} u_t(x, s) - \frac{\mu_2}{(1+s)^{\alpha+1}} u(x, s) \right\} \phi(x, s) dx \\ & = \int_0^t ds \int_{\mathbf{R}^n} |u(x, s)|^p \phi(x, s) dx \end{aligned} \tag{3}$$

for all test functions $\phi \in C_0^\infty(\mathbf{R}^n \times [0, T))$ and for all $t \in [0, T)$.

Theorem 1. *Let $\alpha < 1$, $n \geq 1$ and $p > 1$. Assume that both $f \in H^1(\mathbf{R}^n)$ and $g \in L^2(\mathbf{R}^n)$ are non-negative, at least one of them does not vanish identically. Suppose that u is an energy solution of (1) on $\mathbf{R}^n \times [0, T)$ that satisfies, for some $R \geq 1$,*

$$\text{supp } u \subset \{(x, t) \in \mathbf{R}^n \times [0, T) : |x| \leq t + R\}. \tag{4}$$

Then, there exists a constant $\varepsilon_0 = \varepsilon_0(f, g, R, n, p, \mu_1, \mu_2, \alpha, \beta) > 0$ such that T has to satisfy

$$T \leq 3\zeta(C\varepsilon)$$

for $0 < \varepsilon \leq \varepsilon_0$, where $\zeta = \zeta(\bar{\varepsilon})$ is the larger solution to the equation

$$\bar{\varepsilon} \zeta^{\frac{2}{p-1} - n + \frac{1+\alpha}{4}} \exp\left(\frac{2}{1-\alpha} \sqrt{\mu_2 \exp\left(\frac{\mu_1}{1-\beta}\right) \zeta^{\frac{1-\alpha}{2}}}\right) = 1 \quad (5)$$

and C is a positive constant independent of ε .

Remark 2.1. Let us make some observations:

- The assumption (4) can be replaced by $\text{supp}\{f, g\} \subset \{x \in \mathbf{R}^n : |x| \leq R\}$ when $n = 1, 2$, or $p \leq n/(n-2)$ for $n \geq 3$. This fact is established by local existence of such an energy solution. See Appendix in the last section.
- Since letting $\varepsilon \rightarrow 0$ we have $\zeta \rightarrow +\infty$ in (5), it is not difficult to see that $T \leq c[\log(1/\varepsilon)]^{2/(1-\alpha)}$ for some constant $c > 0$ follows from (5). In fact, this inequality is trivial when the exponent $\delta := 2/(p-1) - n + (1+\alpha)/4$ of the first ζ is non-negative, while ζ^δ can be absorbed by square root of the exponential term when $\delta < 0$.
- It is an open question the optimality of the upper bound of the lifespan in Theorem 1.

3 Kato's type lemma

In order to prove our theorem, we need a slightly different version of the improved Kato's lemma introduced in [15].

Lemma 3.1. Let $p > 1$ and $0 \leq \tilde{T}_0 < T$ be positive constants. Suppose that $A \in C^1([\tilde{T}_0, T])$, $B \in C^1([0, T])$, $m \in C^1([0, T])$ are strictly positive functions, $B = B(t)$ is decreasing and that $m(t)$ is bounded by two constants $\bar{m}, \underline{m} > 0$ ($\underline{m} \leq m(t) \leq \bar{m}$) for $t \geq 0$. Define the function

$$h(t) := B(t)^{1/2} A(t)^{(p-1)/2-\delta}, \quad (6)$$

where δ is a constant such that $0 < \delta < (p-1)/2$ and $h'(t) \geq 0$ for $t \geq \tilde{T}_0$.

