

ON LARGE POTENTIAL PERTURBATIONS OF THE SCHRÖDINGER, WAVE AND KLEIN–GORDON EQUATIONS

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ABSTRACT. We prove a sharp resolvent estimate in scale invariant norms of Amgon–Hörmander type for a magnetic Schrödinger operator on \mathbb{R}^n , $n \geq 3$

$$L = -(\partial + iA)^2 + V$$

with large potentials A, V of almost critical decay and regularity.

The estimate is applied to prove sharp smoothing and Strichartz estimates for the Schrödinger, wave and Klein–Gordon flows associated to L .

1. Introduction. We consider a selfadjoint Schrödinger operator in $L^2(\mathbb{R}^n)$, $n \geq 3$, of the form

$$L = -(\partial + iA)^2 + V \tag{1.1}$$

where $A = (A_1, \dots, A_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the magnetic potential and $V : \mathbb{R}^n \rightarrow \mathbb{R}$ the electric potential. In order to allow a unified treatment of the dispersive equations corresponding to L , we shall always assume $L \geq 0$, although this assumption can be relaxed. We are interested in the dispersive properties for solutions of the equations

$$i\partial_t u + Lu = 0, \quad \partial_t^2 u + Lu = 0, \quad \partial_t^2 u + (L + 1)u = 0, \tag{1.2}$$

associated to the operator L .

The critical behaviour for dispersion appears to be $|A| \lesssim |x|^{-1}$, $|V| \lesssim |x|^{-2}$, and one of our goals is to get as close as possible to this kind of singularity. All the results of the paper are valid under the following assumption (note however that weaker conditions are required in the course of the paper):

Assumption (L). Let $n \geq 3$. The operator L in (1.1) is selfadjoint in $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$, non negative, 0 is not a resonance for L , and writing $w(x) = \langle \log |x| \rangle^\mu \langle x \rangle^\delta$ for some $\delta > 0$, $\mu > 1$,

$$w(x)|x|^2(V - i\partial \cdot A) \in L^\infty, \quad w(x)|x|\widehat{B} \in L^\infty, \quad w(x)|x|A \in L^\infty \cap \dot{H}_{2n}^{1/2} \tag{1.3}$$

where $\widehat{B}_k := \sum_{j=1}^n \frac{x_j}{|x|} (\partial_j A_k - \partial_k A_j)$ is the tangential component of the magnetic field.

It is well known that a resonance at 0 is an obstruction to dispersion. The precise notion required here is the following:

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Definition 1.1 (Resonance). We say that 0 is a *resonance* for the operator L if there exists a nonzero $v \in H^2_{loc}(\mathbb{R}^n \setminus 0) \cap H^1_{loc}$ solution of $Lv = 0$ with the properties

$$|x|^{\frac{n}{2}-2-\sigma}v \in L^2 \quad \text{and} \quad |x|^{\frac{n}{2}-1-\sigma}\partial v \in L^2 \quad \forall \sigma \in (0, \sigma_0). \tag{1.4}$$

for some $\sigma_0 > 0$. The function v is then called a *resonant state* at 0 for L . (If $n \geq 5$ this condition reduces to 0 being an eigenvalue of L).

A standard approach to the problem is based on a uniform estimate for the resolvent operator of L . This approach has a long tradition, starting from the classical theories of Kato, Kato–Kuroda and Agmon. The bulk of the paper (Sections 2–4) is devoted to prove the following estimate, which is sharp even for $L = -\Delta$:

Theorem 1.2 (Resolvent estimate). *Suppose Assumption (L) is verified. Then the resolvent operator $R(z) = (L - z)^{-1}$ satisfies the estimate*

$$\|R(z)f\|_{\dot{X}} + |z|^{\frac{1}{2}}\|R(z)f\|_{\dot{Y}} + \|\partial R(z)f\|_{\dot{Y}} \lesssim \|f\|_{\dot{Y}^*} \tag{1.5}$$

with a constant uniform in z in the complex strip $|\Im z| \leq 1$. In particular, the boundary values of $R(\lambda \pm i\epsilon)$ as $\epsilon \downarrow 0$ are well defined bounded operators from \dot{Y}^* to \dot{X} (or to \dot{Y} provided $\lambda \neq 0$).

See Theorem 4.1 and Corollary 4.6 below. Here the spaces \dot{X}, \dot{Y} have norms

$$\|v\|_{\dot{X}}^2 := \sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |v|^2 dS \quad \|v\|_{\dot{Y}}^2 := \sup_{R>0} \frac{1}{R} \int_{|x|\leq R} |v|^2 dx$$

while \dot{Y}^* is the (pre)dual of \dot{Y} ; note that \dot{Y}^* is an homogeneous version of the Agmon–Hörmander space B [2]. The last property in the statement is also called the *limiting absorption principle* for L . We think that an interesting contribution of the present paper is a conceptually simple proof of (1.5), based on a combination of the multiplier method (for large frequencies) and Fredholm theory (for small frequencies).

With (1.5) at our disposal, the classical Kato’s theory of smoothing operators gives with little effort several smoothing estimates (also known as *local energy decay*) for the Schrödinger flow e^{itL} . Kato’s theory was extended in [11] to include the wave and Klein–Gordon equations. By combining these techniques, we obtain the following scaling invariant estimates:

Theorem 1.3 (Smoothing estimates). *Under Assumption (L), we have:*

$$\begin{aligned} & \| |x|^{-1/2} e^{itL} f \|_{\dot{Y}L_t^2} + \| |D|^{1/2} e^{itL} f \|_{\dot{Y}L_t^2} \leq C \|f\|_{L^2}, \\ & \| |x|^{-1/2} e^{it\sqrt{L}} f \|_{\dot{Y}L_t^2} + \| |D|^{1/2} e^{it\sqrt{L}} f \|_{\dot{Y}L_t^2} \leq C \|f\|_{\dot{H}^{1/2}}, \\ & \| |x|^{-1/2} e^{it\sqrt{L+1}} f \|_{\dot{Y}L_t^2} + \| |D|^{1/2} e^{it\sqrt{L+1}} f \|_{\dot{Y}L_t^2} \leq C \|f\|_{\dot{H}^{1/2}}. \end{aligned}$$

In the $\dot{Y}L_t^2$ norm the order of integration is reversed, but one can easily write these estimates in a more standard (and actually equivalent) form in terms of L^2 weighted norms. Indeed, if ρ is any function such that $\sum_{j \in \mathbb{Z}} \|\rho\|_{L^\infty(|x| \sim 2^j)}^2 < \infty$, we have $\|\rho|x|^{-1/2}v\|_{L^2} \lesssim \|v\|_{\dot{Y}}$, hence the smoothing estimates for Schrödinger can be written

$$\|\rho|x|^{-1}e^{itL}f\|_{L_t^2L^2} + \|\rho|x|^{-\frac{1}{2}}|D|^{\frac{1}{2}}e^{itL}f\|_{L_t^2L^2} \lesssim \|f\|_{L^2}$$

and similarly for the wave and Klein–Gordon equation. A typical example of such a weight is $\rho = \langle \log |x| \rangle^{-\nu}$ for $\nu > 1/2$.

These smoothing estimates, together with the corresponding inhomogeneous ones, are proved in Section 5 and in particular Corollary 5.6, 5.8 and 5.9. Note that if we are in the *Coulomb gauge* $\partial \cdot A = 0$, the last condition in (1.3) is not necessary both for the smoothing estimates and the uniform resolvent estimate (1.5).

As a final application, in Section 6 we prove the full set of Strichartz estimates for the three dispersive equations (1.2). We recall the basic facts for the unperturbed case in dimension $n \geq 3$:

- A couple (p, q) is *Schrödinger admissible* if

$$p \in [2, \infty], \quad q \in [2, \frac{2n}{n-2}], \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2},$$

and *wave admissible* if

$$p \in [2, \infty], \quad q \in [2, \frac{2(n-1)}{n-3}], \quad \frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \quad q \neq \infty.$$

- The *homogeneous Strichartz estimates* are

$$\begin{aligned} \|e^{-it\Delta} f\|_{L_t^p L^q} &\lesssim \|f\|_{L^2}, \quad (p, q) \text{ Schrödinger admissible,} \\ \| |D|^{\frac{1}{q} - \frac{1}{p}} e^{-it|D|} f\|_{L_t^p L^q} &\lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}, \quad (p, q) \text{ wave admissible,} \\ \| \langle D \rangle^{\frac{1}{q} - \frac{1}{p}} e^{it\langle D \rangle} f\|_{L_t^p L^q} &\lesssim \|f\|_{H^{\frac{1}{2}}}, \quad (p, q) \text{ Schrödinger or wave admissible.} \end{aligned}$$

Corresponding inhomogeneous versions of the estimates are also true.

- The previous estimates can be refined using Lorentz norms. At the (*Schrödinger*) *endpoint* $(2, \frac{2n}{n-2})$ one gets

$$\|e^{-it\Delta} f\|_{L_t^2 L^{\frac{2n}{n-2}, 2}} \lesssim \|f\|_{L^2} \tag{1.6}$$

and from this case all the other estimates can be recovered, by interpolating with the conservation of L^2 mass; actually, by real interpolation one obtains estimates in the $L_t^p L^{q, 2}$ norm for every admissible couple (p, q) . A similar situation occurs at the (*wave*) *endpoint* $(2, \frac{2(n-1)}{n-3})$, in dimension $n \geq 4$.

Then in Section 6 we prove:

Theorem 1.4 (Strichartz estimates). *Suppose Assumption (L) is verified. Then we have the estimates*

$$\|e^{itL} f\|_{L_t^2 L^{\frac{2n}{n-2}, 2}} \lesssim \|f\|_{L^2}$$

and hence the full set of $L_t^p L^q$ estimates, for all Schrödinger admissible (p, q) ; moreover, for all non endpoint, wave admissible couple (p, q) we have

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} e^{-it\sqrt{L}} f\|_{L_t^p L^q} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}$$

and for all non endpoint, wave or Schrödinger admissible couple (p, q) we have

$$\| \langle D \rangle^{\frac{1}{q} - \frac{1}{p}} e^{it\sqrt{L+1}} f\|_{L_t^p L^q} \lesssim \|f\|_{H^{\frac{1}{2}}}.$$

These estimates and their nonhomogeneous versions are proved in Theorems 6.3, 6.5 and 6.6 in Section 6. Note that we prove the endpoint estimate for the Schrödinger equation, using a result for the unperturbed Schrödinger flow due to Ionescu and Kenig [24] (which can be refined to Lorentz spaces, as remarked in [34]).

Remark 1.5. In assumption (1.3) the homogeneous Sobolev space $\dot{H}_{2n}^{1/2} = (-\Delta)^{-1/4}(L^{2n})$ is used. The last condition on A may be difficult to check on concrete examples; however, by Sobolev embedding one has

$$w(x)|x|A \in \dot{H}_n^1 \quad \implies \quad w(x)|x|A \in \dot{H}_{2n}^{1/2}$$

so that a stronger, but much easier to check, sufficient assumption is $\partial(w(x)|x|A) \in L^n$.

Remark 1.6. We compare our results with [18], where for the first time smoothing and Strichartz estimates were obtained for Schrödinger equations with large magnetic potentials, in any dimension $n \geq 3$. The assumptions on the coefficients in [18] are

$$|A| + \langle x \rangle |V| \lesssim \langle x \rangle^{-1-\epsilon}, \quad \langle x \rangle^{1+\epsilon'} A \in \dot{H}_{2n}^{1/2}, \quad A \text{ is continuous.}$$

These conditions are largely overlapping with (1.3); we require a stronger condition on \widehat{B} , which is defined as a combination of first derivatives of A , but on the other hand we can consider potentials A, V which are singular at the origin. Other improvements with respect to [18] are

- the endpoint Strichartz estimate for Schrödinger;
- sharp scaling invariant resolvent and smoothing estimates;
- a unified treatment including wave and Klein–Gordon equations.

Last but not least, our proof is ‘elementary’, indeed we use only multiplier methods and Fredholm theory (and standard results from Calderón–Zygmund theory). The only nonelementary result we need is Koch and Tataru’s [31] to exclude embedded eigenvalues for L . One can make the paper self-contained by assuming explicitly that no resonances exist in the spectrum of L . Note that under this additional assumption we can take $\delta = 0$ in (1.3), since the additional decay is used mainly to handle possible embedded resonances (see Lemma 3.3).

Note also that by a gauge transform it is possible to reduce to the case $\partial \cdot A = 0$ i.e. to the *Coulomb gauge*; see Remark 4.3, Corollary 4.6 and Corollary 6.4 for details. However, the quantity \widehat{B} is gauge invariant and the assumption on \widehat{B} can not be removed by a change of gauge.

One disadvantage of the techniques used here is the restriction $n \geq 3$ on the space dimension; the technical reason for this is the failure of the multiplier method in low dimension. The difficulty is not only technical, indeed in $n \leq 2$ even for $L = -\Delta$ the constant functions are resonances at 0. Thus a different approach is needed. This difficulty was overcome in [33] by the use of suitable modified norms which quotient out the constants. Another important advantage of the approach in [33] is that time-dependent coefficients can be handled.

We conclude with a short (and incomplete) summary of earlier results. The case of purely electric potentials $A = 0$ is well understood; the list of papers on this topic is long and here we mention only [26], [10], [7], [16], and the series by Yajima [42], [43], [3] (see also [12]) concerning L^p boundedness of the scattering wave operator. In particular, [38] introduced the strategy of proof used here, based on Kato’s theory (see also [26]).

The case of a *small* magnetic potential A was studied in [20], [39], [19], and in [13] where a comprehensive study was done on the main dispersive equations perturbed with a small magnetic and a large electric potential, including massive

and massless Dirac systems, and [14]. See also [41], [33] where the case of fully variable coefficients is considered.

Smoothing and Strichartz estimates for the Schrödinger equation with a *large* magnetic potential were proved in [18]–[17] (discussed above), and for the wave equation in [11], where the resolvent estimates of [18] were used.

Standard references for Strichartz estimates, at least for the Schrödinger and wave equations, are [21], [22] and [29]. The situation for the Klein–Gordon flow is complicated by the different scaling of $\langle D \rangle$ for small and large frequencies. A complete analysis was made in [32]; a proof for Schrödinger admissible (p, q) can be found in [13], while wave admissible points can be deduced from the precised dispersive estimate of [6].

Remark 1.7. By similar techniques it is possible to prove smoothing and Strichartz estimates also for Dirac systems. This will be part of the joint work [15], concerning the cubic Dirac equation perturbed by a large magnetic potential.

2. The resolvent estimate for large frequencies. We shall make constant use of the dyadic norms

$$\|v\|_{\ell^p L^q} := \left(\sum_{j \in \mathbb{Z}} \|v\|_{L^q(2^j \leq |x| < 2^{j+1})}^p \right)^{1/p}, \tag{2.1}$$

with obvious modification when $p = \infty$. More generally, we denote the mixed radial–angular $L^q L^r$ norms on a spherical ring $C = R_1 \leq |x| \leq R_2$ with

$$\|v\|_{L^q_{|x|} L^r(C)} = \|v\|_{L^q L^r(C)} := \left(\int_{R_1}^{R_2} \left(\int_{|x|=\rho} |v|^r dS \right)^{q/r} d\rho \right)^{1/q},$$

and we define for all $p, q, r \in [1, \infty]$

$$\|v\|_{\ell^p L^q L^r} := \|\{ \|v\|_{L^q L^r(2^j \leq |x| < 2^{j+1})} \}_{j \in \mathbb{Z}}\|_{\ell^p}. \tag{2.2}$$

Clearly, when $q = r$ we have simply $\|v\|_{\ell^p L^q L^q} = \|v\|_{\ell^p L^q}$. With these notations, the Banach norms appearing in (1.5) can be equivalently defined as

$$\|v\|_{\dot{X}}^2 \simeq \| |x|^{-1} v \|_{\ell^\infty L^\infty L^2}^2, \quad \|v\|_{\dot{Y}}^2 \simeq \| |x|^{-1/2} v \|_{\ell^\infty L^2}^2, \quad \|v\|_{\dot{Y}^*} \simeq \| |x|^{1/2} v \|_{\ell^1 L^2}.$$

For large frequencies $|\Re z| \gg 1$, we study the equation

$$\Delta_A v + Wv + iZ \cdot \partial^A v + zv = f \tag{2.3}$$

using a direct approach based on the Morawetz multiplier method. Here $A(x) = (A_1(x), \dots, A_n(x)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $Z(x) = (Z_1(x), \dots, Z_n(x)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $V : \mathbb{R}^n \rightarrow \mathbb{R}$, and we use the notations

$$\hat{x}_j = \frac{x_j}{|x|}, \quad \hat{x} = \frac{x}{|x|}, \quad \partial = (\partial_1, \dots, \partial_n), \quad \partial^A = (\partial_1^A, \dots, \partial_n^A).$$

$$\Delta_A = \sum_{j=1}^n (\partial_j + iA_j(x))^2, \quad \partial_j = \frac{\partial}{\partial x_j}, \quad \partial_j^A = \partial_j + iA_j(x),$$

Recall that, using the convention of implicit summation over repeated indices,

$$B_{jk} = \partial_j A_k - \partial_k A_j, \quad \hat{B}_j = B_{jk} \hat{x}_k, \quad \hat{B} = (\hat{B}_1, \dots, \hat{B}_n).$$

and we call the matrix B the *magnetic field* associated to the potential $A(x)$, and \hat{B} the *tangential part* of the field.

