On the flows associated to selfadjoint operators on metric measure spaces
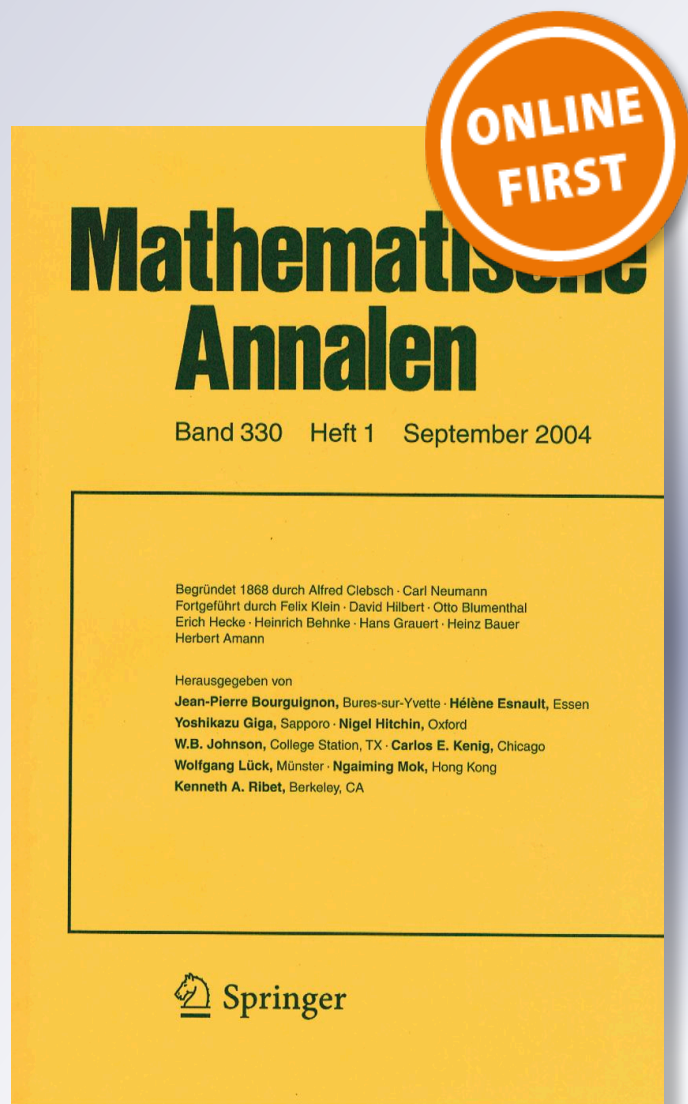
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On the flows associated to selfadjoint operators on metric measure spaces

The Anh Bui[1] · **Piero D'Ancona**[2] · **Xuan Thinh Duong**[1] · **Detlef Müller**[3]

Abstract

Let X be a metric space with a doubling measure satisfying $\mu(B) \gtrsim r_B^n$ for any ball B with any radius $r_B > 0$. Let L be a non negative selfadjoint operator on $L^2(X)$. We assume that e^{-tL} satisfies a Gaussian upper bound and that the flow e^{itL} satisfies a typical $L^1 - L^\infty$ dispersive estimate of the form

$$\|e^{itL}\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}}.$$

Then we prove a similar $L^1 - L^\infty$ dispersive estimate for a general class of flows $e^{it\phi(L)}$, with $\phi(r)$ of power type near 0 and near ∞. In the case of fractional powers $\phi(L) = L^\nu$, $\nu \in (0, 1)$, we deduce dispersive estimates for e^{itL^ν} with data in Sobolev, Besov or Hardy spaces H_L^p with $p \in (0, 1]$, associated to the operator L.

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Contents

1 Introduction

Let (X, d, μ) be a metric space endowed with a nonnegative Borel measure μ satisfying the doubling condition: there exists a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \tag{1}$$

for all $x \in X$, $r > 0$ and all balls $B(x, r) := \{y \in X : d(x, y) < r\}$. In this paper we shall assume in addition that

$$\mu(B(x, r)) \gtrsim r^n \tag{2}$$

for all $x \in X$ and $r > 0$ and for some $n \geq 1$.

We note that the doubling property (1) yields a constant $D > 0$ so that

$$\mu(B(x, \lambda r)) \leq C\lambda^D \mu(B(x, r)), \tag{3}$$

for all $\lambda \geq 1$, $x \in X$ and $r > 0$; and that

$$\mu(B(x, r)) \leq C\left(1 + \frac{d(x, y)}{r}\right)^{\tilde{n}} \mu(B(y, r)), \tag{4}$$

for all $x, y \in X$ and $r > 0$.

The main question discussed here is the following. Suppose a selfadjoint operator L on $L^2(X)$ satisfies an $L^1 - L^\infty$ dispersive estimate of the form

$$\|e^{itL}\|_{L^1 \to L^\infty} \lesssim |t|^{-a}. \tag{5}$$

This is frequently the case for many important operators, notably the Laplacian $L = -\Delta$ and its potential perturbations. Under this assumption, is it possible to deduce similar estimates for the more general class of flows $e^{it\phi(L)}$:

$$\|\psi(L)e^{it\phi(L)}\|_{L^1 \to L^\infty} \lesssim |t|^{-b}? \tag{6}$$

For instance, if we choose $\phi(L) = \sqrt{L}$, we are asking if a dispersive estimate for the wave flow $e^{it\sqrt{L}}$ can be deduced directly from a corresponding estimate for the Schrödinger flow e^{itL}. The possibility of such a reduction is suggested by a suitable *subordination formula* connecting the two flows (see [33]); the formula was applied in [15] to the Hermite operator and the twisted Laplacian. Our main goal here is to extend this idea to more general functions $\phi(L)$ with power–like behaviour near 0 and ∞, and to prove sharper estimates in terms of Sobolev, Hardy or Besov norms, appropriately defined. Of course, the interest in dispersive estimates is justified by their crucial role in a large number of important applications to linear and nonlinear evolution equations and in harmonic analysis.

We shall make the following assumptions on the selfadjoint nonnegative operator L:

(A1) The Schrödinger flow e^{itL} satisfies a dispersive estimate:

$$\|e^{itL}\|_{L^1 \to L^\infty} \lesssim t^{-n/2}, \quad t \in (0, T_0)$$

where $T_0 \in (0, +\infty]$.

(A2) The kernel $p_t(x, y)$ of e^{-tL} admits a Gaussian upper bound: $\exists C, c > 0$ such that for all $x, y \in X$ and $t > 0$,

$$|p_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{ct}\right).$$

Under (A1)–(A2), we can prove rather sharp pointwise decay estimates for the flow $e^{it\phi(L)}$, provided $\phi(r) \sim r^{m_1}$ near 0 and $\phi(r) \sim r^{m_2}$ near ∞, $m_1, m_2 > 0$ (see the precise assumptions (H1), (H2) at the beginning of Sect. 3). Our general estimates for $e^{it\phi(L)}$ are given in Theorem 3.3 for the low frequency regime and Theorem 3.4 for the high frequency regime.

We illustrate our results in the special, and particularly interesting case of fractional powers $\phi(L) = L^\nu$, $\nu \in (0, 1)$ (see Theorems 3.7 and 3.9):

Theorem 1.1 *Assume L satisfies (A1) and (A2), and let $\nu \in (0, 1)$. Then we have:*

(i) *For $s > (1 - \nu)n + \nu$,*

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|(I + L)^{s/2} f\|_{L^1}, \quad |t| < T_0.$$

(ii) *For $p \in (0, 1)$ and $s = n(1/p - \nu) + \nu$,*

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H_L^p}, \quad |t| < T_0.$$

(iii) *Moreover, for $s = (1 - \nu)n + \nu$ we have*

$$\left\|e^{itL^\nu} f\right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^{s,L}}.$$

Here H_L^p is the Hardy space associated to L, which can be defined via square functions or, equivalently, by atomic decompositions; see Sect. 2.2 for details. To our knowledge, the use of Hardy spaces with $p < 1$ is novel in this context (recall however that Beals [5] proved dispersive estimates for H^1 data). The spaces $\dot{B}_{1,1}^{s,L}$ associated to L are a generalization of classical Besov spaces to the setting of metric measure spaces, also new to our knowledge. Their construction and some properties are given at the end of Sect. 3. Note in particular that in some cases (e.g., $L = -\Delta + |x|^2$) we can prove that the new Besov spaces contain properly the standard ones with the same indices.

The decay rate $\sim |t|^{-\frac{n-1}{2}}$ appearing in the Theorem is independent of ν and may be surprising at first. However it is easy to check that one can not expect a better decay for general L; see Sect. 3.2 for a discussion of this phenomenon and related counterexamples. Note however that for $L = -\Delta$ on $L^2(\mathbb{R}^n)$ estimate (iii) reduces to

$$\|e^{it(-\Delta)^{1/2}}\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^{\frac{n+1}{2}}}$$

which is known to be sharp for the wave flow.

The plan of the paper is the following. In Sect. 2 we prove some preliminary kernel estimates and recall the definition of Hardy spaces associated to L, giving an equivalent atomic characterization. In Sect. 3 we prove the crucial subordination formulas, extending the earlier results in [33]; we then apply the formulas to prove general dispersive estimates for the flows $e^{it\phi(L)}$. We also sharpen our results in the case of fractional powers, and in particular we define Besov spaces associated to the operator L. In the final Sect. 4 we examine some applications to concrete operators, namely Hermite and Laguerre operators and the twisted Laplacian, and in particular we improve both the rate of decay and the loss of derivatives compared with the similar estimates in [15].

2 Preliminary results

2.1 Kernel estimates

Let L be a nonnegative, self-adjoint operator on $L^2(X)$ satisfying (A2). Denote by $E_L(\lambda)$ the spectral decomposition of L. Then by spectral theory, for any bounded Borel function $F : [0, \infty) \to \mathbb{C}$ we can define

$$F(L) = \int_0^\infty F(\lambda) dE_L(\lambda)$$

as a bounded operator on $L^2(X)$.

We have the following useful lemma.

Lemma 2.1 (a) *Let $\varphi \in \mathscr{S}(\mathbb{R})$ be an even function. Then for any $N > 0$ there exists C such that*

$$|K_{\varphi(t\sqrt{L})}(x, y)| \leq \frac{C}{\mu(B(x, t)) + \mu(B(y, t))} \left(1 + \frac{d(x, y)}{t}\right)^{-N}, \qquad (7)$$

for all $t > 0$ and $x, y \in X$.

(b) *Let $\varphi \in \mathscr{S}(\mathbb{R})$ be an even function. Then for any $1 \leq p \leq q \leq \infty$ we have*

$$\|\varphi(t\sqrt{L})\|_{L^p \to L^q} \lesssim t^{-(\frac{n}{p} - \frac{n}{q})}, \quad t > 0. \qquad (8)$$

Estimate (7) was proved in [9, Lemma 2.3] for $X = \mathbb{R}^n$. We now give a new proof for the general case. To do this, we need the following estimates.

