

SINGULAR SPACES, GROUPOIDS AND METRICS OF POSITIVE SCALAR CURVATURE

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ABSTRACT. We define and study, under suitable assumptions, the fundamental class, the index class and the rho class of a spin Dirac operator on the regular part of a spin stratified pseudomanifold. More singular structures, such as singular foliations, are also treated. We employ groupoid techniques in a crucial way; however, an effort has been made in order to make this article accessible to readers with only a minimal knowledge of groupoids. Finally, whenever appropriate, a comparison between classical microlocal methods and groupoids methods has been provided.

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1. INTRODUCTION

1.1. Smooth closed manifolds and the Higson-Roe analytic surgery sequence.

Let (X, g) be a spin riemannian manifold with fundamental group $\pi_1(X) =: \Gamma$. We fix a spin structure s . We denote by X_Γ its universal cover. We consider \mathcal{D}_g , the Dirac operator associated to g and s , and \mathcal{D}_g^Γ its Γ -equivariant lift to X_Γ . Thanks to the work of Atiyah, Kasparov, Mishchenko and many others we know that we can attach to these data two K-theory classes:

- the fundamental class $[\mathcal{D}_g] \in K_*(X)$;
- the index class $\text{Ind}(\mathcal{D}_g^\Gamma) \in K_*(C_r^*\Gamma)$

with $*$ = $\dim X$. The index class is particularly interesting in connection with the existence of metrics of positive scalar curvature; indeed, on the one hand the class $\text{Ind}(\mathcal{D}_g^\Gamma) \in K_*(C_r^*\Gamma)$ is independent of the choice of g and, on the other hand, $\text{Ind}(\mathcal{D}_g^\Gamma) = 0$ if g has positive scalar curvature, a result based on a generalization to the C^* -context of the well-known Lichnerowicz argument. See

for example [Ros83, Theorem 1.1] for the details. This means that the class $\text{Ind}(\not{D}_g^\Gamma) \in K_*(C_r^*\Gamma)$ is an *obstruction* to the existence of a metric of positive scalar curvature on X ¹.

There are in fact many equivalent realizations of these classes. One that is particularly elegant and that we would like to single out at this point is due to Nigel Higson and John Roe; it stems from a long exact sequence of K -theory groups for C^* -algebras. The C^* -algebras in question are $C^*(X_\Gamma)^\Gamma$ and $D^*(X_\Gamma)^\Gamma$, obtained as the closures of the Γ -equivariant operators on X_Γ that satisfy a finite propagation property and, in addition, are respectively ‘locally compact’ or ‘pseudolocal’. The former C^* -algebra is an ideal in the latter; thus we have a short exact sequence of C^* -algebras $0 \rightarrow C^*(X_\Gamma)^\Gamma \rightarrow D^*(X_\Gamma)^\Gamma \rightarrow D^*(X_\Gamma)^\Gamma/C^*(X_\Gamma)^\Gamma \rightarrow 0$ which gives rise to a long exact sequence in K -theory known as the *analytic surgery sequence* of Higson and Roe:

$$(1) \quad \cdots \rightarrow K_{m+1}(C^*(X_\Gamma)^\Gamma) \rightarrow K_{m+1}(D^*(X_\Gamma)^\Gamma) \rightarrow K_{m+1}(D^*(X_\Gamma)^\Gamma/C^*(X_\Gamma)^\Gamma) \rightarrow K_m(C^*(X_\Gamma)^\Gamma) \rightarrow \cdots$$

Crucial to the development of the theory are the canonical isomorphisms

$$(2) \quad K_{*+1}(D^*(X_\Gamma)^\Gamma/C^*(X_\Gamma)^\Gamma) = K_*(X) \quad \text{and} \quad K_*(C^*(X_\Gamma)^\Gamma) = K_*(C_r^*\Gamma);$$

using these, the analytic surgery sequence can be rewritten as

$$(3) \quad \cdots \rightarrow K_{m+1}(C_r^*\Gamma) \rightarrow K_{m+1}(D^*(X_\Gamma)^\Gamma) \rightarrow K_m(X) \rightarrow K_m(C_r^*\Gamma) \rightarrow \cdots$$

By employing ellipticity and the finite propagation speed property for the wave operator associated to \not{D}_g^Γ , one can define an element in $K_{*+1}(D^*(X_\Gamma)^\Gamma/C^*(X_\Gamma)^\Gamma)$ and this element corresponds precisely to the fundamental class $[\not{D}_g]$ under the first isomorphism in (2). The index class is then realized as

$$(4) \quad \text{Ind}(\not{D}_g^\Gamma) = \delta[\not{D}_g] \in K_*(C^*(X_\Gamma)^\Gamma) = K_*(C_r^*\Gamma),$$

with $\delta : K_{*+1}(D^*(X_\Gamma)^\Gamma/C^*(X_\Gamma)^\Gamma) \rightarrow K_*(C^*(X_\Gamma)^\Gamma)$ the connecting homomorphism in the analytic surgery sequence.

Assume now that g is of positive scalar curvature; then we know that $\delta[\not{D}_g] = 0$ in $K_*(C^*(X_\Gamma)^\Gamma)$; this means that there exists a *lift* of $[\not{D}_g]$ to $K_{*+1}(D^*(X_\Gamma)^\Gamma)$. In fact, since the Lichnerowicz argument shows that the operator \not{D}_g^Γ is L^2 -invertible, it is easy to see that there exists a natural lift; we call this lift the *rho class of the positive scalar curvature metric g* and denote it by $\rho(g) \in K_{*+1}(D^*(X_\Gamma)^\Gamma)$. This is a (higher) *secondary invariant* of \not{D}_g^Γ . Thus, if g is of positive scalar curvature

$$K_{*+1}(D^*(X_\Gamma)^\Gamma) \longrightarrow K_{*+1}(D^*(X_\Gamma)^\Gamma/C^*(X_\Gamma)^\Gamma) \longrightarrow K_*(C^*(X_\Gamma)^\Gamma)$$

$$\rho(g) \xleftarrow{\cdots\cdots\cdots} [\not{D}_g] \xrightarrow{\delta} 0$$

Summarizing, given an arbitrary metric g we can define

$$(5) \quad [\not{D}_g] \in K_*(X) \quad \text{and} \quad \text{Ind}(\not{D}_g^\Gamma) = \delta[\not{D}_g] \in K_*(C_r^*\Gamma)$$

and if g is of positive scalar curvature metric then we have

$$(6) \quad \rho(g) \in K_{*+1}(D^*(X_\Gamma)^\Gamma)$$

¹One can also consider a more refined object, the class defined by the real Dirac operator in $KO_*(C_{r,\mathbb{R}}^*(\Gamma))$; however, in this paper we shall bound ourselves to the complex case.

as a lift of $[\not{D}_g]$ in (3). It should be noticed that if g is of positive scalar curvature then the classic APS rho invariant and the Cheeger-Gromov rho invariant can be reobtained from $\rho(g)$ by applying suitable traces to $K_{*+1}(D^*(X_\Gamma)^\Gamma)$. This is work of Higson-Roe for the former invariant, see [HR10], and Benamou-Roy for the latter, see [BR15]. Conjecturally, the rho class defined above should also give Lott's higher rho invariant $\rho_{\text{Lott}}(g)$, see [Lot92], an element in the delocalized noncommutative de Rham homology of the Connes-Moscovici algebra of Γ .

Building on fundamental work of Higson and Roe for the signature operator, see [HR05a], [HR05b], [HR05c], it was proved in [PS14] that these three classes can be employed in order to map Stolz' surgery sequence for positive scalar curvature metrics to the analytic surgery sequence (3). In particular, one can prove that the rho class gives well-defined maps

$$(7) \quad \rho : \pi_0(\mathcal{R}^+(X)) \rightarrow K_{\dim X+1}(D^*(X_\Gamma)^\Gamma) \quad \text{and} \quad \rho : \tilde{\pi}_0(\mathcal{R}^+(X)) \rightarrow K_{\dim X+1}(D^*(X_\Gamma)^\Gamma)$$

with $\mathcal{R}^+(X)$ denoting the space of positive scalar curvature metrics on X and $\tilde{\pi}_0(\mathcal{R}^+(X))$ the associated set of concordance classes.

Crucial to the proof of the well-definedness of (7) is the *delocalized Atiyah-Patodi-Singer index theorem* proved in [PS14], a result that establishes a link between the index class of a spin riemannian manifold with boundary, with the boundary having positive scalar curvature, and the rho class of the boundary metric.

An alternative treatment of these results was later given by Xie and Yu, see [XY14], using Yu's localization algebra.

We cannot end this subsection without mentioning the very interesting geometric applications of these various rho invariants to the world of positive scalar curvature metrics. Without entering into the details we list here a number of articles that use (higher) rho invariants in order to *distinguish* metrics of positive scalar curvature up to bordism, concordance or isotopy: Botvinnik-Gilkey [BG95], Leichtnam-Piazza [LP01], Azzali-Wahl [AW], Benamou-Mathai [BM15], Piazza-Schick [PS07], Zeidler [Zei16], Weinberger-Yu [WY15], Xie-Yu [XY17], Barcenas-Zeidler [BZ17], Xie-Yu-Zeidler [XYZ17], Zenobi [Zen17].

1.2. Enter groupoids.

There is yet another approach to the three classes

$$[\not{D}_g], \quad \text{Ind}(\not{D}_g^\Gamma) \quad \text{and} \quad \rho(g)$$

and this alternative approach involves Lie groupoids in a fundamental way. It was developed by Zenobi in his Ph.D. thesis, see [Zen] [Zen16]. The cornerstone of this method is the concept of adiabatic deformation of a Lie groupoid, a concept first introduced by Alain Connes in [Con94] and then developed by many others in different contexts; this will be treated later in the paper, but we want to give at least the basic idea now. First, a couple of preliminary remarks:

(i) the quantization of a symbol defines a Poincaré isomorphism

$$(8) \quad PD : K^*(T^*X) \rightarrow K_*(X),$$

see [Kas95, CS84]. Thus we can equivalently think of the fundamental class $[\not{D}_g]$ as an element in $K^*(T^*X)$ which is in turn isomorphic to $K_*(C_0(T^*X))$.

(ii) The index class $\text{Ind}(\not{D}_g^\Gamma)$ is defined using a Γ -equivariant parametrix with remainders that are given by Γ -invariant smooth kernels on $X_\Gamma \times X_\Gamma$ with compact support in $X_\Gamma \times X_\Gamma / \Gamma := X_\Gamma \times_\Gamma X_\Gamma$. Thus we are led to consider the groupoid $G \rightrightarrows X$,

$$G := X_\Gamma \times_\Gamma X_\Gamma \rightrightarrows X$$

with source and range maps given by $s[x, y] = p(y)$ and $r[x, y] = p(x)$ where $p : X_\Gamma \rightarrow X$ is the projection of the Galois covering. We shall give the necessary definitions later. Consider now the *adiabatic deformation* of $G \equiv X_\Gamma \times_\Gamma X_\Gamma \rightrightarrows X$. This is the groupoid $G_{ad}^{[0,1]}$ over X given, as a set, by

$$TX \times \{0\} \cup X_\Gamma \times_\Gamma X_\Gamma \times (0, 1] \rightrightarrows X$$

with range and source maps given at 0 by the projection $\pi : TX \rightarrow X$ and by r and s introduced above on the rest of the deformation². One can define a natural smooth structure on the set $TX \times \{0\} \cup X_\Gamma \times_\Gamma X_\Gamma \times (0, 1]$. We shall in fact be interested in the groupoid

$$G_{ad}^{[0,1]} := TX \times \{0\} \cup X_\Gamma \times_\Gamma X_\Gamma \times (0, 1] \rightrightarrows X$$

Now, as we shall explain, one can attach a C^* -algebra to a groupoid; the evaluation at 0 for $G_{ad}^{[0,1]}$ then induces a short exact sequence of C^* -algebras

$$0 \rightarrow C_r^*(X_\Gamma \times_\Gamma X_\Gamma) \otimes C_0(0, 1) \rightarrow C_r^*(G_{ad}^{[0,1]}) \xrightarrow{\text{ev}_0} C_r^*(TX) \rightarrow 0$$

with associated long exact sequence in K-theory given by

$$(9) \quad \begin{aligned} \cdots \rightarrow K_*(C_r^*(X_\Gamma \times_\Gamma X_\Gamma) \otimes C_0(0, 1)) &\rightarrow K_*(C_r^*(G_{ad}^{[0,1]})) \xrightarrow{(\text{ev}_0)_*} K_*(C_r^*(TX)) \\ &\xrightarrow{\delta_{ad}} K_{*+1}(C_r^*(X_\Gamma \times_\Gamma X_\Gamma) \otimes C_0(0, 1)) \rightarrow \cdots \end{aligned}$$

In the above long exact sequence $TX \xrightarrow{\pi} X$ is viewed as a groupoid, $TX \rightrightarrows X$, with $r = \pi = s$ and $C_r^*(TX)$ is the C^* -algebra of this groupoid. It is easy to see that, as C^* -algebras, $C_r^*(TX) \simeq C_0(T^*X)$; we obtain in this way natural identifications $K_*(C_r^*(TX)) = K_*(C_0(T^*X)) = K^*(T^*X)$. It is also true, and easy to prove, that if the action of Γ is cocompact then $K_*(C_r^*(X_\Gamma \times_\Gamma X_\Gamma))$ is naturally isomorphic to $K_*(C_r^*\Gamma)$ so that there exists a Bott isomorphism

$$\beta : K_{*+1}(C_r^*(X_\Gamma \times_\Gamma X_\Gamma) \otimes C_0(0, 1)) \rightarrow K_*(C_r^*(X_\Gamma \times_\Gamma X_\Gamma)) = K_*(C_r^*\Gamma).$$

Let us now go back to our spin Dirac operator: we have already stated that its principal symbol defines a class $[\sigma_{\text{pr}}(\not{D}_g)] \in K^*(T^*X) = K_*(C_r^*(TX))$ with the property that

$$(10) \quad PD([\sigma_{\text{pr}}(\not{D}_g)]) = [\not{D}_g] \in K_*(X).$$

Consider the adiabatic index homomorphism, defined as the composition of δ_{ad} with the Bott isomorphism β :

$$(11) \quad \text{Ind}^{\text{ad}} := \beta \circ \delta_{ad} : K_*(C_r^*(TX)) \rightarrow K_{\dim X}(C_r^*(X_\Gamma \times_\Gamma X_\Gamma)).$$

We define the adiabatic index class of \not{D}_g^Γ as

$$(12) \quad \text{Ind}^{\text{ad}}(\not{D}_g^\Gamma) := \beta \circ \delta_{ad}[\sigma_{\text{pr}}(\not{D}_g)] \in K_{\dim X}(C_r^*(X_\Gamma \times_\Gamma X_\Gamma)).$$

It can be proved that the

$$(13) \quad \text{Ind}^{\text{ad}}(\not{D}_g^\Gamma) = \text{Ind}(\not{D}_g^\Gamma) \in K_{\dim X}(C_r^*\Gamma)$$

²Each vector bundle $E \xrightarrow{\pi} X$ gives in a natural way a groupoid, with both the range and the source map equal to π ; the groupoid law is then the vector sum along the fibers of E .

under the identification $K_*(C_r^*(X_\Gamma \times_\Gamma X_\Gamma)) = K_*(C_r^*\Gamma)$. In fact, more is true, in that we have a commutative diagram

$$(14) \quad \begin{array}{ccc} K_*(C_r^*(TX)) & \xrightarrow{\text{Ind}^{ad}} & K_*(C_r^*(X_\Gamma \times_\Gamma X_\Gamma)) \\ \downarrow PD & & \downarrow \cong \\ K_*(X) & \xrightarrow{\mu} & K_*(C_r^*\Gamma) \end{array}$$

where μ is the assembly map and with the first vertical arrow an isomorphism (then (13) follows from (10)). Assume now that g is of positive scalar curvature; then, from (13) or, in fact, directly, we have that $\text{Ind}^{ad}(\not{D}_g^\Gamma) = 0$ in $K_{\dim X}(C_r^*(X_\Gamma \times_\Gamma X_\Gamma))$ and so we finally see that we can define the (adiabatic) rho class as a certain natural lift of $[\sigma_{\text{pr}}(\not{D}_g)] \in K_*(C_0(TX))$ to $K_*(C_r^*(G_{ad}^{[0,1]}))$:

$$(15) \quad \rho^{\text{ad}}(g) \in K_*(C_r^*(G_{ad}^{[0,1]})) \equiv K_*(C_r^*(TX \times \{0\} \cup X_\Gamma \times_\Gamma X_\Gamma \times (0,1)))$$

Thus, in this case,

$$K_*(C_r^*(G_{ad}^{[0,1]})) \xrightarrow{\text{evo}} K_*(C_r^*(TX)) \xrightarrow{\text{Ind}^{ad}} K_{\dim X}(C_r^*(X_\Gamma \times_\Gamma X_\Gamma)) = K_*(C_r^*(\Gamma))$$

$$\rho^{\text{ad}}(g) \xleftarrow{\dots\dots\dots} [\sigma_{\text{pr}}(\not{D}_g)] \xrightarrow{\dots\dots\dots} 0$$

Let us summarize the situation:

by considering the adiabatic deformation of the groupoid $X_\Gamma \times_\Gamma X_\Gamma \rightrightarrows X$ we have obtained the *adiabatic sequence*

$$\dots \rightarrow K_*(C_r^*(G_{ad}^{[0,1]})) \xrightarrow{(\text{evo})^*} K_*(C_r^*(TX)) \xrightarrow{\text{Ind}^{ad}} K_*(C_r^*(X_\Gamma \times_\Gamma X_\Gamma)) \rightarrow \dots$$

The last two groups are the receptacle of (alternative but equivalent versions of) the fundamental class and of the index class respectively, viz. $[\sigma_{\text{pr}}(\not{D}_g)]$ and $\text{Ind}^{ad}(\not{D}_g^\Gamma)$, whereas the first group is the receptacle of the rho class $\rho^{\text{ad}}(g)$ whenever g is of positive scalar curvature. This means that we can consider the three K-theory classes in the Higson-Roe descripton

$$(16) \quad [\not{D}_g] \in K_*(X), \quad \text{Ind}(\not{D}_g^\Gamma) \in K_*(C_r^*\Gamma), \quad \rho(g) \in K_{*+1}(D^*(X_\Gamma)^\Gamma)$$

or, alternatively, the three classes

$$(17) \quad [\sigma_{\text{pr}}(\not{D}_g)] \in K_*(C_0(T^*X)), \quad \text{Ind}^{ad}(\not{D}_g^\Gamma) \in K_*(C_r^*(X_\Gamma \times_\Gamma X_\Gamma)), \quad \rho^{\text{ad}}(g) \in K_*(C_r^*(G_{ad}^{[0,1]}))$$

in the adiabatic deformation picture. It should also be added at this point that a purely groupoid-proof of the delocalized APS index theorem is possible, giving, again, the well-definedness of (7) or, better, the well-definedness of the analogue maps

$$(18) \quad \rho^{\text{ad}} : \pi_0(\mathcal{R}^+(X)) \rightarrow K_{\dim X+1}(C_r^*(G_{ad}^{[0,1]})), \quad \rho^{\text{ad}} : \tilde{\pi}_0(\mathcal{R}^+(X)) \rightarrow K_{\dim X+1}(C_r^*(G_{ad}^{[0,1]}))$$

It is important to notice that the C^* -algebras entering into the adiabatic treatment are much better behaved than their coarse counterparts; for example $C_r^*(G_{ad}^{[0,1]})$ is a separable C^* -algebra whereas $D^*(X_\Gamma)^\Gamma$ is not. This is important whenever one wants to pair the rho class with suitable cyclic cocycles.

We now come to a **fundamental question**:

did we gain anything by bringing groupoids into the picture ?

The answer is *yes*, as we shall argue in the next subsection.

1.3. Why groupoids ?

The positive answer to our fundamental question is based on the following very **general principle**:

groupoid techniques usually apply to large classes of groupoids; they are not specific to a particular example.

And indeed, in [Zen] [Zen16], the above construction is applied to the adiabatic deformation of *any* Lie groupoid, not necessarily the groupoid $X_\Gamma \times_\Gamma X_\Gamma \rightrightarrows X$. In particular, a very general delocalized APS index theorem is proved using groupoid techniques. This gives, for example, the rho class of a metric on a foliation (X, \mathcal{F}) which has positive scalar curvature along the leaves. The whole present article can be seen as another manifestation of this principle: we shall define the rho class of a stratified space and of a singular foliation (under a positive scalar curvature assumption) following the above procedure for suitable groupoids; needless to say, these groupoids depend on the specific geometries that will be considered.

One might rightly wonder why this general principle is true. We single out the following reasons:

- groupoids have very strong functoriality properties; for example, they can be restricted and pulled-back;
- to any Lie groupoid we can associate a C^* -algebra; we can use the functoriality properties of groupoids in order to obtain natural C^* -algebras homomorphisms;
- to any Lie groupoid we can associate a pseudodifferential algebra;
- the combination of these last two items often produce interesting K-theory classes that are defined in great generality.

1.4. Singular structures. In this work we shall see these ideas applied to the following three geometric situations. First, we are given a pseudomanifold ${}^S X$ of depth k with a Thom-Mather stratification. Associated to ${}^S X$ we have its resolution, X , a manifold with fibered corners. See [ALMP12] and also [DLR15]. For this introduction we fix our attention on Thom-Mather pseudomanifolds of depth 1. This means, in particular, that we are given a locally compact metrizable space ${}^S X$ such that

- ${}^S X$ is the union of two smooth manifolds ${}^S X^{\text{reg}}$ and Y , the two strata;
- ${}^S X^{\text{reg}}$ is open and dense in ${}^S X$;
- Y , the singular stratum, is a smooth compact manifold;
- there is an open neighbourhood T_Y of Y in ${}^S X$, with a continuous retraction $\pi: T_Y \rightarrow Y$ and a continuous map $\rho: T_Y \rightarrow [0, +\infty[$ such that $\rho^{-1}(0) = Y$.
- there exists a smooth compact manifold L_Y , called the link associated to Y , such that T_Y is a fiber bundle over Y with fiber $C(L_Y)$, the cone over L_Y .

The resolution X associated to ${}^S X$ is defined as $X := {}^S X \setminus \rho^{-1}([0, 1))$. This is a manifold with boundary $H := \rho^{-1}(1)$ and H is the total space of a fibration $L_Y \rightarrow H \xrightarrow{\phi} Y$ with base Y and with typical fiber the link L_Y . Thus the resolution of a depth-1 Thom-Mather pseudomanifold ${}^S X$ is a manifold with fibered boundary X ; there is a natural identification between the interior of X , $\overset{\circ}{X}$, and ${}^S X^{\text{reg}}$.

We shall introduce a metric structure on our singular space by endowing ${}^S X^{\text{reg}}$, or, equivalently $\overset{\circ}{X}$, with a riemannian metric g . There are many different types of metrics that can be considered; we shall consider (rigid) fibered boundary metrics, firstly studied by Mazzeo and Melrose in [MM98]. These are metrics that in a tubular neighbourhood of the singularity Y , or equivalently, in a collar

neighborhood of the boundary of X , can be written in the following special form:

$$(19) \quad \frac{dx^2}{x^4} + \frac{\phi^* g_Y}{x^2} + h_{H/Y}$$

with x a boundary defining function for ∂X and $h_{H/Y}$ a fiber metric on the boundary fibration $L_Y \rightarrow H \equiv \partial X \xrightarrow{\phi} Y$. The vector fields dual to the metric (19) are given by

$$x^2 \partial_x, \quad x \partial_{y_1}, \dots, x \partial_{y_k}, \quad \partial_{z_1}, \dots, \partial_{z_\ell}$$

with (y_1, \dots, y_k) coordinates on Y and (z_1, \dots, z_ℓ) coordinates along the link L_Y ; these vector fields generate locally the Lie algebra of vector fields on X

$$(20) \quad \mathcal{V}_\Phi(X) = \{\xi \in \mathcal{V}_b(X) \mid \xi|_{\partial X} \text{ is tangent to the fibers of } \phi : \partial X \rightarrow Y \text{ and } \xi x \in x^2 C^\infty(X)\}$$

where $\mathcal{V}_b(X)$ is the Lie algebra of vector fields that are tangent to the boundary. The Lie algebra (20) is finitely generated and projective as a $C^\infty(X)$ -module and thus, according to Serre-Swan, there exists a smooth vector bundle ${}^\Phi TX \rightarrow X$, the Φ -tangent bundle, whose sections are precisely the vector fields in $\mathcal{V}_\Phi(X)$. A fibered boundary metric of product type as in (19) extends as a smooth metric to ${}^\Phi TX \rightarrow X$.