We prove the following result:

Theorem 2.1 (Resolvent estimate for large frequencies). *Let $n \geq 3$. There exists a constant σ_0 depending only on n such that the following holds.*

Assume $v, f : \mathbb{R}^n \rightarrow \mathbb{C}$ satisfy (2.3), W can be split as $W = W_L + W_S$, with $W_L(x), W_S(x), Z_j(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ and

$$\| |x|^{3/2} W_S \|_{\ell^1 L^2 L^\infty} + \| |x| Z \|_{\ell^1 L^\infty} \leq \sigma_0, \quad |\Re z| \geq \sigma_0^{-1} \left[\| |x| \widehat{B} \|_{\ell^1 L^\infty}^2 + \| |x| W_L \|_{\ell^1 L^\infty} \right] + 2. \tag{2.4}$$

Then the following estimate holds for all z as in (2.4) with $|\Im z| \leq 1$

$$\| v \|_{\dot{X}}^2 + (n - 3) \| \frac{v}{|x|^{3/2}} \|_{L^2}^2 + |z| \| v \|_{\dot{Y}}^2 + \| \partial^A v \|_{\dot{Y}}^2 \lesssim \| f \|_{\dot{Y}^*}^2 \tag{2.5}$$

with an implicit constant depending only on n .

Remark 2.2. Under a weak additional assumption on A , the norm $\| \partial^A v \|_{\dot{Y}}$ in (2.5) can be replaced by $\| \partial v \|_{\dot{Y}}$, thanks to the following

Lemma 2.3. *Assume $n \geq 3$ and $A \in \ell^\infty L^n$. Then the following estimate holds*

$$\| \partial v \|_{\dot{Y}} \lesssim (1 + \| A \|_{\ell^\infty L^n}) \left[\| \partial^A v \|_{\dot{Y}} + \| v \|_{\dot{X}} \right]. \tag{2.6}$$

Proof. Let C_j be the spherical shell $2^j \leq |x| \leq 2^{j+1}$ and $\tilde{C}_j = C_{j-1} \cup C_j \cup C_{j+1}$. Let ϕ be a nonnegative cutoff function equal to 1 on C_j and vanishing outside \tilde{C}_j , and let $\phi_j(x) = \phi(2^{-j}x)$. Then we can write

$$\| \partial v \|_{L^2(C_j)} \leq \| \phi_j \partial v \|_{L^2} \leq \| \phi_j \partial^A v \|_{L^2} + \| \phi_j A v \|_{L^2}$$

By Hölder's inequality and Sobolev embedding we have

$$\| \phi_j A v \|_{L^2} \leq \| A \|_{L^n(\tilde{C}_j)} \| \phi_j v \|_{L^2} \lesssim \| A \|_{\ell^\infty L^n} \| \phi_j v \|_{L^{\frac{2n}{n-2}}} \lesssim \| A \|_{\ell^\infty L^n} \| \partial(\phi_j |v|) \|_{L^2}.$$

We expand the last term as

$$\| \partial(\phi_j v) \|_{L^2} \leq \| (\partial \phi_j) |v| \|_{L^2} + \| \phi_j (\partial |v|) \|_{L^2}.$$

We note that $|\partial \phi_j| \lesssim 2^{-j}$ and we recall the pointwise diamagnetic inequality

$$|\partial |v|| \leq |\partial^A v|$$

valid since $A \in L^2_{loc}$. Then we can write

$$\| \partial(\phi_j v) \|_{L^2} \lesssim 2^{-j} \| v \|_{L^2(\tilde{C}_j)} + \| \partial^A v \|_{L^2(\tilde{C}_j)} \lesssim 2^{-j/2} \| v \|_{L^\infty L^2(\tilde{C}_j)} + \| \partial^A v \|_{L^2(\tilde{C}_j)}.$$

Summing up, we have proved

$$\| \partial v \|_{L^2(C_j)} \lesssim (1 + \| A \|_{\ell^\infty L^n}) \left[\| \partial^A v \|_{L^2(\tilde{C}_j)} + 2^{-j/2} \| v \|_{L^\infty L^2(\tilde{C}_j)} \right]$$

Multiplying both sides by $2^{-j/2}$ and taking the sup in $j \in \mathbb{Z}$ we get the claim. \square

2.1. Formal identities. In the course of the proof we shall reserve the symbols

$$\lambda = \Re z, \quad \epsilon = \Im z$$

for the components of the frequency $z = \lambda + i\epsilon$ in (2.3).

We recall two formal identities which are a special case of the identities in [9] (see also [9]): for any real valued weights $\phi(x), \psi(x)$, we have (using implicit summation)

$$\begin{aligned} \Re \partial_j Q_j &= \Re(\Delta_A w + (\lambda + i\epsilon)w) \overline{[\Delta_A, \psi]w} - \frac{1}{2} \Delta^2 \psi |w|^2 + 2\partial_j^A w (\partial_j \partial_k \psi) \overline{\partial_k^A w} \\ &\quad + 2\Im(\overline{w} B_{jk} \partial_j^A w \partial_k \psi) + 2\epsilon \Im(w \partial_j \psi \overline{\partial_j^A w}) \end{aligned} \tag{2.7}$$

and

$$\partial_j P_j = \bar{w} \Delta_A w \phi + |\partial^A w|^2 \phi - \frac{1}{2} \Delta \phi |w|^2 + i \Im(\partial_j^A w \partial_j \phi \bar{w}) \quad (2.8)$$

where the quantities $Q = (Q_1, \dots, Q_n)$ and $P = (P_1, \dots, P_n)$ are defined by

$$Q_j := \partial_j^A w \overline{[\Delta_A, \psi] w} - \frac{1}{2} \partial_j \Delta \psi |w|^2 + \partial_j \psi [\lambda |w|^2 - |\partial^A w|^2]$$

$$P_j := \partial_j^A w \bar{w} \phi - \frac{1}{2} \partial_j \phi |w|^2.$$

Both formulas are easily checked by expanding the terms in divergence form; they are actually Morawetz type identities corresponding to the two multipliers

$$\overline{[\Delta_A, \psi] w} = (\Delta \psi) \bar{w} + 2 \partial \psi \cdot \overline{\partial^A w} \quad \text{and} \quad \phi \bar{w}.$$

If we write equation (2.3) in the form

$$\Delta_A v + (\lambda + i\epsilon)v = g, \quad \text{where} \quad g := f - Wv - iZ \cdot \partial^A v \quad (2.9)$$

and we apply (2.7), (2.8), we obtain

$$\Re \partial_j \{(Q_j + P_j)\} = I_{\nabla v} + I_v + I_\epsilon + I_B + I_g \quad (2.10)$$

where

$$I_{\nabla v} = 2 \partial_j^A v (\partial_j \partial_k \psi) \overline{\partial_k^A v} + \phi |\partial^A v|^2, \quad I_v = -\frac{1}{2} \Delta (\Delta \psi + \phi) |v|^2 - \lambda \phi |v|^2$$

$$I_B = 2 \Im(\bar{v} B_{jk} \partial_j^A v \partial_k \psi), \quad I_\epsilon = 2\epsilon \Im(v \partial_j \psi \overline{\partial_j^A v}),$$

$$I_g = \Re(g \overline{[\Delta_A, \psi] v} + g \bar{v} \phi)$$

In the following we shall integrate these formulas on \mathbb{R}^n and use the fact that the boundary terms vanish after integration. This procedure can be justified in each case e.g. by approximating v with smooth compactly supported functions and then extending the resulting estimates by density. We omit the details which are standard.

2.2. Preliminary estimates. Choosing $\phi = 1$ in (2.8), substituting (2.9) and taking the imaginary part, we get

$$\epsilon |v|^2 = \Im(g \bar{v}) - \Im \partial_j \{\bar{v} \partial_j^A v\}$$

and after integration on \mathbb{R}^n we obtain

$$\epsilon \|v\|_{L^2}^2 = \Im \int g \bar{v}. \quad (2.11)$$

Taking instead the real part of the same identity (also with $\phi = 1$) we obtain

$$|\partial^A v|^2 = \lambda |v|^2 - \Re(g \bar{v}) + \Re \partial_j \{\bar{v} \partial_j^A v\}$$

and after integration

$$\|\partial^A v\|_{L^2}^2 = \lambda \|v\|_{L^2}^2 - \Re \int g \bar{v}. \quad (2.12)$$

In order to estimate the term I_ϵ in (2.10) we use (2.11) and (2.12) as follows:

$$\int I_\epsilon \leq 2|\epsilon| \|\partial \psi\|_{L^\infty} \|v\|_{L^2} \|\partial^A v\|_{L^2} \leq C |\epsilon|^{1/2} \left(\int |g \bar{v}| \right)^{1/2} (\|\lambda\| \|v\|_{L^2}^2 + \int |g \bar{v}|)^{1/2}$$

with $C = 2\|\partial \psi\|_{L^\infty}$, then again by (2.11)

$$\leq C \left(\int |g \bar{v}| \right)^{1/2} (\|\lambda\| \int |g \bar{v}| + |\epsilon| \int |g \bar{v}|)^{1/2}$$

and we arrive at the estimate

$$\int I_\epsilon \leq 2\|\partial\psi\|_{L^\infty}(|\lambda| + |\epsilon|)^{1/2}\|g\bar{v}\|_{L^1}. \quad (2.13)$$

Another auxiliary estimate will cover the (easy) case of negative $\lambda = -\lambda_- \leq 0$. Write the real part of identity (2.8) in the form

$$\lambda_-|v|^2\phi + |\partial^A v|^2\phi - \frac{1}{2}\Delta\phi|v|^2 = \sum_\alpha \partial_j \Re P_j - \Re(g_\alpha \bar{v}_\alpha)\phi$$

and choose the radial weight

$$\phi = \frac{1}{|x| \vee R} \implies \phi' = -\frac{1}{|x|^2} \mathbf{1}_{|x|>R}, \quad \phi'' = -\frac{1}{R^2} \delta_{|x|=R} + \frac{2}{|x|^3} \mathbf{1}_{|x|>R}.$$

Note that

$$-\Delta\phi = \frac{1}{R^2} \delta_{|x|=R} + \frac{n-3}{|x|^3} \mathbf{1}_{|x|>R}.$$

Integrating over \mathbb{R}^n and taking the supremum over $R > 0$ we obtain the estimate for the case of negative $\lambda = -\lambda_- \leq 0$

$$\lambda_- \|v\|_{\dot{Y}}^2 + \|\partial^A v\|_{\dot{Y}}^2 + \frac{1}{2} \|v\|_{\dot{X}}^2 + \frac{n-3}{2} \| |x|^{-3/2} v \|_{L^2}^2 \leq \| |x|^{-1} g\bar{v} \|_{L^1}. \quad (2.14)$$

2.3. The main terms. In the following we assume $|\epsilon| \leq 1$ and $\lambda \geq 2$. We choose in (2.10), for arbitrary $R > 0$,

$$\psi = \frac{R^2 + |x|^2}{2R} \mathbf{1}_{|x| \leq R} + |x| \mathbf{1}_{|x| > R}, \quad \phi = -\frac{1}{R} \mathbf{1}_{|x| \leq R}. \quad (2.15)$$

We have then

$$\psi' = \frac{|x|}{|x| \vee R}, \quad \psi'' = \frac{1}{R} \mathbf{1}_{|x| \leq R}, \quad \Delta\psi + \phi = \frac{n-1}{|x| \vee R}, \quad (2.16)$$

$$\Delta(\Delta\psi + \phi) = -\frac{n-1}{R^2} \delta_{|x|=R} - \frac{(n-1)(n-3)}{|x|^3} \mathbf{1}_{|x|>R}$$

This implies

$$3 \sup_{R>0} \int I_v \geq \frac{n-1}{2} \|v\|_{\dot{X}}^2 + (n-3) \| |x|^{-3/2} v \|_{L^2}^2 + \lambda \|v\|_{\dot{Y}}^2 \quad (2.17)$$

Next we can write, since ψ is radial,

$$2\partial_j^A v (\partial_j \partial_k \psi) \overline{\partial_k^A v} = 2\psi'' |\hat{x} \cdot \partial^A v|^2 + 2\frac{\psi'}{|x|} \left[|\partial^A v|^2 - |\hat{x} \cdot \partial^A v|^2 \right] \geq \frac{2}{R} \mathbf{1}_{|x|<R} |\partial^A v|^2.$$

This implies

$$\sup_{R>0} \int I_{\nabla v} \geq \|\partial^A v\|_{\dot{Y}}^2. \quad (2.18)$$

Further we have, since $B_{jk} \partial_k \psi = B_{jk} \hat{x}_k \psi' = \hat{B}_j \psi'$,

$$|I_B| \leq \frac{2|x|}{|x| \vee R} |v| |\partial^A v| |\hat{B}| \leq 2|v| |\partial^A v| |\hat{B}|$$

which implies

$$\begin{aligned} \int |I_B| &\leq 2 \| |x| \hat{B} \|_{\ell^1 L^\infty} \| |x|^{-1/2} \partial^A v \|_{\ell^\infty L^2} \| |x|^{-1/2} v \|_{\ell^\infty L^2} \\ &= 2 \| |x| \hat{B} \|_{\ell^1 L^\infty} \|\partial^A v\|_{\dot{Y}} \|v\|_{\dot{Y}} \end{aligned}$$

and by Cauchy–Schwartz, for any $\delta > 0$,

$$\int |I_B| \leq \delta \|\partial^A v\|_{\dot{Y}}^2 + \delta^{-1} \| |x| \widehat{B} \|_{\ell^1 L^\infty}^2 \|v\|_{\dot{Y}}^2. \quad (2.19)$$

Finally, since $|\Delta\psi + \phi| \leq (n-1)|x|^{-1}$ and $|\partial\psi| \leq 1$, we have

$$\int |I_g| \leq (n-1) \| |x|^{-1} g\bar{v} \|_{L^1} + 2 \| g\overline{\partial^A v} \|_{L^1}. \quad (2.20)$$

Summing up, by integrating identity (2.10) over \mathbb{R}^n and using estimates (2.13) (2.17), (2.18), (2.19) and (2.20) we obtain (recall that $|\partial\psi| \leq 1$; recall also that $\lambda \geq 2$ and $|\epsilon| \leq 1$ so that $|\epsilon| + |\lambda| \lesssim \lambda$)

$$\begin{aligned} \|v\|_{\dot{X}}^2 + (n-3) \| |x|^{-3/2} v \|_{L^2}^2 + \lambda \|v\|_{\dot{Y}}^2 + \|\partial^A v\|_{\dot{Y}}^2 &\lesssim \\ &\lesssim \delta \|\partial^A v\|_{\dot{Y}}^2 + \delta^{-1} \| |x| \widehat{B} \|_{\ell^1 L^\infty}^2 \|v\|_{\dot{Y}}^2 + \lambda^{1/2} \|g\bar{v}\|_{L^1} + \| |x|^{-1} g\bar{v} \|_{L^1} + \|g\overline{\partial^A v}\|_{L^1} \end{aligned}$$

where $\delta > 0$ is arbitrary and the implicit constant depends only on n . Note now that if δ is chosen small enough with respect to n and we assume

$$\lambda \geq c(n) \| |x| \widehat{B} \|_{\ell^1 L^\infty}^2 \quad (2.21)$$

for a suitably large $c(n)$, we can absorb two terms at the right and we get the estimate

$$\|v\|_{\dot{X}}^2 + \mu_n \left\| \frac{v}{|x|^{3/2}} \right\|_{L^2}^2 + \lambda \|v\|_{\dot{Y}}^2 + \|\partial^A v\|_{\dot{Y}}^2 \leq c_0 \left(\left\| \frac{g\bar{v}}{|x|} \right\|_{L^1} + \|g\overline{\partial^A v}\|_{L^1} + \lambda^{1/2} \|g\bar{v}\|_{L^1} \right) \quad (2.22)$$

where $c_0 \geq 1$ is a constant depending only on n .