Lemma 2.2 *Let $\lambda > 0$. Then we have:*

(a) *For any $N > 0$ and $s = N + D + 1/2$, there exits $C = C(N)$ so that*

$$|F(\lambda\sqrt{L})(x, y)| \leq \frac{C}{\sqrt{\mu(B(x, \lambda))\mu(B(y, \lambda))}} \left(1 + \frac{d(x, y)}{\lambda}\right)^{-N} \|F\|_{W_s^2} \qquad (9)$$

for all $x, y \in \mathbb{R}^n$ and all Borel functions supported in $[1/2, 2]$.

(b) *For any $N > 0$ and $s = 2(N + D + 1)$ there exits $C = C(N)$ so that*

$$|F(\lambda\sqrt{L})(x, y)| \leq \frac{C}{\sqrt{\mu(B(x, \lambda))\mu(B(y, \lambda))}} \left(1 + \frac{d(x, y)}{\lambda}\right)^{-N} \|F\|_{W_s^\infty} \qquad (10)$$

for all smooth functions F supported in $[0, 2]$ with $F^{(2v+1)}(0) = 0$ for all $v \in \mathbb{N}$.

Proof (a) Set $G(x) = F(\sqrt{x})e^x$. Then we have

$$G(\lambda^2 L)e^{-\lambda^2 L} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\lambda^2(1-i\tau)L} \widehat{G}(\tau) d\tau$$

where $\widehat{G}$ is the Fourier transform of G.

This, along with the fact that $F(\lambda\sqrt{L}) = G(\lambda^2 L)e^{-\lambda^2 L}$, implies

$$K_{F(\lambda\sqrt{L})}(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{G}(\tau) p_{\lambda^2(1-i\tau)}(x, y) d\tau. \qquad (11)$$

Recalling [10, Lemma 4.1] we have

$$|p_z(x, y)| \leq \frac{C}{\left[\mu(B(x, \sqrt{\frac{|z|}{\cos\theta}}))\mu(B(y, \sqrt{\frac{|z|}{\cos\theta}}))\right]^{1/2}}$$

$$\times \exp\left(-c\frac{d(x, y)^2}{|z|}\cos\theta\right) \frac{1}{(\cos\theta)^D}$$

for all $x, y \in X$ and $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$ where $\theta = \arg z$.

It follows that

$$|p_{\lambda^2(1-i\tau)}(x,y)| \leq \frac{C}{\left[\mu(B(x,\lambda\sqrt{1+\tau^2}))\mu(B(y,\lambda\sqrt{1+\tau^2}))\right]^{1/2}}$$

$$\times \exp\left[-c\frac{d(x,y)^2}{\lambda^2(1+\tau^2)}\right](1+\tau^2)^{D/2}$$

$$\leq \frac{C}{[\mu(B(x,\lambda))\mu(B(y,\lambda))]^{1/2}}\exp\left[-c\frac{d(x,y)^2}{\lambda^2(1+\tau^2)}\right](1+|\tau|)^D.$$

This, along with (11), implies that

$$|F(\lambda\sqrt{L})(x,y)| \lesssim \int_{\mathbb{R}} |\widehat{G}(\tau)|\frac{1}{\sqrt{\mu(B(x,\lambda))\mu(B(y,\lambda))}}$$

$$\times \exp\left(-c\frac{d(x,y)^2}{\lambda^2(1+\tau^2)}\right)(1+|\tau|)^D d\tau$$

$$\lesssim \frac{1}{\sqrt{\mu(B(x,\lambda))\mu(B(y,\lambda))}}\left(1+\frac{d(x,y)}{\lambda}\right)^{-N}$$

$$\int_{\mathbb{R}} |\widehat{G}(\tau)|(1+|\tau|)^{N+D}d\tau$$

$$\lesssim \frac{1}{\sqrt{\mu(B(x,\lambda))\mu(B(y,\lambda))}}\left(1+\frac{d(x,y)}{\lambda}\right)^{-N}$$

$$\times \left(\int_{\mathbb{R}} |\widehat{G}(\tau)|(1+|\tau|^2)^{N+D+1}d\tau\right)^{1/2}\left(\int_{\mathbb{R}}(1+|\tau|^2)^{-1}d\tau\right)^{1/2}$$

$$\lesssim \frac{1}{\sqrt{\mu(B(x,\lambda))\mu(B(y,\lambda))}}\left(1+\frac{d(x,y)}{\lambda}\right)^{-N}\|G\|_{W_s^2}$$

where $s = N + D + 1$.

Since supp $F \subset [1/2, 2]$, $\|G\|_{W_s^2} \sim \|F\|_{W_s^2}$. Hence, we obtain

$$|F(\lambda\sqrt{L})(x,y)| \lesssim \frac{1}{\sqrt{\mu(B(x,\lambda))\mu(B(y,\lambda))}}\left(1+\frac{d(x,y)}{\lambda}\right)^{-N}\|F\|_{W_s^2}.$$

(b) Since F can be extended to be an even function, we can write $F(x) = f(x^2)$ for all $x \geq 0$ and $\|f\|_{W_s^2} \lesssim \|F\|_{W_{2s}^\infty}$ for every $s > 0$. See [40]. At this stage, arguing similarly to (a) we obtain (10). $\qquad\square$

We are ready to prove Lemma 2.1.

Proof of Lemma 2.1 (a) Let $\psi_0 \in C^\infty(\mathbb{R})$ supported in $[0,2]$ such that $\psi_0 = 0$ on $[0,1]$ and $0 \leq \psi_0 \leq 1$. Set $\psi(\lambda) = \psi_0(\lambda) - \psi_0(2\lambda)$ and $\psi_j(\lambda) = \psi(2^{-j}\lambda)$ for $j \geq 1$. Then we have

$$\sum_{j \geq 0} \psi_j(\lambda) = 1, \lambda > 0.$$

Hence,

$$\varphi(t\sqrt{L}) = \sum_{j \geq 0} \psi_j(t\sqrt{L})\varphi(t\sqrt{L}). \tag{12}$$

By (10) we have

$$|\psi_0(t\sqrt{L})\varphi(t\sqrt{L})(x, y)| \leq \frac{1}{\sqrt{\mu(B(x, \lambda))\mu(B(y, \lambda))}} \left(1 + \frac{d(x, y)}{\lambda}\right)^{-N}. \tag{13}$$

Since supp $\psi \subset [1/2, 2]$, using (9) and (3), we have, for $j \geq 1$,

$$|\psi_j(t\sqrt{L})\varphi(t\sqrt{L})(x, y)| \leq \frac{C}{\sqrt{\mu(B(x, 2^{-j}t))\mu(B(y, 2^{-j}t))}}$$
$$\times \left(1 + \frac{d(x, y)}{2^{-j}t}\right)^{-N} \|h_j\|_{W_s^2}$$
$$\leq \frac{C2^{jD}}{\sqrt{\mu(B(x, t))\mu(B(y, t))}} \left(1 + \frac{d(x, y)}{t}\right)^{-N} \|h_j\|_{W_s^2}$$

where $s = N + D + 1$ and $h_j(\lambda) = \psi(\lambda)\varphi(2^j\lambda)$.

Since $\varphi \in \mathscr{S}(\mathbb{R})$, $\|h_j\|_{W_s^2} \leq C2^{-j(D+1)}$. As a consequence,

$$|\psi_j(t\sqrt{L})\varphi(t\sqrt{L})(x, y)| \leq \frac{C2^{-j}}{\sqrt{\mu(B(x, t))\mu(B(y, t))}} \left(1 + \frac{d(x, y)}{t}\right)^{-N}.$$

This, along with (12) and (13), implies that for each $N > 0$ there exists C such that

$$|\varphi(t\sqrt{L})(x, y)| \leq \frac{C}{\sqrt{\mu(B(x, t))\mu(B(y, t))}} \left(1 + \frac{d(x, y)}{t}\right)^{-N}$$

for all $x, y \in X$ and $t > 0$.

Applying (4), we will obtain (7) as desired. This completes the proof of (a).

(b) From the estimate in part (a) and (2) we have

$$\|\varphi(t\sqrt{L})\|_{L^1 \to L^\infty} \lesssim t^{-n} \quad \text{and} \quad \|\varphi(t\sqrt{L})\|_{L^1 \to L^1} \lesssim 1$$

for all $t > 0$.

By interpolating, we have

$$\|\varphi(t\sqrt{L})\|_{L^1 \to L^q} \lesssim t^{-(n-\frac{n}{q})}$$

for all $t > 0$ and $1 \le q \le \infty$.

On the other hand, from the estimate in part (a) again we have

$$\|\varphi(t\sqrt{L})\|_{L^q \to L^q} \lesssim 1.$$

The last two estimates, in combination with interpolation, imply (8) as desired. □

2.2 Hardy spaces associated to operators

We first recall from [25,27] the definition of the Hardy spaces associated to an operator. Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Gaussian upper bound (A2). Let $0 < p \le 1$. Then the Hardy space $H_L^p(X)$ is defined as the completion of

$$\{f \in L^2(X) : S_L f \in L^p(X)\}$$

under the norm $\|f\|_{H_L^p(X)} = \|S_L f\|_{L^p}$, where the square function S_L is defined as

$$S_L f(x) = \left(\int_0^\infty \int_{d(x,y)<t} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)dt}{\mu(B(x,t))} \right)^{1/2}.$$

Definition 2.3 (*Molecules for L*) Let $\epsilon > 0$, $0 < p \le 1$ and $M \in \mathbb{N}$. A function $m(x)$ is called a $(p, 2, M, L, \epsilon)$-molecule associated to a ball $B \subset X$ of radius r_B if there exists a function $b \in D(L^M)$ such that

(i) $m = L^M b$;
(ii) $\|L^k b\|_{L^2(S_j(B))} \le 2^{-j\epsilon} r_B^{2(M-k)} \mu(2^j B)^{1/2 - 1/p}$ for all $k = 0, 1, \ldots, M$ and $j = 0, 1, 2 \ldots$

where $S_j(B) = 2^j \backslash 2^{j-1} B$ as $j \ge 1$ and $S_0(B) = B$.

The definition of atoms is taken from [25,27] which requires (ii)-(iii). Assumption (ii) in particular can be thought of as a mild *locality* condition on the operator L.