Keeping with the *general principle* stated in the previous subsection, we shall consider related but more general singular structures. As a first generalization of manifolds with fibered boundary we shall consider manifolds with a *foliated* boundary, already studied by Rochon [Roc12]; thus X is a manifold with boundary and there exists a foliation \mathcal{F} on ∂X . If $T\mathcal{F} \subset T(\partial X)$ is the integrable vector bundle that defines $(\partial X, \mathcal{F})$, then we shall consider the Lie algebra of vector fields

$$\mathcal{V}_\mathcal{F}(X) = \{\xi \in \mathcal{V}_b(X) \mid \xi|_{\partial X} \in C^\infty(\partial X, T\mathcal{F}) \text{ and } \xi x \in x^2 C^\infty(X)\}$$

This defines, as before, the \mathcal{F} -tangent bundle, ${}^\mathcal{F}TX \rightarrow X$; a \mathcal{F} -metric $g_\mathcal{F}$ is a metric on \mathring{X} that extends as a smooth metric on ${}^\mathcal{F}TX \rightarrow X$.

There is a further generalization of a depth-1 Thom-Mather space, or, equivalently, of a manifold with fibered boundary; we consider a foliated manifold (X, \mathcal{H}) with non-empty boundary and with \mathcal{H} transverse to the boundary ∂X . We then make the additional assumption that ∂X is foliated by a second foliation \mathcal{F} such that $\mathcal{F} \subset \mathcal{H}|_{\partial X}$. We refer to this geometric situation as a *foliation degenerating on the boundary* or, quite plainly, as a *singular foliation*.

The relevant Lie algebra of vector fields is now given by

$$\mathcal{V}_\mathcal{F}(X, \mathcal{H}) = \{\xi \in C^\infty(X, T\mathcal{H}) \cap \mathcal{V}_b(X), \quad \xi|_{\partial X} \in C^\infty(\partial X, T\mathcal{F}) \text{ and } \xi x \in x^2 C^\infty(X)\}$$

This is a finitely generated projective $C^\infty(X)$ -module and thus, according to Serre-Swan, there exists a vector bundle ${}^\mathcal{F}T\mathcal{H}$ whose sections are precisely given by $\mathcal{V}_\mathcal{F}(X, \mathcal{H})$. An admissible metric $g_\mathcal{F}^\mathcal{H}$ in this situation is a foliated metric on $(\mathring{X}, \mathcal{G}|_{\mathring{X}})$ that extends to a smooth metric on ${}^\mathcal{F}T\mathcal{H}$.

Notice that the three Lie algebras of vector fields in the three singular structures

$$\mathcal{V}_\Phi(X), \quad \mathcal{V}_\mathcal{F}(X), \quad \mathcal{V}_\mathcal{F}(X, \mathcal{H})$$

give rise to three algebras of differential operators, denoted respectively,

$$\text{Diff}_\Phi^*(X), \quad \text{Diff}_\mathcal{F}^*(X), \quad \text{Diff}_\mathcal{F}^*(X, \mathcal{H}).$$

1.5. Main goal of this article. Consider the three singular structures introduced above and consider in particular the resulting tangent bundles endowed with the corresponding admissible metrics

$$(\Phi TX, g_\Phi), \quad (\mathcal{F}TX, g_{\mathcal{F}}), \quad (\mathcal{F}T\mathcal{H}, g_{\mathcal{F}}^{\mathcal{H}}).$$

Make a spin assumption on these bundle and fix in each case a spin structure. Then we have, correspondingly, a spin-Dirac operator in each situation, denoted generically as \mathcal{D} . We have, respectively :

$$\mathcal{D} \in \text{Diff}_{\Phi}^*(X), \quad \mathcal{D} \in \text{Diff}_{\mathcal{F}}^*(X), \quad \mathcal{D} \in \text{Diff}_{\mathcal{F}}^*(X, \mathcal{H})$$

where for simplicity we have omitted the spinor bundle form the notation.

We ask ourselves the following questions:

- (i) can we define a fundamental class $[\mathcal{D}]$?
- (ii) can we define an index class ?
- (iii) if the metric satisfies a positive scalar curvature assumption, is the index class equal to zero ?
- (iv) in the latter case, can we define a rho class ?
- (v) does the rho class descend to the set of path-connected components or to the set of concordance classes of the corresponding space of metrics of positive scalar curvature, as in (7) ?

Needless to say, part of the problem is to understand in which K-theory groups these classes should live.

The main goal of this article is to provide answers to the above questions.

1.6. Results and techniques. We first concentrate on stratified pseudomanifolds, for simplicity now in the depth-1 case. We let ${}^S X$ be the singular pseudomanifold and X its resolution; we denote by Γ the fundamental group of ${}^S X$. We fix a Φ -metric g_Φ with local form near the boundary equal to

$$\frac{dx^2}{x^4} + \frac{\phi^* g_Y}{x^2} + h_{\partial X/Y}.$$

Assume additionally that the vertical tangent bundle $T(\partial X/Y)$ is spin and fix a spin structure for $(T(\partial X/Y), h_{\partial X/Y})$. Building on the Φ -calculus of Mazzeo and Melrose [MM98] and on its generalization to pseudomanifolds of arbitrary depth by Debord, Lescure and Rochon [DLR15], we show that if the metric $h_{\partial X/Y}$ has positive scalar curvature along the fibers, then \mathcal{D} and \mathcal{D}_Γ are *fully elliptic* and there is therefore a well defined fundamental class $[\mathcal{D}] \in K_*({}^S X)$ and an index class $\text{Ind}(\mathcal{D}_\Gamma) \in K_*(C_r^* \Gamma)$. Moreover, if g_Φ is of positive scalar curvature, then $\text{Ind}(\mathcal{D}_\Gamma) = 0$ in $K_*(C_r^* \Gamma)$. We then bring groupoids into the picture and show that these two classes can also be described through a suitable adiabatic groupoid (this groupoid already appears in [DLR15]). For the fundamental class this equivalent description of $[\mathcal{D}] \in K_*({}^S X)$ was already known; see [DLR15]. For the index class we give a very detailed proof of this compatibility, building on ideas of Monthubert-Pierrot, Debord and Skandalis, see [MP97, DS17]. Having given an adiabatic groupoid description of $[\mathcal{D}] \in K_*({}^S X)$ and of $\text{Ind}(\mathcal{D}_\Gamma) \in K_*(C_r^* \Gamma)$ we now **define** the rho class of a positive scalar curvature metric g_Φ as the adiabatic rho class. Using the delocalized APS index theorem for general Lie groupoids, due to Zenobi, we then prove that this rho class enjoys the usual stability properties on the space of positive scalar curvature Φ -metrics; for example, we establish the analogue of (18). The advantage of this approach through groupoids is that it can be extended with a relatively small effort to the other two more singular situations; this is of course in accordance with the general principle put forward in this introduction and it is in fact one of the main reasons to give a groupoid treatment of the three K-theoretic invariants in the stratified setting. In generalizing to the foliated case we shall take advantage of the results contained in the

recent paper of Debord and Skandalis [DS17]; it is also because of their results that we are able to give a unified treatment of the three singular structures. Notice that in the foliated case all the three classes are defined *directly* in the adiabatic context. More information about the content of this paper can be gathered from the description of the single sections given below.

The paper is organized as follows. In Section 2 we recall the basics about stratified pseudomanifolds and their resolutions into manifolds with corners with an iterated fibration structure (briefly, manifolds with fibered corners). Section 3 is devoted to Φ -geometry; thus we introduce the Φ -vector fields on a manifold with fibered corners, the associated Φ -tangent bundle and the metrics that we shall consider, the Φ -metrics. Section 4 is a brief introduction to groupoids and algebroids. In Section 5 we present the groupoid associated to a stratified pseudomanifold. In Section 6 we introduce the Φ -pseudodifferential algebra, both using classical microlocal techniques and as the pseudodifferential algebra associated to the groupoid defined by our stratified pseudomanifold; it is in this section that fully elliptic Φ -pseudodifferential operators are introduced. In Section 7 we employ Φ -pseudodifferential operators in order to define the fundamental class and the index class of a Γ -equivariant fully elliptic Φ -pseudodifferential operator on a stratified pseudomanifold ${}^S X_\Gamma$ endowed with a free stratified cocompact Γ -action³; these are elements in $K_*^\Gamma({}^S X_\Gamma)$ and $K_*(C_r^* \Gamma)$ respectively. In Section 8 we introduce the adiabatic deformation of our groupoid and define the non-commutative symbol and the adiabatic index class associated to a Γ -equivariant fully elliptic Φ -pseudodifferential operator. We also introduce the rho-class of an invertible operator and study its fundamental properties. In Section 9 we state the main theorems relating the fundamental class and the index class defined through classic microlocal methods with the non-commutative symbol and the adiabatic index class defined through the adiabatic deformation of our groupoid. Section 10 is devoted to a detailed proof of the equality between the classical index class and the adiabatic index class in our stratified context. In Section 11 and Section 12 we treat carefully the case of spin Dirac operators and its connections with the world of positive scalar curvature metrics. Finally, in Section 13 we extend all of our results to a class of singular foliations.

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2. SMOOTHLY STRATIFIED SPACES AND THEIR RESOLUTIONS

2.1. Smoothly stratified spaces. We start by recalling the definition of a smoothly stratified pseudomanifold ${}^S X$ with Thom-Mather control data (briefly, a smoothly stratified space). The definition is given inductively.

Definition 1. *A smoothly stratified space of depth 0 is a closed manifold. Let $k \in \mathbb{N}$, assume that the concept of smoothly stratified space of depth $\leq k$ has been defined. A smoothly stratified space ${}^S X$ of depth $k + 1$ is a locally compact, second countable Hausdorff space which admits a locally*

³for example, the universal cover of a stratified pseudomanifold

finite decomposition into a union of locally closed strata $\mathfrak{S} = \{S^j\}$, where each S^j is a smooth (usually open) manifold, with dimension depending on the index j . We assume the following:

- i) If $S^i, S^j \in \mathfrak{S}$ and $S^i \cap \overline{S^j} \neq \emptyset$, then $S^i \subset \overline{S^j}$.
- ii) Each stratum S is endowed with a set of ‘control data’ T_S, π_S and ρ_S ; here T_S is a neighbourhood of S in ${}^S X$ which retracts onto Y , $\pi_S : T_S \rightarrow S$ is a fixed continuous retraction and $\rho_S : T_S \rightarrow [0, 2)$ is a ‘radial function’ on the tubular neighbourhood such that $\rho_S^{-1}(0) = S$. Furthermore, we require that if $Z \in \mathfrak{S}$ and $Z \cap T_S \neq \emptyset$, then

$$(\pi_S, \rho_S) : T_S \cap Z \rightarrow S \times [0, 2)$$

is a proper differentiable submersion.

- iii) If $W, Y, Z \in \mathfrak{S}$, and if $p \in T_Y \cap T_Z \cap W$ and $\pi_Z(p) \in T_Y \cap Z$, then $\pi_Y(\pi_Z(p)) = \pi_Y(p)$ and $\rho_Y(\pi_Z(p)) = \rho_Y(p)$.
- iv) If $Y, Z \in \mathfrak{S}$, then

$$\begin{aligned} Y \cap \overline{Z} \neq \emptyset &\Leftrightarrow T_Y \cap Z \neq \emptyset, \\ T_Y \cap T_Z \neq \emptyset &\Leftrightarrow Y \subset \overline{Z}, Y = Z \text{ or } Z \subset \overline{Y}. \end{aligned}$$

- v) There exist a family of smoothly stratified spaces (with Thom-Mather control data) of depth less than or equal to k , indexed by \mathfrak{S} , $\{L_Y, Y \in \mathfrak{S}\}$, with the property that the restriction $\pi_Y : T_Y \rightarrow Y$ is a locally trivial fibration with fibre the cone $C(L_Y)$ over L_Y (called the link over Y), with atlas $\mathcal{U}_Y = \{(\phi, \mathcal{U})\}$ where each ϕ is a trivialization $\pi_Y^{-1}(\mathcal{U}) \rightarrow \mathcal{U} \times C(L_Y)$, and the transition functions are stratified isomorphisms of $C(L_Y)$ which preserve the rays of each conic fibre as well as the radial variable ρ_Y itself, hence are suspensions of isomorphisms of each link L_Y which vary smoothly with the variable $y \in \mathcal{U}$.

If in addition we let ${}^S X_j$ be the union of all strata of dimensions less than or equal to j , and require that

- vi) ${}^S X = {}^S X_n \supseteq {}^S X_{n-1} = {}^S X_{n-2} \supseteq {}^S X_{n-3} \supseteq \dots \supseteq {}^S X_0$ and ${}^S X \setminus {}^S X_{n-2}$ is dense in ${}^S X$

then we say that \widehat{X} is a stratified pseudomanifold.

The depth of a stratum S is the largest integer k such that there is a chain of strata $S = S^k, \dots, S^0$ with $S^{j-1} \subset \overline{S^j}$ for $1 \leq j \leq k$. A stratum of minimal depth is always a closed manifold. The maximal depth of any stratum in ${}^S X$ is called the depth of ${}^S X$ as a stratified space. (We follow here the convention of [DLR15], given that we shall use heavily this paper.)

We refer to the dense open stratum of a stratified pseudomanifold ${}^S X$ as its regular set, and the union of all other strata as the singular set,

$$\text{reg}({}^S X) := {}^S X \setminus \text{sing}({}^S X), \quad \text{where} \quad \text{sing}({}^S X) = \bigcup_{\substack{S \in \mathfrak{S} \\ \text{depth } S > 0}} S.$$

In this paper, we shall often for brevity refer to a smoothly stratified pseudomanifold with Thom-Mather control data as a *smoothly stratified space*.

2.2. Iterated fibration structures. Let X be a manifold with corners. We assume that each boundary hypersurface $H \subset X$ is embedded in X . This means that there exists a boundary defining function $x_H \in C^\infty(X)$ such that $x_H^{-1}(0) = H$, x_H is positive on $X \setminus H$ and the differential dx_H is nowhere zero on H . In such a situation, one can choose a smooth retraction $r_H : \mathcal{N}_H \rightarrow H$, where \mathcal{N}_H is a (tubular) neighborhood of H in X such that $(r_H, x_H) : \mathcal{N}_H \rightarrow H \times [0, 1)$ is a diffeomorphism on its image. We call $(\mathcal{N}_H, r_H, x_H)$ a tube system for H . A smooth map $f : X \rightarrow Y$ between manifolds with corners is said to be a fibration if it is a locally trivial surjective submersion.

Definition 2. Let X be a compact manifold with corners and H_1, \dots, H_k an exhaustive list of its set of boundary hypersurfaces M_1X . Suppose that each boundary hypersurface H_i is the total space of a smooth fibration $\phi_i: H_i \rightarrow S_i$ where the base S_i is also a compact manifold with corners. The collection of fibrations $\phi = (\phi_1, \dots, \phi_k)$ is said to be an iterated fibration structure if there is a partial order on the set of hypersurfaces such that

- (1) for any subset $I \subset \{1, \dots, k\}$ with $\bigcap_{i \in I} H_i \neq \emptyset$, the set $\{H_i | i \in I\}$ is totally ordered.
- (2) If $H_i < H_j$, then $H_i \cap H_j \neq \emptyset$, $\phi_i: H_i \cap H_j \rightarrow S_i$ is a surjective submersion and $S_{ji} := \phi_j(H_i \cap H_j) \subset S_j$ is a boundary hypersurface of the manifold with corners S_j . Moreover, there is a surjective submersion $\phi_{ji}: S_{ji} \rightarrow S_i$ such that on $H_i \cap H_j$ we have $\phi_{ji} \circ \phi_j = \phi_i$.
- (3) The boundary hypersurfaces of S_j are exactly the S_{ji} with $H_i < H_j$. In particular if H_i is minimal, then S_i is a closed manifold.

We shall refer to a manifold with corners endowed with an iterated fibration structure as a *manifold with fibered corners*. For more on the notion of manifold with fibered corners, a notion due to Richard Melrose, the reader is referred to [ALMP12], [DLR15].

2.3. The resolution of a stratified space. If ${}^S X$ is a smoothly stratified pseudomanifold then, as explained in detail in [ALMP12], it is possible to resolve ${}^S X$ into a manifold with fibered corners X . More precisely, there exists a manifold with fibered corners X and a continuous surjective map $\beta: X \rightarrow {}^S X$ which restrict to a diffeomorphism from the interior of X onto the regular part of ${}^S X$. The construction of X is iterative, by means of radial blow-ups. If ${}^S X$ is normal (i.e. the links are connected), then the boundary hypersurfaces of X correspond bijectively to the strata of ${}^S X$: to each stratum Y there is an associated boundary hypersurface H_Y which is a fibration with base the resolution of the closure of Y , a stratified space itself, and fibers the resolution of the links of Y . If the links are not connected then each stratum contributes to a *collective boundary hypersurface*, see [ALMP16] and references therein. For simplicity we bound ourselves to the normal case; we thus have a bijection between the strata of ${}^S X$ and the boundary hypersurfaces of X .

Notice that the process of replacing ${}^S X$ by X is already present in Thom's seminal work [Tho69], and versions of it have appeared in Verona's 'resolution of singularities' [Ver84] and the 'déplissage' of Brasselet-Hector-Saralegi [BHS91]. These constructions show that any smoothly stratified space can be resolved to a smooth manifold, possibly with corners; the additional information given in [ALMP12] is the existence of an iterated boundary fibration structure on the resolved manifold with corners X .

2.4. Galois coverings. Let Γ be a finitely generated discrete group. Assume now that $\Gamma - {}^S X_\Gamma \rightarrow {}^S X$ is a Galois Γ -covering of a smoothly stratified space ${}^S X$. Since ${}^S X_\Gamma$ and ${}^S X$ are locally homeomorphic we can endow ${}^S X_\Gamma$ with the structure of a smoothly stratified space in such a way that the projection map ${}^S X_\Gamma \rightarrow {}^S X$ be a stratified map. Notice that the strata of ${}^S X_\Gamma$ are the lifts of the strata of ${}^S X$ and that the link of a stratum $Y \subset {}^S X$ is the same as the link of the lifted stratum, Y_Γ , in ${}^S X_\Gamma$. Let X_Γ be the resolution of ${}^S X_\Gamma$; following the inductive procedure that defines the resolution of a stratified space it is not difficult to see that it is possible to lift the action of Γ from ${}^S X_\Gamma$ to X_Γ . Moreover, with this induced action, X_Γ is a Galois covering of X and the interior of X_Γ , which is the regular part of ${}^S X_\Gamma$, is a Galois Γ -cover of the interior of X , which is in turn the regular part of ${}^S X$. In fact, the action of Γ respects the boundary fibrations structures of X_Γ and X : if $H \xrightarrow{\phi} S$ is a boundary hypersurface of X corresponding to a stratum Y of ${}^S X$ and if p denotes the quotient map induced by the action, then there is a commutative diagram

$$(21) \quad \begin{array}{ccc} H_\Gamma & \xrightarrow{p} & H \\ \downarrow \phi_\Gamma & & \downarrow \phi \\ S_\Gamma & \xrightarrow{p} & S \end{array}$$

where $H_\Gamma \xrightarrow{p} H$ induces a diffeomorphism on the fibers of the two fibrations, $H_\Gamma \xrightarrow{\phi_\Gamma} S_\Gamma$ and $H \xrightarrow{\phi} S$, and where S_Γ is the resolution of the closure of Y_Γ .

3. VECTOR FIELDS AND METRICS

3.1. Fibered-corners vector fields. Let X be a manifold with fibered corners as above, with fibered hypersurfaces $\phi_1: H_1 \rightarrow S_1, \dots, \phi_k: H_k \rightarrow S_k$. We assume that H_1, \dots, H_k is an exhaustive list of the boundary hypersurfaces of X . For each i , let x_i be a boundary defining function of the hypersurface H_i . The b -vector fields are the vector fields on X that are tangent to the boundary:

$$(22) \quad \mathcal{V}_b(X) = \{\xi \in C^\infty(X, TX) \ ; \ \xi x_i \in x_i C^\infty(X) \ \forall i\}$$

They form a Lie subalgebra of the Lie algebra of all vector fields on X .

We introduce the space of fibered-corners vector fields, or Φ -vector fields as

$$(23) \quad \mathcal{V}_\Phi(X) = \{\xi \in \mathcal{V}_b(X) \ \xi|_{H_i}, \text{ is tangent to the fibers of } \phi_i: H_i \rightarrow S_i \text{ and } \xi x_i \in x_i^2 C^\infty(X) \ \forall i\}$$

and we point out that they also form a Lie subalgebra.

3.2. The vector bundle ${}^\Phi TX$ associated to $\mathcal{V}_\Phi(X)$. The algebra $\mathcal{V}_\Phi(X)$ is a finitely generated projective $C^\infty(X)$ -module and thus, according to the Serre-Swan theorem, there exists a smooth vector bundle ${}^\Phi TX$ on X and a natural map $i_\Phi: {}^\Phi TX \rightarrow TX$ with the property that

$$i_\Phi(C^\infty(X; {}^\Phi TX)) = \mathcal{V}_\Phi(X)$$

where, with a small abuse of notation, we also use i_Φ for the resulting map on sections. The following properties are crucial:

- $C^\infty(X; {}^\Phi TX)$ has a Lie algebra structure;
- the map i_Φ on sections satisfies $i_\Phi[X, Y] = [i_\Phi X, i_\Phi Y]$ for all $X, Y \in C^\infty(X; {}^\Phi TX)$;
- if $f \in C^\infty(X)$ then $[X, fY] = f[X, Y] + (i_\Phi(X)f)Y$ for all $X, Y \in C^\infty(X; {}^\Phi TX)$.

As we shall see in a moment, it is possible to encode all of the above by saying that ${}^\Phi TX$ is a Lie algebroid over X with anchor map $i_\Phi: {}^\Phi TX \rightarrow TX$.

3.3. Metrics. Consider a smoothly stratified space ${}^S X$ and its regular part $\text{reg}({}^S X)$. Since $\text{reg}({}^S X)$ is diffeomorphic to the interior of X , $\overset{\circ}{X}$, we directly use the symbol $\overset{\circ}{X}$ for this smooth non-compact manifold. We can endow $\overset{\circ}{X}$ with a variety of riemannian metrics.

In [ALMP12] [ALMP16] the main focus was on iterated *incomplete edge* metrics, which are now called *iterated wedge metrics*. Iterated *complete edge* metrics, simply called now *iterated edge metrics* can also be considered: given an iterated wedge metric g_w we can define an iterated edge metric by setting $g_e := \rho^{-2} g_w$ where ρ is the product of all the boundary defining functions x_i , $i \in \{1, \dots, k\}$.

As explained for example in [ALMP12], an iterated edge metric g_e extends as a smooth metric on the edge tangent bundle ${}^eTX \rightarrow X$ which is, by definition, the vector bundle obtained by applying the Serre-Swan theorem to the Lie algebra of edge vector fields

$$(24) \quad \mathcal{V}_e(X) = \{\xi \in \mathcal{V}_b(X) \mid \xi|_{H_i}, \text{ is tangent to the fibers of } \phi_i : H_i \rightarrow S_i \forall i\}$$

(this is indeed a finitely generated projective $C^\infty(X)$ -module).

In this article we shall be interested in *fibered corner metrics*⁴ and these are defined analogously but in terms of $\mathcal{V}_\Phi(X)$ and the associated vector bundle ${}^\Phi TX$:

Definition 3. *A riemannian metric on \mathring{X} is a fibered corner metric if it extends as a smooth bundle metric to ${}^\Phi TX \rightarrow X$.*

We shall not distinguish between the metric on \mathring{X} and its extension to ${}^\Phi TX$.

Notice that a fibered corner metric g_Φ is complete. \mathring{X} , endowed with a fibered corner metric g_Φ , is an example of a manifold with a Lie structure at infinity [ALN07]; \mathring{X} endowed with an iterated edge metric g_e is another example of a manifold with a Lie structure at infinity.

Given a fibered corner metric g_Φ we can also consider the metric $g_{\text{fcusp}} := \rho^{-2}g_\Phi$, which is, by definition, an iterated fibered cusp metric. This is also a complete metric.

Notice that iterated fibered cusp metrics and iterated wedge metrics are also smooth metrics on suitable vector bundles over X ; however, the corresponding vector fields, i.e., the sections of these vector bundles, are not closed under Lie bracket.

Example. (Rigid metrics on depth-1 spaces.)