2.4. Conclusion. We now substitute in estimate (2.22)

$$g = f - W(x)v - Z(x) \cdot \partial^A v$$

(see (2.9)). Consider the terms at the right in (2.22), recalling that

$$W = W_S + W_L.$$

We denote by γ, Γ the quantities

$$\gamma := \| |x|^{3/2} W_S \|_{\ell^1 L^2 L^\infty} + \| |x| Z \|_{\ell^1 L^\infty}, \quad \Gamma := \| |x| W_L \|_{\ell^1 L^\infty}.$$

Then we have

$$\| |x|^{-1} g v \|_{L^1} \leq \| |x|^{-1} W(x) v^2 \|_{L^1} + \| |x|^{-1} Z(x) \cdot \partial^A v \|_{L^1} + \| |x|^{-1} f \bar{v} \|_{L^1}$$

and, for any $\delta > 0$,

$$\begin{aligned} \| |x|^{-1} W(x) v^2 \|_{L^1} &\leq \| |x|^{1/2} W_L \|_{\ell^1 L^2 L^\infty} \|v\|_{\dot{X}} \|v\|_{\dot{Y}} + \| |x| W_S \|_{\ell^1 L^1 L^\infty} \|v\|_{\dot{X}}^2 \\ &\leq (\delta + \gamma) \|v\|_{\dot{X}}^2 + \delta^{-1} \Gamma^2 \|v\|_{\dot{Y}}^2 \end{aligned}$$

$$\| |x|^{-1} Z(x) \cdot \partial^A v \|_{L^1} \leq \| |x| Z \|_{\ell^1 L^\infty} \|v\|_{\dot{Y}} \|\partial^A v\|_{\dot{Y}} \leq \gamma^2 \|\partial^A v\|_{\dot{Y}}^2 + \gamma^2 \|v\|_{\dot{Y}}^2$$

$$\| |x|^{-1} f \bar{v} \|_{L^1} \leq \|f\|_{\dot{Y}^*} \|v\|_{\dot{X}} \leq \delta \|v\|_{\dot{X}}^2 + \delta^{-1} \|f\|_{\dot{Y}^*}^2.$$

In a similar way we have

$$\|g\partial^A v\|_{L^1} \leq \|W(x)v\partial^A v\|_{L^1} + \|Z(x)(\partial^A v)^2\|_{L^1} + \|f\overline{\partial^A v}\|_{L^1},$$

and

$$\begin{aligned} \|W(x)v\partial^A v\|_{L^1} &\leq \| |x| W_L \|_{\ell^1 L^\infty} \|v\|_{\dot{Y}} \|\partial^A v\|_{\dot{Y}} + \| |x|^{3/2} W_S \|_{\ell^1 L^2 L^\infty} \|v\|_{\dot{X}} \|\partial^A v\|_{\dot{Y}} \\ &\leq 2\delta \|\partial^A v\|_{\dot{Y}}^2 + \delta^{-1} \Gamma^2 \|v\|_{\dot{Y}}^2 + \delta^{-1} \gamma^2 \|v\|_{\dot{X}}^2, \end{aligned}$$

$$\|Z(x)(\partial^A v)^2\|_{L^1} \leq \| |x| Z \|_{\ell^1 L^\infty} \|\partial^A v\|_{\dot{Y}}^2 \leq \gamma \|\partial^A v\|_{\dot{Y}}^2,$$

$$\|f\overline{\partial^A v}\|_{L^1} \leq \delta \|\partial^A v\|_{\dot{Y}}^2 + \delta^{-1} \|f\|_{\dot{Y}^*}^2.$$

Finally we have

$$\lambda^{1/2} \|g\bar{v}\|_{L^1} \leq \lambda^{1/2} \|Wv^2\|_{L^1} + \lambda^{1/2} \|Zv\partial^A v\|_{L^1} + \lambda^{1/2} \|f\bar{v}\|_{L^1}$$

and

$$\begin{aligned} \lambda^{1/2} \|Wv^2\|_{L^1} &\leq \lambda^{1/2} \| |x| W_L \|_{\ell^1 L^\infty} \|v\|_{\dot{Y}}^2 + \lambda^{1/2} \| |x|^{3/2} W_S \|_{\ell^1 L^2 L^\infty} \|v\|_{\dot{X}} \|v\|_{\dot{Y}} \\ &\leq (\lambda^{1/2} \Gamma + \lambda \gamma) \|v\|_{\dot{Y}}^2 + \gamma \|v\|_{\dot{X}}^2, \end{aligned}$$

$$\begin{aligned} \lambda^{1/2} \|Zv\partial^A v\|_{L^1} &\leq \lambda^{1/2} \| |x| Z \|_{\ell^1 L^\infty} \|v\|_{\dot{Y}} \|\partial^A v\|_{\dot{Y}} \leq \lambda \gamma \|v\|_{\dot{Y}}^2 + \gamma \|\partial^A v\|_{\dot{Y}}^2, \\ \lambda^{1/2} \|f\bar{v}\|_{L^1} &\leq \delta \lambda \|v\|_{\dot{Y}}^2 + \delta^{-1} \|f\|_{\dot{Y}^*}^2. \end{aligned}$$

Summing up, we get

$$\begin{aligned} \| |x|^{-1} gv \|_{L^1} + \|g\partial^A v\|_{L^1} + \lambda^{1/2} \|g\bar{v}\|_{L^1} &\leq (\delta + 2\gamma + \delta^{-1} \gamma^2) \|v\|_{\dot{X}}^2 \\ &+ (2\delta^{-1} \Gamma^2 + \gamma^2 + \lambda^{1/2} \Gamma + 2\lambda \gamma + \delta \lambda) \|v\|_{\dot{Y}}^2 + (\gamma^2 + 4\delta + 2\gamma) \|\partial^A v\|_{\dot{Y}}^2 + 3\delta^{-1} \|f\|_{\dot{Y}^*}^2. \end{aligned}$$

Recalling that $c_0 \geq 1$ is the constant in (2.22), depending only on n , we require that

$$\delta = \frac{1}{16c_0}, \quad \gamma \leq \frac{1}{16c_0}, \quad |\lambda| \geq 2^8 c_0^2 \Gamma^2 + 2 + c(n) \| |x| \widehat{B} \|_{\ell^1 L^\infty}^2 \quad (2.23)$$

(note that this implies also (2.21) and $\lambda \geq 2$) and one checks that

$$\delta + 2\gamma + \delta^{-1} \gamma^2 \leq \frac{1}{2c_0}, \quad \gamma^2 + 4\delta + 2\gamma \leq \frac{1}{2c_0}$$

and

$$2\delta^{-1} \Gamma^2 + \gamma^2 + \lambda^{1/2} \Gamma + 2\lambda \gamma + \delta \lambda \leq \frac{\lambda}{2c_0}.$$

Thus with the choices (2.23) we have for positive λ

$$\| |x|^{-1} gv \|_{L^1} + \|g\partial^A v\|_{L^1} + \lambda^{1/2} \|g\bar{v}\|_{L^1} \leq \frac{1}{2c_0} \|v\|_{\dot{X}}^2 + \frac{\lambda}{2c_0} \|v\|_{\dot{Y}}^2 + \frac{1}{2c_0} \|\partial^A v\|_{\dot{Y}}^2 + 3\delta^{-1} \|f\|_{\dot{Y}^*}^2$$

and plugging this into (2.22), and absorbing the first three terms at the right from the left side of the inequality, we conclude that

$$\|v\|_{\dot{X}}^2 + \mu_n \frac{v}{|x|^{3/2}} \|_{L^2}^2 + \lambda \|v\|_{\dot{Y}}^2 + \|\partial^A v\|_{\dot{Y}}^2 \leq c_1 \|f\|_{\dot{Y}^*}^2 \quad (2.24)$$

with c_1 a constant depending only on n .

Note that for *negative* λ , starting from estimate (2.14) instead of (2.22) and applying the same argument, we obtain a similar estimate, provided λ satisfies (2.23). Since our assumptions imply $|\epsilon| \leq |\lambda|$, we see that the proof of Theorem 2.1 is concluded.

3. The resolvent estimate for small frequencies. We now consider the remaining case of small frequencies; more precisely, we shall prove an estimate for all z which is uniform for z varying in any bounded region. Define an operator H as

$$Hv := -\Delta v - W(x)v - iA \cdot \partial_j v - i\partial \cdot (A(x)v) \quad (3.1)$$

with $W : \mathbb{R}^n \rightarrow \mathbb{R}$, $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and assume that H is selfadjoint on $L^2(\mathbb{R}^n; \mathbb{C}^N)$. In order to estimate the resolvent operator of H

$$R(z) := (H - z)^{-1} = (-\Delta - W - iA \cdot \partial - i\partial \cdot A - z)^{-1}$$

we use the (Lippmann–Schwinger) formula

$$R(z) = R_0(z)(I - K(z))^{-1}, \quad K(z) := [W + iA \cdot \partial + i\partial \cdot A]R_0(z) \quad (3.2)$$

expressing $R(z)$ in terms of the free resolvent

$$R_0(z) = (-\Delta - z)^{-1}.$$

We recall a few, more or less standard, facts on the free resolvent $R_0(z)$. For $z \in \mathbb{C} \setminus [0, +\infty)$, $R_0(z)$ is a holomorphic map with values in the space of bounded operators $L^2 \rightarrow H^2$ and satisfies an estimate

$$\|R_0(z)f\|_{\dot{X}} + |z|^{\frac{1}{2}}\|R_0(z)f\|_{\dot{Y}} + \|\partial R_0(z)f\|_{\dot{Y}} \lesssim \|f\|_{\dot{Y}^*} \tag{3.3}$$

with an implicit constant independent of z (sharp resolvent estimates can probably be traced back to [30]. A complete proof is given e.g. in [9]; actually (3.3) is a special case of the computations in the previous Section for zero potentials, in which case the proof given above works with no restriction on the frequency). When z approaches the spectrum of the Laplacian $\sigma(-\Delta) = [0, +\infty)$, it is possible to define two limit operators

$$R_0(\lambda \pm i0) = \lim_{\epsilon \downarrow 0} R_0(\lambda \pm i\epsilon), \quad \epsilon > 0, \lambda \geq 0$$

but the two limits are different if $\lambda > 0$. These limits exist in the norm of bounded operators from the weighted L^2_s space with norm $\|\langle x \rangle^s f\|_{L^2}$ to the weighted Sobolev space $H^2_{-s'}$ with norm $\sum_{|\alpha| \leq 2} \|\langle x \rangle^{-s'} \partial^\alpha f\|_{L^2}$, for arbitrary $s, s' > 1/2$ (see [1]). Since these spaces are dense in \dot{Y}^* and \dot{Y} (or \dot{X}) respectively, and estimate (3.3) is uniform in z , one obtains that (3.3) is valid also for the limit operators $R_0(\lambda \pm i0)$. In the following we shall write simply $R_0(z)$, $z \in \overline{\mathbb{C}^\pm}$, to denote either one of the extended operators $R_0(\lambda \pm i\epsilon)$ with $\epsilon \geq 0$, defined on the closed upper (resp. lower) complex half-plane. Note also that the map $z \mapsto R_0(z)$ is continuous with respect to the operator norm of bounded operators $L^2_s \rightarrow H^2_{-s'}$, for every $s, s' > 1/2$, and from this fact one easily obtains that it is also continuous with respect to the operator norm of bounded operators from $\dot{Y}^* \rightarrow H^2_{-s'}$.

Thus in particular

$$R_0(z) : \dot{Y}^* \rightarrow \dot{X}, \quad \partial R_0(z) : \dot{Y}^* \rightarrow \dot{Y}$$

are uniformly bounded operators for all $z \in \overline{\mathbb{C}^\pm}$; note also the formula

$$\Delta R_0(z) = -I - zR_0(z).$$

Moreover, for any smooth cutoff $\phi \in C_c^\infty(\mathbb{R}^n)$ and all $z \in \overline{\mathbb{C}^\pm}$, the map $z \mapsto \phi R_0(z)$ is continuous w.r.to the norm of bounded operators $\dot{Y}^* \rightarrow H^2$, and hence

$$\phi R_0(z) : \dot{Y}^* \rightarrow L^2 \quad \text{and} \quad \phi \partial R_0(z) : \dot{Y}^* \rightarrow L^2 \quad \text{are compact operators.}$$

Similarly one gets that $z \mapsto \phi R_0(z)$ is continuous w.r.to the norm of bounded operators $\dot{Y}^* \rightarrow L^\infty_{|x|} L^2_\omega$ and

$$\phi R_0(z) : \dot{Y}^* \rightarrow L^\infty_{|x|} L^2_\omega \quad \text{is a compact operator.}$$

In order to invert the operator $I - K(z)$ we shall apply Fredholm theory. The essential step is the following compactness result:

Lemma 3.1. *Let $z \in \overline{\mathbb{C}^\pm}$ and assume W, A satisfy*

$$N := \|\lvert x \rvert^{3/2}(W + i(\partial \cdot A))\|_{\ell^1 L^2 L^\infty} + \|\lvert x \rvert A\|_{\ell^1 L^\infty} < \infty. \tag{3.4}$$

Then $K(z) = (W + iA \cdot \partial + i\partial \cdot A)R_0(z)$ is a compact operator on \dot{Y}^ , and the map $z \mapsto K(z)$ is continuous with respect to the norm of bounded operators on \dot{Y}^* .*

Proof. We decompose K as follows. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a cutoff function equal to 1 for $|x| \leq 1$ and to 0 for $|x| \geq 2$. Define for $r > 2$

$$\chi_r(x) = \chi(x/r)(1 - \chi(rx))$$

so that χ_r vanishes for $|x| \geq 2r$ and also for $|x| \leq 1/r$, and equals 1 when $2/r \leq |x| \leq r$. Then we split

$$K = A_r + B_r$$

where

$$A_r(z) = \chi_r \cdot K(z), \quad B_r(z) = (1 - \chi_r) \cdot K(z).$$

First we show that A_r is a compact operator on \dot{Y}^* . Indeed, for $s > 2r > 4$ we have $\chi_r \chi_s = \chi_r$ and we can write

$$A_r = \chi_s A_r = \chi_s (W + i(\partial \cdot A)) \chi_r R_0(z) + 2i \chi_s A \cdot \chi_r \partial R_0(z).$$

By the estimate

$$\|(W + i(\partial \cdot A))v\|_{\dot{Y}^*} \leq \| |x|^{3/2} (W + i(\partial \cdot A)) \|_{\ell^1 L^2 L^\infty} \|v\|_{\dot{X}} \leq N \|v\|_{\dot{X}} \quad (3.5)$$

we see that multiplication by $W + i(\partial \cdot A)$ is a bounded operator from \dot{X} to \dot{Y}^* . Moreover, multiplication by χ_s is a bounded operator $L_{|x|}^\infty L_\omega^2 \rightarrow \dot{X}$ and the operator $\chi_r R_0 : \dot{Y}^* \rightarrow L_{|x|}^\infty L_\omega^2$ is compact as remarked above. A similar argument applies to the second term in A_r , using the estimate

$$\|Av\|_{\dot{Y}^*} \leq \| |x| A \|_{\ell^1 L^\infty} \|v\|_{\dot{Y}} \leq N \|v\|_{\dot{Y}} \quad (3.6)$$

and compactness of $\chi_r \partial R_0 : \dot{Y}^* \rightarrow L^2$. Summing up, we obtain that $A_r : \dot{Y}^* \rightarrow \dot{Y}^*$ is a compact operator. Similarly, we see that $z \mapsto A_r(z)$ is continuous with respect to the norm of bounded operators on \dot{Y}^* .