Assume that $F \in C^2([0, T])$ satisfies

$$F(t) \geq A(t) \quad \text{for } t \geq \tilde{T}_0, \quad (7)$$

$$\{m(t)F'(t)\}' \geq B(t)|F(t)|^p \quad \text{for } t \geq 0, \quad (8)$$

$$F(0), F'(0) \geq 0, \quad F(0) + F'(0) > 0. \quad (9)$$

If $F'(0) = 0$, suppose that there exists a time $\tilde{t} > 0$ such that

$$F(\tilde{t}) \geq 2F(0). \quad (10)$$

Define the time

$$\tilde{T}_1 := \begin{cases} \overline{m}\underline{m}^{-1}F(0)/F'(0) & \text{if } F'(0) \neq 0, \\ \tilde{t} & \text{if } F'(0) = 0. \end{cases}$$

Then, for $\tilde{T} \geq \max\{\tilde{T}_0, \tilde{T}_1\}$ we have $T \leq 3\tilde{T}$ assuming that

$$\tilde{T} h(\tilde{T}) A(\tilde{T})^\delta \geq \delta^{-1} \overline{m} \sqrt{(p+1)/\underline{m}}. \quad (11)$$

Proof. First of all, let us prove that $F(t), F'(t) > 0$ for $t > 0$. We need to consider two cases according to the initial conditions (9) on F .

Case 1: $F'(0) > 0$. From (8) it follows $F'(t) \geq m(0)F'(0)m(t)^{-1} > 0$, and then $F(t) \geq F(0) + m(0)F'(0) \int_0^t m(s)^{-1} ds > 0$ for $t > 0$.

Case 2: $F'(0) = 0$. Then $F(0) > 0$. It follows from (8) evaluated in $t = 0$ that $\{mF'\}'(0) > 0$, which implies $m(t)F'(t) > m(0)F'(0) = 0$ for small $t > 0$. Hence, the fact that $\{mF'\}'(t) \geq 0$ for $t \geq 0$ from (8) yields that $m(t)F'(t) > 0$, that is $F'(t) > 0$, and so $F(t) > F(0) > 0$ for $t > 0$.

Moreover, observe that

$$F(t) \geq 2F(0) \quad \text{for } t \geq \tilde{T}_1. \quad (12)$$

Indeed, if $F'(0) = 0$, it follows by the hypothesis (10) and by the fact that F is increasing. If $F'(0) > 0$, by (8) and because m is bounded, we have $F(t) \geq F(0) + \underline{m}\overline{m}^{-1}F'(0)t$, from which (12) follows.

Multiplying (8) by $m(t)F'(t) > 0$, we get

$$\left(\frac{1}{2} \{m(t)F'(t)\}^2 \right)' \geq m(t)B(t)|F(t)|^p F'(t) \quad \text{for } t > 0.$$

From this inequality, the positivity of F and the facts that B is decreasing and m is bounded, it follows that

$$\begin{aligned} \frac{1}{2}F'(t)^2 &\geq \frac{1}{2}\overline{m}^{-2}\underline{m}^2 F'(0)^2 + \overline{m}^{-2}\underline{m}B(t) \int_0^t F(s)^p F'(s) ds \\ &\geq \frac{\overline{m}^{-2}\underline{m}}{p+1} B(t) F(t)^p \{F(t) - F(0)\} \quad \text{for } t \geq 0, \end{aligned}$$

and so, using equation (12), we get

$$F'(t) \geq \overline{m}^{-1} \sqrt{\underline{m}/(p+1)} B(t)^{1/2} F(t)^{(p+1)/2} \quad \text{for } t \geq \tilde{T}_1. \quad (13)$$

Now, fix $\tilde{T} := \max\{\tilde{T}_0, \tilde{T}_1\}$ and define the function

$$H(t) := \int_{\tilde{T}}^t h(s) ds = \int_{\tilde{T}}^t B(s)^{1/2} A(s)^{(p-1)/2-\delta} ds \quad \text{for } t \geq \tilde{T}.$$