If, for example, ${}^S X$ is a depth-1 stratified space, so that X is a manifold with fibered boundary $\partial X =: H, Z - H \xrightarrow{\phi} S$, then an example of fibered corner metric (in this case, a fibered boundary metric) is a metric that on \mathring{X} , that near $\partial X \equiv H \equiv \{x = 0\}$, can be written as

$$(25) \quad g_\Phi = \frac{dx^2}{x^4} + \frac{\phi^* g_S}{x^2} + g_{H/S}$$

*with g_S a metric on S , the singular locus of ${}^S X$, and $g_{H/S}$ a metric on the vertical tangent bundle of H . A fibered boundary metric with this product structure near the boundary is called **rigid**. Wedge metrics, edge metrics and fibered cusp metrics with the additional property of being rigid would be respectively defined by the following forms near ∂X :*

$$g_w = dx^2 + \phi^* g_S + x^2 g_{H/S}, \quad g_e = \frac{dx^2}{x^2} + \frac{\phi^* g_S}{x^2} + g_{H/S}, \quad g_{\text{fcusp}} = \frac{dx^2}{x^2} + \phi^* g_S + x^2 g_{H/S}.$$

A general rigid fibered corner metric has an iterative description similar to the one in (25): in a collar neighbourhood of H_i it can be written as

$$(26) \quad g_\Phi = \frac{dx_i^2}{x_i^4} + \frac{\phi_i^* g_{S_i}}{x_i^2} + g_{H_i/S_i}$$

with g_{S_i} a fibered corner metric on S_i and g_{H_i/S_i} a family of fibered corners metrics on the fibers of $H_i \xrightarrow{\phi_i} S_i$.

In this article we shall work exclusively with rigid fibered corner metrics.

⁴also called Φ -metrics in the literature

3.4. Densities. The Φ -density bundle ${}^\Phi\Omega$ is the bundle on X with fiber at p equal to $\{u : \Lambda^{\dim X}({}^\Phi T_p X) \rightarrow \mathbb{R} ; u(t\omega) = |t|u(\omega) \forall \omega \in \Lambda^{\dim X}({}^\Phi T_p X), \forall t \neq 0\}$. The volume form associated to a fibered corner metric is a section of the Φ -density bundle.

4. A CRASH COURSE ON LIE GROUPOIDS AND LIE ALGEBROIDS

We refer the reader to the survey [DL10] and the bibliography therein for a detailed overview about groupoids and their role in index theory. Still, we shall now give the fundamental notions that are necessary in order to understand the content of this article.

4.1. Basics.

Definition 4. Let G and $G^{(0)}$ be two sets. A groupoid structure on G over $G^{(0)}$ is given by the following morphisms:

- Two maps: $r, s : G \rightarrow G^{(0)}$, which are respectively the range and source map.
- A map $u : G^{(0)} \rightarrow G$ called the unit map that is a section for both s and r . We can identify $G^{(0)}$ with its image in G .
- An involution: $i : G \rightarrow G, \gamma \mapsto \gamma^{-1}$ called the inverse map. It satisfies: $s \circ i = r$.
- A map $m : G^{(2)} \rightarrow G, (\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$ called the product, where the set

$$G^{(2)} := \{(\gamma_1, \gamma_2) \in G \times G \mid s(\gamma_1) = r(\gamma_2)\}$$

is the set of composable pair. Moreover for $(\gamma_1, \gamma_2) \in G^{(2)}$ we have $r(\gamma_1 \cdot \gamma_2) = r(\gamma_1)$ and $s(\gamma_1 \cdot \gamma_2) = s(\gamma_2)$.

The following properties must be fulfilled:

- The product is associative: for any $\gamma_1, \gamma_2, \gamma_3$ in G such that $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) = r(\gamma_3)$ the following equality holds

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3) .$$

- For any γ in G : $r(\gamma) \cdot \gamma = \gamma \cdot s(\gamma) = \gamma$ and $\gamma \cdot \gamma^{-1} = r(\gamma)$.

We denote a groupoid structure on G over $G^{(0)}$ by $G \rightrightarrows G^{(0)}$, where the arrows stand for the source and target maps.

We will adopt the following notations:

$$G_A := s^{-1}(A), \quad G^B := r^{-1}(B) \quad G_A^B := G_A \cap G^B \quad \text{and} \quad G|_A := G_A^A$$

in particular if $x \in G^{(0)}$, the s -fiber (resp. r -fiber) of G over x is $G_x = s^{-1}(x)$ (resp. $G^x = r^{-1}(x)$).

Definition 5. We call G a Lie groupoid when G and $G^{(0)}$ are second-countable smooth manifolds with $G^{(0)}$ Hausdorff, and the structural homomorphisms are smooth.

Let us see some examples. In the following ones X is a smooth manifold, $\phi : X \rightarrow B$ is a smooth fibration, $p : X_\Gamma \rightarrow X$ is a Galois Γ -covering, $E \rightarrow X$ is a vector bundle:

$G \rightrightarrows G^{(0)}$	r	s	i	m
$X \times X \rightrightarrows X$	$(x, y) \mapsto x$	$(x, y) \mapsto y$	$(x, y) \mapsto (y, x)$	$(x, y) \cdot (y, z) = (x, z)$
$X \times_B X \rightrightarrows X$	$(x, y) \mapsto x$	$(x, y) \mapsto y$	$(x, y) \mapsto (y, x)$	$(x, y) \cdot (y, z) = (x, z)$
$X_\Gamma \times_\Gamma X_\Gamma \rightrightarrows X$	$(\tilde{x}, \tilde{y}) \mapsto p(\tilde{x})$	$(\tilde{x}, \tilde{y}) \mapsto p(\tilde{y})$	$(\tilde{x}, \tilde{y}) \mapsto (\tilde{y}, \tilde{x})$	$(\tilde{x}, \tilde{y}) \cdot (g\tilde{y}, \tilde{z}) = (\tilde{x}, g^{-1}\tilde{z})$
$E \rightrightarrows X$	$(x, \xi) \mapsto x$	$(x, \xi) \mapsto x$	$(x, \xi) \mapsto (x, -\xi)$	$(x, \xi) \cdot (x, \eta) = (x, \xi + \eta)$

4.2. Groupoid C^* -algebras. We can associate to a Lie groupoid G the $*$ -algebra

$$C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$$

of the compactly supported sections of the half densities bundle associated to $\ker ds \oplus \ker dr$, with:

- the involution given by $f^*(\gamma) = \overline{f(\gamma^{-1})}$;
- and the convolution product given by $f * g(\gamma) = \int_{G_{s(\gamma)}} f(\gamma\eta^{-1})g(\eta)$.

For all $x \in G^{(0)}$ the algebra $C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$ can be represented on $L^2(G_x, \Omega^{\frac{1}{2}}(G_x))$ by

$$\lambda_x(f)\xi(\gamma) = \int_{G_x} f(\gamma\eta^{-1})\xi(\eta),$$

where $f \in C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$ and $\xi \in L^2(G_x, \Omega^{\frac{1}{2}}(G_x))$.

Definition 6. *The reduced C^* -algebra of a Lie groupoid G , denoted by $C_r^*(G)$, is the completion of $C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$ with respect to the norm*

$$\|f\|_r = \sup_{x \in G^{(0)}} \|\lambda_x(f)\|_x$$

where $\|\cdot\|_x$ is the operator norm on $L^2(G_x, \Omega^{\frac{1}{2}}(G_x))$.

The full C^ -algebra of G is the completion of $C_c^\infty(G, \Omega^{\frac{1}{2}}(\ker ds \oplus \ker dr))$ with respect to all continuous representations.*

Notation; We shall often omit from the notation the half-densities bundle and consider it as understood.

Example 7. *We give a few examples:*

- $C_r^*(X \times X) \cong C^*(X \times X) \cong \mathbb{K}(L^2(X))$;
- $C_r^*(X \times_B X) \cong C^*(X \times_B X)$ is a field over B of compact operators C^* -algebras;
- $C_r^*(X_\Gamma \times_\Gamma X_\Gamma)$ is Morita equivalent⁵ to $C_r^*(\Gamma)$;
- $C_r^*(E) \cong C^*(E) \cong C_0(E^*)$.

Remark 8. *From now on, if X is a G -invariant closed subset of $G^{(0)}$ (this is also called a saturated closed subset of $G^{(0)}$) we will call $e_X: C_c^\infty(G) \rightarrow C_c^\infty(G|_X)$ the restriction map to X . This gives an exact sequence of full groupoid C^* -algebras*

$$(27) \quad 0 \longrightarrow C^*(G|_{G^{(0)} \setminus X}) \longrightarrow C^*(G) \longrightarrow C^*(G|_X) \longrightarrow 0,$$

see [Ren80]. Notice that in general this is not true for the reduced C^* -algebras: the reader can find examples of this phenomenon in [HLS02]. However, in some examples that we shall consider (not all) the short exact sequence (27) will also hold for the reduced C^* -algebras. For instance this is the case when the groupoid $G|_X$ is amenable.

⁵in particular they have the same K-theory groups

4.3. Lie algebroids.

Definition 9. A Lie algebroid $\mathfrak{A} = (p : \mathfrak{A} \rightarrow TM, [\cdot, \cdot]_{\mathfrak{A}})$ on a smooth manifold M is a vector bundle $\mathfrak{A} \rightarrow M$ equipped with a bracket $[\cdot, \cdot]_{\mathfrak{A}} : C^\infty(M; \mathfrak{A}) \times C^\infty(M; \mathfrak{A}) \rightarrow C^\infty(M; \mathfrak{A})$ on the module of sections of \mathfrak{A} , together with a homomorphism of fiber bundle $p : \mathfrak{A} \rightarrow TM$ from \mathfrak{A} to the tangent bundle TM of M , called the anchor map, fulfilling the following conditions:

- the bracket $[\cdot, \cdot]_{\mathfrak{A}}$ is \mathbb{R} -bilinear, antisymmetric and satisfies the Jacobi identity,
- $[X, fY]_{\mathfrak{A}} = f[X, Y]_{\mathfrak{A}} + p(X)(f)Y$ for all $X, Y \in C^\infty(M; \mathfrak{A})$ and f a smooth function of M ,
- $p([X, Y]_{\mathfrak{A}}) = [p(X), p(Y)]$ for all $X, Y \in C^\infty(M; \mathfrak{A})$.

Here, with a small abuse, we are using the same notation for the bundle map p and for the map induced by p on the sections of the two bundles.

Let G be a Lie groupoid. The tangent space to s -fibers, that is $T_s G := \ker ds$ restricted to the objects of G is $\bigcup_{x \in G^{(0)}} TG_x$ and it has the structure of Lie algebroid on $G^{(0)}$, with the anchor map given by dr . It is denoted by $\mathfrak{A}G$ and we call it the Lie algebroid of G . It is easy to prove that $\mathfrak{A}G$ is isomorphic to the normal bundle of the inclusion $G^{(0)} \hookrightarrow G$.

Given a Lie algebroid $\mathfrak{A} = (p : \mathfrak{A} \rightarrow TM, [\cdot, \cdot]_{\mathfrak{A}})$ on manifold M we can ask whether it can be integrated, i.e. whether it is the Lie algebroid of a Lie groupoid. As clarified in [AM85] [CF03] this is not always possible; however, as we shall see in moment, there are sufficient conditions ensuring the existence of a Lie groupoid integrating a given Lie algebroid.

Notice that if a Lie algebroid is integrable then it can be the Lie algebroid associated to different Lie groupoids; for example if M is a smooth compact manifold with universal cover M_Γ and $\mathfrak{A} = (\text{Id} : TM \rightarrow TM)$ is the Lie algebroid over M given by the identity map, then \mathfrak{A} is the Lie algebroid associated to the pair groupoid $M \times M \rightrightarrows M$ but also to the groupoid $M_\Gamma \times_\Gamma M_\Gamma \rightrightarrows M$.

5. THE GROUPOID ASSOCIATED TO STRATIFIED SPACES

We now go back to the smoothly stratified space ${}^S X$. Let X be its resolution, a manifold with fibered corners. Consider the Lie algebra of vector fields $\mathcal{V}_\Phi(X)$ and the associated Lie algebroid $({}^\Phi TX, i_\Phi)$. We wish to integrate this algebroid to a groupoid. We follow [DLR15].

Let $H_i \xrightarrow{\phi_i} S_i, i \in \{1, \dots, k\}$ be an exhaustive list of the boundary hypersurfaces of X and let x_i be a boundary defining function for H_i . We assume that

$$i < j \Rightarrow H_i < H_j \text{ or } H_i \cap H_j = \emptyset.$$

By means of the Lie algebra of Φ vector fields $\mathcal{V}_\Phi(X)$ we have defined the Lie algebroid ${}^\Phi TX$ with anchor map

$$i_\Phi : {}^\Phi TX \rightarrow TX.$$

Notice that the anchor map is injective (in fact, an isomorphism) when restricted to $\overset{\circ}{X}$, a dense open subset of X . According to a theorem of Debord, see [Deb01, Theorem 2] we thus obtain that the Lie algebroid ${}^\Phi TX$ is integrable. Following [DLR15] we can explicitly exhibit a groupoid $G_\Phi \rightrightarrows X$ integrating the Lie algebroid ${}^\Phi TX$. This is described as follows: over $\overset{\circ}{X}$ the groupoid G_Φ is simply the pair groupoid $\overset{\circ}{X} \times \overset{\circ}{X}$; over the boundary of X the groupoid G_Φ is given by

$$\bigsqcup_{i=1}^k (H_i \times_{\phi_i} {}^\Phi TS_i \times_{\phi_i} H_i)_{|G_i} \times \mathbb{R}$$

with $G_i = H_i \setminus \cup_{j>i} H_j$. The range and source maps are given as follows: if h and h' are points in G_i , with $\phi_i(h) = s = \phi_i(h')$, and if $v \in T_s S_i$ then

$$s(h, v, h', \lambda) = h, \quad r(h, v, h', \lambda) = h'.$$

The composition of two composable elements (h, v, h', λ) and (h', w, h'', μ) is given, by definition, by $(h, v + w, h'', \lambda + \mu)$.

Although only a set for the time being, G_Φ can be given a smooth structure by observing, as in [DLR15], that there is a natural bijection

$$(28) \quad G_\Phi \longleftrightarrow \mathring{X}_\Phi^2 \cup \mathring{\text{ff}}_\Phi$$

with X_Φ^2 the Φ -double space (introduced in the next section) and $\mathring{\text{ff}}_\Phi$ its front face. Using the properties of the Φ -double space it is proved in [DLR15], see page 26 there, and [Les, Section 4.3, Proposition 6], that G_Φ integrates the Lie algebroid $({}^\Phi TX, i_\Phi)$. For a slightly different approach to $G_\Phi \rightrightarrows X$ see also the last section of this paper.

We shall be also concerned with another groupoid integrating the Lie algebroid ${}^\Phi TX$. This is the Lie groupoid $G_\Phi^\Gamma \rightrightarrows X$ which is described as follows. Recall the Galois covering $X_\Gamma \xrightarrow{p} X$ and the structure of the total space X_Γ as a manifold with fibered corners $H_{i,\Gamma}$ where, as discussed in (21), we have

$$(29) \quad \begin{array}{ccc} H_{i,\Gamma} & \xrightarrow{p} & H \\ \downarrow \phi_{i,\Gamma} & & \downarrow \phi_i \\ S_{i,\Gamma} & \xrightarrow{p} & S \end{array}$$

p denoting the quotient map with respect to the induced Γ -action. Consider now $G_{\Phi,\Gamma} \rightrightarrows X_\Gamma$ defined precisely as before. This groupoid is freely acted upon by Γ : if g is an element of Γ and (x, y) is in $\mathring{X}_\Gamma \times \mathring{X}_\Gamma$, then $g \cdot (x, y) = (g \cdot x, g \cdot y)$; if instead (x, ξ, y, t) is an element over the boundary, then $g \cdot (x, \xi, y, t) = (g \cdot x, dg \cdot \xi, g \cdot y, t)$. The quotient is our groupoid $G_\Phi^\Gamma \rightrightarrows X$. Then, by construction and by (29), $G_\Phi^\Gamma \rightrightarrows X$ is equal to $\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma$ over \mathring{X} and is equal to

$$\bigsqcup_{i=1}^k (H_i \times_{\phi_i} {}^\Phi TS_i \times_{\phi_i} H_i)_{|G_i} \times \mathbb{R}$$

on ∂X .

5.1. Notation. In the sequel we shall use exclusively the groupoid $G_\Phi^\Gamma \rightrightarrows X$. We shall often denote the groupoid $G_\Phi^\Gamma \rightrightarrows X$ simply by $G \rightrightarrows X$.

6. PSEUDODIFFERENTIAL OPERATORS

Let $\text{Diff}_\Phi^*(X)$ be the algebra generated by products of elements of the Lie algebra $\mathcal{V}_\Phi(X)$ and of functions in $C^\infty(X)$. The algebra $\text{Diff}_\Phi^*(X)$ is contained in an algebra of pseudodifferential operator, which can be described in two compatible ways.

The first way is purely microlocal and it involves a blown-up double space on which the Schwartz kernel of the pseudodifferential operators can be easily described. The general philosophy underlying this approach is due to Merlose; for this particular Lie algebra of vector fields the corresponding pseudodifferential calculus is due to Mazzeo and Melrose for manifolds with fibered boundary and to Debord, Lescure and Rochon for manifolds with fibered corners.

The second way to enlarge $\text{Diff}_\Phi^*(X)$ to a pseudodifferential algebra is to use the Lie groupoid G_Φ and the pseudodifferential algebra canonically associated to it.

We shall now briefly recall these two points of view.

6.1. The Φ -calculus. In this subsection, following the seminal paper of Mazzeo and Melrose [MM98] and its extension to manifolds with fibered corners given in [DLR15], we recall briefly how the Φ -calculus on a manifold with fibered corners X is defined. According to the general philosophy put forward by Richard Melrose, we define the Φ -pseudodifferential operators by specifying properties of their Schwartz kernels on $X \times X$ and we do this by lifting these kernels to a resolved product space, the Φ -double space. Let $H_i \xrightarrow{\phi_i} S_i$, $i \in \{1, \dots, k\}$, an exhaustive list of the boundary hypersurfaces of X and let x_i be a boundary defining function for H_i . We assume as before that

$$i < j \Rightarrow H_i < H_j \text{ or } H_i \cap H_j = \emptyset.$$

In order to define the Φ -double space we first blow-up the p -submanifolds ⁶ $H_i \times H_i$ in $X \times X$, thus obtaining the b -double space

$$X_b^2 := [X \times X; H_1 \times H_1; H_2 \times H_2, \dots; H_k \times H_k]$$

also denoted $X \times_b X$. Blowing up the corners $H_i \times H_i$ in a different order will result in a diffeomorphic manifold. The submanifolds

$$D_{\phi_i} = \{(h, h') \in H_i \times H_i \text{ s.t. } \phi_i(h) = \phi_i(h')\}$$

of $H_i \times H_i$ can be lifted to X_b^2 where they become p -submanifolds, denoted Δ_{ϕ_i} . The Φ -double space is, by definition, the space obtained by blowing-up in X_b^2 the p -submanifolds $\Delta_{\phi_1}, \dots, \Delta_{\phi_k}$, in this order:

$$X_{\Phi}^2 := [X_b^2; \Delta_{\phi_1}; \dots; \Delta_{\phi_k}]$$

We shall also denote it by $X \times_{\Phi} X$. There are well defined blow-down maps

$$\beta_b : X_b^2 \rightarrow X^2, \quad \beta_{\Phi-b} : X_{\Phi}^2 \rightarrow X_b^2, \quad \beta_{\Phi} := \beta_b \circ \beta_{\Phi-b} : X_{\Phi}^2 \rightarrow X^2;$$

the front faces associated to the boundary hypersurfaces are, by definition, the p -submanifolds

$$\text{ff}_{\phi_i} := \beta_{\Phi-b}^{-1}(D_{\phi_i}).$$

They are boundary hypersurfaces of X_{Φ}^2 . We use the notation $\text{ff}_{\Phi} := \cup_{i=1}^k \text{ff}_{\phi_i}$. Of particular importance is also the lifted diagonal

$$\Delta_{\Phi} := \overline{\beta_{\Phi}^{-1}(\dot{\Delta}_X)}$$

If E and F are vector bundles over X , then the space of Φ -pseudodifferential operator of order m is defined in terms of the conormal distributions at Δ_{Φ} :

$$\Psi_{\Phi}^m(X; E, F) := \{K \in I^m(X_{\Phi}^2; \Delta_{\Phi}; \beta_{\Phi}^* \text{HOM}(E, F) \otimes \pi_R^*(\Omega^{\Phi})); K \equiv 0 \text{ at } \partial X_{\Phi}^2 \setminus \text{ff}_{\Phi}\}$$

where $\equiv 0$ means vanishing of the Taylor series and π_R is the map induced by the projection on the right factor. Classical Φ -pseudodifferential operators are defined in terms of polyhomogeneous conormal distributions; they give the space $\Psi_{\Phi\text{-ph}}^m(X; E, F)$. It is not difficult to show that the Schwartz kernels of the Φ -differential operators of order m lift to the Φ -double space where they define elements in $I_{\text{ph}}^m(X_{\Phi}^2; \Delta_{\Phi}; \beta_{\Phi}^* \text{HOM}(E, F) \otimes \pi_R^*(\Omega^{\Phi}))$; thus

$$\text{Diff}_{\Phi}^m(X; E, F) \subset \Psi_{\Phi\text{-ph}}^m(X; E, F) \subset \Psi_{\Phi}^m(X; E, F).$$

⁶We recall that a p -submanifold D of a manifold with corners M is a smooth submanifold which meets all boundary faces of M transversally, and which is covered by coordinate neighborhoods $\{U, (x, y)\}$ in M such that $D \cap U = \{y_j = 0; j = 1, \dots, \text{codim}(D)\}$

There is a short exact sequence, the principal symbol sequence,

$$0 \rightarrow \Psi_{\Phi}^{m-1}(X; E, F) \rightarrow \Psi_{\Phi}^m(X; E, F) \xrightarrow{\sigma_m} S^{[m]}(\Phi TX, p^* \text{HOM}(E, F)) \rightarrow 0,$$

with $p: \Phi TX \rightarrow X$ the natural projection. In addition to the principal symbol of $P \in \Psi_{\Phi}^m(X; E, F)$ one can consider its boundary symbols $\sigma_{\partial_i}(P)$, $i \in \{1, \dots, k\}$, also called *normal operators*. Defined initially by the restriction of the Schwartz kernel of P to the front face ff_{ϕ_i} , the operator $\sigma_{\partial_i}(P)$ is in fact a ΦNS_i -suspended family of Φ -operators, with $\Phi NS_i = TS_i \times \mathbb{R}$; in formulae

$$\sigma_{\partial_i}(P) \in \Psi_{\Phi-\text{sus}(\Phi NS_i)}(H_i/S_i; E, F)$$

This means that $\sigma_{\partial_i}(P)$ is a smoothly varying family of operators parametrized by S_i , the base of H_i , and such that

$$\sigma_{\partial_i}(P)(s) \in \Psi_{\Phi-\text{sus}(T_s S_i \times \mathbb{R})}(\phi_i^{-1}(s); E, F).$$

In practise this means that $\sigma_{\partial_i}(P)(s)$ is an operator on $T_s S_i \times \mathbb{R} \times \phi_i^{-1}(s)$ which is translation invariant in the $T_s S_i \times \mathbb{R}$ direction and thus with Schwartz kernel in $T_s S_i \times \mathbb{R} \times \phi_i^{-1}(s) \times \phi_i^{-1}(s)$. We shall say that $P \in \Psi_{\Phi}^m(X; E, F)$ is elliptic if $\sigma_m(P)$ is invertible off the zero section of ΦTX ; we shall say that P is fully elliptic if it is elliptic and if, in addition, $\sigma_{\partial_i}(P)$ is invertible for each $i \in \{1, \dots, k\}$ ⁷.

One can prove a composition formula for these operators: if $P \in \Psi_{\Phi}^m(X; E, F)$, $Q \in \Psi_{\Phi}^{\ell}(X; F, G)$ then $Q \circ P \in \Psi_{\Phi}^{m+\ell}(X; E, G)$; moreover the principal symbol and the boundary symbols are multiplicative.

The main result in the Φ -calculus is the following parametrix construction:

if $P \in \Psi_{\Phi}^m(X; E, F)$ is fully elliptic then there exists $Q \in \Psi_{\Phi}^{-m}(X; F, E)$ such that

$$(30) \quad Q \circ P - \text{Id} \in \dot{\Psi}_{\Phi}^{-\infty}(X; E), \quad P \circ Q - \text{Id} \in \dot{\Psi}_{\Phi}^{-\infty}(X; F)$$

where $\dot{\Psi}_{\Phi}^{-\infty}(X; E)$ denotes the Φ -operators of order $(-\infty)$ that vanish of infinite order at all boundary hypersurfaces of X_{Φ}^2 .

Notice that elements in $\dot{\Psi}_{\Phi}^{-\infty}(X; E)$ are in fact defined by smooth kernels in $\dot{C}^{\infty}(X \times X, \text{END}(E))$ where the dot means vanishing of infinite order at the boundary. We point out that in contrast with the b -calculus or the edge calculus, the parametrix of a fully elliptic Φ -operator is an element of Ψ_{Φ}^* ; there is no need to enlarge the calculus. Using this crucial information it is not difficult to show that

$$(31) \quad P \in \Psi_{\Phi}^m(X; E) \Rightarrow P \circ (P^* P + \text{Id})^{-\frac{1}{2}} \in \Psi_{\Phi}^0(X; E)$$

One can introduce Φ -Sobolev spaces and prove the boundedness of $P \in \Psi_{\Phi}^m(X; E, F)$ from $H_{\Phi}^{\ell}(X; E)$ to $H_{\Phi}^{\ell-m}(X; E)$; in particular 0-th order Φ -pseudodifferential operators are L^2 -bounded.