Then to conclude the proof it is sufficient to show that $B_r \rightarrow 0$ in the norm of bounded operators on \dot{Y}^* , uniformly in z , as $r \rightarrow \infty$. We have, as in (3.5)–(3.6),

$$\|B_r v\|_{\dot{Y}^*} \leq N_r (\|R_0\|_{\dot{Y}^* \rightarrow \dot{X}} + \|\partial R_0\|_{\dot{Y}^* \rightarrow \dot{Y}}) \|v\|_{\dot{Y}^*}$$

where

$$N_r := \| |x|^{3/2} (1 - \chi_r) (W + i(\partial \cdot A)) \|_{\ell^1 L^2 L^\infty} + 2 \| |x| (1 - \chi_r) A \|_{\ell^1 L^\infty}.$$

Since $N_r \rightarrow 0$ as $r \rightarrow \infty$, we obtain that $\|B_r\|_{\dot{Y}^* \rightarrow \dot{Y}^*} \rightarrow 0$. \square

We now study the injectivity of $I - K(z) : \dot{Y}^* \rightarrow \dot{Y}^*$. Note that if $f \in \dot{Y}^*$ satisfies

$$(I - K(z))f = 0$$

then setting $v = R_0(z)$ by the properties of $R_0(z)$ we have $v \in H_{loc}^1 \cap \dot{X}$, $\nabla v \in \dot{Y}$, $v \in H_{loc}^2(\mathbb{R}^n \setminus 0)$, $\Delta v \in \dot{Y} + \dot{Y}^*$ (or $\Delta v \in \dot{Y}^*$ if $z = 0$) and if $z \neq 0$ we have also $v \in \dot{Y}$. In particular, v is a solution of the equation

$$(H - z)v = 0.$$

For z outside the spectrum of H it is easy to check that this implies $v = f = 0$:

Lemma 3.2. *Let $W, A, K(z)$ be as in Lemma 3.1 and $H = -\Delta - W - iA \cdot \partial - i\partial \cdot A$. If $f \in \dot{Y}^*$ satisfies*

$$(I - K(z))f = 0$$

for some $z \notin \sigma(H)$, then $f = 0$.

Proof. Let $v = R_0(z)f$, fix a compactly supported smooth function χ which is equal to 1 for $|x| \leq 1$, and for $M > 1$ consider $v_M := v(x)\chi(x/M)$. Then $v_M \in L^2$ and

$$(H - z)v_M = \frac{1}{M} \nabla \chi\left(\frac{x}{M}\right)(2\nabla v + 2iAv) + \frac{1}{M^2} \Delta \chi\left(\frac{x}{M}\right)v =: f_M.$$

We have, for $\delta \in (1, \frac{1}{2})$, using the estimate $|A| \lesssim |x|^{-1}$,

$$\begin{aligned} \|f_M\|_{L^2} &\lesssim M^{\delta-\frac{1}{2}} \left(\| |x|^{-\frac{1}{2}-\delta} \nabla v \|_{L^2(|x| \geq M)} + \| |x|^{-\frac{3}{2}-\delta} v \|_{L^2(|x| \geq M)} \right) \\ &\lesssim M^{\delta-\frac{1}{2}} (\| \nabla v \|_{\dot{Y}} + \| v \|_{\dot{X}}) \end{aligned}$$

uniformly in M , so that $f_M \rightarrow 0$ in L^2 as $M \rightarrow \infty$. Since $v_M = R_0(z)f_M$ and $R_0(z)$ is a bounded operator on L^2 , we conclude that $v = f = 0$. \square

The hard case is of course $z \in \sigma(L)$. Then we have the following result, in which we write simply

$$R_0(\lambda) \quad \text{instead of} \quad R_0(\lambda \pm i0)$$

since the computations for the two cases are identical. Note that this is the only step where we use the additional δ decay of the coefficients.

Lemma 3.3. *Assume W and A satisfy for some $\delta > 0$*

$$|x|^2 \langle x \rangle^\delta (W + i\partial \cdot A) \in \ell^1 L^\infty, \quad |x| \langle x \rangle^\delta A \in \ell^1 L^\infty \tag{3.7}$$

and $H = -\Delta - W - i\partial \cdot A - iA \cdot \partial$ is a non negative selfadjoint operator on L^2 . Let $f \in \dot{Y}^*$ be such that, for some $\lambda \geq 0$,

$$(I - K(\lambda))f = 0, \quad K(\lambda) := (W + i\partial \cdot A + iA \cdot \partial)R_0(\lambda).$$

Then in the case $\lambda > 0$ we have $f = 0$, while in the case $\lambda = 0$ we have $|x|^{n/2}f \in L^2$ and the function $v = R_0(0)f$ belongs to $H_{loc}^2(\mathbb{R}^n \setminus 0) \cap \dot{X}$ with $\partial v \in \dot{Y}$, solves $Lv = 0$ and satisfies $|x|^{\frac{n}{2}-2-\delta'}v \in L^2$ and $|x|^{\frac{n}{2}-1-\delta'}\partial v \in L^2$ for any $\delta' > 0$.

Proof. Defining as in the previous proof $v = R_0(\lambda)f$, we see that v solves

$$\Delta v + \lambda v + g = 0, \quad g := Wv + iA \cdot \partial v + i\partial \cdot Av. \tag{3.8}$$

Then given a radial function $\chi \geq 0$ to be precised later, we apply again identity (2.10) with the choices

$$\psi' = \chi, \quad \phi = -\chi'$$

so that in particular $\Delta\psi + \phi = \frac{n-1}{|x|}\chi$. We integrate the identity on \mathbb{R}^n and, after straightforward computations (see Proposition 3.1 of [9] for a similar argument), we arrive at the following *radiation estimate*:

$$\int \chi' |\partial_S v|^2 + 2\left(\frac{\chi}{|x|} - \chi'\right) |(\partial v)_T|^2 - \frac{n-1}{2} \int \Delta\left(\frac{\chi}{|x|}\right) |v|^2 = \Re \int \chi g \left(\frac{n-1}{|x|} \bar{v} + 2\hat{x} \cdot \overline{\partial_S v}\right) \tag{3.9}$$

where we denoted the ‘Sommerfeld’ gradient of v with

$$\partial_S v := \partial v - i\sqrt{\lambda}\hat{x}v, \quad \hat{x} = x/|x|$$

and the tangential component of ∂v with

$$|(\partial v)_T|^2 := |\partial v|^2 - |\hat{x} \cdot \partial v|^2.$$

We now estimate the right hand side of (3.9). We have

$$\left| \Re \int \chi g \frac{\bar{v}}{|x|} \right| \leq \| \chi (W + i(\partial \cdot A)) |x|^{-1} |v|^2 \|_{L^1} + 2 \| \chi A |\partial v| |x|^{-1} v \|_{L^1}$$

$$\leq \|\chi|x|(W + i(\partial \cdot A))\|_{\ell^1 L^1 L^2} \|v\|_{\dot{X}}^2 + 2\|\chi|x|^{1/2}A\|_{\ell^1 L^2 L^\infty} \|v\|_{\dot{X}} \|\nabla v\|_{\dot{Y}}$$

and similarly

$$\begin{aligned} |\Re \int \chi g \widehat{x} \cdot \overline{\partial_S v}| &\leq \|\chi(W + i(\partial \cdot A))v|\partial_S v\|_{L^1} + \|\chi A \cdot \partial v|\partial_S v\|_{L^1} \\ &\leq \|\chi|x|^{3/2}(W + i(\partial \cdot A))\|_{\ell^1 L^2 L^\infty} \|v\|_{\dot{X}} \|\partial_S v\|_{\dot{Y}} + \|\chi|x|A\|_{\ell^1 L^\infty} \|\partial v\|_{\dot{Y}} \|\partial_S v\|_{\dot{Y}}. \end{aligned}$$

Since the quantities $\|v\|_{\dot{X}}$, $\|\partial v\|_{\dot{Y}}$ and $\|\partial_S v\|_{\dot{Y}} \leq \|\partial v\|_{\dot{Y}} + \sqrt{\lambda}\|v\|_{\dot{Y}}$ are all estimated by $\|f\|_{\dot{Y}^*}$ (recall (3.3)), we conclude

$$\left| \Re \int \chi g \left(\frac{n-1}{|x|} \bar{v} + 2\widehat{x} \cdot \overline{\partial_S v} \right) \right| \lesssim N_\chi^2 \|f\|_{\dot{Y}^*}^2 \tag{3.10}$$

where

$$N_\chi^2 := \|\chi|x|^{3/2}(W + i(\partial \cdot A))\|_{\ell^1 L^2 L^\infty} + \|\chi|x|A\|_{\ell^1 L^\infty}.$$

Finally, if we choose

$$\chi(x) = |x|^\delta \quad \text{with } 0 < \delta \leq 1$$

by (3.9) and (3.10) we obtain, dropping a (nonnegative) term at the left,

$$\| |x|^{(\delta-1)/2} \partial_S v \|_{L^2} + \| |x|^{(\delta-3)/2} v \|_{L^2} \lesssim_\delta N_\delta \|f\|_{\dot{Y}^*} \tag{3.11}$$

where by assumption

$$N_\delta^2 := \| |x|^{3/2+\delta}(W + i(\partial \cdot A)) \|_{\ell^1 L^2 L^\infty} + \| |x|^{1+\delta} A \|_{\ell^1 L^\infty} < \infty.$$

Consider now the following identity, obtained using the divergence formula:

$$\int_{|x|=R} (|\partial v|^2 + \lambda|v|^2 - |\partial_S v|^2) d\sigma = 2\Re \int_{|x|\leq R} i\sqrt{\lambda} \partial \cdot (v \overline{\partial v}) = 2\Re \int_{|x|\leq R} i\sqrt{\lambda} (v \overline{\Delta v})$$

for arbitrary $R > 0$. Substituting $\Delta v = -\lambda v - g$ from (3.8) and dropping two pure imaginary terms, we get

$$\int_{|x|=R} (|\partial v|^2 + \lambda|v|^2 - |\partial_S v|^2) d\sigma = 2\Re \int_{|x|\leq R} (A \cdot \partial v + \partial \cdot Av) \bar{v}.$$

The last term can be written, again by the divergence formula,

$$= 2 \int_{|x|\leq R} \partial \cdot (A|v|^2) = 2 \sum_j \int_{|x|=R} \widehat{x}_j \cdot A|v|^2 d\sigma, \quad \widehat{x}_j = x_j/|x|.$$

By assumption $|A| \lesssim |x|^{-1}$, hence for some $R_0 > 0$ we have $\lambda > 2|A(x)|$ for all $|x| > R_0$, and the term in A can be absorbed at the left of the identity. Summing up, we have proved that

$$\int_{|x|=R} (|\partial v|^2 + \lambda|v|^2) d\sigma \leq 2 \int_{|x|=R} |\partial_S v|^2 d\sigma, \quad R \geq R_0. \tag{3.12}$$

Multiplying both sides by $|x|^{\delta-1}$, integrating in the radial direction from R_0 to ∞ , and using (3.11), we conclude

$$\| |x|^{(\delta-1)/2} \partial v \|_{L^2(|x|\geq R_0)} + \sqrt{\lambda} \| |x|^{(\delta-1)/2} v \|_{L^2(|x|\geq R_0)} \lesssim \|f\|_{\dot{Y}^*}. \tag{3.13}$$

In the case $\lambda > 0$ we have proved that $|x|^{(\delta-1)/2} v \in L^2$ i.e., λ is a resonance, and this is enough to conclude that $v = 0$ by applying one of the available results on the absence of embedded eigenvalues. We shall apply the results from [31] which are particularly sharp. We need to check the assumptions on the potentials required

in [31]. The potential V in [31] is simply $V = z$ in our case, which we are assuming real and > 0 , thus condition A.1 is trivially satisfied. Concerning W we have

$$\|W\|_{L^{n/2}} \leq \| |x|^{-2} \|_{\ell^\infty L^{n/2}} \| |x|^2 W \|_{\ell^{n/2} L^\infty} < \infty$$

by assumption, thus $W \in L^{n/2}$ and condition A.2 in [31] is satisfied. Concerning the potential Z in the notations of [31], which coincides with A here, we have

$$\|A\|_{\ell^\infty L^n} \leq \| |x|^{-1} \|_{\ell^\infty L^n} \| |x| A \|_{\ell^n L^\infty} < \infty$$

thus $A \in \ell^\infty L^n$; moreover a similar computation applied to $\mathbf{1}_{|x|>M} A$ gives

$$\| \mathbf{1}_{|x|>M} A \|_{\ell^\infty L^n} \leq \| |x|^{-1} \|_{\ell^\infty L^n} \| \mathbf{1}_{|x|>M} |x|^{-\delta} \|_{L^\infty} \| |x|^{1+\delta} A \|_{\ell^n L^\infty} \rightarrow 0 \text{ as } M \rightarrow \infty.$$

Thus to check that A satisfies condition A.3 in [31] it remains to check that the low frequency part $S_{<R} A$ of A satisfies A.2 for R large enough. $S_{<R} A$ is obviously smooth. Moreover, it is clear that $|x|A \rightarrow 0$ as $|x| \rightarrow \infty$; in order to prove the same decay property for $S_{<R} A$ we represent it as a convolution with a suitable Schwartz kernel ϕ

$$\phi * A(x) = \int_{|y| \leq \frac{|x|}{2}} A(y) \phi(x-y) + \int_{|y| \geq \frac{|x|}{2}} A(y) \phi(x-y).$$

The first integral is bounded by $C_k \langle x \rangle^{-k}$ for all k . For the second one we write

$$|x| \int_{|y| \geq \frac{|x|}{2}} A(y) \phi(x-y) \leq \int_{|y| \geq \frac{|x|}{2}} |y| A(y) \phi(x-y) = o(|x|).$$

We have thus proved that $|x| S_{<R} A \rightarrow 0$ as $|x| \rightarrow \infty$ (for any fixed R) and hence $A = Z$ satisfies condition A.3. Applying Theorem 8 of [31], we conclude that $v = 0$.

It remains to consider the case $\lambda = 0$. We denote by \dot{L}_s^2 the Hilbert space with norm

$$\|v\|_{\dot{L}_s^2} := \| |x|^s v \|_{L^2}.$$

By the well known Stein–Weiss estimate for fractional integrals in weighted L^p spaces, applied to $R_0(0)v = \Delta^{-1}v = c|x|^{2-n} * v$, we see that $R_0(0)$ is a bounded operator

$$R_0(0) : \dot{L}_s^2 \rightarrow \dot{L}_{s-2}^2 \quad \text{for all } 2 - \frac{n}{2} < s < \frac{n}{2}$$

while $\partial R_0(0) = c(x|x|^{-n}) * v$ is a bounded operator

$$\partial R_0(0) : \dot{L}_s^2 \rightarrow \dot{L}_{s-1}^2 \quad \text{for all } 1 - \frac{n}{2} < s < \frac{n}{2}.$$

Recall also that $R_0(0)$ is bounded from \dot{Y}^* to \dot{X} and $\partial R_0(0)$ is bounded from \dot{Y}^* to \dot{Y} . Moreover from the assumption on W, A it follows that the corresponding multiplication operators are bounded operators

$$W + i(\partial \cdot A) : \dot{X} \rightarrow \dot{L}_{1/2+\delta}^2, \quad W + i(\partial \cdot A) : \dot{L}_{s-2}^2 \rightarrow \dot{L}_{s+\delta}^2 \quad \forall s \in \mathbb{R},$$

$$A : \dot{Y} \rightarrow \dot{L}_{1/2+\delta}^2, \quad A : \dot{L}_{s-1}^2 \rightarrow \dot{L}_{s+\delta}^2 \quad \forall s \in \mathbb{R}.$$

Combining all the previous properties we deduce that $K(0) = (W + i\partial \cdot A + iA \cdot \partial)R_0(0)$ is a bounded operator

$$K(0) : \dot{Y}^* \rightarrow \dot{L}_{1/2+\delta}^2 \quad \text{and} \quad K(0) : \dot{L}_s^2 \rightarrow \dot{L}_{s+\delta}^2, \quad \forall 2 - \frac{n}{2} < s < \frac{n}{2}. \quad (3.14)$$

Since we know that $f \in \dot{Y}^*$ and that $f = K(0)f$, applying (3.14) repeatedly, we obtain in a finite number of steps that $f \in \dot{L}_{n/2}^2$, which in turn implies $v =$

$R_0(0)f \in \dot{L}_s^2$ for all $s < \frac{n}{2} - 2$ and $\partial v = \partial R_0(0)f \in \dot{L}_s^2$ for all $s < \frac{n}{2} - 1$. The proof is concluded. \square

If $K(z)$ is compact and $I - K(z)$ is injective on \dot{Y}^* (under suitable assumptions), it follows from Fredholm theory that $(I - K(z))^{-1}$ is a bounded operator for all $z \in \mathbb{C}$. However we need a bound uniform in z , and to this end it is sufficient to prove that the map $z \mapsto (I - K(z))^{-1}$ is continuous. This follows from a general well known result which we reprove here for the benefit of the reader. Note that $z \mapsto I - K(z)$ is trivially continuous (and holomorphic for $z \notin \sigma(H)$).