Because $0 < \delta < (p-1)/2$, from inequality (13) and from (7) we obtain

$$F'(t)/F(t)^{1+\delta} \geq \bar{m}^{-1} \sqrt{\underline{m}/(p+1)} B(t)^{1/2} A(t)^{(p-1)/2-\delta} \quad \text{for } t \geq \tilde{T}.$$

Integrating this inequality on $[2\tilde{T}, t]$ we get

$$\frac{1}{\delta} \left(\frac{1}{F(2\tilde{T})^\delta} - \frac{1}{F(t)^\delta} \right) \geq \frac{1}{\bar{m}} \sqrt{\frac{\underline{m}}{p+1}} \int_{2\tilde{T}}^t B(s)^{1/2} A(s)^{(p-1)/2-\delta} ds.$$

Neglecting the second term on the left-hand side, from (7) evaluated in $t = 2\tilde{T}$ and recalling the definition of H , we have

$$A(2\tilde{T})^{-\delta} \geq F(2\tilde{T})^{-\delta} \geq \delta \bar{m}^{-1} \sqrt{\underline{m}/(p+1)} [H(t) - H(2\tilde{T})] \quad (14)$$

for $t \geq 2\tilde{T}$. Since h is increasing, we get

$$H(2\tilde{T}) \geq h(\tilde{T}) \int_{\tilde{T}}^{2\tilde{T}} ds = \tilde{T} h(\tilde{T}). \quad (15)$$

Observe moreover that A is increasing, in fact

$$h'(t) = h(t) \{B'(t)/(2B(t)) + [(p-1)/2 - \delta] A'(t)/A(t)\},$$

and so, because $h, A, B > 0$, $h' \geq 0$ and $B' \leq 0$, we get $A' \geq 0$. Then, by equation (15), hypothesis (11) and the monotonicity of A , we have

$$A(2\tilde{T})^\delta H(2\tilde{T}) \geq A(\tilde{T})^\delta \tilde{T} h(\tilde{T}) \geq \delta^{-1} \bar{m} \sqrt{(p+1)/\underline{m}}.$$

Inserting this inequality in (14) we obtain $2H(2\tilde{T}) \geq H(t)$. Since $H''(t) = h'(t) \geq 0$ we have also, integrating two times on $[2\tilde{T}, t]$, that $H(t) \geq H(2\tilde{T}) + H'(2\tilde{T})\{t - 2\tilde{T}\}$. These two relations give us the estimate

$$t \leq 2\tilde{T} + H(2\tilde{T})/H'(2\tilde{T}) \quad \text{for } t > 2\tilde{T}.$$

Finally, observe that

$$(H(t)/H'(t))' = 1 - H(t)H''(t)/(H'(t))^2 \leq 1$$

from which, integrating on $[\tilde{T}, 2\tilde{T}]$, we get

$$H(2\tilde{T})/H'(2\tilde{T}) \leq H(\tilde{T})/H'(\tilde{T}) + 2\tilde{T} - \tilde{T} = \tilde{T},$$

and so we have $t \leq 3\tilde{T}$. Therefore the proof of the lemma is completed.

4 Proof of Theorem 1

Following the idea in [7] and [9], we introduce the multiplier

$$m(t) := \exp\left(\mu_1 \frac{(1+t)^{1-\beta}}{1-\beta}\right).$$

Clearly, $1 \geq m(t) \geq m(0) > 0$ for $t \geq 0$. Let us define the functional

$$F_0(t) := \int_{\mathbf{R}^n} u(x, t) dx,$$

and then $F_0(0) = \varepsilon \int_{\mathbf{R}^n} f(x) dx$, $F_0'(0) = \varepsilon \int_{\mathbf{R}^n} g(x) dx$ are non-negative and do not both equal to zero, due to the hypothesis for the initial data.