We close this subsection with a few comments on the equivariant case: if X is the resolution of ${}^S X$ and X_{Γ} is the resolution of a Galois cover ${}^S X_{\Gamma}$, $\Gamma - {}^S X_{\Gamma} \rightarrow {}^S X$, then we know that X_{Γ} is a Galois cover of X ,

$$(32) \quad \Gamma - X_{\Gamma} \xrightarrow{p} X$$

with an additional property encoded by (21). Consider in this context the diagonal action of Γ on $X_{\Gamma} \times X_{\Gamma}$; there is a lift of this action on $X_{\Gamma} \times_b X_{\Gamma}$ (this is already considered in [LP97]) and on

$$X_{\Gamma} \times_{\Phi} X_{\Gamma}$$

⁷where invertibility is meant for each parameter $s \in S_i$ and for each $i \in \{1, \dots, k\}$ as a map from the space of rapidly decreasing sections of $T_s S_i \times \mathbb{R} \times \phi_i^{-1}(s)$ into itself

and it is therefore possible to define the space of Γ -equivariant Φ -pseudodifferential operators of order m , $\Psi_{\Phi, \Gamma}^m(X_\Gamma)$, and similarly for $\Psi_{\Phi, \Gamma}^m(X_\Gamma, E_\Gamma)$ if E_Γ is a Γ -equivariant complex vector bundle on X_Γ .

An operator is of Γ -compact support if the support of its Schwartz kernel is compact in $X_\Gamma \times_{\Phi} X_\Gamma / \Gamma$. We denote by $\Psi_{\Phi, \Gamma, c}^m(X_\Gamma, E_\Gamma)$ the resulting vector space. Remark that a Γ -equivariant Φ -differential operator is certainly of Γ -compact support. The composition of two Φ -operators of Γ -compact support is well defined and still a Φ -operator of Γ -compact support. We thus obtain an algebra $\Psi_{\Phi, \Gamma, c}^*(X_\Gamma, E_\Gamma)$.

Finally, the boundary symbols of a Γ -equivariant Φ -differential operator D_Γ can be identified with the boundary symbols of the operator D induced on the quotient X . Indeed, if $H_\Gamma \xrightarrow{\phi_\Gamma} S_\Gamma$ is a boundary hypersurface of X_Γ then we know that there is no action of Γ in the fibers of H_Γ ; $\sigma_\partial(D_\Gamma)$ is a S_Γ -family of operators, which is Γ -equivariant. Thus $\sigma_\partial(D_\Gamma)(s)$ is an operator on $T_s S_\Gamma \times \mathbb{R} \times \phi_\Gamma^{-1}(s) \equiv T_{p(s)} S \times \mathbb{R} \times \phi^{-1}(\phi(s))$ which is, in addition, Γ -equivariant with respect to the variable s ; this means that

$$\sigma_\partial(D_\Gamma)(s) = \sigma_\partial(D_\Gamma)(\gamma s);$$

thus, up to the above identifications, we can obtain the boundary operator of D_Γ at the boundary hypersurface $H_\Gamma \rightarrow S_\Gamma$ of X_Γ from the boundary operator of D at the boundary hypersurface $H \rightarrow S$ of X .

This reasoning establishes the following important fact:

Proposition 10. *If D_Γ is a fully elliptic element in $\text{Diff}_{\Phi, \Gamma}^m(X_\Gamma, E_\Gamma)$ then there exists a parametrix Q_Γ of Γ -compact support, $Q_\Gamma \in \Psi_{\Phi, \Gamma, c}^{-m}(X_\Gamma; F_\Gamma, E_\Gamma)$, such that*

$$(33) \quad Q_\Gamma \circ D_\Gamma - \text{Id} \in \dot{\Psi}_{\Phi, \Gamma, c}^{-\infty}(X_\Gamma; E_\Gamma), \quad D_\Gamma \circ Q_\Gamma - \text{Id} \in \dot{\Psi}_{\Phi, \Gamma, c}^{-\infty}(X_\Gamma; F_\Gamma)$$

The same proposition can be stated replacing D_Γ by a fully elliptic element $P \in \Psi_{\Phi, \Gamma, c}^m(X_\Gamma, E_\Gamma)$.

6.2. Pseudodifferential operators on a groupoid G . Given a Lie groupoid G it is possible to define an algebra of G -pseudodifferential operator $\Psi_c^*(G)$, see [MP97, NWX99].

Let us recall briefly the general definition.

Definition 11. *A linear G -operator is a continuous linear map*

$$P: C_c^\infty(G, \Omega^{\frac{1}{2}}) \rightarrow C^\infty(G, \Omega^{\frac{1}{2}})$$

such that:

- P restricts to a continuous family $(P_x)_{x \in G^{(0)}}$ of linear operators $P_x: C_c^\infty(G_x, \Omega^{\frac{1}{2}}) \rightarrow C^\infty(G_x, \Omega^{\frac{1}{2}})$ such that

$$Pf(\gamma) = P_{s(\gamma)} f_{s(\gamma)}(\gamma) \quad \forall f \in C_c^\infty(G, \Omega^{\frac{1}{2}})$$

where f_x denotes the restriction of f to $G_x := s^{-1}(x)$.

- The following equivariance property holds:

$$U_\gamma P_{s(\gamma)} = P_{r(\gamma)} U_\gamma,$$

where U_γ is the map induced on functions by the right multiplication by γ .

A linear G -operator P is pseudodifferential of order m if

- its Schwartz kernel k_P is a distribution on G that is smooth outside $G^{(0)}$;

- for every distinguished chart $\psi : U \subset G \rightarrow \Omega \times s(U) \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$ of G :

$$\begin{array}{ccc} U & \xrightarrow{\psi} & \Omega \times s(U) \\ & \searrow s & \swarrow p_2 \\ & & s(U) \end{array}$$

the operator $(\psi^{-1})^* P \psi^* : C_c^\infty(\Omega \times s(U)) \rightarrow C_c^\infty(\Omega \times s(U))$ is a smooth family parametrized by $s(U)$ of pseudodifferential operators of order m on Ω .

We say that P is smoothing if k_P is smooth and that P is compactly supported if k_P is compactly supported on G .

One can show that the space $\Psi_c^*(G)$ of compactly supported pseudodifferential G -operators is an involutive algebra.

Symbol map. Observe that a pseudodifferential G -operator induces a family of pseudodifferential operators on the s -fibers. So we can define the principal symbol of a pseudodifferential G -operator P as a function $\sigma(P)$ on \mathfrak{S}^*G , the cosphere bundle associated to the Lie algebroid $\mathfrak{A}G$ by

$$\sigma(P)(x, \xi) = \sigma_{pr}(P_x)(x, \xi)$$

where $\sigma_{pr}(P_x)$ is the principal symbol of the pseudodifferential operator P_x on the manifold G_x .

Quantization. Conversely, given a symbol f of order m on \mathfrak{A}^*G we can quantize it to a pseudodifferential G -operator once we have, in addition, the following data:

- (1) A smooth embedding $\theta : U \rightarrow \mathfrak{A}G$, where U is a open set in G containing $G^{(0)}$, such that $\theta(G^{(0)}) = G^{(0)}$, $(d\theta)|_{G^{(0)}} = \text{Id}$ and $\theta(\gamma) \in \mathfrak{A}_{s(\gamma)}G$ for all $\gamma \in U$;
- (2) A smooth compactly supported map $\phi : G \rightarrow \mathbb{R}_+$ such that $\phi^{-1}(1) = G^{(0)}$.

Then a G -pseudodifferential operator $P_{f,\theta,\phi}$ is obtained by the formula:

$$P_{f,\theta,\phi}u(\gamma) = \int_{\gamma' \in G_{s(\gamma)}, \xi \in \mathfrak{A}_{r(\gamma)}^*(G)} e^{-i\theta(\gamma'\gamma^{-1}) \cdot \xi} f(r(\gamma), \xi) \phi(\gamma'\gamma^{-1}) u(\gamma')$$

with $u \in C_c^\infty(G, \Omega^{\frac{1}{2}})$. The principal symbol of $P_{f,\theta,\phi}$ is just the leading part of f .

By definition, operators of zero order $\Psi_c^0(G)$ are a subalgebra of the multiplier algebra $M(C_r^*(G))$ (that is nothing but the algebra of bounded adjointable operators on the $C_r^*(G)$ -module given by $C_r^*(G)$ itself), whereas operators of negative order are actually in $C_r^*(G)$. In what follows we will denote $\overline{\Psi_c^0(G)}$ the C^* -algebra obtained as the closure of 0-order G -pseudodifferential operator in the multiplier algebra $M(C_r^*(G))$. All these definitions and properties immediately extend to the case of operators acting between sections of bundles on $G^{(0)}$ pulled back to G with the range map r . The space of compactly supported pseudodifferential operators on G acting on sections of $r^*E \otimes \Omega^{\frac{1}{2}}$ and taking values in sections of $r^*F \otimes \Omega^{\frac{1}{2}}$ will be denoted $\Psi_c^*(G; E, F)$. If $F = E$ we get an algebra denoted by $\Psi_c^*(G; E)$. Notice that $\Psi_c^0(G; E)$ is a subalgebra of $\mathbb{B}(\mathcal{E})$ where \mathcal{E} is the Hilber $C_r^*(G)$ -module obtained as the closure of $C_c^\infty(G; r^*E \otimes \Omega^{\frac{1}{2}})$ with respect to the obvious $C_r^*(G)$ -norm. Similarly we can define \mathcal{F} from $C_c^\infty(G; r^*F \otimes \Omega^{\frac{1}{2}})$ and one can easily see that $\Psi_c^*(G; E, F)$ is a subalgebra of $\mathbb{B}(\mathcal{E}, \mathcal{F})$.

6.3. Simple ellipticity and K-theory classes. An operator is *elliptic* when its principal symbol is invertible off the zero section of the dual of the algebroid. If P is elliptic then, as in the classical situation, it has a parametrix inverting it modulo $\Psi_c^{-\infty}(G) = C_c^\infty(G)$. See [Vas06].

If $P \in \Psi_c^0(G; E, F)$ is elliptic and Q is a parametrix with remainders R and S in $C_c^\infty(G; r^*E \otimes s^*E^* \otimes \Omega^{\frac{1}{2}}) \subset \mathbb{K}(\mathcal{E})$ and $C_c^\infty(G; r^*F \otimes s^*F^* \otimes \Omega^{\frac{1}{2}}) \subset \mathbb{K}(\mathcal{F})$ respectively, then we can define a class in $KK(\mathbb{C}, C_r^*(G))$ by considering the Kasparov bimodule defined by the operator

$$T := \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$$

and the Hilbert $C_r^*(G)$ -module $\mathcal{H} = \mathcal{E} \oplus \mathcal{F}$.

Let $Y \subset G^{(0)}$ be a closed saturated subset, where we recall that saturated means that Y is a union of G -orbits. Then we can consider $G|_Y$; we assume that

$$(34) \quad 0 \longrightarrow C_r^*(G|_{X \setminus Y}) \longrightarrow C_r^*(G) \longrightarrow C_r^*(G|_Y) \longrightarrow 0$$

is exact. By restriction we obtain a pair $(T|_Y, H|_Y)$ which is a Kasparov $(\mathbb{C}, C_r^*(G|_Y))$ -bimodule. Let us assume that this bimodule is degenerate; then the Kasparov bimodule (T, H) actually defines an element $[T, H]$ in $KK(\mathbb{C}, C_r^*(G|_{X \setminus Y}))$. See [Ska85, Lemma 3.2] for a detailed proof of this fact.

6.4. The algebra $\Psi_c^*(G_\Phi^I)$. Let us go back to stratified pseudomanifolds and let us investigate what a pseudodifferential G_Φ^I -operator P is.

By definition, P is really a family of pseudodifferential operators on the s -fibers of G_Φ^I with an additional equivariant property. Bearing in mind the objects of the groupoid G_Φ^I (they are given in terms of the manifold with fibered corners X) we have, correspondingly, two kinds of families:

- let x be a point in the interior of X , then the s -fiber over x is just \mathring{X}_Γ and P_x is a pseudodifferential Φ -operator. Moreover, since the action of G_Φ^I is transitive when restricted to \mathring{X}_Γ , the equivariance property implies that $P_x = P_y$ for all $x, y \in X \setminus \partial X$; moreover, since the isotropy group $(G_\Phi^I)_x$ (which we recall is defined as $s^{-1}(x) \cap r^{-1}(x)$) is isomorphic to Γ , it follows that P_x is Γ -equivariant for all $x \in X \setminus \partial X$;
- let h be a point in the boundary ∂X of X and let Z be the fiber of ϕ over h ; $s^{-1}(h)$ is isomorphic to $Z \times \mathbb{R}^n \times \mathbb{R}$. So P_h is a pseudodifferential operator on $Z \times \mathbb{R}^n \times \mathbb{R}$ and, by the equivariance property one has that P_h is translation invariant on the euclidean part and that $P_h = P_k$ for all $h, k \in \partial X$ such that $\phi(h) = \phi(k)$. Notice that on the boundary there is no Γ -equivariancy condition.

6.5. Simple ellipticity versus full ellipticity. If a pseudodifferential G_Φ^I -operator is elliptic, then, as observed before, it is invertible modulo $C_c^\infty(G_\Phi^I) \subset C_r^*(G_\Phi^I)$. This C^* -algebra is too big; for example if $\Gamma = \{1\}$ then $C_r^*(G_\Phi)$ (or even $C_c^\infty(G_\Phi)$) is not contained in the algebra of compact operators on L^2 . This means that, in contrast with the closed compact case, an element that is elliptic, it is not in general Fredholm. More generally, an elliptic pseudodifferential G_Φ^I -operator will define an index class in $K_*(C_r^*(G_\Phi^I))$ but this K-theory group is not very interesting for the questions we address in this article. If we want to work with Fredholm operators, or more generally, if we want to define interesting K-theory classes, we must consider operators that are not only elliptic but also have some additional property that ensures, to the very least, their invertibility modulo compacts. On the basis of the microlocal approach explained in the previous section, we know that a sufficient condition is given by the invertibility of the normal families, but at this stage we want to get to this condition in an autonomous way, i.e. within the theory of groupoids. To this end

we begin by observing that when $\Gamma = \{1\}$ the algebra of compact operators on L^2 is realized by the C^* -algebra $C_r^*(\dot{X} \times \dot{X})$, the C^* -algebra of the restriction of G_Φ to $\dot{X} = X \setminus \partial X$. So, we are looking for a condition that would ensure invertibility modulo $C_r^*(\dot{X} \times \dot{X})$. This is when $\Gamma = \{1\}$. If we are in the true equivariant case, i.e. $\Gamma \neq \{1\}$, this condition translates to being invertible modulo $C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma)$, the C^* -algebra of the restriction of G_Φ^Γ to \dot{X} ; indeed, it is easy to see that $K_*(C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma)) = K_*(C_r^*\Gamma)$ and the latter is certainly an interesting K -theory group.

The additional condition we are looking for is precisely *full ellipticity* and our immediate goal now is to get to this condition within the groupoid approach.

Let us denote by ∂G_Φ^Γ the restriction of G_Φ^Γ to ∂X and let

$$\sigma_\partial: \overline{\Psi_c^0(G_\Phi^\Gamma)} \rightarrow \overline{\Psi_c^0(\partial G_\Phi^\Gamma)}$$

be the morphism of C^* -algebras induced by the restriction to the boundary. Recall that here the closure of the groupoid pseudodifferential $*$ -algebra is taken inside the multiplier algebra of the groupoid C^* -algebra. See the end of Section 6.2.

Then $C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma) = \ker(\sigma) \cap \ker(\sigma_\partial) = \ker(\sigma \oplus \sigma_\partial)$, where $\sigma: \overline{\Psi_c^0(G_\Phi^\Gamma)} \rightarrow C(\mathfrak{S}^*G_\Phi^\Gamma)$ is given by the principal symbol. If we denote by $\sigma_{f.e.}$ the morphism induced by the universal property associated to the following pull-back diagram

$$\begin{array}{ccc} \overline{\Psi_c^0(G_\Phi^\Gamma)} & \xrightarrow{\sigma} & C(\mathfrak{S}^*G_\Phi^\Gamma) \\ \sigma_{f.e.} \searrow & & \downarrow \\ \overline{\Psi_c^0(\partial G_\Phi^\Gamma)} \times_{\partial X} C(\mathfrak{S}^*G_\Phi^\Gamma) & \longrightarrow & C(\mathfrak{S}^*G_\Phi^\Gamma) \\ \downarrow & & \downarrow \\ \overline{\Psi_c^0(\partial G_\Phi^\Gamma)} & \longrightarrow & C(\mathfrak{S}^*\partial G_\Phi^\Gamma) \end{array}$$

(Note: The diagram shows a pull-back square with a curved arrow σ from the top-left to the top-right, a dashed arrow $\sigma_{f.e.}$ from the top-left to the middle-left, a solid arrow σ_∂ from the top-left to the bottom-left, and solid arrows forming the square.)

then we obtain the following exact sequence of C^* -algebras

$$(35) \quad 0 \longrightarrow C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma) \longrightarrow \overline{\Psi_c^0(G_\Phi^\Gamma)} \xrightarrow{\sigma_{f.e.}} \overline{\Psi_c^0(\partial G_\Phi^\Gamma)} \times_{\partial X} C(\mathfrak{S}^*G_\Phi^\Gamma) \longrightarrow 0.$$

Definition 12. A pseudodifferential G_Φ^Γ -operator $P \in \Psi_c^0(G_\Phi)$ is fully elliptic if the element $\sigma_{f.e.}(P)$ is invertible in $\overline{\Psi_c^0(\partial G_\Phi^\Gamma)} \times_{\partial X} C(\mathfrak{S}^*G_\Phi^\Gamma)$. We call $\sigma_{f.e.}(P)$ the full symbol of P .

From now on we will briefly denote the C^* -algebra $\overline{\Psi_c^0(\partial G_\Phi^\Gamma)} \times_{\partial X} C(\mathfrak{S}^*G_\Phi^\Gamma)$ of full symbols by Σ .

We postpone the treatment of the K -theory classes associated to a fully elliptic Φ -operator to a later section.

6.6. Comparing $\Psi_\Phi^*(X)$ and $\Psi_c^*(G_\Phi)$ and a fundamental remark. The inclusion of G_Φ into $X \times_\Phi X$ given by (28), together with the remarks made in subsection 6.4, lead to the following inclusion of $*$ -algebras

$$\Psi_c^*(G_\Phi) \rightarrow \Psi_\Phi^*(X)$$

with $\Psi_c^*(G_\Phi)$ identified with the subalgebra of $\Psi_\Phi^*(X)$ made of operators with Schwartz kernel of compact support in $X_\Phi^2 \cup \mathring{\mathfrak{f}}_\Phi$. Notice, in particular, that this induces a bijection at the level of differential operators. Up to this identification, one immediately deduces that if P is an operator in $\Psi_c^*(G_\Phi)$, then $\sigma_\partial(P)$ corresponds to the totality of normal operators for P ; indeed, both of them

are given by restriction to the front faces. See also the last section of this paper for more on this point. Furthermore, it also follows in an obvious way that the notions of fully ellipticity in the two contexts are compatible.

Remark 13. *As we have just observed, $\Psi_c^*(G_\Phi)$ is smaller than $\Psi_\Phi^*(X)$ because of the support condition. In particular the parametrrix of a fully elliptic differential operator is in $\Psi_\Phi^*(X)$ but not in $\Psi_c^*(G_\Phi)$, given that the parametrrix involves the inverses of the normal operators and these are not compactly supported. This is usually seen as a drawback of the groupoid approach: the algebra $\Psi_c^*(G_\Phi)$ seems to be too small to be of any interest in conjunction with index theory (indeed, index theory is based on the construction of a parametrrix). We explain why, in the present context, this is not so:*

we are following a K -theoretic approach to index theory; K -theory works well for C^ -algebras, given that for C^* -algebras we have additional results such as Bott periodicity. Because of this, in the groupoid approach to index theory we are really interested in $\overline{\Psi_c^0(G_\Phi)}$, a C^* -algebra, rather than in $\Psi_c^0(G_\Phi)$, a $*$ -algebra. Now, and this is the crucial remark,*

$$\overline{\Psi_c^0(G_\Phi)} = \overline{\Psi_\Phi^*(X)}$$

which means that the difference between $\Psi_c^(G_\Phi)$ and $\Psi_\Phi^*(X)$ will disappear once we take closures. Similar considerations can be given for the relationship between $\Psi_c^\Gamma(G_\Phi)$ and $\Psi_{\Phi,\Gamma,c}^*(X_\Gamma)$*

Needless to say, there are situations, typically involving spectral theory, where the difference between the two calculi are indeed relevant.

7. K -THEORY CLASSES: THE MICROLOCAL APPROACH

Let ${}^S X$ be a smoothly stratified space of dimension n and let ${}^S X_\Gamma$ be a Galois covering of it. Let X and X_Γ be the respective resolutions. Let P be a Γ -equivariant fully elliptic Φ -operator on X_Γ ; we assume that P has Γ -compact support. Our next goal is to define

- a K -homology class $[P]$ in $K_n^\Gamma({}^S X_\Gamma)$;
- an index class $\text{Ind}(P)$ in $K_n(C_r^*(\Gamma))$.

There are two way to do it: the first one is classic and employs a parametrrix, the second one employs groupoid techniques. In this section we shall briefly explain the classic, microlocal approach, in the next one we shall explain the groupoid approach, with a final subsection devoted to the compatibility between the two points of view. The case of Γ -equivariant Dirac operators will be treated in a separate section.

7.1. The fundamental class of a fully elliptic operator in $K_*^\Gamma({}^S X_\Gamma)$. We follow [DLR15, Section 11] but we perform constructions directly in the equivariant case. Let $P \in \Psi_{\Phi,\Gamma,c}^0(X_\Gamma)$ be a Γ -equivariant fully elliptic Φ -pseudodifferential operator of Γ -compact support. Let $Q \in \Psi_{\Phi,\Gamma,c}^0(X_\Gamma)$ be a parametrrix for P . Consider $\mathbf{H} = L_{g_\Phi}^2(X_\Gamma) \oplus L_{g_\Phi}^2(X_\Gamma)$ where g_Φ is a Γ -equivariant Φ -metric on X_Γ . Consider the subalgebra $C_\Phi^\infty(X_\Gamma) \subset C^\infty(X_\Gamma)$ of smooth functions that are constant along the fibers of $H_{i,\Gamma} \xrightarrow{\phi_{i,\Gamma}} S_{i,\Gamma}$ for each $i \in \{1, \dots, k\}$. There is a dense inclusion $C_\Phi^\infty(X_\Gamma) \subset C({}^S X_\Gamma)$. We consider the bounded operator on \mathbf{H} defined by multiplication by $f \in C_\Phi^\infty(X_\Gamma)$, denoted $\mathbf{m}(f)$. This is a Φ -operator of order 0 and it is obviously of Γ -compact support. We obtain in this way a representation $\mathbf{m} : C({}^S X_\Gamma) \rightarrow \mathbb{B}(\mathbf{H})$. Let

$$\mathbf{P} = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix},$$

a Γ -equivariant bounded operator on \mathbf{H} . Then, following closely the proof given in [DLR15, Section 11], one obtains the following Γ -equivariant K-homology class

$$(36) \quad [\mathbf{H}, \mathbf{m}, \mathbf{P}] \in K_*^\Gamma(\mathcal{S}X_\Gamma).$$

We shall denote this K-homology class simply as $[P] \in K_*^\Gamma(\mathcal{S}X_\Gamma)$.

7.2. The index class in $K_*(C_r^*\pi_1(\mathcal{S}X))$. Let P and Q be as in the previous subsection. We first assume that $\mathcal{S}X$ is even dimensional. We can define in the usual way a $C_r^*\Gamma$ -Hilbert module \mathcal{E} , see for example [BCH94]. Let $R, S \in \dot{\Psi}_{\Phi, \Gamma, c}^{-\infty}(X_\Gamma)$ be the remainders given by Proposition 10. They certainly define elements in $\mathbb{K}(\mathcal{E})$. We consider the following idempotents in the unitalization of $\mathbb{K}(\mathcal{E})$:

$$(37) \quad p := \begin{pmatrix} R^2 & R(I+R)Q \\ SP & I-S^2 \end{pmatrix}, \quad p_0 := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

and we set, by definition,

$$(38) \quad \text{Ind}_\Gamma(P) := [p] - [p_0]$$

The class in (38) is an element in $K_0(\mathbb{K}(\mathcal{E})) \simeq K_0(C_r^*\Gamma)$. See for example [CM90] for the motivation behind this definition.