Definition 2.4 (*Hardy spaces associated to L*) Given $\epsilon > 0$, $0 < p \le 1$ and $M \in \mathbb{N}$, we say that $f = \sum \lambda_j m_j$ is a molecule $(p, 2, M, L, \epsilon)$-representation if $\{\lambda_j\}_{j=0}^\infty \in \ell^p$, each m_j is a $(p, 2, M, L, \epsilon)$-atom, and the sum converges in $L^2(X)$. The space $H_{L,\text{mol},M,\epsilon}^p(X)$ is then defined as the completion of

$$\left\{ f \in L^2(X) : f \text{ has a molecule } (p, 2, M, L, \epsilon)\text{-representation} \right\},$$

with the norm given by

$$\|f\|_{H^p_{L,\text{mol},M,\epsilon}(X)}^p$$
$$= \inf\left\{\sum|\lambda_j|^p : f = \sum\lambda_j m_j \text{ is a molecule } (p, 2, M, L, \epsilon)\text{-representation}\right\}.$$

Theorem 2.5 [17] *Let $\epsilon > 0$, $p \in (0, 1]$ and $M > \frac{n(2-p)}{4p}$. Then the Hardy spaces $H^p_{L,\text{mol},M,\epsilon}(X)$ and $H^p_L(X)$ coincide and have equivalent norms.*

We note that if $L = -\Delta$ on $L^2(\mathbb{R}^n)$, then $H^p_L(\mathbb{R}^n)$ reduces to the standard Hardy space $H^p(\mathbb{R}^n)$ on $\mathbb{R}^n$ for $p \in (0, 1]$. In general, depending on the choice of the operator L, it may happen that either $H^p(\mathbb{R}^n) \subset H^p_L(\mathbb{R}^n)$, or $H^p_L(\mathbb{R}^n) \subset H^p(\mathbb{R}^n)$, or $H^p(\mathbb{R}^n) \neq H^p_L(\mathbb{R}^n)$ without inclusions. See for example [16,18]. We examine now in more detail the case of the Hermite operator $L = -\Delta + |x|^2$ on $\mathbb{R}^n$, for which the Hardy spaces can be characterized via a new atomic decompositions as follows.

Let $\rho(x) = \min\{1, |x|^{-1}\}$ for $x \in \mathbb{R}^n$. Let $p \in (0, 1]$. A function a is called a (p, ∞, ρ)-*atom* associated to the ball $B(x_0, r)$ if

(i) $\text{supp}\, a \subset B(x_0, r)$;
(ii) $\|a\|_{L^\infty} \leq |B(x_0, r)|^{-1/p}$;
(iii) $\int x^\alpha a(x)dx = 0$ for all $|\alpha| \leq \lfloor n(1/p - 1)\rfloor$ if $r < \rho(x_0)/4$.

The Hardy space $H^p_{at,\rho}(\mathbb{R}^n)$ is then defined as the set of all functions f which can be expressed in the form $f = \sum_j \lambda_j a_j$ where $(\lambda_j)_j \in \ell^p$ and a_j are (p, ∞, ρ)-atoms. Its norm is given by

$$\|f\|_{H^p_{at,\rho}(\mathbb{R}^n)}^p := \inf\left\{\sum_j|\lambda_j|^p : f = \sum_j\lambda_j a_j\right\}$$

where the infimum is taken over all possible atomic decompositions of f. From the definition, it is obvious that $H^p(\mathbb{R}^n) \subsetneq H^p_{at,\rho}(\mathbb{R}^n)$ for all $p \in (0, 1]$; more importantly, we have $H^1_{at,\rho}(\mathbb{R}^n) \equiv H^1_L(\mathbb{R}^n)$ (see for instance [20]), thus the Hardy space associated to the Hermite operator is *larger* than the standard one.

We recall an important result in [17, Proposition 3.27] which plays an important role in the sequel.

Proposition 2.6 *Let $\epsilon > 0$, $p \in (0, 1]$ and $M > \frac{n(2-p)}{4p}$. Suppose that $f = \sum_{j=1}^N \lambda_j a_j$ for some $N \in \mathbb{N}$, where $a'_j s$ are $(p, 2, 2M, L, \epsilon)$ molecules and $\sum_{j=1}^N|\lambda_j|^p < \infty$. Then there exists a representation $f = \sum_{j=1}^K \gamma_j m_j$ for some $K \in \mathbb{N}$ where $m'_j s$ are $(p, 2, M, L, \sigma)$ molecules for some $\sigma > 0$ such that*

$$\sum_{j=1}^K|\gamma_j|^p \sim \|f\|_{H^p_L(X)}.$$

3 Dispersive estimates for $e^{it\phi(L)}$

3.1 Subordination formulas and dispersive estimates for $e^{it\phi(L)}$

Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function. We denote by (H1) and (H2) the following assumptions on ϕ:

(H1) There exists $0 < m_1 \leq 1$ such that

$$\phi'(r) \sim r^{m_1-1} \quad \text{and} \quad |\phi''(r)| \gtrsim r^{m_1-2}, \quad r \geq 1.$$

(H2) There exists $m_2 > 0$ such that

$$\phi'(r) \sim r^{m_2-1} \quad \text{and} \quad |\phi''(r)| \gtrsim r^{m_2-2}, \quad 0 < r < 1.$$

The following two results are crucial for the rest of the paper, and are strongly inspired by ideas from [33].

Theorem 3.1 *Assume ϕ satisfies (H1) and g is a C^∞ function supported in $[1/2, 2]$. Then there exist $c_0 > 1$, and functions and $\rho_t(x, \lambda)$ and $a_t(s, \lambda)$ defined on $\mathbb{R}^2$ for each t satisfying*

$$\operatorname{supp} \rho_t(\cdot, \lambda) \subset [\lambda^2/5, 5\lambda^2] \quad \text{and} \quad |\rho_t(x, \lambda)| \leq C(k, \|g\|_{C^k}, \phi)(t\lambda^{2m_1})^{-k}, k \geq 0,$$
$$(14)$$

and

$$\operatorname{supp} a_t(\cdot, \lambda) \subset [2c_0^{-1}, 2c_0] \quad \text{and} \quad |a_t(s, \lambda)| \leq C(\|g\|_{C^1}, \phi), \tag{15}$$

so that

$$g(\lambda^{-1}\sqrt{x})e^{it\phi(x)} = \rho_t(x, \lambda) + \sqrt{t\lambda^{2m_1}}\eta(\lambda^{-2}x) \int e^{ixt\lambda^{2m_1-2}s} a_t(s, \lambda)ds \tag{16}$$

for all $x, t > 0$ and $\lambda \geq 1$, where $\eta \in C^\infty(\mathbb{R})$ is supported in $[1/5, 5]$ and $\eta \equiv 1$ on $[1/4, 4]$.

Proof Let $k \in \mathbb{Z}$ and $t > 0$. For $\lambda \geq 1$ we denote by $\Psi_\lambda(\xi)$ the Fourier transform of $g(\lambda^{-1}\sqrt{x})e^{it\phi(x)}$, i.e.,

$$\Psi_\lambda(\xi) = \int g(\lambda^{-1}\sqrt{x})e^{it\phi(x)}e^{-ix\xi}dx$$
$$= \lambda^2 \int g(\sqrt{u})e^{i[t\phi(\lambda^2 u)-\lambda^2 u\xi]}du. \tag{17}$$

Let $\tau \in C^\infty(\mathbb{R})$ supported in $[2c_0^{-1}, 2c_0]$ with $\tau \equiv 1$ in $[c_0^{-1}, c_0]$ where c_0 will be determined later. Then by the Fourier inversion formula we have

$$g(\lambda^{-1}\sqrt{x})e^{it\phi(x)} = \eta(\lambda^{-2}x) \int \left(1 - \tau\left(\frac{\xi}{t\lambda^{2m_1-2}}\right)\right) \Psi_\lambda(\xi)e^{i\xi x}d\xi$$

$$+ \eta(\lambda^{-2}x) \int \tau\left(\frac{\xi}{t\lambda^{2m_1-2}}\right) \Psi_\lambda(\xi)e^{i\xi x}d\xi$$

$$=: \rho_t(x, \lambda) + A_t(x, \lambda)$$

where $\eta \in C^\infty(\mathbb{R})$ is supported in $[1/5, 5]$ and $\eta \equiv 1$ on $[1/4, 4]$.

Observe that

$$\partial_u[t\phi(\lambda^2 u) - \lambda^2 u\xi] = \lambda^2 t\phi'(\lambda^2 u) - \lambda^2 \xi$$

We note that the integrand in the expression for $\rho_t(x, \lambda)$ is supported where either $\xi < c_0^{-1}t\lambda^{2m_1-2}$ or $\xi > c_0 t\lambda^{2m_1-2}$. In this situation, by (H1) we can choose c_0 large enough so that

$$|\partial_u[t\phi(\lambda^2 u) - \lambda^2 u\xi]| \gtrsim (\lambda^2|\xi| + t\lambda^{2m_1})$$

Hence, by integration by parts in (17), we have for these ξ that

$$|\Psi_\lambda(\xi)| \le C_{k,g,\phi}\lambda^2(\lambda^2|\xi| + t\lambda^{2m_1})^{-k}, \quad \forall k \ge 0, \lambda \ge 1.$$

This implies

$$|\rho_t(x, \lambda)| \le C_{k,g,\phi}(t\lambda^{2m_1})^{-k}, k \ge 0,$$

which proves (14).

We now estimate the term $A_t(x, \lambda)$. By a change of variables, we have

$$A_t(x, \lambda) = t\lambda^{2m_1-2}\eta(\lambda^{-2}x) \int \tau(s)\Psi_\lambda(t\lambda^{2m_1-2}s)e^{ixt\lambda^{2m_1-2}s}ds$$

$$= t\lambda^{2m_1}\eta(\lambda^{-2}x) \int \tau(s)e^{ixt\lambda^{2m_1-2}s} \int g(\sqrt{u})e^{i[t\phi(\lambda^2 u)-t\lambda^{2m_1}us]}duds$$

$$= \sqrt{t\lambda^{2m_1}}\eta(\lambda^{-2}x) \int e^{ixt\lambda^{2m_1-2}s}a_t(s, \lambda)ds$$

where

$$a_t(s, \lambda) = \sqrt{t\lambda^{2m_1}} \int \tau(s)g(\sqrt{u})e^{i[t\phi(\lambda^2 u)-t\lambda^{2m_1}us]}du.$$

It is clear that supp $a(\cdot, \lambda) \subset [2c_0^{-1}, 2c_0]$. Moreover, on the support of g we have $1/4 < u < 4$. In this situation, by (H1) we have

$$\left| \frac{\partial^2}{\partial u^2} [t\phi(\lambda^2 u) - t\lambda^{2m_1} us] \right| \gtrsim t\lambda^{2m_1}.$$

Hence, by van der Corput's Lemma in [38] we obtain

$$|a_t(x, \lambda)| \lesssim 1.$$

This implies $|a(s, \lambda)| \lesssim 1$ for all $s \in [2c_0^{-1}, 2c_0]$ and $\lambda \geq 1$. This proves (15). □

In a similar way we have the following result:

Theorem 3.2 *Assume ϕ satisfies (H2) and g is a C^∞ function supported in $[1/2, 2]$. Then there exist $c_0 > 1$, and functions and $\rho_t(x, \lambda)$ and $a_t(s, \lambda)$ defined on $\mathbb{R}^2$ for each t satisfying*

$$|\rho_t(x, \lambda)| \leq C(k, \|g\|_{C^k}, \phi)(t\lambda^{2m_2})^{-k}, \quad k \geq 0, \tag{18}$$

and

$$\text{supp } a_t(\cdot, \lambda) \subset [2c_0^{-1}, 2c_0] \quad and \quad |a_t(s, \lambda)| \leq C(\|g\|_{C^1}, \phi). \tag{19}$$

so that

$$g(\lambda^{-1}\sqrt{x})e^{it\phi(x)} = \rho_t(x, \lambda) + \sqrt{t\lambda^{2m_2}}\eta(\lambda^{-2}x) \int e^{ixt\lambda^{2m_2-2}s} a_t(s, \lambda)ds$$

for all $x, t > 0$ and $0 < \lambda < 1$, where $\eta \in C^\infty(\mathbb{R})$ is supported in $[5, 1/5]$ and $\eta \equiv 1$ on $[1/4, 4]$.