Lemma 3.4. *Let X_1, X_2 be two Banach spaces, K_j, K compact operators from X_1 to X_2 , and assume the sequence $K_j \rightarrow K$ in the operator norm as $j \rightarrow \infty$. If $I - K_j, I - K$ are invertible with bounded inverses, then $(I - K_j)^{-1} \rightarrow (I - K)^{-1}$ in the operator norm.*

Proof. Let $\phi \in X_2$ and let $c_j := \|(I - K_j)^{-1}\phi\|_{X_1}$. If by contradiction $c_j \rightarrow \infty$, then defining $\psi_j = (I - K_j)^{-1}\phi \cdot c_j^{-1}$ and $\phi_j = \phi \cdot c_j^{-1}$ we would have

$$\|\psi_j\|_{X_1} = 1, \quad \|\phi_j\|_{X_2} \rightarrow 0, \quad \phi_j = (I - K_j)\psi_j.$$

The last identity can be written

$$\psi_j = \phi_j + (K_j - K)\psi_j + K\psi_j.$$

The first two terms at the right tend to 0, and the third one converges, by possibly passing to a subsequence, since K is compact; let $\psi = \lim K\psi_j$. By the previous identity we see that also ψ_j converges to ψ so that $\|\psi\| = 1$ and $\psi = K\psi$, which contradicts the invertibility of $I - K$.

We have thus proved that, for any $\phi \in X_2$, the sequence $\chi_j := (I - K_j)^{-1}\phi$ is bounded in X_1 . Write this identity in the form

$$\chi_j = \phi + K\chi_j + (K_j - K)\chi_j$$

and note as before that $K\chi_j$ is a relatively compact sequence; let χ be any one of its limit points. Letting $j \rightarrow \infty$ we get $\chi = \phi + K\chi$, i.e., $(I - K_j)^{-1}\phi \rightarrow (I - K)^{-1}\phi$. Applying the uniform boundedness principle we get the claim. \square

We finally sum up the previous results. We shall need to assume that 0 is not a resonance, in the sense of Definition 1.1. Note that in Lemma 3.3 we proved in particular that if $f \in \dot{Y}^*$ satisfies $f = K(0)f$, then $v = R_0(0)f$ is a resonant state at 0.

Theorem 3.5. *Assume the operator H defined in (3.1) is non negative and selfadjoint on L^2 , with W and A satisfying (3.7) for some $\delta > 0$. In addition, assume that 0 is not a resonance for H , in the sense of Definition 1.1.*

Then $I - K(z)$ is a bounded invertible operator on \dot{Y}^ , with $(I - K(z))^{-1}$ bounded uniformly for z in bounded subsets of $\overline{\mathbb{C}}^\pm$. Moreover, the resolvent operator $R(z) = (H - z)^{-1}$ satisfies the estimate*

$$\|R(z)f\|_{\dot{X}} + |z|^{\frac{1}{2}}\|R(z)f\|_{\dot{Y}} + \|\partial R(z)f\|_{\dot{Y}} \leq C(z)\|f\|_{\dot{Y}^*} \tag{3.15}$$

for all $z \in \overline{\mathbb{C}}^\pm$, where $C(z)$ is a continuous function of z .

Proof. It is sufficient to combine Lemmas 3.1, 3.2, 3.3, 3.4 and apply Fredholm theory in conjunction with assumption (1.4), to prove the claims about $I - K(z)$; note that (3.7) include the assumptions of Lemmas 3.1–3.4. Finally, using the representation (3.2) and the free estimate (3.3) we obtain (3.15). \square

4. The full resolvent estimate. In this Section and the following ones we shall freely use a few results from classical harmonic analysis, in particular the basic properties of Muckenhoupt classes A_p and Lorentz spaces. For more details see e.g. [23], [25] and [40].

Consider the operator L defined by

$$Lv = -\Delta_A v + Vv \tag{4.1}$$

and the resolvent equation

$$Lv - zv = f. \tag{4.2}$$

We put together the estimates of the previous Sections to obtain:

Theorem 4.1 (Resolvent estimate). *Let $n \geq 3$. Assume the operator L defined in (4.1) is selfadjoint and non negative on $L^2(\mathbb{R}^n)$, with domain $H^2(\mathbb{R}^n)$. Assume $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy for some $\delta > 0$:*

$$|x|^2 \langle x \rangle^\delta (V - i\partial \cdot A), \quad |x| \langle x \rangle^\delta A \quad \text{and} \quad |x| \langle x \rangle^\delta \widehat{B} \quad \text{belong to} \quad \ell^1 L^\infty. \tag{4.3}$$

Moreover, assume 0 is not a resonance for L , in the sense of Definition 1.1.

Then for all $z \in \overline{\mathbb{C}^\pm}$ with $|\Im z| \leq 1$ the resolvent operator $R(z) = (L - z)^{-1}$ satisfies the following estimate uniform in z :

$$\|R(z)f\|_{\dot{X}} + |z|^{\frac{1}{2}} \|R(z)f\|_{\dot{Y}} + \|\partial R(z)f\|_{\dot{Y}} \lesssim \|f\|_{\dot{Y}^*}. \tag{4.4}$$

Proof. The proof is obtained by combining the estimates of Theorems 2.1 and 3.5. In order to apply Theorem 3.5, we write L in the form

$$Lv = -\Delta v + (V + |A|^2) - iA \cdot \partial v - i\partial \cdot (Av)$$

which coincides with H defined in (3.1) with the choice $W = -V - |A|^2$. The assumptions of Theorem 3.5, see (3.7), are satisfied if

$$|x|^2 \langle x \rangle^\delta (V + |A|^2 - i\partial \cdot A) \quad \text{and} \quad |x| \langle x \rangle^\delta A \quad \text{belong to} \quad \ell^1 L^\infty$$

and these conditions are implied by (4.3), with a possibly different δ . This proves (4.4) for z in any bounded set.

In order to apply Theorem 2.1 we note that the operator L is already in the form required for (2.3), choosing $Z = 0$ and $W = -V$. We check assumption (2.4): the assumptions on \widehat{B} (and $Z = 0$) are satisfied. Next we split $W = -V$ as

$$V_r = \mathbf{1}_{|x| \leq r} V, \quad V'_r = V - V_r$$

and we note that from $|x|^2 V \in \ell^1 L^\infty$ it follows that

$$\||x|^{3/2} V_r\|_{\ell^1 L^2 L^\infty} \leq \||x|^2 V_r\|_{\ell^1 L^\infty} \rightarrow 0 \quad \text{as} \quad r \rightarrow 0.$$

On the other hand, $|x| V'_r \in \ell^1 L^\infty$ for any r . Thus if we choose $W_S = -V_r$, $W_L = -V'_r$ for r sufficiently small, then (2.4) are satisfied. This proves (4.4) for all sufficiently large z belonging to the strip $|\Im z| \leq 1$, with a constant independent of z , and the proof is concluded. \square

Corollary 4.2. *Under the assumptions of Theorem 4.1, for all $z \in \overline{\mathbb{C}^\pm}$ with $|\Im z| \leq 1$ the resolvent operator $R(z) = (L - z)^{-1}$ satisfies the following estimate, uniform in z :*

$$\||D|^{1/2} R(z) |D|^{1/2} f\|_{\dot{Y}} + \||x|^{-1/2} R(z) |x|^{-1/2} f\|_{\dot{Y}} \lesssim \|f\|_{\dot{Y}^*}. \tag{4.5}$$

Proof. Recall that $|x|^{-s}$ is in the Muckenhoupt class A_2 if and only if $|s| < n$, and this implies that the Riesz operator $Rv := \mathcal{F}^{-1}(\widehat{v}(\xi)\xi/|\xi|)$ (where both $\mathcal{F}v$ and \widehat{v} denote Fourier transform) satisfies the weighted estimate

$$\| |x|^{-s} Rv \|_{L^2} \lesssim \| |x|^{-s} v \|_{L^2} \quad (4.6)$$

for all $|s| < n/2$. Introduce the weighted dyadic norms

$$\|v\|_{\ell^q(2^{-js})L^2} := \|2^{-js}v\|_{L^2(C_j)}\|_{\ell_j^q} = \left(\sum_{j \in \mathbb{Z}} 2^{-qjs} \|v\|_{L^2(C_j)}^q \right)^{1/q}$$

where $C_j \subseteq \mathbb{R}^n$ is the ring $2^j \leq |x| < 2^{j+1}$ as usual. Then (4.6) can be written

$$\|Rv\|_{\ell^2(2^{-js})L^2} \lesssim \|v\|_{\ell^2(2^{-js})L^2} \quad \forall |s| < \frac{n}{2}. \quad (4.7)$$

We recall now the real interpolation formula: if $q_0, q_1, q \in (0, \infty]$, $\theta \in (0, 1)$, $s_0 \neq s_1 \in \mathbb{R}$,

$$(\ell^{q_0}(2^{-js_0})L^2, \ell^{q_1}(2^{-js_1})L^2)_{\theta, q} \simeq \ell^q(2^{-js})L^2, \quad s = (1-\theta)s_0 + \theta s_1$$

(Theorem 5.6.1 in [5]). If we apply the formula with $q_0 = q_1 = 2$, $q = \infty$, $s_0 = 1/2 - \epsilon$, $s_1 = 1/2 + \epsilon$ with $\epsilon > 0$ and $\theta = 1/2$, we obtain

$$(\ell^2(2^{-j(1/2-\epsilon)})L^2, \ell^2(2^{-j(1/2+\epsilon)})L^2)_{1/2, \infty} = \ell^\infty(2^{-j/2})L^2 \simeq \dot{Y}.$$

Then, interpolating the inequalities (4.7) for $s = 1/2 \pm \epsilon$ with $\epsilon > 0$ small, we obtain

$$\|Rv\|_{\dot{Y}} \lesssim \|v\|_{\dot{Y}} \quad (4.8)$$

i.e., the Riesz operator is bounded on \dot{Y} . By duality, R is also bounded on \dot{Y}^* .

Exactly the same argument applies to the Calderón–Zygmund operators $|D|^{iy}$, $y \in \mathbb{R}$, which are defined via the formula $|D|^{iy}v := \mathcal{F}^{-1}(|\xi|^{iy}\widehat{v}(\xi))$, thus we have for all $y \in \mathbb{R}$

$$\||D|^{iy}v\|_{\dot{Y}} \lesssim \|v\|_{\dot{Y}}. \quad (4.9)$$

with a norm growing polynomially in $y \in \mathbb{R}$ (like $|y|^{n/2}$ at most). The same property holds for $|D|^{iy} : \dot{Y}^* \rightarrow \dot{Y}^*$.

Now we can write, by (4.8) and (4.4),

$$\||D|R(z)f\|_{\dot{Y}} = \|R \cdot \partial R(z)f\|_{\dot{Y}} \lesssim \|f\|_{\dot{Y}^*}$$

uniformly in z . Thus $|D|R(z) : \dot{Y}^* \rightarrow \dot{Y}$ is bounded, uniformly in z , and by duality the same holds for $R(z)|D|$.

We now apply Stein–Weiss interpolation to the analytic family of operators

$$T_w := |D|^{1-w}R(z)|D|^w, \quad w \text{ in the complex strip } 0 \leq \Im w \leq 1.$$

Indeed, writing

$$T_{iy} = |D|^{-iy}|D|R(z)|D|^{iy}, \quad T_{1+iy} = |D|^{-iy} \cdot R(z)|D| \cdot |D|^{iy}$$

and using the previous steps, we see that $T_w : \dot{Y}^* \rightarrow \dot{Y}$ is a bounded operator for $\Re w = 0$ and $\Re w = 1$, uniformly in $y = \Im w$, which implies $T_w : \dot{Y}^* \rightarrow \dot{Y}$ is a bounded operator for all w in the strip. Taking $w = 1/2$ we prove the first part of (4.5).

Consider now the second part of (4.5). Recalling that $\||x|^{-1}v\|_{\dot{Y}} \leq \|v\|_{\dot{X}}$, from (4.4) we have in particular

$$\||x|^{-1}R(z)f\|_{\dot{Y}} \lesssim \|f\|_{\dot{Y}^*}$$

and hence by duality

$$\|R(z)|x|^{-1}f\|_{\dot{Y}} \lesssim \|f\|_{\dot{Y}^*}.$$

Interpolating between these estimates as in the first part of the proof, we obtain (4.5). \square

Remark 4.3. The weight $\langle x \rangle^\delta$ with $\delta > 0$ in assumption (4.3) is required only to exclude resonances embedded or at the threshold, using Lemma 3.3. If we assume a priori the condition

$$(L + \lambda)v = 0, \quad v \in H_{loc}^2 \cap \dot{Y}, \quad \lambda \geq 0 \implies v = 0 \quad (4.10)$$

then Lemma 3.3 is no longer necessary and Theorem 4.1 holds with $\delta = 0$.

Remark 4.4 (Gauge transformation). If we apply a change of gauge

$$u = e^{i\phi(x)}v$$

the magnetic Laplacian transforms as follows:

$$\Delta_A(e^{i\phi}v) = e^{i\phi}\Delta_{\tilde{A}}v, \quad \tilde{A} = A + \partial\phi. \quad (4.11)$$

In particular, if we choose

$$\phi(x) = \Delta^{-1}\partial \cdot A \implies \partial \cdot \tilde{A} = 0$$

we see that we can gauge away the term $\partial \cdot A$ with an appropriate choice of ϕ in Theorem 4.1, although the details require some work. Note also that the magnetic field B is *gauge invariant*, since $\partial_j\partial_k\phi - \partial_k\partial_j\phi = 0$.

It will be useful to prepare estimates for the gauge transform in Sobolev spaces.

Lemma 4.5 (Boundedness of the gauge transform). *Assume $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $\partial\phi \in L^{n,\infty}$. Then we have*

$$\|e^{i\phi}v\|_{\dot{H}_p^s} \simeq \|v\|_{\dot{H}_p^s}, \quad \|e^{i\phi}v\|_{\dot{H}_{p'}^{-s}} \simeq \|v\|_{\dot{H}_{p'}^{-s}}$$

for all $s \in [0, 1]$ and $1 < p < \frac{n}{s}$ i.e. $\frac{n}{n-s} < p' < \infty$.