Classic arguments, see for example [BCH94], [Bla98], [WO93], show that our index class in $K_0(C_r^*\Gamma)$ is the image of the K-homology class $[P]$, defined by P , through the composition of the assembly map $\mu_\Gamma : K_0^\Gamma(\mathcal{S}X_\Gamma) \rightarrow KK(\mathbb{C}, C_r^*\Gamma)$ with the isomorphism $KK(\mathbb{C}, C_r^*\Gamma) \simeq K_0(C_r^*\Gamma)$. With a small abuse of notation we write

$$(39) \quad \mu_\Gamma[P] = \text{Ind}_\Gamma(P).$$

The odd case can be treated by similar methods.

8. K-THEORY CLASSES: THE GROUPOID APPROACH

8.1. The adiabatic groupoid: an introduction. Let X be a closed smooth manifold. Consider the pair groupoid $X \times X \rightrightarrows X$. Its smooth convolution algebra $C_c^\infty(X \times X, \Omega^{\frac{1}{2}}(\ker dr \oplus \ker ds))$ of smooth compactly supported half-densities on $X \times X$ is nothing but the $*$ -algebra of the smoothing operators on $L^2(X, \Omega^{\frac{1}{2}})$ and $C_r^*(X \times X)$, its reduced C^* -algebra, is thus isomorphic to the algebra of compact operators $\mathbb{K}(L^2(X, \Omega^{\frac{1}{2}}))$.

The Lie algebroid of $X \times X$ is given by the tangent bundle TX : it is a Lie groupoid and, by means of the Fourier transform, its C^* -algebra $C_r^*(TX)$ is isomorphic to $C_0(T^*X)$ (notice that 0th order symbols on X are bounded multipliers of this algebra). By Poincaré duality, see [CS84], we know that $K_*(C_0(T^*X))$ is isomorphic to $KK^*(C(X), \mathbb{C})$, the K-homology of X .

Following Alain Connes we now consider a new groupoid, the tangent groupoid of the smooth manifold X . As a set it is given by

$$(X \times X)_{ad}^{[0,1]} := TX \times \{0\} \sqcup X \times X \times (0, 1] \rightrightarrows X \times [0, 1],$$

This set can be equipped with a suitable smooth structure that we shall now describe. On $TX \times \{0\}$ and $X \times X \times (0, 1]$ we have the usual topology; the gluing of this two pieces is described in terms of convergence in the following way: we say that $(x_n, y_n, t_n) \in X \times X \times (0, 1]$ converges to $(x, \xi, 0) \in TX \times \{0\}$ if t_n goes to 0, x_n, y_n tends to x and the tangent vector $(x_n, (y_n - x_n)/t_n)$ converges to (x, ξ) , see [Con94]. In Section 13 we will describe this smooth structure in an alternative way for more general situations, see Example 36.

The evaluation at 0 defines a C^* -algebra homomorphism $\text{ev}_0: C_r^*((X \times X)_{ad}^{[0,1]}) \rightarrow C_r^*(TX)$, which induces a short exact sequence of C^* -algebras:

$$(40) \quad 0 \longrightarrow C_r^*(X \times X) \otimes C_0(0,1) \longrightarrow C_r^*((X \times X)_{ad}^{[0,1]}) \longrightarrow C_r^*(TX) \longrightarrow 0.$$

Consider the long exact sequence in K-theory:

$$\dots \longrightarrow K_*(C_r^*(X \times X) \otimes C_0(0,1)) \longrightarrow K_*(C_r^*((X \times X)_{ad}^{[0,1]})) \longrightarrow K_*(C_r^*(TX)) \xrightarrow{\delta_{ad}} \dots$$

As already explained in the introduction, we define the adiabatic index homomorphism Ind^{ad} as the composition of δ_{ad} and the Bott isomorphism β :

$$\text{Ind}^{ad} = \beta \circ \delta_{ad} : K_*(C_r^*(TX)) \rightarrow K_*(C_r^*(X \times X)) = K_*(\mathbb{K})$$

(needless to say, $K_1(\mathbb{K}) = 0$).

We have the following important result, see [Con94] [MP97] for a proof:

Proposition 14. *Under the identification of $K_0(C_r^*(TX))$ with $K^0(T^*X)$ and of $K_0(\mathbb{K})$ with \mathbb{Z} , the adiabatic index homomorphism is equal to the Atiyah-Singer analytic index homomorphism.*

8.2. The adiabatic groupoid: beyond an introduction. Both for explaining the compatibility result and for later use in our treatment of rho classes, we are now going to treat more in detail the adiabatic connecting homomorphism.

One can prove the following:

- let us consider the C^* -algebra homomorphism $\text{ev}_0: C_r^*((X \times X)_{ad}^{[0,1]}) \rightarrow C_r^*(TX)$ induced by evaluation at 0. Since the kernel of this homomorphism is a cone, which is K-contractible, the element $[\text{ev}_0]$ of $KK(C_r^*((X \times X)_{ad}^{[0,1]}), C_r^*(TX))$ is a KK-equivalence, namely there exists an inverse element $[\text{ev}_0]^{-1} \in KK(C_r^*(TX), C_r^*((X \times X)_{ad}^{[0,1]}))$.
- If σ is the symbol of an elliptic 0-order pseudodifferential operator P on X , then we can describe the image of its class through $[\text{ev}_0]^{-1}$ in the following way:
take the pull-back of σ to $T^*X \times [0,1]$, the dual Lie algebroid of $(X \times X)_{ad}^{[0,1]}$; this is the symbol $\sigma \times id_{[0,1]}$; it produces an elliptic pseudodifferential operator P_{ad} on $(X \times X)_{ad}^{[0,1]}$, whose restriction at 1 is the pseudodifferential operator P and whose restriction at 0 is the Fourier transform of σ ; thus we see that $[\sigma] \otimes_{C_r^*(TX)} [\text{ev}_0]^{-1} = [P_{ad}]$, where on the right hand side we have the class associated to the elliptic operator P_{ad} , as explained in subsection 6.3;
- if we denote by $\text{ev}_1: C_r^*((X \times X)_{ad}^{[0,1]}) \rightarrow C_r^*(X \times X)$ the evaluation at 1, then the KK-element

$$(41) \quad [\text{ev}_0]^{-1} \otimes_{C_r^*((X \times X)_{ad}^{[0,1]})} [\text{ev}_1] \in KK(C_r^*(TX), C_r^*(X \times X))$$

is, up to the Bott isomorphism, equal to the boundary morphism δ_{ad} of (9); it follows that $\text{Ind}^{ad}: K_*(C_r^*(TX)) \rightarrow K_*(C_r^*(X \times X)) = K_*(\mathbb{K})$ is given by

$$(42) \quad \text{Ind}^{ad} = [\text{ev}_0]^{-1} \otimes [\text{ev}_1].$$

Remark 15. *Similarly if we have a Galois Γ -covering $X_\Gamma \rightarrow X$, we can consider the Lie groupoid $X_\Gamma \times_\Gamma X_\Gamma \rightrightarrows X$ and its adiabatic deformation*

$$TX \times \{0\} \sqcup X_\Gamma \times_\Gamma X_\Gamma \times (0,1] \rightrightarrows X \times [0,1].$$

Using the same arguments as above, taking the Γ -equivariant pseudodifferential operator associated to an equivariant symbol σ , we can define the adiabatic index class in $K_(C_r^*(X_\Gamma \times_\Gamma X_\Gamma)) \cong$*

$K_*(C_r^*(\Gamma))$, using the boundary morphism associated to the following exact sequence

$$(43) \quad 0 \longrightarrow C_r^*(X_\Gamma \times_\Gamma X_\Gamma) \otimes C_0(0, 1) \longrightarrow C_r^*((X_\Gamma \times_\Gamma X_\Gamma)_{ad}^{[0,1]}) \longrightarrow C_r^*(TX) \longrightarrow 0 .$$

Moreover this class is proved to be equal to the index class we defined using an equivariant parametrix. In the following we will adapt this construction to the context of fully elliptic operators on singular manifolds.

Remark 16. *This construction extends to any Lie groupoid G . We refer the reader to Section 13 for the details.*

8.3. The noncommutative tangent bundle and the noncommutative symbol. Let us now move to the case of a singular manifold ${}^S X$. Let X be its resolution. Following the general philosophy we have explained in the closed case, we are led to consider the adiabatic deformation of the Lie groupoid $G_\Phi^\Gamma \rightrightarrows X$.

From now on we denote the Lie groupoid $G_\Phi^\Gamma \rightrightarrows X$ simply by $G \rightrightarrows X$.

We recall that the Lie algebroid of $G \rightrightarrows X$ is ${}^\Phi TX$ and that the Fourier transform of the symbol of a Γ -equivariant 0th-order Φ -pseudodifferential operator of Γ -compact support defines a class in $K_*(C_r^*({}^\Phi TX)) \simeq K_*(C_0({}^\Phi T^*X))$. In contrast with the closed case this K-group is not isomorphic to the K-homology of ${}^S X$. Indeed Debord, Lescure and Rochon [DLR15] building on work of Debord and Lescure [DL09], proved that there is another Lie groupoid $T^{NC}X \rightrightarrows X \times \{0\} \cup \partial X \times (0, 1)$, built up from the adiabatic deformation of G , such that the K-theory of its C^* -algebra is isomorphic to the K-homology group $K_*({}^S X)$.

The idea is that the right objects to consider are not elliptic symbols but elliptic symbols together with the non-commutative symbols, i.e. the normal operators.

Definition 17. *We define $T^{NC}X$, the non-commutative tangent bundle of X , in the following way: first we take the adiabatic deformation of G*

$$G_{ad}^{[0,1]} := {}^\Phi TX \times \{0\} \cup G \times (0, 1] \rightrightarrows X \times [0, 1],$$

then we restrict to $\mathring{X} \times \{0\} \cup \partial X \times [0, 1)$.

We obtain the disjoint union of $T(X \setminus \partial X)$, the tangent bundle of the interior of X , and of the adiabatic deformation of the restriction of G to ∂X , ∂G , open at 1:

$$(44) \quad T\mathring{X} \cup (\partial G)_{ad}^{[0,1]} \rightrightarrows \mathring{X} \times \{0\} \cup \partial X \times [0, 1) .$$

We shall see that this last piece of information, i.e. the fact that the deformation is open at 1, is directly linked to the full ellipticity of the operator. The C^* -algebra of the groupoid $T^{NC}X$ fits into the following exact sequence, obtained by restriction:

$$(45) \quad 0 \longrightarrow C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma) \otimes C_0(0, 1) \longrightarrow C_r^*(G_{ad}^{[0,1]}) \longrightarrow C_r^*(T^{NC}X) \longrightarrow 0 .$$

Let σ be the symbol of a fully elliptic pseudodifferential G -operator P of order 0. As we saw in the previous subsection for the closed case, it defines a pseudodifferential $G_{ad}^{[0,1]}$ -operator P_{ad} that restricts to the Fourier transform of σ at 0 and to P at 1.

Recall from Section 6.4 that a G -operator, with $G = G_\Phi^\Gamma$, is a family of operators parametrized by X : the restriction of P to the interior $X \setminus \partial X$ corresponds to a Γ -equivariant pseudodifferential Φ -operator P of Γ -compact support whereas the restriction of P to ∂X corresponds to the collection of all normal families of P .

It is crucial to observe that P_{ad} , the adiabatic operator associated to a fully elliptic operator, not only defines a class in $KK^*(\mathbb{C}, C_r^*(G_{ad}^{[0,1]}))$ but also a class in $KK^*(\mathbb{C}, C_r^*(G_{ad}^{[0,1],F}))$, where $C_r^*(G_{ad}^{[0,1],F})$ is the restriction of $G_{ad}^{[0,1]}$ to $X \times [0, 1] \setminus \partial X \times \{1\}$ ⁸. Indeed, using the discussion given at the end of Subsection 6.3, this is true precisely because P is fully elliptic: its restriction to $\partial X \times \{1\}$, which is given by the normal families of P , is invertible and we can find a parametrix \mathcal{Q} of P whose normal operators are the inverses of the normal operators of P . Summarizing, from a fully elliptic operator $P \in \Psi_c^0(G)$ we have defined a class $[P_{ad}^F]$ in $KK^*(\mathbb{C}, C_r^*(G_{ad}^{[0,1],F}))$. We now observe that there is KK-equivalence between $C_r^*(G_{ad}^{[0,1],F})$ and $C_r^*(T^{NC}X)$ induced by the restriction of $G_{ad}^{[0,1],F}$ to $T^{NC}X$ (indeed, the kernel of the restriction homomorphism is $C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma \times (0, 1]) \cong C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma) \otimes C_0((0, 1])$ which is K-contractible). Summarizing, the following definition is well-posed:

Definition 18. *Let P be a fully elliptic operator with symbol σ . The restriction of the operator P_{ad}^F to $\mathring{X} \times \{0\} \cup \partial X \times [0, 1]$ defines a class*

$$[\sigma_{nc}(P)] \in K_*(C_r^*(T^{NC}X)).$$

We call this class the noncommutative symbol of P .

Generalizing the discussion in the previous subsection, see in particular (42), we now define the adiabatic index of P as the image of $[\sigma_{nc}(P)]$ through the boundary morphism associated to (45):

Definition 19. *The adiabatic index of a non-commutative symbol $[\sigma_{nc}(P)] \in K_*(C_r^*(T^{NC}X))$ is defined as its image through the following composition*

$$(46) \quad K_*(C_r^*(T^{NC}X)) \xrightarrow{[\text{ev}_0]^{-1}} K_*(C_r^*(G_{ad}^{[0,1],F})) \xrightarrow{[\text{ev}_1]} K_*(C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma)).$$

Thus

$$(47) \quad \text{Ind}^{\text{ad}} := [\text{ev}_1] \circ [\text{ev}_0]^{-1}$$

We set

$$(48) \quad \text{Ind}^{\text{ad}}(P) := [\text{ev}_1] \circ [\text{ev}_0]^{-1}([\sigma_{nc}(P)]) \in K_*(C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma)) \simeq K_*(C_r^*\Gamma).$$

8.4. Rho classes and delocalized APS index theorem. Let us now assume that there is a homotopy $\{P_t\}$ from $P_0 = P$ to an invertible operator P_1 , so that the adiabatic index class of P in $K_*(C_r^*\Gamma)$ is zero. Thus the concatenation of P_{ad} and P_t , after a suitable reparametrization, defines a $G_{ad}^{[0,1]}$ -operator such that its restriction to $X \times \{1\}$ defines a degenerate Kasparov cycle. Again, using the discussion given at the end of Subsection 6.3, it follows that this operator actually defines a class in $KK(\mathbb{C}, C_r^*(G_{ad}^{[0,1]}))$.

Definition 20. *We define $\rho(P, \{P_t\})$ as the class in $KK(\mathbb{C}, C_r^*(G_{ad}^{[0,1]}))$ that we have just constructed. It depends of course on P but also on the homotopy $\{P_t\}$. If P itself is invertible then we omit the constant path in the notation and write simply $\rho(P)$.*

Notice that this definition is a particular case of a general definition for Lie groupoids given in [Zen16]. It is proved in that paper that these rho-classes are linked to Atiyah-Patodi-Singer index classes on groupoids with boundary, a result that generalizes to the groupoid context the delocalized APS index theorem for Galois coverings proved in [PS14]. In this article we shall only

⁸here the superscript F stands for *Fredholm*

consider the following special case of this very general version of the delocalized APS index theorem.

Consider a stratified pseudomanifold ${}^S X$ and the associated cylinder ${}^S X \times [0, 1]$. Let ${}^S X_\Gamma$ be its universal covering. Let X_Γ denote, as usual, the resolved manifold associated to ${}^S X_\Gamma$. Let G be the groupoid G_Φ^Γ . We consider the Lie groupoid \mathcal{G} given, by definition, by $G \times b[0, 1] \rightrightarrows X \times [0, 1]$, that is, the product of G with the b -groupoid of $[0, 1]$, see [Mon99]. There is a natural notion of full ellipticity for \mathcal{G} . Now, let us consider \mathcal{G}_{ad}^F , defined as the restriction of \mathcal{G}_{ad} to $X \times [0, 1] \times [0, 1] \setminus \partial(X \times [0, 1]) \times \{1\}$. A fully elliptic operator A on $X \times [0, 1]$ defines, as before, a class $[A_{ad}^F] \in KK^*(\mathbb{C}, C_r^*(\mathcal{G}_{ad}^F))$. If we restrict the groupoid to the boundary of the cylinder, we obtain the groupoid $G_{ad}^{[0,1]} \times \{0, 1\} \times \mathbb{R} \rightrightarrows X \times \{0, 1\}$.

We assume that the normal operators associated to the boundary hypersurfaces $X \times \{i\}$ of the cylinder are of the form $A_i + \partial$ where $A_i \in \Psi_c^0(G)$ is invertible for $i = 1, 2$ and ∂ is a translation invariant operator on \mathbb{R} that generates $KK^1(\mathbb{C}, C_0(\mathbb{R}))$.

Since the restriction of the groupoid $X \times \{0, 1\}$ is equal to the product of the two groupoids $G_{ad}^{[0,1]} \times \{0, 1\}$ and \mathbb{R} , it is easy to see that the image of $[A_{ad}^F]$ through the restriction $ev_{\{0,1\}}$ to $X \times \{0, 1\}$ is the exterior Kasparov product of $\rho(A_0) \oplus -\rho(A_1) \in KK^{*-1}(\mathbb{C}, C_r^*(G_{ad}^{[0,1]}) \oplus C_r^*(G_{ad}^{[0,1]}))$ with $[\partial]$. Instead, if we evaluate the class $[A_{ad}^F]$ at the adiabatic deformation parameter $t = 1$, we obtain the class $\text{Ind}^{ad}(A)$ in $KK^*(\mathbb{C}, C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma \times (0, 1) \times (0, 1)))$, which we identify with to $KK^*(\mathbb{C}, C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma))$ through Bott isomorphism.

Consider the two following homomorphisms:

- $\alpha: KK^*(\mathbb{C}, C_r^*(G_{ad}^{[0,1]}) \oplus C_r^*(G_{ad}^{[0,1]})) \otimes C_0(\mathbb{R}) \rightarrow KK^{*+1}(\mathbb{C}, C_r^*(G_{ad}^{[0,1]}))$, which is the composition of the sum $a \oplus b \mapsto a + b$ and the Bott periodicity isomorphism;
- $\beta: KK^*(\mathbb{C}, C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma)) \rightarrow KK^{*+1}(\mathbb{C}, C_r^*(G_{ad}^{[0,1]}))$, which is the composition of Bott periodicity with the obvious injection $\iota: C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma \times (0, 1)) \rightarrow C_r^*(G_{ad}^{[0,1]})$.

The following result is a special case of [Zen16, Theorem 3.6]

Theorem 21. *If A is as above, then $\alpha(ev_{\{0,1\}}[A_{ad}^F]) = \iota_* \circ \beta(ev_1[A_{ad}^F])$. Put it differently, the following delocalized APS index formula holds:*

$$\rho(A_0) - \rho(A_1) = \iota_* \circ \beta(\text{Ind}^{ad}(A)) \in KK^{*+1}(\mathbb{C}, C_r^*(G_{ad}^{[0,1]})).$$

In particular if $\text{Ind}^{ad}(A)$ vanishes, then $\rho(A_0) = \rho(A_1)$.

9. COMPATIBILITY OF K-THEORY CLASSES: RESULTS

The following result is discussed in the work of Debord, Lescure and Rochon [DLR15]:

Theorem 22 (Poincaré duality). *The K -theory of $C_r^*(T^{NC} X)$ is isomorphic to the equivariant K -homology of the stratified manifold ${}^S X_\Gamma$. Moreover, under this isomorphism, the equivariant K -homology class $[P] \in K_*^\Gamma({}^S X_\Gamma)$ of a fully elliptic operator P in $\Psi_c^0(G_\Phi^\Gamma)$ corresponds to the class $[\sigma_{nc}(P)] \in K_*(C_r^*(T^{NC} X))$ defined above.*

Now we want to compare the index class defined through the adiabatic deformation and the index class defined in the classical way:

Theorem 23. *Let $P \in \Psi_c^0(G_\Phi^\Gamma)$ be a fully elliptic Φ -operator. Then the adiabatic index class and the index class defined in (38) through the Connes-Skandalis projector, are equal. In formulae:*

$$(49) \quad \text{Ind}^{ad}(P) = \text{Ind}_\Gamma(P) \quad \text{in} \quad K_*(C_r^*\Gamma)$$

In fact, more is true, in that there is a commutative diagram with vertical maps isomorphisms of abelian groups:

$$(50) \quad \begin{array}{ccc} K_*(C_r^*(T^{NC}X)) & \xrightarrow{\text{Ind}^{\text{ad}}} & K_*(C_r^*(\overset{\circ}{X}_\Gamma \times \overset{\circ}{X}_\Gamma)) \\ \text{PD} \downarrow & & \downarrow \simeq \\ K_*^\Gamma(SX_\Gamma) & \xrightarrow{\mu_\Gamma} & K_*(C_r^*\Gamma) \end{array}$$

Consequently, if $[\sigma_{nc}(P)] \in K_*(C_r^*(T^{NC}X))$ is the non-commutative symbol of a fully elliptic Γ -equivariant Φ -operator P and $[P] \in K_*^\Gamma(SX_\Gamma)$ is its K-homology class then, identifying the groups on the right hand side through the right vertical isomorphism, we have

$$(51) \quad \text{Ind}^{\text{ad}}(P) \equiv [ev_1] \circ [ev_0]^{-1}([\sigma_{nc}(P)]) = \mu_\Gamma(\text{PD}([\sigma_{nc}(P)])) \equiv \mu_\Gamma([P]) \quad \text{in} \quad K_*(C_r^*\Gamma)$$

10. COMPATIBILITY OF K-THEORY CLASSES: PROOFS

The Poincaré duality isomorphism is proved in great detail in [DLR15], building on [DL09]. We therefore concentrate on the proof of Theorem 23.

10.1. Preliminaries to the proof of Theorem 23: relative K-Theory. We begin by giving some basic definitions and results concerning relative K-theory and excision. Let A and B be C^* -algebras and let $\phi : A \rightarrow B$ a homomorphism of C^* -algebras. We define $K_*(\phi)$ as the homotopy classes of triples (p, q, z) , where p, q are projections in $\mathbb{M}_\infty(A)$ and z is an invertible element in $\mathbb{M}_\infty(B)$ such that $z\phi(p)z^{-1} = \phi(q)$.

This is a special case of the notion, treated in [Ska], of the K-theory group associated to an element in $KK(A, B)$, that in this case is the element associated to ϕ . It is proved in this article that there is an isomorphism of abelian groups

$$(52) \quad K_*(\phi) \simeq K_*(C_\phi)$$

where C_ϕ denotes the mapping cone of $\phi : A \rightarrow B$:

$$C_\phi := \{(a, f) \in A \oplus C_0([0, 1], B) \ ; \ \phi(a) = f(0)\}.$$

The relative K-theory group of a morphism fits into the following long exact sequence:

$$(53) \quad \cdots \rightarrow K_*(B \otimes C_0(0, 1)) \rightarrow K_*(\phi) \rightarrow K_*(A) \xrightarrow{\phi} K_*(B) \rightarrow \cdots$$

It turns out that if ϕ is surjective then there is an excision isomorphism

$$K_*(\phi) \xrightarrow{\text{ex}} K_*(\text{Ker}(\phi)).$$

Finally if we have a commutative diagram

$$(54) \quad \begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \alpha \downarrow & & \downarrow \beta \\ A' & \xrightarrow{\phi'} & B' \end{array}$$

then there is an associated group homomorphism

$$(55) \quad K_*(\phi) \xrightarrow{(\alpha, \beta)_*} K_*(\phi').$$

The diagram (54) induces a mapping of the long exact sequence (53) for $A \xrightarrow{\phi} B$ to the one for $A' \xrightarrow{\phi'} B'$ and it is clear from the five-lemma that if α_* and β_* are isomorphism of K-theory groups, then so is $K_*(\phi) \xrightarrow{(\alpha, \beta)_*} K_*(\phi')$.

10.2. More preliminaries: the class $\text{Ind}_\Gamma(P) \in K_*(C_r^*\Gamma)$ through relative K-Theory. We set

$$(56) \quad \Sigma := \overline{\Psi^0(G_\Phi^\Gamma)} / C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma).$$

We define a morphism $\mathbf{m} : C(X) \rightarrow \Sigma$ as the composition

$$C(X) \xrightarrow{m} \overline{\Psi^0(G_\Phi^\Gamma)} \xrightarrow{\sigma_{f.e.}} \Sigma$$

with the first arrow given by the multiplication operator and the second arrow denoting the morphism already considered in (35).