Taking into account of (4), we choose the test function $\phi = \phi(x, s)$ in the definition of energy solution (3) such that it satisfies $\phi \equiv 1$ in $\{(x, s) \in \mathbf{R}^n \times [0, t] : |x| \leq s + R\}$, to get

$$\begin{aligned} & \int_{\mathbf{R}^n} u_t(x, t) dx - \int_{\mathbf{R}^n} u_t(x, 0) dx + \int_0^t ds \int_{\mathbf{R}^n} \frac{\mu_1}{(1+s)^\beta} u_t(x, s) dx \\ &= \int_0^t \int_{\mathbf{R}^n} \frac{\mu_2}{(1+s)^{\alpha+1}} u(x, s) dx + \int_0^t ds \int_{\mathbf{R}^n} |u(x, s)|^p dx, \end{aligned}$$

which yields, by taking derivative with respect to t ,

$$F_0''(t) + \frac{\mu_1}{(1+t)^\beta} F_0'(t) = \frac{\mu_2}{(1+t)^{\alpha+1}} F_0(t) + \int_{\mathbf{R}^n} |u(x, t)|^p dx. \quad (16)$$

Multiplying both sides of (16) with $m(t)$ yields

$$\{m(t)F_0'(t)\}' = m(t) \frac{\mu_2}{(1+t)^{\alpha+1}} F_0(t) + m(t) \int_{\mathbf{R}^n} |u(x, t)|^p dx. \quad (17)$$

Integrating the previous equation twice on $[0, t]$, we obtain

$$\begin{aligned} F_0(t) &\geq F_0(0) + F_0'(0) \int_0^t \frac{ds}{m(s)} + \mu_2 \int_0^t \frac{ds}{m(s)} \int_0^s \frac{m(\sigma)F_0(\sigma)}{(1+\sigma)^{\alpha+1}} d\sigma \\ &\quad + \int_0^t \frac{ds}{m(s)} \int_0^s m(\sigma) d\sigma \int_{\mathbf{R}^n} |u(x, \sigma)|^p dx \quad \text{for } t \geq 0. \end{aligned} \quad (18)$$

By a comparison argument, we observe $F_0(t) > 0$ for $t > 0$, and consequently also $F_0'(t) > 0$ for $t > 0$ by an integration of (17). In fact, if $F_0(0) > 0$, then F_0 is strictly positive for at least small times. Supposing

that t_0 is the smallest zero time of F_0 , calculating (18) in t_0 we get a contradiction. If $F_0(0) = 0$, then $F_0'(0) > 0$ and so $F_0'(t) > 0$ for at least small time; due to the fact that F_0 is strictly increasing we then conclude that it is positive for at least small time $t > 0$. Supposing that $t_0 > 0$ is the smallest zero point of F_0 , calculating (18) in t_0 we get again a contradiction.

Moreover observe that if $F_0'(0) = 0$, neglecting the last term on the right-hand side of (18), and noting that F is increasing and m is bounded, we have

$$F_0(t) \geq F_0(0) + 2^{-1}m(0)\mu_2 F_0(0)(1+t)^{-\max\{0,\alpha+1\}}t^2$$

and so $F_0(\tilde{t}) \geq 2F_0(0)$, if we choose $\tilde{t} = \tilde{t}(\mu_1, \mu_2, \alpha, \beta) > 0$ such that

$$2^{-1}m(0)\mu_2(1+\tilde{t})^{-\max\{0,\alpha+1\}}\tilde{t}^2 = 1.$$

Now we need estimates for $\{mF_0'\}'$ and F_0 . Neglecting the first term on the right-hand side of (17) and applying Hölder's inequality for the last term, there exists $C_1 = C_1(n, p, R) > 0$ such that, for $t \geq 0$,

$$\{m(t)F_0'(t)\}' \geq m(0)C_1(1+t)^{-n(p-1)}|F_0(t)|^p =: B(t)|F_0(t)|^p \quad (19)$$