Proof The proof of this theorem is similar to that of Theorem 3.1 and we omit details. □

We now apply the previous Theorems to obtain a high frequency (resp. low frequency) dispersive estimate for the flow $e^{it\phi(L)}$:

Theorem 3.3 *Assume L satisfies (A1) and (A2), ϕ satisfies (H1), and $\psi \in C^\infty(\mathbb{R})$ is supported in $[1/2, 2]$. Then we have*

$$|\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)}f| \lesssim |t|^{-\frac{n-1}{2}}\lambda^{(1-m_1)n+m_1}\|f\|_{L^1}, \quad \lambda \geq 1, \ |t| < T_0. \tag{20}$$

Proof From Theorem 3.1 and spectral theory, there exist functions ρ, a and η as in Theorem 3.1 so that

$$\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)} = \rho_t(L, \lambda) + \sqrt{t\lambda^{2m_1}}\eta(\lambda^{-2}L) \int e^{it\lambda^{2m_1-2}sL} a_t(s, \lambda)ds$$

$$= \rho_t(L, \lambda) + A_{t,\lambda}(L).$$

We first estimate the term related to $A_{t,\lambda}(L)$. By using (A1), (A2) and (15) we have

$$\|A_{t,\lambda}(L)\|_{L^1 \to L^\infty} \lesssim \sqrt{\lambda^{2m_1}t}[\lambda^{2m_1-2}t]^{-n/2} \int_{2c_0^{-1}}^{2c_0} s^{-n/2}|a_t(s,\lambda)|ds$$

$$\lesssim t^{-\frac{n-1}{2}}\lambda^{(1-m_1)n+m_1}.$$

We now take care of the term $\rho_t(L,\lambda)$. Let $\varphi \in C^\infty(\mathbb{R})$ with supp $\varphi \subset [1/6, 6]$ and $\varphi \equiv 1$ in $[1/5, 5]$. Since $\rho_t(\cdot, \lambda)$ is supported in $[\lambda^2/5, 5\lambda^2]$, we have

$$\rho_t(L,\lambda) = \varphi(\lambda^{-1}\sqrt{L})\rho_t(L,\lambda)\varphi(\lambda^{-1}\sqrt{L}).$$

Therefore,

$$\|\rho_t(L,\lambda)\|_{L^1 \to L^\infty} \leq \|\varphi(\lambda^{-1}\sqrt{L})\|_{L^1 \to L^2}\|\rho_t(L,\lambda)\|_{L^2 \to L^2}\|\varphi(\lambda^{-1}\sqrt{L})\|_{L^2 \to L^\infty}.$$

Using (8) we have

$$\|\varphi(\lambda^{-1}\sqrt{L})\|_{L^1 \to L^2} \lesssim \lambda^{n/2}, \quad \text{and} \quad \|\varphi(\lambda^{-1}\sqrt{L})\|_{L^2 \to L^\infty} \lesssim \lambda^{n/2}. \qquad (21)$$

By (14) we deduce

$$\|\rho_t(L,\lambda)\|_{L^2 \to L^2} \leq \|\rho_t(\cdot,\lambda)\|_{L^\infty} \lesssim (\lambda^{2m_1}t)^{-\frac{n-1}{2}}.$$

Therefore,

$$\|\rho_t(L,\lambda)\|_{L^1 \to L^\infty} \lesssim \lambda^n(\lambda^{2m_1}t)^{-\frac{n-1}{2}} = t^{-\frac{n-1}{2}}\lambda^{(1-m_1)n+m_1}.$$

Summing up, we have proved that

$$\|\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)}\|_{L^1 \to L^\infty} \lesssim t^{-\frac{n-1}{2}}\lambda^{(1-m_1)n+m_1}, \quad \lambda \geq 1. \qquad (22)$$

This completes our proof. □

For the low frequency estimate we have:

Theorem 3.4 *Assume L satisfies (A1) and (A2), ϕ satisfies (H2), and $\psi \in C^\infty(\mathbb{R})$ is supported in $[1/2, 2]$. Then we have*

$$|\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)}f| \lesssim |t|^{-\frac{n-1}{2}}\lambda^{(1-m_2)n+m_2}\|f\|_{L^1}, \quad 0 < \lambda < 1, \ |t| < T_0. \ (23)$$

Proof If $T_0 = \infty$, then we can argue similarly to the proof of Theorem 3.3. It remains to consider the case $T_0 < \infty$. Arguing as for (21) we obtain

$$|\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)}f| \lesssim \lambda^n\|f\|_{L^1}.$$

Now we note that

$$\lambda^n \lesssim |t|^{-\frac{n-1}{2}} \lambda^{(1-m_2)n+m_2} \quad \Longleftrightarrow \quad |t| \lambda^{2m_2} \lesssim 1$$

and indeed we have

$$|t| \lambda^{2m_2} \leq T_0 \cdot 1 = T_0$$

which concludes the proof. □

Combining Theorems 3.3 and 3.4 we conclude:

Corollary 3.5 *Let L satisfy (A1) and (A2) and let ϕ satisfy (H1) and (H2) with $m_1 = m_2 = m$. Assume that $\psi \in C^\infty(\mathbb{R})$ supported in $[1/2, 2]$ then we have*

$$|\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)}f| \lesssim |t|^{-\frac{n-1}{2}} \lambda^{(1-m)n+m} \|f\|_{L^1}, \quad \lambda > 0, |t| < T_0. \quad (24)$$

3.2 A case study: dispersive estimates for fractional Schrödinger semigroups

We consider in more detail the following special case of Corollary 3.5:

Proposition 3.6 *Let L satisfy (A1) and (A2), and let $\nu \in (0, 1)$. Assume that $\psi \in C^\infty(\mathbb{R})$ is supported in $[1/2, 2]$. Then we have*

$$|\psi(\lambda^{-1}\sqrt{L})e^{itL^\nu}f| \lesssim |t|^{-\frac{n-1}{2}} \lambda^{(1-\nu)n+\nu} \|f\|_{L^1}, \quad \lambda > 0, |t| < T_0. \quad (25)$$

Readers familiar with the stationary phase method may suspect that the estimate (25) is not sharp. Indeed, for the family of flows $e^{it(-\Delta)^\nu}$ the decay rate is governed by the phase

$$\phi(t, x, \xi) = x \cdot \xi + t|\xi|^{2\nu},$$

more precisely by the curvature of the surface $\nabla_\xi \phi = 0$. In the case $\nu = 1/2$, corresponding to the wave equation, the surface is degenerate and one gets a decay rate $\sim |t|^{-\frac{n-1}{2}}$, while for $\nu \neq 1/2$ one expects better decay, at least for suitable data. However, it is not difficult to check that for nonhomogeneous phases this simple picture is wrong: for arbitrary $\nu \in (0, 1)$, one can construct operators L such that e^{itL} has a decay rate $\sim t^{-\frac{n}{2}}$ while e^{itL^ν} decays like $\sim t^{-\frac{n-1}{2}}$ at best in general. Thus, in this sense, our result is sharp.

We show the explicit construction for $\nu = 1/3$. Define a piecewise C^1 function $a_0 : [0, +\infty) \to [0, +\infty)$ as follows:

$$a_0(s) = \begin{cases} s^2/2 & \text{if } 0 \leq s \leq 1/2, \\ s^3 & \text{if } 1/2 \leq s \leq 1, \\ 3s^2 - 2 & \text{if } s \geq 1 \end{cases} \quad \text{so that} \quad a_0'(s) = \begin{cases} s & \text{if } 0 \leq s \leq 1/2, \\ 3s^2 & \text{if } 1/2 \leq s \leq 1, \\ 6s & \text{if } s \geq 1. \end{cases}$$

Note that a' is strictly increasing with (positive) jumps at $s = 1/2$ and $s = 1$, and that for all $s \neq 1/2, 1$ we have

$$s \leq a_0'(s) \leq 6s, \qquad 1 \leq a_0''(s) \leq 6.$$

We can obviously modify a_0 in two small neighbourhoods of the jump points and obtain a new C^∞ function $a : [0, +\infty) \to [0, +\infty)$ such that for all $s \geq 0$

$$s \leq a'(s) \leq Ks, \qquad 1 \leq a''(s) \leq K$$

for some positive constant $K > 6$, and in addition

$$a(s) = \begin{cases} s^2/2 & \text{if } 0 \leq s \leq 1/2 - \delta, \\ s^3 & \text{if } 1/2 + \delta \leq s \leq 1 - \delta, \\ 3s^2 - 2 & \text{if } s \geq 1 + \delta \end{cases} \qquad \delta = 1/100.$$

Note that we have also the symbol type property

$$|a^{(j)}(s)| \lesssim s^{2-j}, \qquad j = 0, 1, 2, \ldots.$$

We then define the oscillatory integral with a radial phase

$$I(t, x) = \int e^{i(x \cdot \xi + t\phi(\xi))} \psi(\xi) d\xi \quad \text{with } \phi(\xi) = a(|\xi|), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n$$

where $\psi \in C_c^\infty(\mathbb{R}^n)$ is a fixed test function; note that ψ satisfies trivially the symbol property

$$|\partial^\alpha \psi(\xi)| \lesssim |\xi|^{-|\alpha|}.$$

By the stationary phase method one gets the uniform estimate

$$|I(t, x)| \leq C|t|^{-\frac{n}{2}}, \qquad \forall t \neq 0, \quad x \in \mathbb{R}^n. \tag{26}$$

For instance, this is a special case of Theorem 3.1 in [26] (with the choices $m_1 = m_2 = 2$, $b_1 = b_2 = 0$ so that $v_1 = v_2 = 0$). The assumptions of the Theorem require, besides the symbol properties of ϕ, ψ, that $\phi(\xi)$ satisfy

$$|\nabla \psi| \simeq 1, \qquad C_1 \leq |\det D^2 \phi| \leq C_2$$

for some constants $C_1, C_2 > 0$; these conditions are easily verified from the above construction.