Proof. Let $Tv := e^{i\phi(x)}v$ be the multiplication operator. T is an isometry of L^p into itself for all $p \in [1, \infty]$. Moreover

$$\|Tv\|_{\dot{H}_p^1} \simeq \|\partial Tv\|_{\dot{H}_p^1} \leq \|v\partial\phi\|_{L^p} + \|\partial v\|_{L^p} \lesssim \|\partial\phi\|_{L^{n,\infty}}\|v\|_{\frac{np}{n-p},p} + \|\partial v\|_{L^p}$$

and by Sobolev embedding in Lorentz spaces

$$\|v\|_{\frac{np}{n-p},p} \lesssim \|v\|_{\dot{H}_p^1}$$

valid for $1 < p < n$, we deduce that T is a bounded operator on \dot{H}_p^1 provided $1 < p < n$. Thus by complex interpolation we obtain that T is bounded on \dot{H}_p^s provided $1 < p < n/s$, and since T^{-1} i.e. multiplication by $e^{-i\phi}$ enjoys the same property, the first claim is proved. The second claim follows by duality. \square

We can now give a version of Theorem 4.1 improved with the use of the gauge transform, as mentioned in Remark 4.3:

Corollary 4.6. *Let $n \geq 3$. Assume the operator L defined in (4.1) is selfadjoint and non negative on $L^2(\mathbb{R}^n)$, with domain $H^2(\mathbb{R}^n)$. Assume $V : \mathbb{R}^n \rightarrow \mathbb{R}$, $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy for some $\delta > 0$*

$$|x|^2\langle x \rangle^\delta(V - i\partial \cdot A - i\Delta\phi) \quad \text{and} \quad |x|\langle x \rangle^\delta(|A| + |\partial\phi| + |B|) \quad \text{belong to} \quad \ell^1 L^\infty. \quad (4.12)$$

Moreover, assume 0 is not a resonance for L , in the sense of Definition 1.1. Then estimates (4.4) and (4.5) are valid.

Proof. We apply the gauge transformation (4.11). By assumption the new potential $\tilde{A} = A + \partial\phi$ satisfies (4.3), while the magnetic field B does not change, since $\partial_j(\partial_k\phi) - \partial_k(\partial_j\phi) = 0$. Thus we are in position to apply Theorem 4.1 and we obtain that the resolvent operator $\tilde{R}(z) = (\tilde{L} - z)^{-1}$, where $\tilde{L} = -\Delta_{\tilde{A}} + V$, satisfies estimate (4.4). Since

$$\tilde{R}(z) = e^{i\phi}R(z)e^{-i\phi},$$

this gives immediately the uniform boundedness of $R(z) : \dot{Y}^* \rightarrow \dot{X}$ and $|z|^{1/2}R(z) : \dot{Y}^* \rightarrow \dot{Y}$. For the derivative term, we have

$$\|\partial R(z)f\|_{\dot{Y}} \leq \|\partial\tilde{R}(z)e^{i\phi}f\|_{\dot{Y}} + \|(\partial\phi)\tilde{R}(z)e^{i\phi}f\|_{\dot{Y}}.$$

The first term is bounded by \dot{Y}^* thanks to the estimate for $\tilde{R}(z)$. For the second term, we note that the assumptions on ϕ imply $|\partial\phi| \lesssim |x|^{-1}$ and hence we can write

$$\|(\partial\phi)\tilde{R}(z)e^{i\phi}f\|_{\dot{Y}} \lesssim \| |x|^{-1}\tilde{R}(z)e^{i\phi}f \|_{\dot{Y}} \leq \|\tilde{R}(z)e^{i\phi}f\|_{\dot{X}} \lesssim \|f\|_{\dot{Y}^*}$$

and the proof of (4.4) for $R(z)$ is concluded. The second estimate (4.5) is proved by duality and interpolation as in the proof of Corollary 4.2. \square

5. Smoothing estimates. Using the Kato smoothing theory, the resolvent estimates of the previous section can be converted into estimates for the time-dependent Schrödinger flow with little effort. The theory was initiated in [27] and took the final form in [28] (see also [37], [35]); it was further expanded in [11] to include in the general theory also the wave and Klein–Gordon flows. Here we follow the formulation¹ of [11].

Let $\mathcal{H}, \mathcal{H}_1$ be two Hilbert spaces and H a selfadjoint operator in \mathcal{H} . Denote with $R(z) = (H - z)^{-1}$ the resolvent operator of H , and with $\Im R(z) = 2^{-1}(R(z) - R(z)^*)$ its imaginary part.

Definition 5.1 (Smoothing operator). A closed operator A from \mathcal{H} to \mathcal{H}_1 with dense domain $D(A)$ is called:

(i) *H-smooth*, with constant a , if $\exists \epsilon_0$ such that for every $\epsilon, \lambda \in \mathbb{R}$ with $0 < |\epsilon| < \epsilon_0$ the following uniform bound holds:

$$|(\Im R(\lambda + i\epsilon)A^*v, A^*v)_{\mathcal{H}}| \leq a\|v\|_{\mathcal{H}_1}^2, \quad v \in D(A^*); \tag{5.1}$$

(ii) *H-supersmooth*, with constant a , if in place of (5.1) one has

$$|(R(\lambda + i\epsilon)A^*v, A^*v)_{\mathcal{H}}| \leq a\|v\|_{\mathcal{H}_1}^2, \quad v \in D(A^*). \tag{5.2}$$

The following result is proved in Lemma 3.6 and Theorem 5.1 of [27] (see also Theorem XIII.25 in [37]). Here $L^2\mathcal{H}$ denotes the space of L^2 functions on \mathbb{R} with values in \mathcal{H} :

Theorem 5.2. *Let $A : \mathcal{H} \rightarrow \mathcal{H}_1$ be a closed operator with dense domain $D(A)$. Then A is H -smooth with constant a if and only if, for any $v \in \mathcal{H}$, one has $e^{-itH}v \in D(A)$ for almost every t and the following estimate holds:*

$$\|Ae^{-itH}v\|_{L^2\mathcal{H}_1} \leq 2a^{\frac{1}{2}}\|v\|_{\mathcal{H}}. \tag{5.3}$$

¹We take the chance to correct a couple of typos in [11], in the definition of smoothing operators and in the statement of Theorem 5.3.

Thus H -smoothness is equivalent to the smoothing estimate (5.3) for the homogeneous flow e^{-itH} . In a similar way, H -supersmoothness is equivalent to a nonhomogeneous estimate:

Theorem 5.3 ([11]). *Let $A : \mathcal{H} \rightarrow \mathcal{H}_1$ be a closed operator with dense domain $D(A)$. Assume A is H -supersmooth with constant a . Then $e^{-itH}v \in D(A)$ for almost any $t \in \mathbb{R}$ and any $v \in \mathcal{H}$; moreover, for any step function $h(t) : \mathbb{R} \rightarrow D(A^*)$, $Ae^{-i(t-s)H}A^*h(s)$ is Bochner integrable in s over $[0, t]$ (or $[t, 0]$) and satisfies, for all $\epsilon \in (-\epsilon_0, \epsilon_0)$, the estimate*

$$\|e^{-|\epsilon|t} \int_0^t Ae^{-i(t-s)H}A^*h(s)ds\|_{L^2\mathcal{H}_1} \leq 2a\|e^{-|\epsilon|t}h(t)\|_{L^2\mathcal{H}_1}. \tag{5.4}$$

Conversely, if (5.4) holds, then A is H -supersmooth with constant $2a$.

The extension to the wave and Klein–Gordon groups is the following:

Theorem 5.4 ([11]). *Let $\nu \in \mathbb{R}$ with $H + \nu \geq 0$ and let P be the orthogonal projection onto $\ker(H + \nu)^\perp$. Assume A and $A(H + \nu)^{-\frac{1}{4}}P$ are closed operators with dense domain from \mathcal{H} to \mathcal{H}_1 .*

(i) *If A is H -smooth with constant a , then $A(H + \nu)^{-\frac{1}{4}}P$ is $\sqrt{H + \nu}$ -smooth with constant $C = (\pi + 3)a$. In particular, we have the estimate*

$$\|Ae^{-it\sqrt{H+\nu}}v\|_{L^2\mathcal{H}_1} \leq 2C^{\frac{1}{2}}\|(H + \nu)^{\frac{1}{4}}v\|_{\mathcal{H}}, \quad \forall v \in D((H + \nu)^{\frac{1}{4}}). \tag{5.5}$$

(ii) *If A is H -supersmooth with constant a , then $A(H+\nu)^{-\frac{1}{4}}P$ is $\sqrt{H+\nu}$ -supersmooth with constant $C = (\pi + 3)a$. In particular, we have the estimate*

$$\left\| \int_0^t Ae^{-i(t-s)\sqrt{H+\nu}}(H + \nu)^{-\frac{1}{2}}PA^*h(s)ds \right\|_{L^2\mathcal{H}_1} \leq C\|h\|_{L^2\mathcal{H}_1} \tag{5.6}$$

for any step function $h : \mathbb{R} \rightarrow D((H + \nu)^{-\frac{1}{4}}PA^*)$.

We can now recast the resolvent estimates of Corollary 4.2 in the framework of the Kato–Yajima theory:

Corollary 5.5. *Let $\rho(x)$ be an arbitrary function in ℓ^2L^∞ . Assume the operator L defined by (4.1) satisfies the assumptions of either Theorem 4.1 or Corollary 4.6. Then the operators*

$$\rho(x)|x|^{-1} \quad \text{and} \quad \rho(x)|x|^{-1/2}|D|^{1/2}$$

are L -supersmooth (and hence L -smooth), with a constant of the form $C\|\rho\|_{\ell^2L^\infty}^2$. If in addition $|x|\partial\rho \in \ell^2L^\infty$, then the operator

$$|D|^{1/2}\rho|x|^{-1/2}$$

is also L -supersmooth, with a constant $C\|\rho\| + \|x|\partial\rho\|_{\ell^2L^\infty}^2$.

Proof. Note that

$$\|\rho|x|^{-1}v\|_{L^2} \leq \|\rho\|_{\ell^2L^\infty}\| |x|^{-1}v\|_{\ell^\infty L^2} = \|\rho\|_{\ell^2L^\infty}\| |x|^{-1/2}v\|_{\dot{Y}}.$$

Thus we have, by (4.5),

$$\begin{aligned} \|\rho|x|^{-1}R(z)|x|^{-1/2}f\|_{L^2} &\lesssim \|\rho\|_{\ell^2L^\infty}\|f\|_{\dot{Y}^*} = \|\rho\|_{\ell^2L^\infty}\|\rho\rho^{-1}|x|^{1/2}f\|_{\ell^1L^2} \\ &\leq \|\rho\|_{\ell^2L^\infty}^2\|\rho^{-1}|x|^{1/2}f\|_{L^2} \end{aligned}$$

which can be written

$$\|\rho|x|^{-1}R(z)|x|^{-1}\rho g\|_{L^2} \lesssim \|\rho\|_{\ell^2L^\infty}^2\|g\|_{L^2}.$$

The proof for $\rho(x)|x|^{-1/2}|D|^{1/2}$ is similar. For the last operator, we first note that for any $y \in \mathbb{R}$

$$\begin{aligned} \||D|^{1+iy}\rho|x|^{-1/2}R(z)f\|_{L^2} &\simeq \|\partial(\rho|x|^{-1/2}R(z)f)\|_{L^2} \\ &\leq \|\rho|x|^{-1/2}\partial R(z)f\|_{L^2} + \|(\partial(\rho|x|^{-1/2}))R(z)f\|_{L^2}. \end{aligned}$$

The first term at the right can be bounded by the \dot{Y} norm and hence by $\|f\|_{\dot{Y}^*}$ thanks to (4.4). The second term is bounded by

$$\|(|x|\partial\rho - \frac{\widehat{x}}{2}\rho)|x|^{-3/2}R(z)f\|_{L^2} \lesssim \| |x|\partial\rho + \rho \|_{\ell^2 L^\infty} \| |x|^{-1}R(z)f \|_{\dot{Y}}$$

and hence again by $\|f\|_{\dot{Y}^*}$, using the inequality $\| |x|^{-1}v \|_{\dot{Y}} \leq \|v\|_{\dot{X}}$ and again (4.4). In conclusion we have

$$\||D|^{1+iy}\rho|x|^{-1/2}R(z)f\|_{L^2} \leq C(\|\rho\|_{\ell^2 L^\infty} + \| |x|\partial\rho \|_{\ell^2 L^\infty})\|f\|_{\dot{Y}^*}$$

which implies that the operator

$$|D|^{1+iy}\rho|x|^{-1/2}R(z)|x|^{-1/2}\rho$$

is bounded on L^2 , with a constant $C(\|\rho + |x|\partial\rho\|_{\ell^2 L^\infty})\|\rho\|_{\ell^2 L^\infty}$ where C is independent of z or y . By duality the same holds for the operator

$$\rho|x|^{-1/2}R(z)|x|^{-1/2}\rho|D|^{1+iy}.$$

Hence we can apply Stein–Weiss interpolation to the analytic family of operators

$$T_w = |D|^{1-w}\rho|x|^{-1/2}R(z)|x|^{-1/2}\rho|D|^w$$

with w in the complex strip $0 \leq \Re w \leq 1$, as in the proof of Corollary 4.2. Taking $w = 1/2$ we conclude the proof. \square

Then applying the abstract theory we obtain immediately:

Corollary 5.6 (Smoothing for Schrödinger). *Under the assumptions of either Theorem 4.1 or Corollary 4.6, we have for any $\rho > 0$ in $\ell^2 L^\infty$*

$$\|\rho|x|^{-1}e^{itL}f\|_{L_t^2 L^2} + \|\rho|x|^{-1/2}|D|^{1/2}e^{itL}f\|_{L_t^2 L^2} \leq C\|\rho\|_{\ell^2 L^\infty}\|f\|_{L^2} \tag{5.7}$$

$$\left\| \rho|x|^{-1} \int_0^t e^{i(t-t')L}F(t')dt' \right\|_{L_t^2 L^2} \leq C\|\rho\|_{\ell^2 L^\infty}^2\|\rho^{-1}|x|F\|_{L_t^2 L^2} \tag{5.8}$$

$$\left\| \rho|x|^{-1/2}|D|^{1/2} \int_0^t e^{i(t-t')L}F(t')dt' \right\|_{L_t^2 L^2} \leq C\|\rho\|_{\ell^2 L^\infty}^2\|\rho^{-1}|x|^{1/2}|D|^{-1/2}F\|_{L_t^2 L^2} \tag{5.9}$$

with a constant C independent of ρ .

If in addition $|x|\partial\rho \in \ell^2 L^\infty$, then the previous estimates hold with the operator $\rho|x|^{-1/2}|D|^{1/2}$ replaced by $|D|^{1/2}\rho|x|^{-1/2}$ in (5.7) and (5.9), the operator $\rho^{-1}|x|^{1/2}|D|^{-1/2}$ replaced by $|D|^{-1/2}\rho|x|^{1/2}$ in the last term in (5.9), and the norm $\|\rho\|_{\ell^2 L^\infty}$ replaced by $\|\rho + |x|\partial\rho\|_{\ell^2 L^\infty}$. In particular we have

$$\||D|^{1/2}\rho|x|^{-1/2}e^{itL}f\|_{L_t^2 L^2} \lesssim \|f\|_{L^2}. \tag{5.10}$$

Before stating the corresponding estimates for the wave equation we prove a simple bound for the powers of the operator L :

Lemma 5.7. *Assume $-L = (\partial+iA)^2 - V$ is selfadjoint and non negative in $L^2(\mathbb{R}^n)$, $n \geq 3$, and that $|V| + |A|^2 \lesssim |x|^{-2}$. Then for all $0 \leq s \leq 1$ we have*

$$\|L^{s/2}v\|_{L^2} \lesssim \||D|^s v\|_{L^2}, \quad \||D|^{-s}v\|_{L^2} \lesssim \|L^{-s/2}v\|_{L^2}. \tag{5.11}$$

Proof. The second estimate is equivalent to the first one by duality. It is sufficient to prove the first estimate for $s = 1$ and then interpolate with the trivial case $s = 0$. When $s = 1$ we have

$$\|L^{1/2}v\|_{L^2} = (Lv, v) \lesssim \|\partial v\|_{L^2}^2 + \||x|^{-1}v\|_{L^2}^2 \lesssim \| |D|v\|_{L^2}^2$$

by Hardy’s inequality. □

For the wave flow $e^{it\sqrt{L}}$ and the Klein–Gordon flow $e^{it\sqrt{L+1}}$ we have then:

Corollary 5.8 (Smoothing for Wave–K–G). *Under the assumptions of either Theorem 4.1 or Corollary 4.6, we have for any $\rho > 0$ in ℓ^2L^∞*

$$\|\rho|x|^{-1}e^{it\sqrt{L}}f\|_{L_t^2L^2} + \|\rho|x|^{-1/2}|D|^{1/2}e^{it\sqrt{L}}f\|_{L_t^2L^2} \leq C\|\rho\|_{\ell^2L^\infty}\|L^{1/4}f\|_{L^2} \quad (5.12)$$

where the last term can be estimated by $C'\|\rho\|_{\ell^2L^\infty}\|f\|_{\dot{H}^{1/2}}$,

$$\left\| \rho|x|^{-1} \int_0^t \frac{e^{i(t-t')\sqrt{L}}}{\sqrt{L}} F(t') dt' \right\|_{L_t^2L^2} \leq C\|\rho\|_{\ell^2L^\infty} \|\rho^{-1}|x|F\|_{L_t^2L^2} \quad (5.13)$$

$$\left\| \rho|x|^{-1/2}|D|^{1/2} \int_0^t \frac{e^{i(t-t')\sqrt{L}}}{\sqrt{L}} F(t') dt' \right\|_{L_t^2L^2} \leq C\|\rho\|_{\ell^2L^\infty} \|\rho^{-1}|x|^{1/2}|D|^{-1/2}F\|_{L_t^2L^2} \quad (5.14)$$

with constants C, C' independent of ρ .