Fix $t_0 > 0$ to be chosen later and consider the auxiliary function

$$J(t) = \varepsilon J_0 + \varepsilon J_1(t - t_0) + m(0)\mu_2 \int_{t_0}^t ds \int_{t_0}^s \frac{J(\sigma)}{(1+\sigma)^{\alpha+1}} d\sigma \quad \text{for } t \geq t_0,$$

where $J_0 := \frac{1}{2}\|f\|_{L^1(\mathbf{R}^n)}$, $J_1 := \frac{1}{2}m(0)\|g\|_{L^1(\mathbf{R}^n)}$. By the similar way as above, we get by comparison argument that $F_0(t) \geq J(t)$ for $t \geq t_0$. Setting for the simplicity $c := m(0)\mu_2$, $q := 1 - \alpha$, the function J satisfies

$$J''(t) = c(1+t)^{q-2}J(t) \quad \text{for } t \geq t_0,$$

with $J(t_0) = \varepsilon J_0$, $J'(t_0) = \varepsilon J_1$. One can check that the solution of this ordinary differential equation is

$$J(t) = \varepsilon c_+ J_+(t) + c_- J_-(t)$$

where, setting $B_{1/q}^+ := I_{1/q}$ and $B_{1/q}^- := K_{1/q}$ the modified Bessel functions respectively of the first and second kind with order $1/q$, we have

$$\begin{aligned} J_{\pm}(t) &:= (1+t)^{1/2} B_{1/q}^{\pm} \left(2 \frac{\sqrt{c}}{|q|} (1+t)^{q/2} \right), \\ c_{\pm} &:= \pm \frac{2}{q} (1+t_0)^{-1/2} [(1+t_0)J_1 - J_0] B_{1/q}^{\mp} \left(\frac{2\sqrt{c}}{|q|} (1+t_0)^{q/2} \right) \\ &\quad + J_0 \frac{2\sqrt{c}}{|q|} (1+t_0)^{(q-1)/2} B_{1+1/q}^{\mp} \left(\frac{2\sqrt{c}}{|q|} (1+t_0)^{q/2} \right). \end{aligned}$$

Observe that $c_+ > 0$ at least for $t_0 > 0$ (independent of ε) large enough. From the asymptotic expansions of the modified Bessel functions (see Section 9.7 in [1]), when $t > 0$ is large we have that

$$J_{\pm}(t) = \frac{\sqrt{\pi \mp 1} q}{2c^{1/4}} (1+t)^{\frac{1}{2} - \frac{q}{4}} \exp\left(\pm 2 \frac{\sqrt{c}}{q} (1+t)^{\frac{q}{2}}\right) \left[1 + O\left((1+t)^{-\frac{q}{2}}\right)\right].$$

Consequently, we can find constants $C_2, T_1 > 0$ independent of ε , such that, for every $t \geq T_1$, the following estimate holds:

$$F_0(t) \geq \varepsilon C_2 (1+t)^{\frac{1+\alpha}{4}} \exp\left(2 \frac{\sqrt{m(0)\mu_2}}{1-\alpha} (1+t)^{\frac{1-\alpha}{2}}\right) =: A(t). \quad (20)$$

Thanks to estimates (19) and (20), we can apply Lemma 3.1. Fix $\delta := (p-1)/4$ and, using the definition (6) of h , observe that

$$h'(t) = B^{1/2}(t) A^{(p-1)/2 - \delta}(t) (1+t)^{-1} \times \left\{ -\frac{n(p-1)}{2} + \left(\frac{p-1}{2} - \delta\right) \left[\frac{1+\alpha}{4} + \sqrt{m(0)\mu_2} (1+t)^{\frac{1-\alpha}{2}} \right] \right\},$$

so we can find a time $T_2 = T_2(n, p, \mu_1, \mu_2, \alpha, \beta) \geq 0$ such that $h'(t) > 0$ for $t \geq T_2$. Then we can choose $\tilde{T}_0 = \max\{T_1, T_2\}$.