From estimate (26) one obtains immediately the dispersive estimate

$$\|e^{itL} f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \|f\|_{L^1} \tag{27}$$

for the flow associated to the operator

$$L = \phi(D) \quad \text{i.e.} \quad Lf = \mathcal{F}^{-1}(\phi(\xi)\widehat{f}).$$

Indeed, one can split f in frequency on the ball $|\xi| \leq 10$, and to this first piece one applies (26); on the remaining piece, the operator L coincides with $-3\Delta - 2$ and hence the estimate follows from the standard dispersive estimate for the Schrödinger equation.

However, the flow associated to $L^{1/3}$ can not decay faster than $|t|^{-\frac{n-1}{2}}$ in general. Indeed, for functions f with

$$\operatorname{supp}\widehat{f} \subset \{1/2 + \delta < |\xi| < 1 - \delta\}$$

we have

$$Lf = (-\Delta)^{3/2} f \quad \Longrightarrow \quad e^{itL^{1/3}} f = e^{it|D|} f$$

so that $e^{itL^{1/3}} f$ is precisely the wave flow applied to f, for which the sharp decay rate is known to be $\sim |t|^{-\frac{n-1}{2}}$.

It is clear that a similar construction can be performed for any power L^ν, $\nu \in (0, 1)$.

By a suitable frequency decomposition, we can deduce from 3.6 the following dispersive estimates with a loss of derivatives:

Theorem 3.7 *Let L satisfy* (A1) *and* (A2), *and let* $\nu \in (0, 1)$.

(i) *For* $s > (1 - \nu)n + \nu$,

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|(I + L)^{s/2} f\|_{L^1}, \quad |t| < T_0.$$

(ii) *For* $p \in (0, 1)$ *and* $s = n(1/p - \nu) + \nu$,

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H_L^p}, \quad |t| < T_0.$$

Proof (i) Let $\varphi_0 \in C^\infty[0, \infty)$ so that $\varphi_0(\lambda) = 1$ for $\lambda \in [0, 1]$ and $\operatorname{supp}\varphi_0 \subset [0, 2]$. Let $\varphi(\lambda) = \varphi_0(\lambda) - \varphi_0(2\lambda)$ and set $\varphi_j(\lambda) = \varphi(2^{-j}\lambda)$, $j \geq 1$. Then we have $\operatorname{supp}\varphi \subset [1/2, 2]$ and

$$\sum_{j \geq 0} \varphi_j(\lambda) = 1, \quad \forall \lambda \geq 0$$

so that

$$(I + L)^{-s/2} = \sum_{j \geq 0} \varphi_j(\sqrt{L})(I + L)^{-s/2}$$

$$= \sum_{j \geq 0} 2^{-js} \left[2^{js} \varphi_j(\sqrt{L})(I + L)^{-s/2} \right]$$

$$=: \sum_{j \geq 0} 2^{-js} \widetilde{\varphi}_j(\sqrt{L})$$

It follows that

$$e^{itL^\nu} f = \sum_{j \geq 0} 2^{-js} \widetilde{\varphi}_j(\sqrt{L}) e^{itL^\nu} (I + L)^{s/2} f.$$

Hence,

$$\|e^{itL^\nu} f\|_{L^\infty} \leq \sum_{j \geq 0} 2^{-js} \|\widetilde{\varphi}_j(\sqrt{L}) e^{itL^\nu} (I + L)^{s/2} f\|_{L^\infty}$$

$$= \sum_{j \geq 0} I_j$$

For $j \geq 1$, by applying Proposition 3.6 we have

$$\sum_{j \geq 1} I_j \lesssim \sum_{j \geq 1} 2^{-js} |t|^{-\frac{n-1}{2}} 2^{j[(1-\nu)n+\nu]} \|(I + L)^{s/2} f\|_{L^1}$$

$$\lesssim |t|^{-\frac{n-1}{2}} \|(I + L)^{s/2} f\|_{L^1}$$

as long as $s > (1 - \nu)n + \nu$.

For $j = 0$ we have

$$\widetilde{\varphi}_0(\sqrt{L}) e^{itL^\nu} (I + L)^{s/2} f = \sum_{k \leq 1} \psi(2^{-k}\sqrt{L}) \widetilde{\varphi}_0(\sqrt{L}) e^{itL^\nu} (I + L)^{s/2} f$$

where $\psi \in C_0^\infty(\mathbb{R})$ is a partition of unity function, i.e., $\psi \in \mathcal{S}(\mathbb{R})$ such that supp $\psi \subset [1/2, 2]$ and

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j}\lambda) = 1 \text{ on } (0, \infty).$$

By applying Proposition 3.6 we have

$$I_0 \lesssim \sum_{k \leq 1} |t|^{-\frac{n-1}{2}} 2^{k[(1-\nu)n+\nu]} \|\widetilde{\varphi}_0(\sqrt{L})(I + L)^{s/2} f\|_{L^1}$$

$$\lesssim |t|^{-\frac{n-1}{2}} \|\widetilde{\varphi}_0(\sqrt{L})(I + L)^{s/2} f\|_{L^1}.$$

Since $\widetilde{\varphi}_0(\lambda)$ can be extended to be an even function in $\mathscr{S}(\mathbb{R})$, then by (8) we have

$$\|\widetilde{\varphi}_0(\sqrt{L})(I + L)^{s/2} f\|_{L^1} \lesssim \|(I + L)^{s/2} f\|_{L^1}.$$

As a consequence,

$$I_0 \lesssim |t|^{-\frac{n-1}{2}} \|(I + L)^{s/2} f\|_{L^1}.$$

Therefore,

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|(I + L)^{s/2} f\|_{L^1}, s > (1 - \nu)n + \nu.$$

(ii) Let $\{\psi(2^{-j}\cdot)\}_{j\in\mathbb{Z}}$ with $\mathrm{supp}\,\psi \subset [1/2, 2]$ be a partition of unity function on $\mathbb{R}\backslash\{0\}$. We then have

$$\begin{aligned}
e^{itL^\nu} f &= \sum_{j\in\mathbb{Z}} \psi(2^{-j}\sqrt{L})e^{itL^\nu} f \\
&= \sum_{j\in\mathbb{Z}} L^{-s/2}\psi(2^{-j}\sqrt{L})e^{itL^\nu} L^{s/2} f
\end{aligned}$$

where $s = n(1/p - \nu) + \nu$.

Let $L^{s/2} f \in H_L^p(X) \cap L^2(X)$. Then, similarly to classical results [5,31], from Theorem 2.5 and Proposition 2.6 it suffices to prove (ii) for all f such that $L^{s/2} f$ has a finite molecule presentation, i.e.,

$$L^{s/2} f = \sum_{\ell=0}^{N} \lambda_\ell a_\ell$$

for some $N \in \mathbb{N}$, where a_j are $(p, 2, M, L, \epsilon)$ molecules with $\epsilon > 0$, $M > \frac{n(2-p)}{2p}$, and

$$\|L^{s/2} f\|_{H_L^p} \sim \left(\sum_{\ell=1}^{N} |\lambda_\ell|^p\right)^{1/p}.$$

From the molecule presentation of $L^{s/2} f$ we have

$$e^{itL^\nu} f = \sum_{\ell=0}^{N} \lambda_\ell \sum_{j\in\mathbb{Z}} L^{-s/2}\psi(2^{-j}\sqrt{L})e^{itL^\nu} a_\ell$$

so that

$$\|e^{itL^\nu} f\|_{L^\infty} \leq \sum_{\ell=0}^{N} |\lambda_\ell| \sum_{j\in\mathbb{Z}} \|L^{-s/2}\psi(2^{-j}\sqrt{L})e^{itL^\nu} a_\ell\|_{L^\infty}. \tag{28}$$

We now claim that

$$\sum_{j \in \mathbb{Z}} \|L^{-s/2}\psi(2^{-j}\sqrt{L})e^{itL^\nu}a\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \tag{29}$$

for all $(p, 2, M, L, \epsilon)$ molecules with $M > \frac{n(2-p)}{2p}$ and $\epsilon > 0$.

Once (29) is proved, the estimate in (ii) follows immediately. Indeed, from (28) and (29), we have

$$\|e^{itL^\nu}f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \sum_{\ell=0}^{\infty} |\lambda_\ell| \le |t|^{-\frac{n-1}{2}}\left(\sum_{\ell}|\lambda_\ell|^p\right)^{1/p} \sim |t|^{-\frac{n-1}{2}}\|L^{s/2}f\|_{H_L^p}$$

which proves (ii).

So, it suffices to prove (29). To do this, suppose that a is a $(p, 2, M, L, \epsilon)$ molecule associated to a ball B with $M > \frac{n(2-p)}{2p}$ and $\epsilon > 0$. Then there exists $b \in D(L^M)$ so that $a = L^M b$ satisfies (i) and (ii) in Definition 2.3. We now split the sum on the left hand side of (29) into 2 parts as follows:

$$\sum_{j \in \mathbb{Z}} \|L^{-s/2}\psi(2^{-j}\sqrt{L})e^{itL^\nu}a\|_{L^\infty} = \sum_{j \ge -\log_2 r_B} \|L^{-s/2}\psi(2^{-j}\sqrt{L})e^{itL^\nu}a\|_{L^\infty}$$
$$+ \sum_{j < -\log_2 r_B} \|L^{-s/2}\psi(2^{-j}\sqrt{L})e^{itL^\nu}a\|_{L^\infty}$$
$$=: E_1 + E_2.$$

Let us take care of E_1 first. To do this, we write

$$E_1 = \sum_{j \ge -\log_2 r_B^2} \|L^{-s/2}\psi(2^{-j}\sqrt{L})e^{itL^\nu}a\|_{L^\infty}$$
$$= \sum_{j \ge -\log_2 r_B} 2^{-js}\|2^{js}L^{-s/2}\psi(2^{-j}\sqrt{L})e^{itL^\nu}a\|_{L^\infty}.$$

Applying Proposition 3.6 we deduce that

$$E_1 \lesssim \sum_{j \ge -\log_2 r_B} 2^{-js}2^{j[(1-\nu)n+\nu]}|t|^{-\frac{n-1}{2}}\|a\|_{L^1}$$
$$\lesssim \sum_{j \ge -\log_2 r_B} 2^{-jn(1/p-1)}|t|^{-\frac{n-1}{2}}\|a\|_{L^1}. \tag{30}$$

From (ii) in Definition 2.3 and (2) we have

$$\|a\|_{L^1} = \sum_{k=0}^{\infty} \|a\|_{L^1(S_k(B))} \leq \sum_{k=0}^{\infty} \mu(2^k B)^{1/2} \|a\|_{L^2(S_k(B))} \quad \text{(due to Hölder's inequality)}$$

$$\leq \sum_{k=0}^{\infty} 2^{-k\epsilon} \mu(2^k B)^{1-1/p}$$

$$\leq \sum_{k=0}^{\infty} 2^{-k(\epsilon+n(1/p-1))} r_B^{-n(1/p-1)}$$

which implies $\|a\|_{L^1} \lesssim r_B^{-n(1/p-1)}$.