Now let E, F be two vector bundles on X , let $(\sigma, N(P_\sigma))$ be a fully elliptic non-commutative symbol between E and F . These data define a class

$$(57) \quad [P]_{rel} = \left[\begin{pmatrix} 1_E & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1_F \end{pmatrix}; \begin{pmatrix} 0 & (\sigma, N(P_\sigma))^{-1} \\ (\sigma, N(P_\sigma)) & 0 \end{pmatrix} \right]$$

in $K_*(\mathbf{m})$, the K-theory of the morphism \mathbf{m} , that is a pair of $C(X)$ -modules that are isomorphic by means of $(\sigma, N(P_\sigma))$ when we see them as Σ -modules through \mathbf{m} .

Since the following diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{\mathbf{m}} & \Sigma \\ M \downarrow & & \downarrow \text{id} \\ \overline{\Psi_c^0(G_\Phi)} & \xrightarrow{\sigma_{f.e.}} & \Sigma \end{array}$$

is commutative, we have a map

$$M_* : K_*(\mathbf{m}) \rightarrow K_*(\sigma_{f.e.}).$$

Since $\sigma_{f.e.}$ is a quotient map we see that $K_*(\sigma_{f.e.})$ is isomorphic, through the inverse of the excision map, to $K_*(\text{Ker}(\sigma_{f.e.})) = K_*(C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma))$.

The inverse of the excision map is given by the following two steps: first we represent any class in $K_*(\sigma_{f.e.})$ by a relative cycle $[p, q, z]$ with respect to $\mathbf{q} : C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma)^+ \rightarrow \mathbb{C}$, the quotient map from the unitization of $C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma)$ to \mathbb{C} ; then $[p, q, z]$ is sent to $[p] - [q]$, which, by definition of $K_*(\mathbf{q})$, is a class in $K_*(C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma))$.

More explicitly, let us consider the class (57). If we take the following invertible lift

$$T = \begin{pmatrix} 1_E - QP & -(1_E - QP)Q + Q \\ P & 1_F - PQ \end{pmatrix} \quad \text{of} \quad \begin{pmatrix} 0 & (\sigma, N(P_\sigma))^{-1} \\ (\sigma, N(P_\sigma)) & 0 \end{pmatrix},$$

where Q is a full parametrix of P , and if T_t is a path through invertible elements from $1_E \oplus 1_F$ to T (that always exists by the Theorem of Kuiper), then

$$\left[T_t \begin{pmatrix} 1_E & 0 \\ 0 & 0 \end{pmatrix} T_t^{-1}, \begin{pmatrix} 0 & 0 \\ 0 & 1_F \end{pmatrix}; \begin{pmatrix} 0 & (\sigma, N(P_\sigma))^{-1} \\ (\sigma, N(P_\sigma)) & 0 \end{pmatrix} \sigma_{f.e.}(T_t^{-1}) \right]$$

is a homotopy from the cycle in (57) to a relative cycle with respect to \mathbf{q} , whose image in $K_*(C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma))$ is given by:

$$(58) \quad T \begin{pmatrix} 1_E & 0 \\ 0 & 0 \end{pmatrix} T^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & 1_F \end{pmatrix} \in \mathbb{M}_2(C_r^*(\dot{X}_\Gamma \times_\Gamma \dot{X}_\Gamma)).$$

Definition 24. *The relative analytical index Ind^{rel} of a fully elliptic non-commutative symbol $(\sigma, N(P_\sigma))$ with coefficients in two vector bundle E, F is given by the following composition*

$$K_*(\mathfrak{m}) \xrightarrow{M_*} K_*(\sigma_{f.e.}) \xrightarrow{\text{ex}} K_*(C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma)).$$

Remark 25. *It is a classic fact that the boundary map $\partial_{f.\text{ull}}$ in the K -theory sequence associated to*

$$0 \longrightarrow C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma) \longrightarrow \overline{\Psi_c^0(G_\Phi^\Gamma)} \xrightarrow{\sigma_{f.e.}} \Sigma \longrightarrow 0$$

sends a fully elliptic symbol $(\sigma, N(P_\sigma))$ to the class (58).

Consider now the K -theory homomorphism $\psi : K_1(\Sigma) \rightarrow K_0(\mathfrak{m})$, which is the natural morphism that associates to an element $z \in \text{GL}_k(\Sigma)$ the relative cycle $(1_k, 1_k, z)$; then the composition of ψ with the relative index homomorphism Ind^{rel} of Definition 24 is easily seen to be equal to the above boundary map. Put it differently, the following diagram

$$\begin{array}{ccc} K_{*+1}(\Sigma) & & \\ \downarrow & \searrow \partial_{f.\text{ull}} & \\ K_*(\mathfrak{m}) & \xrightarrow{M_*} K_*(\sigma_{f.e.}) \xrightarrow{\text{ex}} & K_*(C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma)) \end{array}$$

is commutative.

10.3. Proof of Theorem 23. We begin by introducing some useful notation:

- $\Sigma^F := \overline{\Psi_c^0(G_{ad}^{[0,1]})}/C_r^*(G_{ad}^{[0,1],F})$ and $\sigma^F : \overline{\Psi_c^0(G)} \rightarrow \Sigma^F$ is the associated quotient map;
- $T_{[0,1]}^{\text{NC}} X$ is defined as the restriction of $G_{ad}^{[0,1]}$ to $\mathring{X} \times \{0\} \cup \partial X \times [0, 1]$;
- $\Sigma^{\text{NC}} := \overline{\Psi_c^0(T_{[0,1]}^{\text{NC}} X)}/C_r^*(T^{\text{NC}} X)$ and $\sigma^{\text{NC}} : \overline{\Psi_c^0(T_{[0,1]}^{\text{NC}} X)} \rightarrow \Sigma^{\text{NC}}$ is the associated quotient map;
- $M^{ad} : C(X \times [0, 1]) \rightarrow \overline{\Psi_c^0((G_\Phi^\Gamma)_{ad})}$ and $M^{\text{NC}} : C(\mathring{X} \times \{0\} \cup \partial X \times [0, 1]) \rightarrow \overline{\Psi_c^0(T_{[0,1]}^{\text{NC}} X)}$ are given by multiplication operators.
- $\mathfrak{m}^F : C(X \times [0, 1]) \rightarrow \Sigma^F$ is the composition of M^{ad} with the quotient map σ^F ;
- $\mathfrak{m}^{\text{NC}} : C(\mathring{X} \times \{0\} \cup \partial X \times [0, 1]) \rightarrow \Sigma^{\text{NC}}$ is the composition of M^{NC} with the quotient map σ^{NC} .

All these algebras and maps, together with the inverse of the excision isomorphisms, denoted ex , fit into the following diagram:

$$(59) \quad \begin{array}{ccccc} K_*(\mathfrak{m}^{\text{NC}}) & \xleftarrow{(\text{ev}_0)_*} & K_*(\mathfrak{m}^F) & \xrightarrow{(\text{ev}_1)_*} & K_*(\mathfrak{m}) \\ \downarrow (M^{\text{NC}}, \text{id})_* & & \downarrow (M^{ad}, \text{id})_* & & \downarrow (M, \text{id})_* \\ K_*(\sigma^{\text{NC}}) & \xleftarrow{(\text{ev}_0)_*} & K_*(\sigma^F) & \xrightarrow{(\text{ev}_1)_*} & K_*(\sigma_{f.e.}) \\ \downarrow \text{ex} & & \downarrow \text{ex} & & \downarrow \text{ex} \\ K_*(C_r^*(T^{\text{NC}} X)) & \xleftarrow{(\text{ev}_0)_*} & K_*(C_r^*((G_\Phi^\Gamma)_F)) & \xrightarrow{(\text{ev}_1)_*} & K_*(C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma)) \end{array}$$

Here, with an abuse of notation that we have already employed we denote by ev_0 the homomorphism induced by restriction from $X \times [0, 1]$ to $\mathring{X} \times \{0\} \cup \partial X \times [0, 1]$ and by ev_1 the homomorphism induced by restriction from $X \times [0, 1]$ to $\mathring{X} \times \{1\}$. Notice that we have used (54) and (55) repeatedly

and that with an abuse of notation we write $(\text{ev}_0)_*$ instead of $(\text{ev}_0, \text{ev}_0)_*$. By functoriality of all these homomorphisms it is easy to see that this diagram commutes.

Observe that the composition $(\text{ev}_1)_* \circ (\text{ev}_0)_*^{-1}$ in the bottom row is the adiabatic index homomorphism Ind^{ad} , whereas the composition $\text{ex} \circ M_*$ in the right column is the relative index homomorphism Ind^{rel} .

We shall now prove that the two homomorphisms appearing in the upper row are isomorphisms and the same is true for the homomorphism $(M^{\text{NC}}, \text{id})_*$, in the vertical left column.

Consider first

$$(60) \quad K_*(\mathfrak{m}^F) \xrightarrow{(\text{ev}_1, \text{ev}_1)_*} K_*(\mathfrak{m}) \quad \text{and} \quad K_*(\mathfrak{m}^F) \xrightarrow{(\text{ev}_0, \text{ev}_0)_*} K_*(\mathfrak{m}^{\text{NC}}).$$

They are induced, respectively, by the following squares, see (54):

$$\begin{array}{ccc} C(X \times [0, 1]) & \xrightarrow{\mathfrak{m}^F} & \Sigma^F \\ \text{ev}_1 \downarrow & & \downarrow \text{ev}_1 \\ C(X) & \xrightarrow{\mathfrak{m}} & \Sigma \end{array} \quad \begin{array}{ccc} C(X \times [0, 1]) & \xrightarrow{\mathfrak{m}^F} & \Sigma^F \\ \text{ev}_0 \downarrow & & \downarrow \text{ev}_0 \\ C(\mathring{X} \times \{0\} \cup \partial X \times [0, 1]) & \xrightarrow{\mathfrak{m}^{\text{NC}}} & \Sigma^{\text{NC}} \end{array}$$

Since the vertical morphisms are obviously surjective and the kernels of the are isomorphic to cones over suitable C^* -algebras, we deduce that the vertical maps induce isomorphisms in K-theory. Consequently, by the argument given at the end of Subsection 10.1, we see that the morphisms appearing in (60) are indeed isomorphisms.

Consider next $K_*(\mathfrak{m}^{\text{NC}}) \xrightarrow{(M^{\text{NC}}, \text{id})_*} K_*(\sigma^{\text{NC}})$, which is induced by the following square:

$$\begin{array}{ccc} C(\mathring{X} \times \{0\} \cup \partial X \times [0, 1]) & \xrightarrow{\mathfrak{m}^{\text{NC}}} & \Sigma^{\text{NC}} \\ M^{\text{NC}} \downarrow & & \downarrow \text{id} \\ \overline{\Psi_c^0(T_{[0,1]}^{\text{NC}} X)} & \xrightarrow{\sigma^{\text{NC}}} & \Sigma^{\text{NC}} \end{array}$$

Our goal is to show that $(M^{\text{NC}}, \text{id})_*$ is an isomorphism. To this end we first observe that, since id is an isomorphism, it suffices to prove that M^{NC} induces a K-theory isomorphism (see again the remark at the end of Subsection 10.1). Next we remark that we have a mapping-cone long exact sequence associated to M^{NC} , viz.

$$\dots \rightarrow K_*(\overline{\Psi_c^0(T_{[0,1]}^{\text{NC}} X)} \otimes C_0(0, 1)) \rightarrow K_*(C_{M^{\text{NC}}}) \rightarrow K_*(C(\mathring{X} \times \{0\} \cup \partial X \times [0, 1])) \xrightarrow{M_*^{\text{NC}}} K_*(\overline{\Psi_c^0(T_{[0,1]}^{\text{NC}} X)}) \rightarrow \dots$$

Hence, it suffices to prove that $K_*(C_{M^{\text{NC}}})$ is trivial. To show this we consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_0(\partial X \times (0, 1]) & \longrightarrow & C(\mathring{X} \times \{0\} \cup \partial X \times [0, 1]) & \longrightarrow & C(X \times \{0\}) \longrightarrow 0 \\ & & \downarrow M^\partial \times \text{id}_{(0,1]} & & \downarrow M^{\text{NC}} & & \downarrow M^T \\ 0 & \longrightarrow & \overline{\Psi_c^0(\partial G_\Phi^T \times (0, 1])} & \longrightarrow & \overline{\Psi_c^0(T_{[0,1]}^{\text{NC}} X)} & \longrightarrow & \overline{\Psi_c^0(\Phi T X)} \longrightarrow 0 \end{array}$$

where the two vertical arrows

$$C_0(\partial X \times (0, 1]) \xrightarrow{M^\partial \times \text{id}_{(0,1]}} \overline{\Psi_c^0(\partial G_\Phi^T \times (0, 1])} \quad C(X \times \{0\}) \xrightarrow{M^T} \overline{\Psi_c^0(\Phi T X)}$$

are given by obvious multiplication operators. The diagram induces the following short exact sequence of mapping-cone C^* -algebras :

$$0 \rightarrow C_{M^\partial \times \text{id}_{(0,1]}} \rightarrow C_{M^{\text{NC}}} \rightarrow C_{M^T} \rightarrow 0$$

Now we observe that $C_{M^\partial \times \text{id}_{(0,1]}}$ is isomorphic to $C_{M^\partial} \otimes C_0(0,1]$ which is K -contractible; next, using the fact that $\overline{\Psi_c^0(\Phi TX)}$ is isomorphic to the algebra of continuous functions over the ball bundle of ${}^\Phi T^*X$, we see that M^T is an homotopy equivalence of C^* -algebras and thus its mapping cone is K -contractible. Summarizing, $K_*(C_{M^\partial \times \text{id}_{(0,1]}})$ and $K_*(C_{M^T})$ are both equal to the trivial group and therefore so is $K_*(C_{M^{\text{NC}}})$, which is what we needed to prove in order to establish that $(M^{\text{NC}}, \text{id})_* : K_*(\mathfrak{m}^{\text{NC}}) \rightarrow K_*(\sigma^{\text{NC}})$ is an isomorphism.

Notice in particular that we can connect $K_*(C_r^*(T^{\text{NC}}X))$ to $K_*(\mathfrak{m})$ by means of an isomorphism, denoted here Θ , obtained by travelling up on the the left column and then right on the top row of diagram (59). We end the proof of Theorem 23 by showing that if P is a fully elliptic element in $\Psi_c^0(G_\Phi^r)$ then $\Theta(\sigma_{\text{nc}}(P)) = [P]_{\text{rel}}$. To see this we define an element in $K_*(\mathfrak{m}^F)$ which is mapped to $[P]_{\text{rel}}$ through ev_1 and it is mapped to $\sigma_{\text{nc}}(P)$ by the homomorphism that goes left and then down on the external part of the diagram starting with $K_*(\mathfrak{m}^F)$. This element is nothing but the class of $(\sigma(P) \otimes \text{id}_{[0,1]}, N(P)) \in K_*(\mathfrak{m}^F)$. The image of this element through the homomorphism induced by ev_0 is given by simple restriction, whereas its image through M_*^{NC} amounts to a change of coefficients in the modules associated to E and F (from continuous functions to pseudodifferential operators). Finally, that the image of this element under excision is given by $\sigma_{\text{nc}}(P)$ is proved by proceeding as in Subsection 10.2, once we identify $KK(\mathbb{C}, C_r^*(T^{\text{NC}}X))$ with $K_*(C_r^*(T^{\text{NC}}X))$.

Remark 26. *The proof of Theorem 23 follows closely the proof of the analogous Theorem in [MP97] for the analytical index of elliptic operators on Lie groupoids, in the case where the objects of the Lie groupoid form a closed manifold. Moreover our result gives a generalization of [CRLM14, Theorem 4.1] from the case of a manifold boundary to the case of a manifold with fibered corners. Yet in another direction our proof generalizes Propostion 3.8 in [DS17]; notice that our arguments here build directly on those given there. More precisely our proof generalizes [DS17, Propostion 3.8] to the case in which (in their notations) Ind_{G_F} , where G_F is here the restriction of the Lie groupoid $\text{Blup}_{r,s}^+$ to the boundary, is not invertible and it identifies, by means of the isomorphism Θ in the proof of Theorem 23, the relative index $\text{ind}_{\text{rel}} : K_*(\mu) \rightarrow K_*(C^*(G_W))$ and the morphism $\text{ind}_G^W : K_*(C^*(\mathcal{A}_W G)) \rightarrow K_*(C^*(G_W))$, defined in [DS17, Section 3.3.3].*

11. FULLY ELLIPTIC SPIN DIRAC OPERATORS

11.1. Spin Dirac operators. Let ${}^S X$ be a stratified pseudomanifold of dimension n , X its resolution, ${}^\Phi TX$ the associated Φ -tangent bundle and g_Φ a rigid Φ -metric. We shall say that ${}^S X$ is spin if its regular part, $\text{reg}({}^S X) = \overset{\circ}{X}$, is a spin manifold, meaning that $T\overset{\circ}{X}$ is a spin bundle. We fix a spin structure associated to g_Φ and get a (complex) spinor bundle \mathcal{S} on $\overset{\circ}{X}$. We can equivalently assume that the bundle ${}^\Phi TX$ is spin; indeed $\overset{\circ}{X}$ and X are homotopically equivalent. A spin structure for $(T^\Phi X, g_\Phi)$ gives, by restriction, a spin structure for $(\text{reg}({}^S X), g_\Phi)$; conversely, we can always extend a spin structure $(\text{reg}({}^S X), g_\Phi)$ to a spin structure on $(T^\Phi X, g_\Phi)$. Summarizing: we do spin geometry on ${}^S X$ by passing to the algebroid $T^\Phi X$, see below for the details.

This brings us to the following general treatment.

We define generalized Dirac operators on a Lie groupoid $G \rightrightarrows X$ with Lie algebroid $\mathfrak{A}(G)$ as follows: let g be a metric on $\mathfrak{A}(G)$, by pull-back it defines a G -invariant metric on $\ker ds$ along the s -fibers of G . Let ∇ be the fiber-wise Levi-Civita connection associated to this metric.

Definition 27. *Let $\text{Cliff}(\mathfrak{A}(G))$ be the Clifford algebra bundle over X associated to the metric g . Let S be a bundle of Clifford modules over $\text{Cliff}(\mathfrak{A}(G))$ and let $c(X)$ denotes the Clifford multiplication by $X \in \text{Cliff}(\mathfrak{A}(G))$. Assume that S is equipped with a metric g_S and a compatible connection ∇^S such that:*

- *Clifford multiplication is skew-symmetric, that is*

$$\langle c(X)s_1, s_2 \rangle + \langle s_1, c(X)s_2 \rangle = 0$$

for all $X \in C^\infty(X, \mathfrak{A}(G))$ and $s_1, s_2 \in C^\infty(X, S)$;

- *∇^S is compatible with the Levi-Civita connection ∇ , namely*

$$\nabla_X^S(c(Y)s) = c(\nabla_X Y)s + c(Y)\nabla_X^S(s)$$

for all $X, Y \in C^\infty(X, \mathfrak{A}(G))$ and $s \in C^\infty(X, S)$.

The Dirac operator associated to these data is defined as

$$\mathcal{D}_S: s \mapsto \sum_{\alpha} c(e_{\alpha})\nabla_{\alpha}^S(s)$$

for $s \in C^\infty(X, S)$ and $\{e_{\alpha}\}_{\alpha \in A}$ a local orthonormal frame.

Particular cases of this construction are given by generalized Dirac operators on manifolds with a Lie structure at infinity, which are treated in detail in [ALN07] [ALN04] [LN01] and in the recent preprint [BS16]. In particular, one can define in this generality the notion of spin Lie manifold and that of associated spin Dirac operator. We recall briefly the very natural definitions in the particular case treated here.

Let ${}^S X$ be a stratified space of dimension n , with resolution X . As already anticipated we say that ${}^S X$ is spin if the vector bundle ${}^{\Phi}TX$ is spin⁹. If g_{Φ} is a fibered corner metric on ${}^{\Phi}TX$ then a Φ -spin structure associated to g_{Φ} is a $\text{Spin}(n)$ -principal bundle $P_{\text{Spin}}({}^{\Phi}TX)$ together with a 2-fold covering map to the $\text{SO}(n)$ -principal bundle of orthonormal frames of $({}^{\Phi}TX, g_{\Phi})$. We have a natural notion of spinor bundle \mathcal{S} , obtained from $P_{\text{Spin}}({}^{\Phi}TX)$ through the complex spinor representation of $\text{Spin}(n)$. The spinor bundle is an example of a bundle of Clifford modules $W \rightarrow X$ for the Φ -cotangent bundle ${}^{\Phi}T^*X$ (briefly, a Φ -Clifford module); this notion can of course be given in general, without the spin-assumption. On a Φ -Clifford module $W \rightarrow X$ we can talk about a Clifford Φ -connection $\nabla^W \in \text{Diff}_{\Phi}^1(X; W, W \otimes {}^{\Phi}T^*X)$ where the adjective Clifford refers to the usual compatibility with the Levi-Civita connection associated to g_{Φ} ¹⁰. To the Clifford module structure and the given connection there is associated in a natural way a generalized Dirac operator $D \in \text{Diff}_{\Phi}^1(X, W)$

$$(61) \quad D := c \circ \nabla^W$$

where we have denote by c the map $C^\infty(X, W \otimes {}^{\Phi}T^*X) \rightarrow C^\infty(X, W)$ induced by the Clifford action on W . We can write as usual the action of D as

$$(62) \quad D(s) = \sum_{\alpha} c(e^{\alpha})\nabla_{\alpha}^W(s)$$

⁹this is a topological condition and could be equivalently imposed on TX , given that ${}^{\Phi}TX$ and TX are isomorphic (albeit in a non-natural way).

¹⁰since $C^\infty(X, {}^{\Phi}TX)$ has the structure of a Lie algebra, the usual definition of Levi-Civita connection can be given

for $s \in C^\infty(X, W)$ and $\{e_\alpha\}$ a local orthonormal frame of ${}^\Phi TX$ with dual basis $\{e^\alpha\}$.

If W is the spinor bundle and the Clifford connection is the one induced by the Levi-Civita connection on ${}^\Phi TX$ then we obtain the spin Dirac operator $\mathcal{D} \in \text{Diff}_\Phi^1(X, \mathcal{S})$.

Through the bijection between $\text{Diff}_\Phi^1(X; \mathcal{S})$ and $\text{Diff}^1(G_\Phi; r^*\mathcal{S})$ we obtain also the Dirac operator on the groupoid G_Φ , denoted $\mathcal{D} \in \text{Diff}^1(G_\Phi; r^*\mathcal{S})$. In fact, as explained in detail in [LN01, Proposition 6.1] we can define directly the Dirac operator on G_Φ by lifting the Clifford module structure of \mathcal{S} to $r^*\mathcal{S}$ and by using the Levi-Civita connection on G_Φ associated to the pull-back of g_Φ to a G_Φ -invariant metric on $\ker ds$. Through the usual formula (61) we define in this way an element $\mathcal{D} \in \text{Diff}^1(G_\Phi; \mathcal{S})$ which corresponds to $\mathcal{D} \in \text{Diff}_\Phi^1(X, \mathcal{S})$ through the bijection between $\text{Diff}_\Phi^1(X; \mathcal{S})$ and $\text{Diff}^1(G_\Phi; r^*\mathcal{S})$.

Remark 28. *The above procedure can be applied in general, to any Lie groupoid with a spin Lie algebroid; in particular we also get in this way a Dirac operator on the adiabatic deformation $(G_\Phi)_{ad}^{[0,1]}$.*

The operator $\mathcal{D} \in \text{Diff}^1(G_\Phi; r^*\mathcal{S})$ defines an unbounded regular operator $\overline{\mathcal{D}}$ on the Hilbert $C_r^*(G_\Phi)$ -Hilbert module \mathcal{E} given, by definition, by the closure of $C_c^\infty(G_\Phi, r^*\mathcal{S} \otimes \Omega^{1/2})$ in the norm associated to the $C_r^*(G_\Phi)$ -valued scalar product $\langle \xi, \eta \rangle(\gamma) = \int_{G_s(\gamma)} \langle \xi(\gamma), \eta(\gamma^{-1}\gamma') \rangle_{\mathcal{S}}$. In the sequel, adopting a widely used abuse of notation, we shall not distinguish $\overline{\mathcal{D}}$ from \mathcal{D} . As usual, \mathcal{D} admits a bounded inverse if there exists a bounded $C_r^*(G_\Phi)$ -Hilbert module operator Q such that $\mathcal{D} \circ Q = \text{Id} = Q \circ \mathcal{D}$.

11.2. Fully elliptic spin Dirac operators. We want to describe, first of all, the normal families of \mathcal{D} explicitly. Let $H_i \xrightarrow{\phi_i} S_i$ be one of the fibered boundary hypersurfaces of X . The restriction of G_Φ to $G_i := H_i \setminus \bigcup_{i < j} H_j$ is equal to $(H_i \times_{S_i} {}^\Phi TS_i \times_{S_i} H_i)|_{G_i} \times \mathbb{R}$ and the normal family $\sigma_{\partial_i}(\mathcal{D})$ relative to ϕ_i is just the Dirac operator of this groupoid, endowed with the Clifford structure it inherits from the Clifford structure of G_Φ .