Let us set $\tilde{T} \equiv \zeta - 1$ and $C = C_3[\delta m(0)\sqrt{C_1}/(2\sqrt{p+1})]^{2/(p-1)}$, where $\zeta \equiv \zeta(\bar{\varepsilon})$ is the larger solution to (5) with $\bar{\varepsilon} = C\varepsilon$. There exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, we have $\tilde{T} \equiv \zeta - 1 \geq \max\{\tilde{T}_0, \tilde{T}_1\}$, where \tilde{T}_1 , independent of ε , is defined as in the statement of the Lemma. We can also suppose $\tilde{T} \geq 1$, so that $\zeta - 1 \geq \zeta/2$. Therefore, one can check that (11) holds and so the maximal existence time T of F_0 satisfies $T \leq 3\tilde{T} \leq 3\zeta$. The proof of the Theorem 1 is completed.

5 Appendix

In this section we are going to show the local existence and finite speed of propagation property for energy solution to our problem, as stated in the second point of Remark 2.1. In the following, the positive constant C may vary from line to line. We assume that $p \leq n/(n-2)$ when $n \geq 3$.

Let us denote the function space

$$B_{T,K} := \left\{ \phi \in C([0, T], H^1(\mathbf{R}^n)) \cap C^1([0, T], L^2(\mathbf{R}^n)) : \text{supp } \phi \subset \{(x, t) \in \mathbf{R}^n \times [0, T] : |x| \leq t + R\}, \|\phi\|_{B_{T,K}} \leq K \right\},$$

where T, R, K are fixed positive constants and

$$\|\phi\|_{B_{T,K}} := \sup_{t \in [0, T]} E_\phi^{1/2}(t), \quad E_\phi(t) := \frac{1}{2} \int_{\mathbf{R}^n} (\phi_t^2 + |\nabla \phi|^2) dx.$$

It can be proved that $B_{T,K}$ is a Banach space.

Consider the following Cauchy problem for $v \in B_{T,K}$

$$\begin{cases} u_{tt} - \Delta u = |v|^p + m^2(t)v - b(t)v_t =: F_v(x, t), & \text{in } \mathbf{R}^n \times (0, T), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbf{R}^n, \end{cases} \quad (21)$$

where we set for the simplicity

$$m^2(t) = \frac{\mu_2}{(1+t)^{\alpha+1}}, \quad b(t) = \frac{\mu_1}{(1+t)^\beta}.$$

However, all the calculations below are trivially generalized for any $m^2, b \in C([0, T])$.

We want to show that the map

$$M: v \mapsto u = Mv, \quad v \in B_{T,K},$$

is a contraction. Note that, for $v \in B_{T,K}$, by Gagliardo-Nirenberg inequality and Poincaré inequality, we have

$$\|v\|_{L^{2p}(\mathbf{R}^n)} \leq C \|v\|_{L^2(\mathbf{R}^n)}^{1-\theta(2p)} \|\nabla v\|_{L^2(\mathbf{R}^n)}^{\theta(2p)}, \quad \theta(2p) := n \left(\frac{1}{2} - \frac{1}{2p} \right),$$

for $p \leq n/(n-2)$ when $n \geq 3$, and

$$\|v\|_{L^2(\mathbf{R}^n)} \leq C(t+R) \|\nabla v\|_{L^2(\mathbf{R}^n)},$$

which imply

$$\|v\|_{L^{2p}(\mathbf{R}^n)} \leq C(t+R)^{1-\theta(2p)} \|\nabla v\|_{L^2(\mathbf{R}^n)} \leq C(t+R)^{1-\theta(2p)} E_v^{1/2}. \quad (22)$$

In particular, for fixed $T > 0$, we can check that

$$F_v(x, t) \in L^2(\mathbf{R}^n \times [0, T]).$$

Let us start proving that the map M is onto. Firstly, we show the finite speed propagation of the energy solution, i.e.