Therefore,

$$E_1 \lesssim \sum_{j \geq -\log_2 r_B} [2^{-j} r_B^{-1}]^{n(1/p-1)} |t|^{-\frac{n-1}{2}}$$

$$\lesssim |t|^{-\frac{n-1}{2}}.$$

For the term E_2 inserting $a = L^M b$ into the expression of E_2 we have

$$E_2 = \sum_{j < -\log_2 r_B} \|L^{M-s/2} \psi(2^{-j}\sqrt{L}) e^{itL^\nu} b\|_{L^\infty}$$

$$= \sum_{j < -\log_2 r_B} 2^{j(2M-s)} \|2^{-j(2M-s)} L^{M-s/2} \psi(2^{-j}\sqrt{L}) e^{itL^\nu} b\|_{L^\infty}.$$

By Proposition 3.6 we have

$$E_2 \lesssim \sum_{j < -\log_2 r_B} 2^{j(2M-s)} 2^{j[(1-\nu)n+\nu]} |t|^{-\frac{n-1}{2}} \|b\|_{L^1}$$

$$\lesssim \sum_{j < -\log_2 r_B} 2^{j[2M-n(1/p-1)]} |t|^{-\frac{n-1}{2}} \|b\|_{L^1}.$$

On the other hand, arguing similarly to (30) we have

$$\|b\|_{L^1} \lesssim r_B^{[2M-n(1/p-1)]}.$$

Therefore,

$$E_2 \lesssim \sum_{j < -\log_2 r_B} [2^j r_B]^{[2M-n(1/p-1)]} |t|^{-\frac{n-1}{2}}$$

$$\lesssim |t|^{-\frac{n-1}{2}}$$

as long as $M > \frac{n(2-p)}{2p} > \frac{n(1-p)}{2p}$.

From the estimates of E_1 and E_2 we obtain (29). This competes our proof. $\qquad\square$

Some comments on Theorem 3.7 are in order:

(i) An estimate like (i) in Theorem 3.7 was proved earlier in [15, Theorem 1.2] for the special case of the twisted Laplacian L on $\mathbb{R}^{2d}$, with $d \geq 2$. Note however that the estimate in [15] was

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim \left(|t| + |t|^{\frac{s-2d}{2\nu}}\right) \|L^{s/2} f\|_{L^1} \tag{31}$$

for s in the range $2d > s > 2d - 2\min\{\nu, 1-\nu\}$. Due to the spectral gap in this situation, the term $(I + L)^{s/2}$ in (i) of Theorem 3.7 can be replaced by $L^{s/2}$. Thus, even in this special case, estimate (i) in Theorem 3.7 improves (31) giving both a better range for s and a better decay.

(ii) The estimate (ii) in Theorem 3.7 is new. To the best of our knowledge, this is the first $H_L^p - L^\infty$ dispersive estimate in the literature.

(iii) In the particular case when $L = -\Delta$, estimate (25) can be improved as follows: for all $\nu \in (0, 1)\setminus\{1/2\}$,

$$|\psi(\lambda^{-1}\sqrt{L})e^{itL^\nu} f| \lesssim |t|^{-\frac{n}{2}}\lambda^{(1-\nu)n} \|f\|_{L^1}, \quad \lambda > 0, \ |t| < \infty. \tag{32}$$

See for example [24, Theorem 1]. Thus, following the proof of Theorem 3.7, we obtain that for $\nu \in (0, 1)\setminus\{1/2\}$

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \|(I + L)^{s/2} f\|_{L^1}, \quad s > n(1 - \nu), \tag{33}$$

and

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \|L^{s/2} f\|_{H^p}, \quad p \in (0, 1), \ s = n(1/p - \nu), \tag{34}$$

where H^p is a classical Hardy space. To our knowledge, the estimates (33) and (34) are new.

We now focus on the special case $\nu = 1/2$, which corresponds to the wave–type flow $e^{it\sqrt{L}}$. Arguing similarly to Theorem 3.7 we obtain:

Theorem 3.8 *Let L satisfy* (A1) *and* (A2).

(i) *For $s > \frac{n-1}{2}$, we have*

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|(I + L)^{s/2} f\|_{L^1}, \quad |t| < T_0.$$

(ii) *For $p \in (0, 1)$ and $s = n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$, we have*

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H_L^p}, \quad |t| < T_0.$$

Note that the decay $|t|^{-\frac{n-1}{2}}$ in Theorem 3.8 is sharp, as shown by the well known dispersive estimates for the free wave equation corresponding to the choice $L = -\Delta$. For $L = -\Delta + V$ with small potentials $V \in \mathscr{S}(\mathbb{R}^n)$, $n \geq 3$, it was proved in [5] (see also [30]) that

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|(I-\Delta)^{s/2} f\|_{H^1}, \ s = \frac{n-1}{2}. \tag{35}$$

We see that estimate (ii) in Theorem 3.8 is new for $0 < p < 1$ even when $L = -\Delta$.

We also mention that for Schrödinger operators $L = -\Delta + V$ our assumptions (A1), (A2) are known to hold for several classes of potentials $V(x)$ on $\mathbb{R}^n$, both locally and globally in time. Concerning the global in time case, heat kernel estimates (A2) hold in any dimension under very mild assumptions on V, e.g. for Kato class potentials (see e.g. [14,36]). Dispersive estimates for the Schrödinger propagator of the form (A1) are known for rather general classes of potentials (see [1,13,28,41,42]); sharp results are known only for dimension $n = 1$ [13] and $n = 3$ [4].

Note that Theorems 3.7 and 3.8 give dispersive estimates corresponding to $s > n(1 - \nu) + \nu$. In order to prove estimates for the critical case $s = n(1 - \nu) + \nu$, we need a new version of homogeneous Besov spaces. We conclude this section by proving a version of dispersive estimates for initial data in Besov spaces generated by L. Fix a Littlewood–Paley dyadic partition of unity $\Psi = \{\psi_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}$, and define for all $s \in \mathbb{R}$, $1 \leq p, q < \infty$ the *Besov space* $\dot{B}^{s,L}_{p,q}(X)$ as the completion of the set

$$\left\{ f \in L^2(X) : \|f\|_{\dot{B}^{s,L}_{p,q}} < \infty \right\}$$

for the norm $\| \cdot \|_{B^{s,L}_{p,q}}$ given by

$$\|f\|_{\dot{B}^{s,L}_{p,q}} := \left\{ \sum_{j\in\mathbb{Z}} \left(2^{js} \|\psi_j(\sqrt{L})f\|_{L^p} \right)^q \right\}^{1/q}.$$

We note that this definition is independent of the choice of Ψ. Indeed, suppose that $\Phi = \{\varphi_k\}_{k\in\mathbb{Z}}$ is another dyadic partition of unity. Then for $j \in \mathbb{Z}$ and $f \in L^2$ we have

$$\psi_j(\sqrt{L})f = \sum_{k=j-2}^{j+2} \psi_j(\sqrt{L})\varphi_k(\sqrt{L})f.$$

By (8) we have

$$2^{js}\|\psi_j(\sqrt{L})f\|_{L^p} \lesssim \sum_{k=j-2}^{j+2} 2^{js}\|\varphi_k(\sqrt{L})f\|_{L^p} \sim \sum_{k=j-2}^{j+2} 2^{ks}\|\varphi_k(\sqrt{L})f\|_{L^p} \tag{36}$$

and this implies, as claimed,

$$\left\{\sum_{j\in\mathbb{Z}}\left(2^{js}\|\psi_j(\sqrt{L})f\|_{L^p}\right)^q\right\}^{1/q} \sim \left\{\sum_{j\in\mathbb{Z}}\left(2^{js}\|\varphi_j(\sqrt{L})f\|_{L^p}\right)^q\right\}^{1/q}.$$

We can now prove our dispersive estimate for Besov initial data. Note that for wave type flows $e^{it\sqrt{L}}$ we recover the sharp derivative loss $s = \frac{n-1}{2}$ which is well know for the wave equation:

Theorem 3.9 *Let L satisfy (A1) and (A2), and let $v \in (0, 1)$. Then we have*

$$\left\|e^{itL^v}f\right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}}\|f\|_{\dot{B}^{(1-v)n+v,L}_{1,1}}. \tag{37}$$

In the particular case $v = \frac{1}{2}$ we get

$$\left\|\frac{e^{it\sqrt{L}}}{\sqrt{L}}f\right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}}\|f\|_{\dot{B}^{\frac{n-1}{2},L}_{1,1}}. \tag{38}$$

Proof Let $\Psi = \{\psi_j\}$ be a Littlewood–Paley partition of unity. Then we have

$$\psi_j(\sqrt{L}) = \sum_{k=j-2}^{j+2}\psi_j(\sqrt{L})\psi_k(\sqrt{L})$$

so that

$$\psi_j(\sqrt{L})e^{itL^v}f = \sum_{k=j-2}^{j+2}\psi_j(\sqrt{L})\psi_k(\sqrt{L})e^{itL^v}f.$$

This implies

$$\|\psi_j(\sqrt{L})e^{itL^v}f\|_{L^\infty}$$
$$= \sum_{k=j-2}^{j+2}\|\psi_k(\sqrt{L})e^{itL^v}\psi_j(\sqrt{L})f\|_{L^\infty}$$
$$\lesssim \sum_{k=j-2}^{j+2}|t|^{-\frac{n-1}{2}}2^{k((1-v)n+v)}\|\psi_j(\sqrt{L})f\|_{L^1} \sim |t|^{-\frac{n-1}{2}}2^{j((1-v)n+v)}\|\psi_j(\sqrt{L})f\|_{L^1}$$

where we used (25) in the second inequality. As a consequence,

$$
\begin{aligned}
\|e^{itL^v} f\|_{L^\infty} &\leq \sum_{j\in\mathbb{Z}} \|\psi_j(\sqrt{L})e^{itL^v} f\|_{L^\infty} \\
&\lesssim \sum_{j\in\mathbb{Z}} |t|^{-\frac{n-1}{2}} 2^{j((1-v)n+v)} \|\psi_j(\sqrt{L})f\|_{L^1} \\
&\sim |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^{(1-v)n+v,L}}
\end{aligned}
$$

which proves (37).

The proof of (38) can be done in the same manner, hence we omit details. This completes our proof.

$\square$

4 Some applications

In this section we will apply the obtained results in Sect. 3 to study dispersive estimates for Hermite operators, twisted Laplacians and Laguerre operators. It is worth noticing that the list of applications is not exhaustive, since we just intend to show the generality of our theory. Apart from these applications, one can find more applications in other setting such as Schrödinger operators with smooth potentials.

The estimates in Theorems 3.7, 3.8 and 3.9 hold true for all operators below. Hence, we will not list all of them, but concentrate on some specific estimates.