If in addition $|x|\partial\rho \in \ell^2L^\infty$, then the previous estimates hold with the operator $\rho|x|^{-1/2}|D|^{1/2}$ replaced by $|D|^{1/2}\rho|x|^{-1/2}$ in (5.12) and (5.14), $\rho^{-1}|x|^{1/2}|D|^{-1/2}$ replaced by $|D|^{-1/2}\rho|x|^{1/2}$ in the last term in (5.14), and the norm $\|\rho\|_{\ell^2L^\infty}$ replaced by $\|\rho + |x|\partial\rho\|_{\ell^2L^\infty}$. In particular we have

$$\||D|^{1/2}\rho|x|^{-1/2}e^{it\sqrt{L}}f\|_{L_t^2L^2} \lesssim \|L^{1/4}f\|_{L^2}. \quad (5.15)$$

The same estimates hold if we replace L by $L+1$ everywhere; in this case the last term in (5.12) must be estimated by $C'\|\rho\|_{\ell^2L^\infty}\|f\|_{\dot{H}^{1/2}}$ (nonhomogeneous norm).

Proof. Estimate (5.12) follows from Corollary 5.5, (5.5) and (5.11) with $s = 1/2$. Estimates (5.13), (5.14) are a direct application of (5.6). The other claims are proved in a similar way. □

We note that the previous smoothing estimates can be put into a scale invariant (but equivalent) form. Indeed, one has the equivalence

$$\|v\|_{\dot{Y}} = \sup_{\|\rho\|_{\ell^\infty L^2}=1} \|\rho|x|^{-1/2}v\|_{L^2} \quad (5.16)$$

which is obtained choosing $\rho = \mathbf{1}_{C_j}$ with $C_j = \{x: 2^j \leq |x| < 2^{j+1}\}$ and taking the supremum over $j \in \mathbb{Z}$. Thus we obtain the following result:

Corollary 5.9. *Under the assumptions of either Theorem 4.1 or Corollary 4.6, we have*

$$\||x|^{-1/2}e^{itL}f\|_{\dot{Y}L_t^2} + \||D|^{1/2}e^{itL}f\|_{\dot{Y}L_t^2} \leq C\|f\|_{L^2}, \quad (5.17)$$

$$\||x|^{-1/2}e^{it\sqrt{L}}f\|_{\dot{Y}L_t^2} + \||D|^{1/2}e^{it\sqrt{L}}f\|_{\dot{Y}L_t^2} \leq C\|f\|_{\dot{H}^{1/2}}, \quad (5.18)$$

$$\||x|^{-1/2}e^{it\sqrt{L+1}}f\|_{\dot{Y}L_t^2} + \||D|^{1/2}e^{it\sqrt{L+1}}f\|_{\dot{Y}L_t^2} \leq C\|f\|_{\dot{H}^{1/2}}. \quad (5.19)$$

6. Strichartz estimates. We first prove a simple extension to Lorentz spaces of the Muckenhoupt–Wheeden weighted fractional integration estimate. In the course of the proof of Strichartz estimates we shall actually need only (6.1) and (6.3), but we included the next two Lemmas to give a simple alternative proof of the Hörmander–Plancherel identities in the Appendix of [18], which were crucial to their result.

Lemma 6.1 (Weighted Sobolev embedding). *For all $1 < p \leq q < \infty$ the following inequality holds:*

$$\|vg\|_{L^q} \leq C\|v|D|^\alpha g\|_{L^p}, \quad \frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q} \tag{6.1}$$

for all weights $v \in A_{2-\frac{1}{p}} \cap RH_q$ or, equivalently, such that $v^q \in A_{1+\frac{q}{p}}$. More generally, for any $r \in [1, \infty]$ and p, q, α as above we have the inequality in weighted Lorentz norms

$$\|vg\|_{L^{q,r}} \leq C\|v|D|^\alpha g\|_{L^{p,r}} \tag{6.2}$$

provided the weight v satisfies $v^{q+\epsilon} \in A_{1+\frac{q}{p}-\epsilon}$ for some $\epsilon > 0$.

Proof. Estimate (6.1) for $v \in A_{2-\frac{1}{p}} \cap RH_q$ is due to Muckenhoupt and Wheeden [36] (see also [4]). The equivalent condition on the weight is easy to check, see e.g. [25]. To prove the last statement, fix $\delta > 0$ sufficiently small and write

$$p_+ = \frac{p}{1-\delta p}, \quad p_- = \frac{p}{1+\delta p} \implies \frac{1}{p_\pm} = \frac{1}{p} \mp \delta \quad p_- < p < p_+$$

and similarly for q_\pm . Then (6.1) holds for the couples (p_+, q_+) and (p_-, q_-) with α unchanged, and hence by real interpolation we get (6.2), provided the weight v satisfies

$$v^{q_\pm} \in A_{1+\frac{q_\pm}{(p_\pm)'}}$$

which are implied by $w^{q_+} \in A_{1+\frac{q_+}{(p_-)'}}$ thanks to the inclusion properties of A_p classes. □

Lemma 6.2. *Let $\sigma \in L^1_{loc}(\mathbb{R}^n)$ be such that $\sigma^2 \in A_2$ and $|\partial\sigma^{-1}| \lesssim \sigma^{-1}|x|^{-1}$. Then the following operator is bounded on L^2 :*

$$|D|^{-1/2}\sigma^{-1}|D|^{1/2}\sigma. \tag{6.3}$$

If in addition $\sigma^{-n-\epsilon} \in A_{1+\frac{n}{n-2}-\epsilon}$ for some $\epsilon > 0$ then the following operator is also bounded on L^2 :

$$|D|^{1/2}\sigma^{-1}|D|^{-1/2}\sigma. \tag{6.4}$$

Proof. We prove (6.4) first. Consider the analytic family of operators

$$U_z = |D|^z\sigma|D|^{-z}\sigma^{-1}, \quad z \in \mathbb{C}, \quad 0 \leq \Im z \leq 1.$$

For $z = iy, y \in \mathbb{R}$, we have

$$\|U_{iy}f\|_{L^2} = \|\sigma|D|^{-iy}\sigma^{-1}f\|_{L^2} \lesssim \|\sigma\sigma^{-1}f\|_{L^2} = \|f\|_{L^2}$$

where we used the well-known fact that $|D|^{iy}$ is bounded in weighted L^2 if the weight is in A_2 ; note also that the implicit constant grows at most polynomially in y (actually $\lesssim |y|^{n/2}$, see e.g. [8]). On the other hand, for $z = 1 + iy, y \in \mathbb{R}$ we can write

$$\begin{aligned} \|U_{1+iy}f\|_{L^2} &\simeq \|\partial(\sigma^{-1}|D|^{-1-iy}\sigma f)\|_{L^2} \\ &\leq \| |x|^{-1}\sigma^{-1}|D|^{-1-iy}\sigma f\|_{L^2} + \|\sigma^{-1}\partial|D|^{-1-iy}\sigma f\|_{L^2}. \end{aligned}$$

Since $\sigma^{-2} \in A_2$, we have

$$\|\sigma^{-1}\partial|D|^{-1-iy}\sigma f\|_{L^2} \simeq \|\sigma^{-1}|D||D|^{-1-iy}\sigma f\|_{L^2} \simeq \|f\|_{L^2}.$$

On the other hand,

$$\| |x|^{-1}\sigma^{-1}|D|^{-1-iy}\sigma f\|_{L^2} \lesssim \|\sigma^{-1}|D|^{-1-iy}\sigma f\|_{L^{\frac{2n}{n-2},2}} \lesssim \|\sigma^{-1}|D|^{-iy}f\|_{L^2} \simeq \|f\|_{L^2}$$

using (6.2) with the choice $q = \frac{2n}{n-2}$, $p = r = 2$ and $\alpha = 1$, provided $\sigma^{-n-\epsilon} \in A_{1+\frac{n}{n-2}-\epsilon}$. By Stein–Weiss interpolation we obtain that U_z is bounded in L^2 for all values in the strip, and this gives the claim taking $z = 1/2$.

Consider now the operator (6.3), or equivalently its adjoint, which we denote also by

$$T = \sigma|D|^{1/2}\sigma^{-1}|D|^{-1/2}.$$

To prove that T is bounded on L^2 , we use the analytic family of operators

$$U_w = \sigma|D|^{1/2}\sigma^{-1}$$

for w in the complex strip $0 \leq \Re w \leq 1$. The operator

$$U_{iy} = \sigma|D|^{iy}\sigma^{-1}$$

for $y \in \mathbb{R}$ is bounded on L^2 with a growth at most polynomial in $|y|$ as $|y| \rightarrow \infty$, provided $\sigma \in A_2$. On the other hand, for $w = 1 + iy$ we can write

$$\|U_{1+iy}f\|_{L^2} = \|\sigma|D|^{iy}|D|\sigma^{-1}f\|_{L^2} \leq \|\sigma\partial(\sigma^{-1}f)\|_{L^2}$$

and if we have the property

$$|\partial\sigma^{-1}| \lesssim \sigma^{-1}|x|^{-1} \tag{6.5}$$

we can continue

$$\|U_{1+iy}f\|_{L^2} \lesssim \| |x|^{-1}f\|_{L^2} + \|\partial f\|_{L^2} \lesssim \|\partial f\|_{L^2} \simeq \|f\|_{\dot{H}^1}$$

by Hardy’s inequality; again, the implicit constant grows at most polynomially in y . By Stein–Weiss complex interpolation we deduce the estimate

$$\|U_{1/2}f\|_{L^2} = \|\sigma|D|^{1/2}\sigma^{-1}f\|_{L^2} \lesssim \|f\|_{\dot{H}^{1/2}}$$

which implies

$$\|Tf\|_{L^2} = \|\sigma|D|^{1/2}\sigma^{-1}|D|^{-1/2}f\|_{L^2} \lesssim \|f\|_{L^2}. \quad \square$$

To handle endpoint Strichartz estimates we resort to a mixed endpoint Strichartz–smoothing estimate for the free flow, due to Ionescu and Kenig [24]:

$$\left\| \int_0^t e^{-i(t-t')\Delta} F(t') dt' \right\|_{L_t^2 L^{2^*}} \lesssim \min_{j=1,\dots,n} \| |D_j|^{-1/2} F \|_{L_t^2 L_{x_j}^1 L_{x_j'}^2}, \quad 2^* = \frac{2n}{n-2} \tag{6.6}$$

where the norm at the right are L^1 with respect to one of the coordinates and L^2 with respect to the remaining coordinates. By an easy modification of the argument in [24], as observed by Mizutani [34] one can refine (6.6) to an estimate in the Lorentz norm $L^{2^*,2}$

$$\left\| \int_0^t e^{-i(t-t')\Delta} F(t') dt' \right\|_{L_t^2 L^{2^*,2}} \lesssim \min_{j=1,\dots,n} \| |D_j|^{-1/2} F \|_{L_t^2 L_{x_j}^1 L_{x_j'}^2}; \tag{6.7}$$

moreover, if $w > 0$ is such that $w^2 \in A_2(\mathbb{R}^n)$ and there exists C such that $\int w^{-2} dx_j < C$ for $j = 1, \dots, n$ (uniformly in the remaining variables), then we have

$$\sum_{j=1}^n \| |D_j|^{-1/2} F \|_{L_t^2 L_{x_j}^1 L_{x_j'}^2} \lesssim \sum_{j=1}^n \| w |D_j|^{-1/2} F \|_{L_t^2 L^2} \simeq \| w |D|^{-1/2} F \|_{L_t^2 L^2}$$

by the usual weighted estimates for singular integrals. Thus (6.6) implies the estimate

$$\left\| \int_0^t e^{-i(t-t')\Delta} F(t') dt' \right\|_{L_t^2 L^{2^*,2}} \lesssim \| w |D|^{-1/2} F \|_{L_t^2 L^2} \tag{6.8}$$

for any weight w as above.

Theorem 6.3. *Let $n \geq 3$. Assume the operator L defined in (4.1) is selfadjoint and non negative in $L^2(\mathbb{R}^n)$, with domain $H^2(\mathbb{R}^n)$. Assume $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy for some $\delta > 0$ and $\mu > 1$*

$$\langle \log |x| \rangle^\mu \langle x \rangle^\delta |x|^2 (V - i\partial \cdot A) \in L^\infty, \quad \langle \log |x| \rangle^\mu \langle x \rangle^\delta |x| \widehat{B} \in L^\infty \tag{6.9}$$

and

$$\langle \log |x| \rangle^\mu \langle x \rangle^\delta |x| A \in L^\infty \cap \dot{H}_{2n}^{1/2}. \tag{6.10}$$

Moreover, assume 0 is not a resonance for L , in the sense of Definition 1.1.

Then the Schrödinger flow e^{itL} satisfies the endpoint Strichartz estimate

$$\| e^{itL} f \|_{L_t^2 L^{\frac{2n}{n-2},2}} \lesssim \| f \|_{L^2} \tag{6.11}$$

and the corresponding estimates in $L_t^p L^q$ for all Schrödinger admissible (p, q) ; and the nonhomogeneous estimates

$$\left\| \int_0^t e^{i(t-t')L} F(t') dt' \right\|_{L_t^p L^q} \lesssim \| F \|_{L_{L^{\tilde{p}'}, L^{\tilde{q}'}}} \tag{6.12}$$

for all Schrödinger admissible couples (p, q) and (\tilde{p}, \tilde{q}) such that $p < \tilde{p}'$.

Proof. Since the assumptions of Corollary 4.6 are satisfied, the smoothing estimates (5.7) and (5.10) are valid. The flow $u = e^{itL} f$ is the solution of

$$u(0) = f, \quad i\partial_t u + Lu \equiv i\partial_t u - \Delta_A u - Vu = 0.$$

By Duhamel's formula we can represent u in the form

$$u = e^{-it\Delta} f - i \int_0^t e^{-i(t-t')\Delta} \left[2i\partial \cdot (Au) + (V - i\partial \cdot A - |A|^2)u \right] dt'. \tag{6.13}$$

We compute the $L_t^2 L^{2^*,2}$ norm of u . To the first term in (6.13) we apply (1.6). For the remaining terms we use (6.8) and we are led to estimate

$$I = \| \sigma |D|^{-1/2} \tilde{V} u \|_{L_t^2 L^2}, \quad \tilde{V} := V - i\partial \cdot A - |A|^2,$$

$$II = \| \sigma |D|^{-1/2} \partial \cdot (Au) \|_{L_t^2 L^2} \simeq \| \sigma |D|^{1/2} (Au) \|_{L_t^2 L^2}$$

where σ is any weight on \mathbb{R}^n such that

$$\sigma^2 \in A_2, \quad \int \sigma^{-2} dx_j < \infty \quad j = 1, \dots, n. \tag{6.14}$$

The quantity I can be estimated via the weighted Sobolev embeddings (6.1): with the choices $\alpha = 1/2$, $p = \frac{2n}{n+1}$ and $q = r = 2$ we obtain

$$\| \sigma u \|_{L^2} \lesssim \| \sigma |D|^{1/2} u \|_{L^{\frac{2n}{n+1},2}} \tag{6.15}$$

provided

$$\sigma^{2+\epsilon} \in A_{2-\frac{1}{n}-\epsilon} \tag{6.16}$$

for some $\epsilon > 0$ small. Then we have, by Hölder inequality,

$$I \lesssim \|\sigma \tilde{V} u\|_{L^2_t L^{\frac{2n}{n+1}, 2}} \leq \|\sigma \rho^{-1} |x| \tilde{V}\|_{L^{2n, \infty}} \|\rho |x|^{-1} u\|_{L^2_t L^2} \lesssim \|\sigma \rho^{-1} |x| \tilde{V}\|_{L^{2n, \infty}} \|f\|_{L^2};$$

in the last step we used the smoothing estimate (5.7).