$$\text{supp } u \subset \{(x, t) \in \mathbf{R}^n \times [0, T) : |x| \leq t + R\},$$

by using the density argument similarly to [14]. By density of $C_0^\infty(\mathbf{R}^n)$ in $L^2(\mathbf{R}^n)$, we can approximate the energy data f, g by sequences of smooth and compactly supported functions $\{f_m\}_{m \in \mathbf{N}}, \{g_m\}_{m \in \mathbf{N}}$ in the energy norm $H^1(\mathbf{R}^n)$ and $L^2(\mathbf{R}^n)$ respectively. Noting that $F_v(x, t)$ has compact support, we can find also a sequence of smooth and compactly supported functions $\{F_{v,m}\}_{m \in \mathbf{N}}$ converging to F_v in the norm $L^2(\mathbf{R}^n \times [0, T])$. Let u_m be the smooth solution of the problem

$$\begin{cases} (u_m)_{tt} - \Delta u_m = F_{v,m}(x, t) & \text{in } \mathbf{R}^n \times (0, T) \\ u(x, 0) = \varepsilon f_m(x), \quad u_t(x, 0) = \varepsilon g_m(x) & \text{in } \mathbf{R}^n. \end{cases} \quad (23)$$

Fix $(x_0, t_0) \in \mathbf{R}^n \times (0, T)$ with $|x_0| \geq t_0 + R$ and set

$$C_{(x_0, t_0)} := \{(x, t) \in \mathbf{R}^n \times [0, T) : |x - x_0| \leq t_0 - t\},$$

the backward cone with vertex in (x_0, t_0) . Then, denote the energy on a time-section of the cone as

$$E_{t_0-t}(t, u(t)) := \frac{1}{2} \int_{B_{t_0-t}(x_0)} (u_t^2 + |\nabla u|^2) dx, \quad (24)$$

where $B_r(x_0) := \{x \in \mathbf{R}^n : |x - x_0| \leq r\}$. The standard space-time divergence form yields a local energy inequality

$$E_{t_0-t}^{1/2}(t, u_m(t)) \leq E_{t_0}^{1/2}(0, u_m(0)) + \int_0^t \|F_{v,m}(\cdot, s)\|_{L^2(B_{t_0-s}(x_0))} ds.$$

Applying the previous inequality to the difference $u_m - u_n$ of two solutions of (23), we have that $\{u_m(\cdot, t)\}_{m \in \mathbf{N}}$ is a Cauchy sequence in the norm (24) uniformly in $t \in [0, t_0]$. Hence the limit u is an energy solution satisfying

$$E_{t_0-t}^{1/2}(t, u(t)) \leq E_{t_0}^{1/2}(0, u(0)) + \int_0^t \|F_v(\cdot, s)\|_{L^2(B_{t_0-t}(x_0))} ds,$$

which gives us the fact that

$$f(x) \equiv g(x) \equiv 0 \quad \text{in } C_{x_0, t_0} \cap \{t = 0\} \quad \text{and} \quad F_v \equiv 0 \quad \text{in } C_{(x_0, t_0)}$$

and Poincaré inequality imply

$$u \equiv 0 \quad \text{in } C_{(x_0, t_0)}.$$

Next, we show that $\|Mv\|_{B_{T,K}} \leq K$. It is easy to obtain

$$\frac{\partial}{\partial t} \left(\frac{1}{2} (u_t^2 + |\nabla u|^2) \right) = \operatorname{div}(u_t \nabla u) + |v|^p u_t + m^2(t) v u_t - b(t) v_t u_t. \quad (25)$$

Exploiting (22) we get the estimate

$$\begin{aligned} \int_{\mathbf{R}^n} |v|^p |u_t| dx &\leq \left(\int_{\mathbf{R}^n} |v|^{2p} dx \right)^{1/2} \sqrt{2} E_u^{1/2}(t) \\ &\leq C(t+R)^{p\{1-\theta(2p)\}} E_v^{p/2}(t) E_u^{1/2}(t). \end{aligned}$$