4.1 Hermite operators

Let $L = -\Delta + |x|^2$ be the Hermite operator on $\mathbb{R}^n$ with $n \geq 1$. Let $p_t(x, y)$ denote the kernel of the semigroup e^{-tL}. It is clear that $p_t(x, y)$ enjoys the Gaussian upper bound (A2), and we have an explicit representation for the kernel $p_t(x, y)$:

$$
p_t(x, y) = \frac{1}{\pi^{n/2}} \left(\frac{e^{-2t}}{1-e^{-4t}}\right)^{n/2} \exp\left(-\frac{1}{4}\frac{1+e^{-2t}}{1-e^{-2t}}|x-y|^2 - \frac{1}{4}\frac{1-e^{-2t}}{1+e^{-2t}}|x+y|^2\right)
$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. See for example [39].

It is well-known that for any $\delta > 0$ there exists $C > 0$ so that

$$
\|e^{itL}\|_{L^1 \to L^\infty} \leq \frac{C}{t^{n/2}}, \quad |t| < \pi/2 - \delta.
$$

Therefore, the Hermite operator L satisfies the conditions (A1) and (A2).

As a consequence of Theorems 3.7 and 3.8, we have

Proposition 4.1 *Let $L = -\Delta + |x|^2$ be the Hermite operator on $\mathbb{R}^n$ with $n \geq 1$. Then we have*

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H_L^p}, \quad |t| < \pi/2 - \delta$$

for $p \in (0, 1)$ and $s = n(1/p - \nu) + \nu$; and

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H_L^p}, \ |t| < \pi/2 - \delta$$

for $p \in (0, 1)$ and $s = n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$.

As mentioned in Sect. 2.2 that $H^p(\mathbb{R}^n) \subsetneq H_L^p(\mathbb{R}^n)$, hence the estimate above is sharper than the following estimate:

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H^p}, \quad |t| < \pi/2 - \delta$$

for$p \in (0, 1)$and$s = n(1/p - \nu) + \nu$; and

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H^p}, \ |t| < \pi/2 - \delta$$

for $p \in (0, 1)$ and $s = n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$.

Proposition 4.2 *For each $0 < s < 2$ we have $\dot{B}_{1,1}^{s,L}(\mathbb{R}^n) \hookrightarrow \dot{B}_{1,1}^s(\mathbb{R}^n)$.*

Proof Let $\Psi = \{\psi_j\}$ be a partition of unity, $\psi_j(s) = \psi(2^{-j}s)$. Then we have

$$\psi(2^{-j}\sqrt{-\Delta})f = \sum_{k \in \mathbb{Z}} \psi(2^{-j}\sqrt{-\Delta})\psi(2^{-k}\sqrt{L})f$$

so that

$$\sum_{j \in \mathbb{Z}} 2^{js} \psi(2^{-j}\sqrt{-\Delta})f = \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k \in \mathbb{Z}} \psi(2^{-j}\sqrt{-\Delta})\psi(2^{-k}\sqrt{L})f.$$

It follows that

$$\sum_{j \in \mathbb{Z}} 2^{js} \|\psi(2^{-j}\sqrt{-\Delta})f\|_{L^1} \leq \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k \in \mathbb{Z}} \|\psi(2^{-j}\sqrt{-\Delta})\psi(2^{-k}\sqrt{L})f\|_{L^1}$$

$$\leq \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k \geq j} \|\psi(2^{-j}\sqrt{-\Delta})\psi(2^{-k}\sqrt{L})f\|_{L^1}$$

$$+ \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k < j} \|\psi(2^{-j}\sqrt{-\Delta})\psi(2^{-k}\sqrt{L})f\|_{L^1}$$

$$=: I_1 + I_2.$$

By using (8) we have

$$I_1 \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j} 2^{(j-k)s} 2^{ks} \|\psi(2^{-k}\sqrt{L})f\|_{L^1}$$

$$\lesssim \sum_{k \in \mathbb{Z}} 2^{ks} \|\psi(2^{-k}\sqrt{L})f\|_{L^1} = \|f\|_{\dot{B}^{s,L}_{1,1}}.$$

To estimate the term I_2, by (8) we have

$$I_2 = \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k < j} \|(-\Delta)^{-1}\psi(2^{-j}\sqrt{-\Delta})(-\Delta)L^{-1}L\psi(2^{-k}\sqrt{L})f\|_{L^1}$$

$$\lesssim \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k < j} 2^{-2j} \|(-\Delta)L^{-1}L\psi(2^{-k}\sqrt{L})f\|_{L^1}$$

$$\lesssim \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k < j} 2^{-2j} \|L\psi(2^{-k}\sqrt{L})f\|_{L^1}$$

$$+ \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k < j} 2^{-2j} \||x|^2 L^{-1}L\psi(2^{-k}\sqrt{L})f\|_{L^1}$$

$$\lesssim \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k < j} 2^{-2(j-k)} \|\widetilde{\psi}(2^{-k}\sqrt{L})f\|_{L^1}$$

$$+ \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k < j} 2^{-2(j-k)} \||x|^2 L^{-1}\widetilde{\psi}(2^{-k}\sqrt{L})f\|_{L^1}$$

where $\widetilde{\psi}(x) = x^2 \psi(x)$.

It was proved in [2] that $|x|^2 L^{-1}$ is bounded on L^1. Hence,

$$I_2 \lesssim \sum_{j \in \mathbb{Z}} 2^{js} \sum_{k < j} 2^{-2(j-k)} \|\widetilde{\psi}(2^{-k}\sqrt{L})f\|_{L^1}$$

$$= \sum_{j \in \mathbb{Z}} \sum_{k < j} 2^{-(2-s)(j-k)} 2^{ks} \|\widetilde{\psi}(2^{-k}\sqrt{L})f\|_{L^1}$$

$$\lesssim \sum_{k \in \mathbb{Z}} 2^{ks} \|\widetilde{\psi}(2^{-k}\sqrt{L})f\|_{L^1}.$$

Arguing similarly to the proof of (36) we have

$$\sum_{k \in \mathbb{Z}} 2^{ks} \|\widetilde{\psi}(2^{-k}\sqrt{L})f\|_{L^1} \lesssim \|f\|_{\dot{B}^{s,L}_{1,1}}.$$

Hence, $I_2 \lesssim \|f\|_{\dot{B}^{s,L}_{1,1}}$. This completes our proof. $\qquad\square$

Proposition 4.3 *We have $\dot{B}^0_{1,1} \subsetneq \dot{B}^{0,L}_{1,1}$.*

Proof We claim that if $\int_0^\infty \|t^2 L e^{-t^2 L} f\|_{L^1} t^{-1} dt < \infty$, then $f \in \dot{B}_{1,1}^{0,L}$ and

$$\|f\|_{\dot{B}_{1,1}^{0,L}} \lesssim \int_0^\infty \|t^2 L e^{-t^2 L} f\|_{L^1} \frac{dt}{t}. \tag{39}$$

Indeed, let $\psi_j(s) = \psi(2^{-j} s)$ be a partition of unity. By functional calculus we have for $j \in \mathbb{Z}$

$$\psi(2^{-j} \sqrt{L}) f = c \int_0^\infty (t^2 L)^2 e^{-t^2 L} \psi(2^{-j} \sqrt{L}) f \frac{dt}{t}$$

where $c = \left[\int_0^\infty z^2 e^{-z} \frac{dz}{z} \right]^{-1}$. Hence

$$\sum_{j \in \mathbb{Z}} \|\psi(2^{-j} \sqrt{L}) f\| \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \int_{2^{-k-1}}^{2^{-k}} \|(t^2 L)^2 e^{-t^2 L} \psi(2^{-j} \sqrt{L}) f\|_{L^1} \frac{dt}{t}$$

$$\lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j} \int_{2^{-k-1}}^{2^{-k}} \|(t^2 L)^2 e^{-t^2 L} \psi(2^{-j} \sqrt{L}) f\|_{L^1} \frac{dt}{t} \tag{40}$$

$$+ \sum_{j \in \mathbb{Z}} \sum_{k < j} \int_{2^{-k-1}}^{2^{-k}} \|(t^2 L)^2 e^{-t^2 L} \psi(2^{-j} \sqrt{L}) f\|_{L^1} \frac{dt}{t}$$

$$=: I_1 + I_2.$$

For the term I_1 we have

$$I_1 \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j} \int_{2^{-k-1}}^{2^{-k}} t^2 2^{2j} \|t^2 L e^{-t^2 L} \tilde{\psi}(2^{-j} \sqrt{L}) f\|_{L^1} \frac{dt}{t}$$

$$\lesssim \sum_{j \in \mathbb{Z}} \sum_{k \geq j} \int_{2^{-k-1}}^{2^{-k}} 2^{2(j-k)} \|t^2 L e^{-t^2 L} \tilde{\psi}(2^{-j} \sqrt{L}) f\|_{L^1} \frac{dt}{t}$$

$$\lesssim \sum_{k \in \mathbb{Z}} \int_{2^{-k-1}}^{2^{-k}} \|t^2 L e^{-t^2 L} f\|_{L^1} \frac{dt}{t}$$

where $\tilde{\psi}(x) = x^2 \psi(x)$. Here we have applied (8). Thus

$$I_1 \lesssim \int_0^\infty \|t^2 L e^{-t^2 L} f\|_{L^1} \frac{dt}{t}. \tag{41}$$

Similarly, for $\overline{\psi}(x) = x^{-2}\psi(x)$, we have

$$I_2 \lesssim \sum_{j\in\mathbb{Z}} \sum_{k<j} \int_{2^{-k-1}}^{2^{-k}} t^{-2} 2^{-2j} \|(t^2 L)^3 e^{-t^2 L} \overline{\psi}(2^{-j}\sqrt{L}) f\|_{L^1} \frac{dt}{t}$$

$$\lesssim \sum_{k\in\mathbb{Z}} \int_{2^{-k-1}}^{2^{-k}} \|(t^2 L)^3 e^{-t^2 L} f\|_{L^1} \frac{dt}{t}.$$

We can write

$$(t^2 L)^3 e^{-t^2 L} f = 8 \left(\frac{t^2}{2} L\right)^2 e^{-\frac{t^2}{2} L} \left[\frac{t^2}{2} L e^{-\frac{t^2}{2} L} f\right].$$

Since the kernel of $\left(\frac{t^2}{2} L\right)^2 e^{-\frac{t^2}{2} L}$ satisfies the Gaussian upper bound (see for example [12]), we have

$$\|(t^2 L)^3 e^{-t^2 L} f\|_{L^1} \lesssim \left\|\frac{t^2}{2} L e^{-\frac{t^2}{2} L} f\right\|_{L^1}.$$

As a consequence,

$$I_2 \lesssim \sum_{k\in\mathbb{Z}} \int_{2^{-k-1}}^{2^{-k}} \left\|\frac{t^2}{2} L e^{-\frac{t^2}{2} L} f\right\|_{L^1} \frac{dt}{t} \lesssim \int_0^\infty \|t^2 L e^{-t^2 L} f\|_{L^1} \frac{dt}{t}.$$

This, along with (41) and (40), implies

$$\sum_{j\in\mathbb{Z}} \|\psi(2^{-j}\sqrt{L})\| \lesssim \int_0^\infty \|t^2 L e^{-t^2 L} f\|_{L^1} \frac{dt}{t}$$

which proves the claim (39).