To estimate the quantity II , we first commute the multiplication operator σ with $|D|^{1/2}$. This is possible since the operator

$$T = \sigma |D|^{1/2} \sigma^{-1} |D|^{-1/2}$$

is bounded in L^2 by Lemma 6.2, provided

$$|\partial \sigma^{-1}| \lesssim \sigma^{-1} |x|^{-1}. \tag{6.17}$$

Thus we have, for any $\rho \in \ell^2 L^\infty$,

$$II \lesssim \| |D|^{1/2} \sigma(Au) \|_{L^2_t L^2} = \| |D|^{1/2} (\sigma \rho^{-1} |x|^{1/2} A)(\rho |x|^{-1/2} u) \|_{L^2_t L^2}$$

and using the Kato–Ponce inequality

$$\begin{aligned} &\lesssim \|\sigma \rho^{-1} |x|^{1/2} A\|_{L^\infty} \| |D|^{1/2} \rho |x|^{-1/2} u \|_{L^2_t L^2} \\ &\quad + \| |D|^{1/2} \sigma \rho^{-1} |x|^{1/2} A \|_{L^{2n}} \|\rho |x|^{-1/2} u\|_{L^2_t L^{\frac{2n}{n+1}}}. \end{aligned}$$

We use Sobolev embedding for the last term, and then smoothing estimate (5.10), and we arrive at

$$II \lesssim \|\sigma \rho^{-1} |x|^{1/2} A\|_{L^\infty \cap \dot{H}^{1/2}_{2n}} \|f\|_{L^2}.$$

Summing up, we have proved

$$\|e^{itL} f\|_{L^p L^q} \lesssim (1 + \|\sigma \rho^{-1} |x|^{1/2} A\|_{L^\infty \cap \dot{H}^{1/2}_{2n}} + \|\sigma \rho^{-1} |x| \tilde{V}\|_{L^{2n, \infty}}) \cdot \|f\|_{L^2}.$$

If we now add the condition

$$\rho \lesssim \sigma |x|^{-1/2} \tag{6.18}$$

then we can write

$$\|\sigma \rho^{-1} |x| A^2\|_{L^{2n, \infty}} \lesssim \|\sigma \rho^{-1} |x|^{3/2} A^2\|_{L^\infty} \lesssim \|\sigma \rho^{-1} |x|^{1/2} A\|_{L^\infty}^2$$

and the previous estimate simplifies to

$$\|e^{itL} f\|_{L^p L^q} \lesssim (1 + \|\sigma \rho^{-1} |x|^{1/2} A\|_{L^\infty \cap \dot{H}^{1/2}_{2n}}^2 + \|\sigma \rho^{-1} |x| (V - i\partial \cdot A)\|_{L^{2n, \infty}}) \cdot \|f\|_{L^2}.$$

If we choose

$$\sigma = \rho^{-1} |x|^{1/2}, \quad \rho = \langle \log |x| \rangle^{-\nu}, \quad \nu > 1/2$$

we see that $\rho \in \ell^2 L^\infty$, and (6.14), (6.16) (provided ϵ is small enough), (6.17) and (6.18) are satisfied, as it follows by direct inspection using the basic properties of Muckenhoupt classes (see e.g. [25]). Keeping into account the assumptions on the coefficients (6.9)–(6.10), we have proved (6.11).

The full range of indices (p, q) is obtained immediately by interpolation with the conservation of L^2 mass, and the nonhomogeneous estimate (6.12) is proved by a standard application of the TT^* method and the Christ–Kiselev Lemma, which is possible as long as $\tilde{p}' < p$. \square

By a gauge transformation we obtain the following slightly more general result:

Corollary 6.4. *Let $n \geq 3$ and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\partial\phi \in L^{n,\infty}$. Assume the operator L defined in (4.1) is selfadjoint and non negative in $L^2(\mathbb{R}^n)$, with domain $H^2(\mathbb{R}^n)$. Assume $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy for some $\delta > 0$ and $\mu > 1$*

$$\langle \log |x| \rangle^\mu \langle x \rangle^\delta |x|^2 (V - i\partial \cdot A - i\Delta\phi) \in L^\infty, \quad \langle \log |x| \rangle^\mu \langle x \rangle^\delta |x| \widehat{B} \in L^\infty$$

and

$$\langle \log |x| \rangle^\mu \langle x \rangle^\delta |x| (A + \partial\phi) \in L^\infty \cap \dot{H}_{2n}^{1/2}.$$

Moreover, assume 0 is not a resonance for L , in the sense of Definition 1.1.

Then the conclusion of Theorem 6.3 are still valid for the Schrödinger flow e^{itL} .

Proof. Applying the gauge transformation

$$u = e^{i\phi(x)} \tilde{u}$$

and recalling (4.11) we see that \tilde{u} is a solution of

$$\tilde{u}(0) = e^{i\phi} f, \quad i\partial_t \tilde{u} - \Delta_{\tilde{A}} \tilde{u} - V\tilde{u} = 0, \quad \tilde{A} := A + \partial\phi.$$

By assumption, V and \tilde{A} satisfy the conditions of Theorem 6.3 hence Strichartz estimates are valid for \tilde{u} . Since Lebesgue and Lorentz norms of u and \tilde{u} coincide, the proof is concluded. \square

Theorem 6.5 (Strichartz for Wave). *Under the assumptions of Theorem 6.3, or more generally the assumptions of Corollary 6.4, the wave flow $e^{it\sqrt{L}}$ satisfies the estimates*

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} e^{it\sqrt{L}} f \|_{L_t^p L^q} \lesssim \| f \|_{\dot{H}^{\frac{1}{2}}}, \quad \| |D|^{\frac{1}{q} - \frac{1}{p}} e^{it\sqrt{L}} L^{-1/2} f \|_{L_t^p L^q} \lesssim \| f \|_{\dot{H}^{-\frac{1}{2}}} \quad (6.19)$$

and

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} \int_0^t e^{i(t-t')\sqrt{L}} F(t') dt' \|_{L_t^p L^q} \lesssim \| |D|^{\frac{1}{p} - \frac{1}{q}} F \|_{L_{t'}^{\tilde{p}} L^{\tilde{q}}}. \quad (6.20)$$

for any wave admissible, non endpoint couples (p, q) and (\tilde{p}, \tilde{q}) .

Proof. When $\phi \neq 0$, as in the previous proof, we perform a gauge transform $u = e^{i\phi} \tilde{u}$; note that by Lemma 4.5 the transformation $u \mapsto \tilde{u}$ is bounded on \dot{H}_p^s and on $\dot{H}_{p'}^{-s}$ for $p < n/s$, $s \in [0, 1]$ since $\partial\phi \in L^{n,\infty}$. Thus it is sufficient to prove the Strichartz estimates for \tilde{u} . In the following we shall write $\tilde{A} = A + \partial\phi$, but we shall omit for simplicity the tilde from \tilde{u} and from $\tilde{L} = -\Delta_{\tilde{A}} + V$. Note that the wave flow satisfies the smoothing estimates (5.12) and (5.15).

The function $u = e^{it\sqrt{L}}$ solves

$$\partial_t^2 u = -Lu = \Delta u + 2i\partial \cdot (\tilde{A}u) + (V - |\tilde{A}|^2)u,$$

with data

$$u(0) = f, \quad \partial_t u(0) = i\sqrt{L}f.$$

Thus u can be represented as

$$u = \cos(t|D|)f + i \frac{\sin(t|D|)}{|D|} \sqrt{L}f + \int_0^t \frac{\sin((t-t')|D|)}{|D|} (2i\partial \cdot (\tilde{A}u) + (V - |\tilde{A}|^2)u) dt'. \quad (6.21)$$

To the first term we apply directly the free estimate

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} e^{-it|D|} f \|_{L_t^p L^q} \lesssim \| f \|_{\dot{H}^{\frac{1}{2}}} \quad (6.22)$$

To the second term we apply (6.22), obtaining

$$\| |D|^{-1} \sin(t|D|) \sqrt{L}f \|_{L_t^p L^q} \lesssim \| |D|^{-\frac{1}{2}} \sqrt{L}f \|_{L^2}. \quad (6.23)$$

Since $|V| + |\tilde{A}|^2 \lesssim |x|^{-2}$, by Hardy's inequality we have

$$\|L^{\frac{1}{2}}f\|_{L^2}^2 \leq \|(\partial + iA)v\|_{L^2}^2 + \| |V|^{\frac{1}{2}}f \|_{L^2}^2 \lesssim \|f\|_{\dot{H}^1}^2$$

and by interpolation and duality we obtain

$$\|L^{\frac{s}{2}}f\|_{L^2} \lesssim \|f\|_{\dot{H}^s}, \quad \|f\|_{\dot{H}^{-s}} \lesssim \|L^{-\frac{s}{2}}f\|_{L^2}, \quad 0 \leq s \leq 1. \quad (6.24)$$

Applying these inequalities to (6.23) we get

$$\| |D|^{-1} \sin(t|D|) \sqrt{L} f \|_{L_t^p L^q} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}.$$

Next we consider the last term in (6.21); more generally, we shall estimate two integrals of the form

$$\int_0^t \frac{e^{i(t-t')|D|}}{|D|} \partial \cdot (\tilde{A}u) dt', \quad \int_0^t \frac{e^{i(t-t')|D|}}{|D|} (V - |\tilde{A}|^2) u dt'.$$

Since we are in the non endpoint case, by a standard application of the Christ–Kiselev Lemma, it will sufficient to estimate the two untruncated integrals

$$I = e^{it|D|} \int \frac{e^{-it'|D|}}{|D|} (V - |\tilde{A}|^2) u dt', \quad II = e^{it|D|} \int \frac{e^{-it'|D|}}{|D|} \partial \cdot (\tilde{A}u) dt'.$$

If we first apply (6.22) then the dual smoothing estimate (5.12) in the elementary case $L = -\Delta$, we obtain

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} e^{it|D|} \int \frac{e^{-it'|D|}}{|D|} F(t') dt' \|_{L_t^p L^q} \lesssim \|\rho^{-1} |x|^{1/2} |D|^{-1/2} F\|_{L_t^2 L^2}. \quad (6.25)$$

This gives

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} II \|_{L_t^p L^q} \lesssim \|\rho^{-1} |x|^{1/2} |D|^{-1/2} \partial \cdot (\tilde{A}u)\|_{L_t^2 L^2} \lesssim \|\rho^{-1} |x|^{1/2} |D|^{1/2} (\tilde{A}u)\|_{L_t^2 L^2}$$

since $\rho^{-2}|x| \in A_2$. We can commute $\rho^{-1}|x|^{1/2}$ with $|D|^{1/2}$ as in the proof of Theorem 6.3; we get

$$\lesssim \| |D|^{1/2} \rho^{-1} |x|^{1/2} (\tilde{A}u)\|_{L_t^2 L^2} = \| |D|^{1/2} (\rho^{-2} |x| \tilde{A}) (\rho |x|^{-1/2} u)\|_{L_t^2 L^2}$$

and by the Kato–Ponce inequality

$$\lesssim \|\rho^{-2} |x| \tilde{A}\|_{L^\infty} \| |D|^{1/2} \rho |x|^{-1/2} u \|_{L_t^2 L^2} + \| |D|^{1/2} \rho^{-2} |x| \tilde{A}\|_{L^{2n}} \|\rho |x|^{-1/2} u\|_{L_t^2 L^{\frac{2n}{n-1}}}.$$

Applying (6.1) to the last term and recalling assumptions (6.19), (6.20) we obtain

$$\lesssim \| |D|^{1/2} \rho |x|^{-1/2} u \|_{L_t^2 L^2} + \|\rho |x|^{-1/2} |D|^{1/2} u\|_{L_t^2 L^2} \lesssim \|L^{1/4} f\|_{L^2} \lesssim \|f\|_{\dot{H}^{1/2}}$$

by the smoothing estimates (5.12) and (5.15).

For the remaining term I we have, again by (6.25).

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} I \|_{L_t^p L^q} \lesssim \|\rho^{-1} |x|^{1/2} |D|^{-1/2} (V - |\tilde{A}|^2) u\|_{L_t^2 L^2}.$$

Then we repeat exactly the same steps as in the estimate of the term I in the proof of Theorem 6.3 (with $\sigma = \rho^{-1}|x|^{1/2}$), and we arrive at

$$\lesssim \|\rho |x|^{-1/2} |D|^{1/2} u\|_{L_t^2 L^2} \lesssim \|L^{1/4} f\|_{L^2} \lesssim \|f\|_{L^2}.$$

using (5.12). Summing up, we have proved the first estimate in (6.19).

The proof of the second estimate in (6.19) is completely identical: indeed, it is sufficient to notice that $u = \sin(t\sqrt{L})L^{-1/2}$ solves

$$\partial_t^2 u = -Lu, \quad u(0) = 0, \quad \partial_t u(0) = f.$$

The proof of the nonhomogeneous estimate (6.20) follows as usual by a TT^* argument and the Christ–Kiselev Lemma (since $\tilde{p}' < 2 < p$). \square

Theorem 6.6 (Strichartz for K–G). *Under the assumptions of Theorem 6.3, or of Corollary 6.4, the Klein–Gordon flow $e^{it\sqrt{L+1}}$ satisfies the estimates*

$$\|\langle D \rangle^{\frac{1}{q} - \frac{1}{p}} e^{it\sqrt{L+1}} f\|_{L_t^p L^q} \lesssim \|f\|_{H^{\frac{1}{2}}}, \quad \|\langle D \rangle^{\frac{1}{q} - \frac{1}{p}} e^{it\sqrt{L+1}} L^{-1/2} f\|_{L_t^p L^q} \lesssim \|f\|_{H^{-\frac{1}{2}}} \quad (6.26)$$

and

$$\|\langle D \rangle^{\frac{1}{q} - \frac{1}{p}} \int_0^t e^{i(t-t')\sqrt{L+1}} F(t') dt'\|_{L_t^p L^q} \lesssim \|\langle D \rangle^{\frac{1}{p} - \frac{1}{q}} F\|_{L^{\tilde{p}'} L^{\tilde{q}'}}. \quad (6.27)$$

for any wave or Schrödinger admissible, non endpoint couples (p, q) and (\tilde{p}, \tilde{q}) .

Proof. The proof is almost identical to the proof of Theorem 6.5, and is based on the estimate

$$\|\langle D \rangle^{\frac{1}{q} - \frac{1}{p}} e^{it\sqrt{1-\Delta}} f\|_{L_t^p L_x^q} \lesssim \|f\|_{H^{\frac{1}{2}}(\mathbb{R}^m)}$$

which holds both if the couple (p, q) is wave admissible and if it is Schrödinger admissible. A complete proof for Schrödinger admissible (p, q) can be found e.g. in the Appendix of [13], while for wave admissible indices the proof is obtained starting from the estimate

$$j \geq 1, \quad \phi_j \in \mathcal{S}, \quad \text{spt } \widehat{\phi}_j = \{|\xi| \sim 2^j\} \quad \implies \quad \|e^{it\sqrt{1-\Delta}} \phi_j\|_{L^\infty(\mathbb{R}^m)} \lesssim |t|^{-\frac{m-1}{2}} 2^{\frac{m+1}{2}}$$

(see [6]) and then applying the usual Ginibre–Velo procedure. \square

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