Moreover, it is trivial that

$$\int_{\mathbf{R}^n} |v| |u_t| dx \leq C(t+R) E_v^{1/2}(t) E_u^{1/2}(t)$$

and

$$\int_{\mathbf{R}^n} |v_t| |u_t| dx \leq 2E_v^{1/2}(t) E_u^{1/2}(t).$$

Integrating (25) over $\mathbf{R}^n \times [0, t]$ and using the divergence theorem and the estimates above, we obtain

$$E_u(t) \leq E_u(0) + C \int_0^t a_K(s) E_u(s)^{1/2} ds,$$

where

$$a_K(t) := K^p(t+R)^{p\{1-\theta(2p)\}} + K(t+R)m^2(t) + Kb(t),$$

which yields, by Bihari's inequality, that for some positive constant γ

$$\begin{aligned} E_u(t)^{1/2} &\leq E_u^{1/2}(0) + C \int_0^t a_K(s) ds \\ &\leq E_u^{1/2}(0) + C \max\{K, K^p\} T(1+T)^\gamma. \end{aligned}$$

Hence we can choose K large enough and T small enough such that

$$E_u^{1/2}(0) \leq \frac{K}{2} \quad \text{and} \quad C \max\{K, K^p\} T(1+T)^\gamma \leq \frac{K}{2},$$

and then $E_u^{1/2}(t) \leq K$.

Finally, we can prove the contraction of the map M in a similar way. Fixed $v_1, v_2 \in B_{T,K}$, let

$$u_1 = Mv_1, \quad u_2 = Mv_2 \quad \text{and} \quad \bar{u} = u_1 - u_2, \quad \bar{v} = v_1 - v_2.$$

We have that \bar{u} satisfies the problem

$$\begin{cases} \bar{u}_{tt} - \Delta \bar{u} = |v_1|^p - |v_2|^p + m^2(t)\bar{v} - b(t)\bar{v}_t, & \text{in } \mathbf{R}^n \times (0, T), \\ u(x, 0) = u_t(x, 0) \equiv 0, & x \in \mathbf{R}^n, \end{cases}$$

and the equation

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} (\bar{u}_t^2 + |\nabla \bar{u}|^2) \right) \\ = \operatorname{div}(\bar{u}_t \nabla \bar{u}) + (|v_1|^p - |v_2|^p) \bar{u}_t + m^2(t) \bar{v} \bar{u}_t - b(t) \bar{v}_t \bar{u}_t. \end{aligned} \quad (26)$$

Observe that, by (22), it holds

$$\begin{aligned} \int_{\mathbf{R}^n} \left| |v_1|^p - |v_2|^p \right| |\bar{u}_t| dx &\leq C \int_{\mathbf{R}^n} |v_1 - v_2| (|v_1| + |v_2|)^{p-1} |\bar{u}_t| dx \\ &\leq C \|\bar{v}\| (\|v_1\| + \|v_2\|)^{p-1} \|\bar{u}_t\|_{L^2} \\ &\leq C \|\bar{v}\|_{L^{2p}} (\|v_1\|_{L^{2p}} + \|v_2\|_{L^{2p}})^{p-1} \|\bar{u}_t\|_{L^2}, \\ &\leq CK^{p-1} (t+R)^{p\{1-\theta(2p)\}} E_v^{1/2} E_u^{1/2}. \end{aligned}$$

Then, integrating (26) on $\mathbf{R}^n \times [0, t]$, exploiting again the Bihari's inequality and proceeding similarly as above, we reach the estimate

$$\|\bar{u}\|_{B_{T,K}} \leq C \max\{1, K^{p-1}\} T(1+T)^\gamma \|\bar{v}\|_{B_{T,K}},$$

from which, choosing T small enough, we infer that M is a contraction. The proof of the desired local existence is now completed.

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