Bearing in mind the G_Φ -invariance this means that the following formula holds

$$(63) \quad \sigma_{\partial_i}(\mathcal{D})(h) = (D_{\phi_i})_s + \mathcal{D}_{\mathbb{R}^n, g_s} + \mathcal{D}_{\mathbb{R}}, \quad h \in H, \quad \phi_i(h) = s$$

where D_{ϕ_i} is a vertical family of generalized Dirac operator on $\phi_i : H_i \rightarrow S_i$ and $(D_{\phi_i})_s$ is the operator of this family on the fiber $Z_s := \phi_i^{-1}(s)$; we are identifying ${}^\Phi T_s S_i$ with \mathbb{R}^n and denoting by g_s the metric g_{S_i} at $s \equiv \phi(h)$; $\mathcal{D}_{\mathbb{R}^n, g_s}$ is an euclidian Dirac operator; $\mathcal{D}_{\mathbb{R}}$ denotes the canonical Dirac operator on the flat real line.

Lemma 29. *If the base of the fibration $H_i \xrightarrow{\phi_i} S_i$ is assumed to be spin then the vertical tangent bundle ${}^\Phi T(H_i/S_i)$ is spin and the vertical family D_{ϕ_i} is a family of spin Dirac operators: $D_\Phi \equiv \mathcal{D}_\Phi$.*

Proof. As we are assuming that M is spin, we also obtain that H_i is spin. Since, by assumption, S_i is spin, then

$$Z_i \rightarrow H_i \xrightarrow{\phi_i} S_i$$

is a fibration of spin manifolds. See [LM89] □

Proposition 30. *If for each $i \in \{1, \dots, k\}$ D_{ϕ_i} is a family of invertible Dirac operators, then \mathcal{D} is fully elliptic.*

Proof. Recall that \mathcal{D} is fully elliptic if $\sigma_{\partial_i}(\mathcal{D})$ is a family of invertible operators for all $i \in I$. We shall prove this assertion by showing that $\sigma_{\partial_i}(\mathcal{D})^2$ is a family of strictly positive operators for all $i \in I$. Let $h \in H_i$ and let $s := \phi_i(h) \in S_i$. First recall that $\sigma_{\partial_i}(\mathcal{D})(h)$ is an operator on $Z_s \times \mathbb{R}^n \times \mathbb{R}$ and that the metric on this manifold is a product-type metric $g_{Z_s} \oplus g_{\phi(h)} \oplus d\sigma^2$. Because of that $(D_{\phi_i})_{\phi_i(h)}$ and $\mathcal{D}_{\mathbb{R}^n, g_{\phi(h)}} + \mathcal{D}_{\mathbb{R}}$ anticommute. Thus we have the following identity

$$\sigma_{\partial_i}(\mathcal{D})(h)^2 = (D_{\phi_i})_s^2 + (\mathcal{D}_{\mathbb{R}^n, g_s} + \mathcal{D}_{\mathbb{R}})^2$$

which, by the strict positivity of $(D_{\phi_i})_s^2$, implies the strict positivity of $\sigma_{\partial_i}(\mathcal{D})(h)^2$. \square

Proposition 31. *If the strata are spin and the metrics on the links have positive scalar curvature, then \mathcal{D} is fully elliptic.*

Proof. By Proposition 30 we just have to prove that D_{ϕ_i} is invertible for all $i \in \{1, \dots, k\}$. Let us fix such an $i \in \{1, \dots, k\}$. From the assumption and Lemma 29 we know that the fibers of $H_i \xrightarrow{\phi_i} S_i$ are spin and that $D_{\phi_i} \equiv \mathcal{D}_{\phi_i}$, a family of spin Dirac operators. Thus we can apply the Schrödinger-Lichnerowicz-Weizenböck formula and obtain that $(\mathcal{D}_{\phi_i})_s$ is invertible for each $s \in S_i$. The proposition is proved. \square

12. PRIMARY AND SECONDARY K-THEORY CLASSES ASSOCIATED TO A SPIN DIRAC OPERATOR

12.1. The fundamental class associated to a fully elliptic Dirac operator. In this subsection we want to define the K-homology class associated to a fully elliptic spin Dirac operator \mathcal{D} ; we do this by producing a class in $K_*(C_r^*(T^{NC}X))$. To this end we shall consider the Dirac operator \mathcal{D}_{ad} associated to the adiabatic deformation groupoid $(G_{\Phi})_{ad}^{[0,1]}$, see Remark 28. In fact, we shall work directly in the Γ -equivariant case and denote the resulting Γ -equivariant Dirac operator by \mathcal{D}^{Γ} . As usual we employ the simple notation $G \rightrightarrows X$ for the groupoid G_{Φ}^{Γ} . The adiabatic deformation is therefore denoted, as usual, as $G_{ad}^{[0,1]}$.

We shall take the bounded transform of our operator, $\psi(\mathcal{D}^{\Gamma})$, with $\psi(x) = x/\sqrt{1+x^2}$. This is not an element in $\Psi_c^0(G)$ but rather in an extended version of it, according to Vassout [Vas06]: one adds to $\Psi_c^0(G)$ an algebra of "smoothing operators" which is holomorphically closed and obtains in this way an algebra which is contained densely in $\overline{\Psi_c^0(G)}$. We refer to [Vas06, Section 4] for the details. According to one of the main results in [Vas06], this algebra contains the bounded transform $\psi(\mathcal{D}^{\Gamma})$. In particular, we see that $\psi(\mathcal{D}^{\Gamma}) \in \overline{\Psi_c^0(G)}$. Similarly, we have that $\psi(\mathcal{D}_{ad}^{\Gamma}) \in \overline{\Psi_c^0(G_{ad}^{[0,1]})}$.

In order to define a class in $K_*(C_r^*(T^{NC}X))$ we would like to proceed as in Subsection 8.3. Thus we would like to produce a class in the K-theory of $C_r^*(G_{ad}^{[0,1],F})$; notice however that the natural candidate, namely $\psi(\mathcal{D}_{ad}^{\Gamma})$, does not produce a KK-cycle which is *degenerate* on $\partial X \times \{1\}$, see Subsection 6.3; indeed the restriction of $\psi(\mathcal{D}_{ad}^{\Gamma})^2 - \text{Id}$ to $\partial X \times \{1\}$, i.e. $\psi(N(\mathcal{D}^{\Gamma}))^2 - \text{Id}$, is not equal to 0. To fix this problem we shall now homotope $\psi(\mathcal{D}^{\Gamma})$ to an operator that has this property. Recall the fully elliptic symbol $\sigma_{f.e.}$ and the short exact sequence (35)

$$0 \longrightarrow C_r^*(\mathring{X}_{\Gamma} \times_{\Gamma} \mathring{X}_{\Gamma}) \longrightarrow \overline{\Psi_c^0(G)} \xrightarrow{\sigma_{f.e.}} \Sigma \longrightarrow 0.$$

where we recall that $\Sigma := \overline{\Psi_c^0(\partial G)} \times_{\partial X} C(\mathfrak{S}^*G)$. Now observe that $\sigma_{f.e.}(\psi(\mathcal{D}^{\Gamma})) = \psi(\sigma_{f.e.}(\mathcal{D}^{\Gamma}))$. Moreover consider the family of functions $\psi_s(t) = \psi(t/(1-s))$ for $s \in [0, 1]$ and observe that $\psi_1(t) = \text{sign}(t)$. Since \mathcal{D}^{Γ} is fully elliptic, $\psi(\sigma_{f.e.}(\mathcal{D}^{\Gamma}))$ is invertible and $\psi_s(\sigma_{f.e.}(\mathcal{D}^{\Gamma}))$ is a homotopy through invertible elements with the property that $\psi_1(\sigma_{f.e.}(\mathcal{D}^{\Gamma}))$ is well defined and equal to the

sign of the full symbol of \mathcal{D}^F . By the surjectivity of $\sigma_{f.e.}$ we can lift $\psi_s(\sigma_{f.e.}(\mathcal{D}^F))$ to a path P_s of elliptic operators in $\overline{\Psi_c^0(G)}$ such that $P_0 = \psi(\mathcal{D}^F)$ and P_1 is a fully elliptic operator with normal operator $N(P_1) = \text{sign}(N(\mathcal{D}^F))$. Observe in particular that the restriction to ∂X of $P_1^2 - \text{Id}$ is equal to 0.

Now, up to the reparametrization of the adiabatic deformation from $[0, 1]$ to $[0, 1/2]$ and of the homotopy P_s from $[0, 1]$ to $[1/2, 1]$, the concatenation of these two elements gives an operator on the adiabatic deformation such that the restriction to $\partial X \times \{1\}$ defines a degenerate cycle. As already explained in Subsection 8.3, this provides us with a class $[\mathcal{D}_{ad}^F]$ in the K-theory of $C_r^*(G_{ad}^{[0,1],F})$.

By restricting this operator to $\mathring{X} \times 0 \cup \partial X \times [0, 1]$ we obtain the desired class $[\sigma_{nc}(\mathcal{D}^F)]$ in $K_*(C_r^*(T^{NC}X))$, $*$ = dim X . In the even dimensional case we are of course using the well-known fact that the spinor bundle is \mathbb{Z}_2 -graded and the Dirac operator is odd with respect to this grading. Since $\psi(x) = x/\sqrt{1+x^2}$ is an odd function, we also have that $\psi(\mathcal{D}^F)$ is odd. Hence, it is easy to see that the Kasparov definition of *even* bimodules applies directly, which is why $[\sigma_{nc}(\mathcal{D}^F)] \in K_0(C_r^*(T^{NC}X))$ if X is even dimensional.

Remark 32. *Following closely [DLR15] one can check that the Poincaré dual of $[\sigma_{nc}(\mathcal{D}^F)] \in K_*(C_r^*(T^{NC}X))$ in $K_*^\Gamma(SX_\Gamma)$ is equal to the class defined, as in Subsection 7.1, by P_1 .*¹¹

12.2. The index class of a fully elliptic Dirac operator as an obstruction. Recall Definition 46; following the notations of the previous subsection, the index class $\text{Ind}^{ad}(\mathcal{D})$ is given by the evaluation at 1 of the class $[\mathcal{D}_{ad}^F]$. This class in $KK^*(\mathbb{C}, C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma))$ is represented by the operator P_1 , also defined in the previous subsection.

Proposition 33. *If the Γ -equivariant metric g_Φ has positive scalar curvature everywhere, then $\text{Ind}^{ad}(\mathcal{D})$ vanishes.*

Proof. By the Schrödinger-Lichnerowicz formula, we know that \mathcal{D} is invertible. Following the notations of the previous section, we can define P_s as the path $\psi_s(\mathcal{D})$, for $s \in [0, 1]$; this is well defined up to $s = 1$ since \mathcal{D} is invertible. Moreover we have that $P_1 = \text{sign}(\mathcal{D})$, which induces a degenerated cycle over $C_r^*(\mathring{X}_\Gamma \times_\Gamma \mathring{X}_\Gamma)$. This completes the proof. \square

We see the above result as an obstruction result: if the vertical metrics on the fibrations, which are assumed to be of positive scalar curvature, can be extended to a fibered corners metric of positive scalar curvature, then $\text{Ind}^{ad}(\mathcal{D})$ vanishes.

12.3. The rho-class of an invertible Dirac operator and its properties.

We assume that the Γ -equivariant fibered corners metric g_Φ has positive scalar curvature everywhere; as already remarked we then have that the operator \mathcal{D} is invertible.

Consider the path $P_s := \psi_s(\mathcal{D})$ defined in the previous subsection and observe that, after the usual reparametrization, the concatenation of $\psi(\mathcal{D}_{ad})$ and P_s defines a $(\mathbb{C}, C_r^*(G_{ad}^{[0,1]}))$ -bimodule that, when we restrict it to the adiabatic deformation parameter $t = 1$, is degenerate on the whole manifold X .

Definition 34. *The rho class associated to the metric g_Φ is the class $\rho(g_\Phi) \in KK(\mathbb{C}, C_r^*(G_{ad}^{[0,1]}))$ defined by concatenation of $\psi(\mathcal{D}_{ad})$ and P_s .*

¹¹here we are implicitly extending the reasoning explained in Subsection 7.1 to the closure of $\Psi_c^0(G)$; this extension does not pose any problem.

We shall now study the stability properties of this class. Let $\mathcal{R}_\Phi^+(X)$ be the set of Φ -metrics on X that are of positive scalar curvature. Recall that two Φ -metrics g_0 and g_1 are concordant if there exists a positive scalar curvature Φ -metric g on $X \times [0, 1]$, of product type near the boundary, such that the restriction of g to $X \times \{i\}$ is equal to g_i (for $i = 0, 1$). We denote by $\tilde{\pi}_0(\mathcal{R}_\Phi^+(X))$ the set of concordance classes of psc Φ -metrics. We denote by $\pi_0(\mathcal{R}_\Phi^+(X))$ the connected components of $\mathcal{R}_\Phi^+(X)$.

Proposition 35. *The application $\rho: \tilde{\pi}_0(\mathcal{R}_\Phi^+(X)) \rightarrow K_*(C_r^*(G_{ad}^{[0,1]}))$ given by*

$$[g] \mapsto \rho(g)$$

is well defined.

Similarly, the rho class gives a well defined application $\rho: \pi_0(\mathcal{R}_\Phi^+(X)) \rightarrow K_(C_r^*(G_{ad}^{[0,1]}))$.*

Proof. This is a direct application of the delocalized APS index theorem stated in Theorem 21. \square

13. SINGULAR FOLIATIONS

13.1. Blow-up constructions for Lie groupoids. Following the recent work of Debord and Skandalis, see [DS17], we shall now reobtain the groupoid associated to a manifold with fibered boundary and to the Φ -tangent bundle, as a blow-up construction in the groupoid context.

First, we define the deformation to the normal cone. Let Y be a smooth compact manifold and let X be a submanifold of Y . The deformation to the normal cone $DNC(Y, X)$ is obtained by gluing $N_X^Y \times \{0\}$ with $Y \times \mathbb{R}^*$, where N_X^Y denoted the normal bundle of X in Y . The smooth structure of $DNC(Y, X)$ is described by use of any exponential map $\theta: U' \rightarrow U$ which is a diffeomorphism from an open neighborhood U' of the 0-section in N_X^Y to an open neighborhood U of X . See [, Section 4.1] for the details.

The group \mathbb{R}^* acts on $DNC(Y, X)$ by

$$\lambda \cdot ((x, \xi), 0) = ((x, \lambda^{-1}\xi), 0), \quad \lambda \cdot (y, t) = (y, \lambda t) \quad \text{with } (x, \xi) \in N_X^Y, (y, t) \in Y \times \mathbb{R}^*.$$

Given a commutative diagram of smooth maps

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow f & & \downarrow f \\ X' & \hookrightarrow & Y' \end{array}$$

where the horizontal arrows are inclusions of submanifolds, we naturally obtain a smooth map $DNC(f): DNC(Y, X) \rightarrow DNC(Y', X')$. This map is defined by $DNC(f)(y, \lambda) = (f(y), \lambda)$ for $y \in Y$ and $\lambda \in \mathbb{R}$ and $DNC(f)(x, \xi, 0) = (f(x), f_N(\xi), 0)$ for $(x, \xi) \in N_X^Y$, where $f_N: N_X^Y \rightarrow N_{X'}^{Y'}$ is the linear map induced by the differential df . This map is equivariant with respect to the action of \mathbb{R}^* .

The action of \mathbb{R}^* is free and locally proper on $DNC(Y, X) \setminus X \times \mathbb{R}$ and we define $Blup(Y, X)$ as the quotient space of this action.

If $H \rightrightarrows H^{(0)}$ is a closed subgroupoid of a Lie groupoid $G \rightrightarrows G^{(0)}$, then $DNC(G, H)$ is a Lie groupoid over $DNC(G^{(0)}, H^{(0)})$ where the source and target maps are simply given by $DNC(s)$ and $DNC(r)$ as defined above.

Example 36. *Let $H = G^{(0)} \rightrightarrows G^{(0)}$ be the trivial groupoid, then N_H^G is just $\mathfrak{A}G$, the Lie algebroid of G , and $DNC^+(G, H)$ is the adiabatic deformation groupoid $G_{ad}^{[0, +\infty)}$.*

On the other hand, $Blup(G, H)$ is not a Lie groupoid over $Blup(G^{(0)}, H^{(0)})$, since the $Blup$ construction is not functorial. But $Blup(G, H)$ contains the dense open subset

$$Blup_{r,s}(G, H) := \left(DNC(G, H) \setminus (H \times \mathbb{R} \cup DNC(s)^{-1}(H^{(0)} \times \mathbb{R}) \cup DNC(r)^{-1}(H^{(0)} \times \mathbb{R})) \right) / \mathbb{R}^*$$

that is a Lie groupoid over $Blup(G^{(0)}, H^{(0)})$.

We shall be also interested in a variant of this construction: we consider $DNC(G, H) \rightrightarrows DNC(G^{(0)}, H^{(0)})$ and define $DNC^+(G, H)$ as its restriction to $(N_{H^{(0)}}^{G^{(0)}})^+ \times \{0\} \cup G^{(0)} \times \mathbb{R}_+^*$ with $(N_{H^{(0)}}^{G^{(0)}})^+$ denoting the positive normal bundle¹². We also define $Blup^+(G, H)$ as the quotient of $DNC^+(G, H) \setminus H \times \mathbb{R}_+$ by the action of \mathbb{R}_+^* . We obtain in this way the groupoid

$$Blup_{r,s}^+(G, H) \rightrightarrows Blup^+(G^{(0)}, H^{(0)}).$$

13.2. Revisiting the groupoid associated to a manifold with fibered boundary.

Let now M be a smooth manifold with fibred boundary $\partial M \xrightarrow{\phi} B$. For example, M is the resolution of a depth-1 stratified pseudomanifold ${}^S M$ with singular stratum B .

For simplicity we assume that ∂M is connected.

Our first goal is to recover in this groupoid context the b -stretched product appearing in the definition of the b -calculus, see [Mel93]. Consider $G := M \times M \rightrightarrows M$ and $H := \partial M \times \partial M \rightrightarrows \partial M$. Our present goal is to explicitly describe the groupoid

$$Blup_{r,s}^+(G, H) \rightrightarrows Blup^+(G^{(0)}, H^{(0)}).$$

We have, by definition,

$$DNC(G, H) = \partial M \times \partial M \times \mathbf{R}^2 \times \{0\} \cup M \times M \times \mathbb{R}^*,$$

and

$$DNC(G^{(0)}, H^{(0)}) = \partial M \times \mathbf{R} \times \{0\} \cup M \times \mathbb{R}^*$$

where we use the bold face to distinguish the fibers of the normal bundle. The groupoid structure of $DNC(G, H)$ is explicitly given by the following:

- $s((x, y), (n_1, n_2), 0) = (y, n_2, 0)$, $r((x, y), (n_1, n_2), 0) = (x, n_1, 0)$;
 $s((x, y), t) = (y, t)$, $r((x, y), t) = (x, t)$;
- $m(((x, y), (n_1, n_2), 0), ((y, z), (n_2, n_3), 0)) = ((x, z), (n_1, n_3), 0)$
 $m(((x, y), t), ((y, z), t)) = ((x, z), t)$.

This gives also the groupoid structure for $DNC^+(G, H)$ where, by definition, $DNC^+(G, H)$ is the restriction of $DNC(G, H)$ to $DNC^+(G^{(0)}, H^{(0)})$, which is $\partial M \times \mathbf{R}^+ \times \{0\} \cup M \times \mathbb{R}_+^*$; thus

$$DNC^+(G, H) = \partial M \times \partial M \times \mathbf{R}_+^2 \times \{0\} \cup M \times M \times \mathbb{R}_+^*,$$

Now,

$$Blup^+(G, H) := \left(DNC^+(G, H) \setminus (\partial M \times \partial M \times \{0\} \times \{0\} \cup \partial M \times \partial M \times \mathbb{R}_+^*) \right) / \mathbb{R}_+^*$$

and hence we see that

$$Blup^+(G, H) = \partial M \times \partial M \times \mathbf{S}_+^1 \cup (M \times M \setminus \partial M \times \partial M)$$

which is the b -stretched product of Melrose, denoted $[M \times M; \partial M \times \partial M]$ ¹³.

¹²where, for $h \in H^{(0)}$, $(N_{H^{(0)}}^{G^{(0)}})_h^+$ is defined by $(\mathbb{R}^n)_+ := \mathbb{R}_+^n$ once we fix a linear isomorphism $(N_{H^{(0)}}^{G^{(0)}})_h$ with \mathbb{R}^n

¹³Following the definitions one can prove that this identification gives a diffeomorphism between $Blup^+(G, H)$ and $[M \times M; \partial M \times \partial M]$

Finally, observing that $\partial M \times \partial M \times \{\underline{0}\} \times \{0\} \cup \partial M \times \partial M \times \mathbb{R}_+^*$ is nothing but $H \times \mathbb{R}_+$, we obtain $Blup_{r,s}^+(G, H)$ by removing from $Blup^+(G, H)$ the image through the quotient map of the subspaces

$$\partial M \times \partial M \times (\{\underline{0}\} \times \mathbf{R}_+) \times \{0\} \cup H \times M \times \mathbb{R}_+^* \subset DNC^+(G, H) \setminus H \times \mathbb{R}_+$$

$$\partial M \times \partial M \times (\mathbf{R}_+ \times \{\underline{0}\}) \times \{0\} \cup M \times H \times \mathbb{R}_+^* \subset DNC^+(G, H) \setminus H \times \mathbb{R}_+$$

(these are the points where

$$DNC(s) : DNC^+(G, H) \setminus H \times \mathbb{R}_+ \rightarrow DNC^+(G^{(0)}, H^{(0)})$$

and

$$DNC(r) : DNC^+(G, H) \setminus H \times \mathbb{R}_+ \rightarrow DNC^+(G^{(0)}, H^{(0)})$$

are not defined). We see therefore that

$$Blup_{r,s}^+(G, H) = \partial M \times \partial M \times \mathring{\mathbf{S}}_+^1 \cup \mathring{M} \times \mathring{M}$$

This gives an identification, in the category of smooth manifold with corners, of $Blup_{r,s}^+(G, H)$ with the b-stretched product of Melrose with the left and right boundaries removed.

Remark 37. *Observe that the closed subgroupoid $\partial M \times \partial M \times \mathring{\mathbf{S}}_+^1$ of $Blup_{r,s}^+(G, H)$ is isomorphic to $\partial M \times \partial M \times \mathbb{R}$, that is just the product of the pair groupoid with the additive group \mathbb{R} , through the isomorphism of Lie groupoids given by*

$$(x, y, [t, s]) \mapsto (x, y, \log(t) - \log(s)).$$

Observe also that the product of $(x, y, [t, s])$ and $(y, z, [u, v])$, that is $(x, z, [tu, vs])$, is sent to

$$(x, z, \log(tu) - \log(sv)) = (x, z, \log(t) + \log(u) - \log(s) - \log(v)) = (x, y, \log(t) - \log(s)) \cdot (y, z, \log(u) - \log(v)).$$

In this way we recover the Lie groupoid structure that corresponds to the translation invariance of the indicial operator of a b-operator, which is the restriction of the Schwartz kernel of the operator to the front face of the b-stretched product of Melrose.

Next, we want to recover through this construction the Φ -double space and the associated groupoid, as explained in Section 5. To this end we consider what we have got so far, namely

$$Blup^+(M \times M, \partial M \times \partial M), \quad Blup_{r,s}^+(M \times M, \partial M \times \partial M) \rightrightarrows M,$$

Notice that $\partial M \times_B \partial M \rightrightarrows M$ is a Lie subgroupoid of $Blup_{r,s}^+(M \times M, \partial M \times \partial M)$. We can first consider

$$Blup^+(Blup^+(M \times M, \partial M \times \partial M), \partial M \times_B \partial M);$$

with arguments similar to those given above, one can see without difficulty that this space is the Φ -double space considered in Subsection 6.1. Let us investigate what happens on the boundary: the inward normal space of $\partial M \times_B \partial M$ in the b-calculus groupoid is

$$\partial M \times_B \partial M \times_B TB \times \mathbf{R} \times \mathbf{R}_+ \rightrightarrows B \times \mathbf{R}_+$$

where the structure morphisms are given by:

- the source map $s_N(x, y, \xi, u, v) = (y, v)$ and the range map $r_N(x, y, \xi, u, v) = (x, v)$;
- the multiplication $m((x, y, \xi, u, v), (y, z, \eta, w, v)) = (x, y, \xi + \eta, u + w, v)$.