We now turn to prove $\dot{B}_{1,1}^0 \subsetneq \dot{B}_{1,1}^{0,L}$. Indeed, if $f \in \dot{B}_{1,1}^0$, it was proved in [8] that

$$\int_0^\infty \|t^2 L e^{-t^2 L} f\|_{L^1} \frac{dt}{t} \lesssim \|f\|_{\dot{B}_{1,1}^0}.$$

Therefore, applying (39) we have $f \in \dot{B}_{1,1}^{0,L}$ and

$$\|f\|_{\dot{B}_{1,1}^{0,L}} \lesssim \int_0^\infty \|t^2 L e^{-t^2 L} f\|_{L^1} \frac{dt}{t} \lesssim \|f\|_{\dot{B}_{1,1}^0}.$$

This completes the proof. □

We would like to illustrate the advantage of dispersive estimates in Theorem 3.9 and of the new Besov spaces. In our situation $L = -\Delta + |x|^2$, if $n = 1$, estimate (38) becomes

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^{0,L}}. \tag{42}$$

In this situation, the estimate (42) is sharper than the following estimate

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^0},$$

since by Proposition 4.3, $\dot{B}_{1,1}^0 \subset \dot{B}_{1,1}^{0,L}$.

Although in the general case $n \geq 2$ it is not clear whether the following estimate holds true

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^{\frac{n-1}{2}}},$$

for the particular case $n = 2, 3, 4$ we can find the spaces $\dot{B}_{1,1}^{\frac{n-1}{2},L} \subset \dot{B}_{1,1}^{\frac{n-1}{2}}$ (see Proposition 4.2) such that

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^{\frac{n-1}{2},L}}.$$

We do not know at present if there exists $f \in \dot{B}_{1,1}^{\frac{n-1}{2}}$ such that $\frac{e^{it\sqrt{L}}}{\sqrt{L}} f \notin L^\infty$.

4.2 Twisted Laplacians

Consider the twisted Laplacian on $\mathbb{R}^n$ with $n = 2d, d \in \mathbb{N}$:

$$L = -\frac{1}{2} \sum_{j=1}^d \left[(\partial_{x_j} - iy_j)^2 - (\partial_{y_j} + ix_j)^2 \right].$$

It is well-known that the kernel $p_t(x, y)$ of e^{-tL} admits a Gaussian upper bound

$$|p_t(x, y)| \leq \frac{C}{t^d} \exp\left(-\frac{|x - y|^2}{ct} \right).$$

See [32].

Moreover, by continuation of the corresponding heat kernel in [32] we obtain

$$\|e^{itL}\|_{L^1 \to L^\infty} \lesssim t^{-d}, \quad |t| < \pi/2 - \delta.$$

Therefore, the twisted Laplacian L satisfies (A1) and (A2). As a consequence, all of the estimates in Theorems 3.7, 3.8 and 3.9 hold true for the twisted Laplacian.

4.3 Laguerre operators

Consider the space $X = (0, \infty)^n$ equipped with the Euclidean distance d and measure μ given by $d\mu(x) = d\mu_1(x) \dots d\mu_n(x)$ where $d\mu_k = x_k^{2\alpha_k+1} dx_k, \alpha_k > -1$ for $k = 1, \dots, n$.

It is easy to see that

$$\mu(B(x, r)) \sim \prod_{k=1}^{n} (r + x_k)^{2\alpha_k+1} r \tag{43}$$

where $B(x, r) = \{y \in X : |x - y| < r\}$ is the ball centered in $x = (x_1, x_2, \dots, x_n)$ with radius r. It follows that the measure μ satisfies the doubling condition (1). Moreover, if $\alpha_k > -1/2$ for all k, then we have

$$\mu(B(x, r)) \gtrsim r^N, \quad N = 2n + \sum_{k=1}^{n} 2\alpha_k \geq 1$$

for all $x \in X$ and $r > 0$.

In this section, we always assume that $\alpha_k > -1/2$ for all k.

For $m \in \mathbb{N}$ and $\alpha_k > -1/2, k = 1, 2, \dots, n$, consider the Laguerre functions ψ_k^α which are defined by

$$\psi_m^{\alpha_k}(x) = \left(\frac{2m!}{\Gamma(m + \alpha_k + 1)} \right)^{1/2} L_m^{\alpha_k}(x^2) e^{-x^2/2}, x \in \mathbb{R}, \tag{44}$$

where $L_m^{\alpha_k}$ are the m-th Laguerre polynomials. See for example [29].

We set $\psi_m^\alpha(x) = \prod_{k=1}^{n} \psi_m^{\alpha_k}(x_k)$ for each $k = 1, 2, \dots, n$ and $x = (x_1, x_2, \dots, x_n)$. It is well known that the Laguerre functions form an orthonormal basis for $L^2(X, d\mu)$. We now consider the Laguerre operator L, defined by

$$L = -\Delta - \sum_{k=1}^{n} \frac{2\alpha_k + 1}{x_k} \frac{d}{dx_k} + |x|^2. \tag{45}$$

It is well known that

$$L\psi_m^\alpha = \lambda_{m,\alpha} \psi_m^\alpha, \quad \lambda_{m,\alpha} = (4m + 2)n + \sum_{k=1}^{n} 2\alpha_k.$$

Then the operator L has a non-negative self-adjoint extension which is still denoted by L with domain

$$D(L) = \{f \in L^2(X, d\mu) : \sum_m \lambda_{m,\alpha}^2 |\langle f, \psi_m^\alpha \rangle_{d\mu}|^2 < \infty\}.$$

Moreover, the heat kernel $p_t(x, y)$ associated to the semigroup e^{-tL} is given by

$$p_t(x, y) = \prod_{k=1}^n \frac{2e^{-2t}}{1 - e^{-4t}}$$
$$\times \exp\left(-\frac{1}{2}\frac{1 + e^{-4t}}{1 - e^{-4t}}(x_k^2 + y_k^2)\right)(x_k y_k)^{-\alpha_k} I_{\alpha_k}\left(\frac{2e^{-2t}}{1 - e^{-4t}}x_k y_k\right),$$
(46)

for all $t > 0$, $x, y \in X$ and I_{α_k} being the Bessel function. See for example [29].

Theorem 4.4 *Let $\alpha_k > -1/2$ for all $k = 1, \ldots, n$ and let L be the Laguerre operator defined by (45). Then we have*

$$0 < p_t(x, y) \lesssim \frac{C}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{|x - y|^2}{ct}\right)$$
(47)

for all $x, y \in X$ and $t > 0$.
  Moreover,

$$\|e^{itL}\|_{L^1 \to L^\infty} \lesssim t^{-N/2}, \quad |t| < \pi/2 - \delta$$
(48)

where $N = 2n + \sum_{k=1}^n 2\alpha_k$.
  Hence, the Laguerre operator satisfies (A1) and (A2).

Proof We refer to [6, Lemma 3.1] for the proof of the Gaussian upper bounds (46) and [37, Lemma 3.5] for the proof of the dispersive estimate (48). □

Definition 4.5 Let $p \in (\frac{N}{N+1}, 1]$. A function a is called a p-atom associated to the ball B if

  (i) $\operatorname{supp} a \subset B$
  (ii) $\|a\|_{L^\infty(X)} \le \mu(B)^{-1/p}$
 (iii) $\int a(x)\, d\mu(x) = 0$.

  To define the Hardy space H_{CW}^p for p below 1, we need to introduce the Lipschitz spaces $\mathfrak{L}_\alpha$. We say that the function $f \in \mathfrak{L}_\alpha$ if there exists a constant $c > 0$, such that

$$|f(x) - f(y)| \le c|B|^\alpha$$

for all ball B and $x, y \in B$, and the best constant c can be taken to be the norm of f and is denoted by $\|f\|_{\mathfrak{L}_\alpha}$.

Definition 4.6 (*Hardy spaces of Coifman and Weiss*) Let $\frac{N}{N+1} < p \leq 1$. We say that a function $f \in H_{CW}^p(X)$ if $f \in L^1(X)$ for $p = 1$, or $f \in \mathfrak{L}_{1/p-1}^*$ for $p < 1$, and there exists a sequence $(\lambda_j)_{j\in\mathbb{N}} \in \ell^p$ and a sequence of p-atoms $(a_j)_{j\in\mathbb{N}}$ such that $f = \sum_j \lambda_j a_j$ in $L^1(X)$ for $p = 1$, and $f = \sum_j \lambda_j a_j$ in $\mathfrak{L}_{1/p-1}^*$ for $p < 1$. We set

$$\|f\|_{H_{CW}^p} = \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{1/p} : f = \sum_j \lambda_j a_j\right\}.$$

From Theorems 3.7 and 3.8, we have

Proposition 4.7 *Let* $\alpha_k > -1/2$ *for all* $k = 1, \ldots, n$ *and let* L *be the Laguerre operator defined by* (45). *Then we have*

$$\|e^{itL^\nu}f\|_{L^\infty} \lesssim |t|^{-\frac{N-1}{2}} \|L^{s/2}f\|_{H_L^p}, \quad |t| < \pi/2 - \delta$$

for $p \in (0, 1)$ *and* $s = N(1/p - \nu) + \nu$; *and*

$$\left\|\frac{e^{it\sqrt{L}}}{\sqrt{L}}f\right\|_{L^\infty} \lesssim |t|^{-\frac{N-1}{2}}\|L^{s/2}f\|_{H_L^p}, |t| < \pi/2 - \delta$$

for $p \in (0, 1)$ *and* $s = N(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$.

From [7, Theorem 6.6] we have $H^p(\mathbb{R}^n) \subsetneqq H_L^p(\mathbb{R}^n)$ for $\frac{N}{N+1} < p \leq 1$. Hence the estimates above are sharper than the following estimates:

$$\|e^{itL^\nu}f\|_{L^\infty} \lesssim |t|^{-\frac{N-1}{2}}\|L^{s/2}f\|_{H_{CW}^p}, \quad |t| < \pi/2 - \delta$$

for $p \in (\frac{N}{N+1}, 1)$ and $s = N(1/p - \nu) + \nu$; and

$$\left\|\frac{e^{it\sqrt{L}}}{\sqrt{L}}f\right\|_{L^\infty} \lesssim |t|^{-\frac{N-1}{2}}\|L^{s/2}f\|_{H_{CW}^p}, \quad |t| < \pi/2 - \delta$$

for $p \in (\frac{N}{N+1}, 1)$ and $s = N(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$.

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