Now, as before, we remove the preimage by s_N and r_N of $M \times_B M \times \{0\}$ and we quotient by the action of \mathbb{R}_+^* . If we consider the representative elements of the classes in this quotient with $v = 1$, then this groupoid is clearly given by $\partial M \times_B TB \times_B \partial M \times \mathbb{R}$. This leads to the isomorphism

$$\text{Blup}_{r,s}^+(\text{Blup}_{r,s}^+(M \times M, \partial M \times \partial M), \partial M \times_B \partial M) \cong G_\Phi.$$

Remark 38. *If M_Γ is Γ covering of M which is also a manifold with fibered boundary $\partial M_\Gamma \rightarrow B_\Gamma$, with the Γ -action satisfying (21), then we can re-obtain the groupoid G_Φ^T : this is the groupoid*

$$\text{Blup}_{r,s}^+(\text{Blup}_{r,s}^+(M_\Gamma \times M_\Gamma, \partial M_\Gamma \times \partial M_\Gamma), \partial M_\Gamma \times_{B_\Gamma} \partial M_\Gamma) / \Gamma.$$

13.3. The groupoid associated to a manifold with foliated boundary.

Keeping with the general philosophy that treating a problem with groupoids solves, with minor work, a number of specific geometric problems, we now move to more singular situations. As already observed, when M is a manifold with fibered boundary $\partial M \xrightarrow{\phi} B$, the groupoid $\text{Blup}_{r,s}^+(\text{Blup}_{r,s}^+(M \times M, \partial M \times \partial M), \partial M \times_B \partial M)$, *id est*, up to isomorphism, the groupoid

$$G_\Phi = \overset{\circ}{M} \times \overset{\circ}{M} \cup \partial M \times_B TB \times_B \partial M \times \mathbb{R},$$

integrates the algebroid ${}^\Phi TM \rightarrow M$ defined by the Lie algebra of fibered boundary vector fields on M :

$$\mathcal{V}_\Phi(M) = \{\xi \in \mathcal{V}_b(M) \mid \xi|_{\partial M}, \text{ is tangent to the fibers of } \phi : \partial M \xrightarrow{\phi} B \text{ and } \xi x \in x^2 C^\infty(M)\}$$

Assume now that ∂M is foliated by \mathcal{F} and consider the \mathcal{F} -tangent bundle, ${}^\mathcal{F}TM$, defined through the Serre-Swan theorem starting from the Lie algebra of vector fields

$$\mathcal{V}_\mathcal{F}(M) = \{\xi \in \mathcal{V}_b(M) \mid \xi|_{\partial M} \in C^\infty(\partial M, T\mathcal{F}) \text{ and } \xi x \in x^2 C^\infty(M)\}$$

See [Roc12]. This defines, as in the previous case, an integrable algebroid and we shall now present a groupoid that integrates it.

To this end we first observe that the blow-up construction that we have briefly explained in the previous section works equally well if $H \rightrightarrows H^{(0)}$ is a Lie groupoid with a (possibly non-injective) immersion ι into $G \rightrightarrows G^{(0)}$; we require the additional property that ι induces an embedding $H^{(0)} \rightarrow G^{(0)}$. See [HS87, Remark 3.19]¹⁴. Notice that in this generality already the deformation to the normal cone, denoted here $DNC^+(H \xrightarrow{\iota} G)$, could be non-Hausdorff, even if H and G are Hausdorff; similarly, the groupoid $\text{Blup}_{r,s}^+(H \xrightarrow{\iota} G) \rightrightarrows \text{Blup}^+(G^{(0)}, H^{(0)})$ is such that $\text{Blup}_{r,s}^+(H \xrightarrow{\iota} G)$ is in general non-Hausdorff (whereas, from our additional hypothesis, we know that $\text{Blup}^+(G^{(0)}, H^{(0)})$ is Hausdorff, as it is required from the definition of Lie groupoid). Thanks to the work of Connes, see [Con82], one is nevertheless able to:

- define the C^* -algebra of such a groupoid
- consider the associated pseudodifferential algebra.

We shall come back to these points momentarily.

Consider now the holonomy groupoid $\text{Hol}(\partial M, \mathcal{F}) \rightrightarrows \partial M$ of the foliated manifold $(\partial M, \mathcal{F})$.

Proposition 39. *Let X be a smooth manifold and let \mathcal{F} and \mathcal{F}' be two foliations such that $\mathcal{F} \subset \mathcal{F}'$, then there exists an immersion of Lie groupoids*

$$\iota : \text{Hol}(X, \mathcal{F}) \rightarrow \text{Hol}(X, \mathcal{F}').$$

¹⁴For example, if $\iota : X \rightarrow Y$ is an immersion, then we consider the deformation to the normal cone defined via the bundle $N_{X \xrightarrow{\iota} Y}$ whose fiber at $x \in X$ is $T_{\iota(x)}Y/d\iota(T_x X)$.

Proof. Let $x \in X$ and let L_x and L'_x be the leaves of \mathcal{F} and \mathcal{F}' respectively that pass through x . Let $p_x: \tilde{L}_x \rightarrow L_x$ be the $\text{Hol}(L_x, x)$ -covering of L_x and let $p'_x: \tilde{L}'_x \rightarrow L'_x$ be the $\text{Hol}(L'_x, x)$ -covering of L'_x . We can lift the map $j_x \circ p_x: \tilde{L}_x \rightarrow L'_x$, where j_x is the obvious inclusion, to a map $\iota_x: \tilde{L}_x \rightarrow \tilde{L}'_x$ if

$$(j_x \circ p_x)_\#(\pi_1(\tilde{L}_x, x)) \subset (p'_x)_\#(\pi_1(\tilde{L}'_x, x))$$

inside $\pi_1(L'_x)$. But this is clear from the commutativity of the following diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(\tilde{L}_x, x) & \xrightarrow{(p_x)_\#} & \pi_1(L_x, x) & \longrightarrow & \text{Hol}(L_x, x) \longrightarrow 1 \\ & & \downarrow & & \downarrow (j_x)_\# & & \downarrow \\ 1 & \longrightarrow & \pi_1(\tilde{L}'_x, x) & \xrightarrow{(p'_x)_\#} & \pi_1(L'_x, x) & \longrightarrow & \text{Hol}(L'_x, x) \longrightarrow 1 \end{array}$$

Observe that the last vertical line is well defined since the group

$$\text{Hol}(L_x, x) \simeq \text{Hol}(L'_x, x) \times \text{Hol}_{L'_x}(L_x, x),$$

where $\text{Hol}_{L'_x}(L_x, x)$ is the holonomy group of the leaf L_x inside L'_x ; so the last vertical arrow is just the projection to the first factor. Now it is easy to check that the collection of $\iota_x: \text{Hol}(X, \mathcal{F})_x \rightarrow \text{Hol}(X, \mathcal{F}')_x$ gives a smooth map $\iota: \text{Hol}(X, \mathcal{F}) \rightarrow \text{Hol}(X, \mathcal{F}')$ and that it is an immersion. \square

Corollary 40. *Let (X, \mathcal{F}) a smooth foliated manifold with X compact. Then there exists an immersion of Lie groupoids*

$$\iota: \text{Hol}(X, \mathcal{F}) \rightarrow X \times X$$

Given a manifold M with foliated boundary $(\partial M, \mathcal{F})$ as above, we can first take $\text{Blup}_{r,s}^+(M \times M, \partial M \times \partial M) \rightrightarrows M$ and then consider

$$(64) \quad \text{Blup}_{r,s}^+ \left(\text{Hol}(\partial M, \mathcal{F}) \xrightarrow{\iota} \text{Blup}_{r,s}^+(M \times M, \partial M \times \partial M) \right)$$

explicitly described, up to a groupoid isomorphism, as:

$$(65) \quad \mathring{M} \times \mathring{M} \cup r^* N\mathcal{F} \times \mathbb{R},$$

with $N\mathcal{F} := T(\partial M)/T\mathcal{F}$ and r denoting the range map for the holonomy groupoid $\text{Hol}(\partial M, \mathcal{F})$. Here the groupoid structure is the pairs groupoid structure on $\mathring{M} \times \mathring{M}$, as usual, and on $r^* N\mathcal{F} \times \mathbb{R}$ is given by

$$(66) \quad (\gamma, \xi, t) \cdot (\gamma', \eta, s) = (\gamma \cdot \gamma', \xi + \gamma(\eta), t + s).$$

for $\gamma, \gamma' \in \text{Hol}(\partial M, \mathcal{F})$, $\xi \in N_{r(\gamma)}\mathcal{F}$, $\eta \in N_{r(\gamma')}\mathcal{F}$ and $t, s \in \mathbb{R}$. Notice that, since a foliation is locally given by a fibration and all the construction we used so far in this section are local, it is easy to prove that the normal bundle of the immersion $\text{Hol}(\partial M, \mathcal{F}) \xrightarrow{\iota} \text{Blup}_{r,s}^+(M \times M, \partial M \times \partial M)$ is $r^* N\mathcal{F} \times \mathbb{R} \times \mathbb{R}$ and, exactly as for the Φ -calculus groupoid, that (64) is isomorphic to (65).

13.4. The groupoid associated to a foliation degenerating on the boundary.

We can more generally consider a foliated manifold (M, \mathcal{H}) with non-empty boundary and with \mathcal{H} transverse to the boundary. We make the assumption that ∂M is foliated by a foliation \mathcal{F} such that $\mathcal{F} \subset \mathcal{H}|_{\partial M}$. We assume that M has dimension n , \mathcal{H} has dimension p and \mathcal{F} has dimension q with $q \leq p - 1$. In the previous subsection we have treated the case $p = n$.

Let $T\mathcal{F}$ be the tangent bundle of \mathcal{F} , a subbundle of $T(\partial M)$; let $T\mathcal{H}$ be the tangent bundle of \mathcal{H} and consider the Lie algebra of vector fields

$$\mathcal{V}_{\mathcal{F}}(M, \mathcal{H}) = \{ \xi \in C^\infty(M, T\mathcal{H}) \cap \mathcal{V}_b(M), \quad \xi|_{\partial M} \in C^\infty(\partial M, T\mathcal{F}) \quad \text{and} \quad \xi x \in x^2 C^\infty(M) \}$$

This is a finitely generated projective $C^\infty(M)$ -module and thus, according to Serre-Swan theorem, there exists a vector bundle ${}^{\mathcal{F}}T\mathcal{H}$ whose sections are precisely given by $\mathcal{V}_{\mathcal{F}}(M, \mathcal{H})$. Recall, see [Hil02, Section 5], that $\text{Hol}(\partial M, \mathcal{H}|_{\partial M})$ is a closed and open subgroupoid of $\text{Hol}(M, \mathcal{H})|_{\partial M}$ and, by Proposition 39 above, that there exists an immersion

$$\text{Hol}(\partial M, \mathcal{F}) \xrightarrow{j} \text{Hol}(\partial M, \mathcal{H}|_{\partial M})$$

that, together, give an immersion

$$\iota : \text{Hol}(\partial M, \mathcal{F}) \rightarrow \text{Hol}(M, \mathcal{H})|_{\partial M}.$$

We can thus consider $\text{Blup}_{r,s}^+(\text{Hol}(M, \mathcal{H}), \text{Hol}(M, \mathcal{H})|_{\partial M}) \rightrightarrows M$ and then

$$\text{Blup}_{r,s}^+(\text{Hol}(\partial M, \mathcal{F}) \xrightarrow{\iota} \text{Blup}_{r,s}^+(\text{Hol}(M, \mathcal{H}), \text{Hol}(M, \mathcal{H})|_{\partial M}))$$

which is described as:

$$\text{Hol}(M, \mathcal{H})|_{M^\circ} \cup r^*N_{\mathcal{F}}^{\mathcal{H}} \times \mathbb{R}$$

with $N_{\mathcal{F}}^{\mathcal{H}} := T\mathcal{H}|_{\partial M}/T\mathcal{F}$ and r denoting the range map for the holonomy groupoid $\text{Hol}(\partial M, \mathcal{F})$. Analogously to the previous cases, the groupoid structure is given by that one of the holonomy groupoid in the interior and by (66) on the boundary.

Remark 41. *We can equally make these constructions with the monodromy groupoid of our foliations, i.e. with $\text{Mon}(\partial M, \mathcal{F})$, $\text{Mon}(M, \mathcal{H})$. In this case the analogue of Proposition 39 already appears in [MM03, Proposition 6.2].*

13.5. (Pseudo)-differential operators, K-theory classes and positive scalar curvature.

Let (M, \mathcal{H}) be a foliated manifold with boundary with the foliation \mathcal{H} degenerating into $(\partial M, \mathcal{F})$ at the boundary. The vector fields in $\mathcal{V}_{\mathcal{F}}(M, \mathcal{H})$, together with $C^\infty(M)$, generate an algebra of differential operators on M that we denote $\text{Diff}_{\mathcal{F}}^*(M, \mathcal{H})$. In the case of M being a manifold with foliated boundary we use the notation $\text{Diff}_{\mathcal{F}}^*(M)$ as in [Roc12]. If $D \in \text{Diff}_{\mathcal{F}}^*(M, \mathcal{H})$ then D lifts uniquely to a differential operator on the groupoid

$$(67) \quad G_{\mathcal{F}}(M, \mathcal{H}) := \text{Hol}(M, \mathcal{H})|_{M^\circ} \cup r^*N_{\mathcal{F}}^{\mathcal{H}} \times \mathbb{R}$$

with $N_{\mathcal{F}}^{\mathcal{H}} := T\mathcal{H}|_{\partial M}/T\mathcal{F}$ and r denoting the range map for the holonomy groupoid $\text{Hol}(\partial M, \mathcal{F})$. With a customary abuse of notation we keep the same notation for this lift. The normal operator $N(D)$ is, by definition, the restriction of (the lift) of the operator to $r^*N_{\mathcal{F}}^{\mathcal{H}} \times \mathbb{R}$. This algebra of differential operators is of course a subalgebra of $\Psi^*(G_{\mathcal{F}}(M, \mathcal{H}))$ where, as in Section 12, we have enlarged the algebra of compactly supported operators pseudodifferential operators, $\Psi_c^*(G_{\mathcal{F}}(M, \mathcal{H}))$, by the addition of the smoothing operators as in [Vas06].

We now proceed to define all the objects that have been defined previously in the case of a manifold with fibered boundary and see how to get interesting K-theory classes. We shall be brief.

We consider the algebra $\Psi^0(G_{\mathcal{F}}(M, \mathcal{H}))$. Tangential ellipticity for elements in this algebra is defined as usual, in terms of ${}^{\mathcal{F}}T\mathcal{H}$, whereas full ellipticity is defined by requiring the normal operator to be invertible where invertibility is meant in the algebra $\Psi^0(G_{\mathcal{F}}(M, \mathcal{H})|_{\partial M}) \equiv \Psi^0(r^*N_{\mathcal{F}}^{\mathcal{H}} \times \mathbb{R})$. Similar definitions can be given for an operator of positive order. In order to define the fundamental class and the index class of a fully elliptic operator as well as the rho class of an invertible operator, we introduce the relevant groupoids and associated C^* -algebras:

- the adiabatic deformation $G_{\mathcal{F}}(M, \mathcal{H})_{ad}^{[0,1]} = {}^{\mathcal{F}}T\mathcal{H} \times \{0\} \cup G_{\mathcal{F}}(M, \mathcal{H}) \times [0, 1]$ and its restriction to $M \times [0, 1)$, denoted $G_{\mathcal{F}}(M, \mathcal{H})_{ad}^{[0,1]}$;

- the noncommutative tangent bundle ${}^{\mathcal{F}}T\mathcal{H}^{NC}$ defined as the restriction of $G_{\mathcal{F}}(M, \mathcal{H})_{ad}^{[0,1]}$ to $\mathring{M} \times \{0\} \cup \partial M \times [0, 1)$; it is equal to ${}^{\mathcal{F}}T\mathcal{H} \cup r^*\mathcal{N} \times \mathbb{R} \times (0, 1)$;
- the exact sequence of C^* -algebras:

$$(68) \quad 0 \longrightarrow C^*(G_{\mathcal{F}}(M, \mathcal{H})|_{\mathring{M}} \otimes C_0(0, 1)) \longrightarrow C^*(G_{\mathcal{F}}(M, \mathcal{H})_{ad}^{[0,1]}) \longrightarrow C^*({}^{\mathcal{F}}T\mathcal{H}^{NC}) \longrightarrow 0.$$

Let g be a foliated metric on $(\mathring{M}, \mathcal{H})$, i.e. a metric on $T\mathcal{H}|_{\mathring{M}}$. We shall say that g is an *admissible* metric if g extends to a metric on the Lie algebroid ${}^{\mathcal{F}}T\mathcal{H}$.

We shall work with special admissible metrics, that we call, as usual, *rigid* and that we proceed to define. Recall that the dimension of the leaves of \mathcal{H} is p and that the dimension of the leaves of \mathcal{F} is q , with $q \leq p - 1$. Let U be a distinguished neighbourhood for (M, \mathcal{H}) with $U \cap \partial M \neq \emptyset$. We can assume that U is homeomorphic to $[0, 1)_x \times F^q \times T^{p-q-1} \times S^{n-p}$ where all these sets are open sets in euclidean spaces of the right dimension. We shall also briefly write $U \simeq [0, 1)_x \times F \times T \times S$. Notice that in this notation T^{p-q-1} is a local transversal for \mathcal{F} inside $\mathcal{H}|_{\partial M}$ and S^{n-p} is a local transversal for \mathcal{H} .

We shall say that the admissible metric g is rigid if for each distinguished chart as above $g|_{\mathring{U}}$ is given by a family of metrics $g(s)$ that can be written as

$$g(s) = \frac{dx^2}{x^4} + \frac{g_T(s)}{x^2} + g_F(t, s).$$

Notice that g defines in a natural way a foliated metric $g_{\mathcal{F}}$ on $(\partial M, \mathcal{F})$, locally given by the $g_F(t, s)$.

When necessary, we denote by $g_{\mathcal{F}}^{\mathcal{H}}$ an admissible rigid metric. If we are considering the foliated boundary case, then we employ the notation $g_{\mathcal{F}}$ for an admissible rigid metric in this context. The simple notation g will also be used.

Remark 42. *Whereas a stratified space always carries a rigid fibered corner metric $g_{\mathcal{F}}$, see [ALMP12, Proposition 3.1], it is not true in general that the class of admissible rigid metrics $g_{\mathcal{F}}^{\mathcal{H}}$ is non-empty. Indeed let L be a leaf of the restriction of \mathcal{H} to the boundary and notice that L is foliated by \mathcal{F} , then the restriction of g to L would be a bundle-like metric with respect to \mathcal{F} and there are examples of foliated manifolds that do not admit bundle-like metrics.*

Let us now assume that $T\mathcal{H}$ and $T\mathcal{F}$ have a spin structure. Then we can consider the spin Dirac $G_{\mathcal{F}}(M, \mathcal{H})$ -operator \mathcal{D} associated to g , as in Remark 28.

If the induced metric $g_{\mathcal{F}}$ has positive scalar curvature along the leaves of \mathcal{F} , then as in Proposition 30, we can prove that $N(\mathcal{D})$ is invertible and so that \mathcal{D} is fully elliptic. Following the general procedure used in the previous sections we can then define the following classes and prove the properties of these listed below

- the noncommutative symbol

$$\sigma_{nc}(\mathcal{D}) \in K_*(C^*({}^{\mathcal{F}}T\mathcal{H}^{NC}))$$

defined as in Definition 18;

- the (adiabatic) index class

$$\text{Ind}^{ad}(\mathcal{D}) \in K_*(C^*(G_{\mathcal{F}}(M, \mathcal{H})|_{\mathring{M}}))$$

obtained as the image of $\sigma_{nc}(\mathcal{D})$ through the connecting homomorphism associated to (68); observe that this homomorphism has an explicit form as in (46);

- the index class given by the connecting homomorphism associated to the analogue of (35);
- the equality of these two index classes, proved as in Theorem 23;

- if g has positive scalar curvature everywhere along the leaves of $(\overset{\circ}{M}, \mathcal{H})$ then we can define a rho class

$$\rho(g) \in K_*(C^*(G_{\mathcal{F}}(M, \mathcal{H}))_{ad}^{[0,1]})$$

defined as in Definition 20;

- if $\mathcal{R}_{\mathcal{F}}^+(M, \mathcal{H})$ denotes the set of admissible rigid metrics of positive scalar curvature and $\tilde{\pi}_0(\mathcal{R}_{\mathcal{F}}^+(M, \mathcal{H}))$ the associated set of concordance classes. then the rho class gives a well defined map

$$\rho : \tilde{\pi}_0(\mathcal{R}_{\mathcal{F}}^+(M, \mathcal{H})) \rightarrow K_*(C^*(G_{\mathcal{F}}(M, \mathcal{H}))_{ad}^{[0,1]}).$$

This is proved as in Theorem 21 following the delocalized APS index theorem for general Lie groupoids established in [Zen16, Theorem 3.6]

14. APPENDIX: NON-HAUSDORFF GROUPOIDS

For the benefit of the reader, let us discuss in some detail the non Hausdorff case. At first glance it would seem rather involved, but in fact it is not. To get a clear intuition of what happens when we blow-up a non injective immersion, it is enough consider the local situation, in particular we are going to dissect the following very simple situation. Let us consider the immersion of Lie groupoids

$$\iota : \mathbb{Z}_2 \rightarrow \mathbb{R} \times \mathbb{R}$$

which sends every element of the group \mathbb{Z}_2 to $(0, 0)$. The deformation to the normal cone of this immersion is

$$\iota^* \mathcal{N}_{\iota(\mathbb{Z}_2)}^{\mathbb{R} \times \mathbb{R}} \cup \mathbb{R} \times \mathbb{R} \times \mathbb{R}^*$$

that is nothing else than the quotient of $\mathbb{Z}_2 \times DNC(\mathbb{R} \times \mathbb{R}, \{(0, 0)\})$ by the equivalence relation such that $([0], (x, y), t) \sim ([1], (x, y), t)$ for $t \neq 0$. As a set it is equal to

$$\mathbb{Z}_2 \times \mathbf{R}^2 \times \{0\} \cup \mathbb{R} \times \mathbb{R} \times \mathbb{R}^*.$$

Notice that the non-Hausdorff smooth structure is quotient structure induced by that of the two copies of $DNC(\mathbb{R} \times \mathbb{R}, \{(0, 0)\})$ and this one is given by imposing that the map $\psi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow DNC(\mathbb{R} \times \mathbb{R}, \{(0, 0)\})$

$$\psi(x, y, t) = \begin{cases} (x, y, 0) & \text{if } t = 0 \\ (x, ty, t) & \text{if } t \neq 0 \end{cases}$$

is smooth. Moreover it is easy to prove that the Lie algebroid of $DNC(\mathbb{Z}_2 \xrightarrow{\iota} \mathbb{R} \times \mathbb{R})$ is isomorphic to $\mathbb{R} \times \mathbf{R}$ and that the module of its sections is generated by the vector field $t \frac{\partial}{\partial t}$ over \mathbb{R} .

Now if we proceed to the *Blup* construction, we have to remove suitable subsets and then quotient by the action of \mathbb{R}^* , as explained in 13. Actually we are going to restrict our attention to the $Blup_{r,s}^+$ construction and we obtain the following non-Hausdorff Lie groupoid

$$\mathbb{Z}_2 \times \mathring{\mathbf{S}}^1 \cup \mathbb{R}_+^* \times \mathbb{R}_+^* \equiv \mathbb{Z}_2 \times \mathbb{R} \cup \mathbb{R}_+^* \times \mathbb{R}_+^*$$

that is nothing but the quotient of $\mathbb{Z}_2 \times Blup_{r,s}^+(\mathbb{R} \times \mathbb{R}, \{(0, 0)\})$ by the equivalence relation such that $([0], (x, y)) \sim ([1], (x, y))$ for all $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, with the induced smooth structure just as for the deformation to the normal cone.

Notice that, following [Mon, Section 2.2.2], since the pseudodifferential operators are defined, up to elements in the C^* -algebra, as conormal distribution supported near the diagonal, it turns out that $\Psi_c(\mathbb{Z}_2 \times \mathbb{R} \cup \mathbb{R}_+^* \times \mathbb{R}_+^*)$ is equal to $\Psi_c(\{[0]\} \times \mathbb{R} \cup \mathbb{R}_+^* \times \mathbb{R}_+^*) \equiv \Psi_c(Blup_{r,s}^+(\mathbb{R} \times \mathbb{R}, \{(0, 0)\}))$ up to elements in $C^*(\mathbb{Z}_2 \times \mathbb{R} \cup \mathbb{R}_+^* \times \mathbb{R}_+^*)$.

In other words, since the objects and the s -fibers are Hausdorff manifolds, if we see a pseudo-differential operator on the groupoid as a family of equivariant pseudo-differential operators on the

s -fibers, parametrized by the manifold of the objects, the property of being not Hausdorff of the whole groupoid is not involved in this definition.

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