Rendiconti Sem. Mat. Univ. Pol. Torino Vol. 78, 1 (2020), 125 – 141

## Alessandro Gambini, Remis Tonon, Alessandro Zaccagnini, with an addendum by Jacques Benatar and Alon Nishry

# SIGNED HARMONIC SUMS OF INTEGERS WITH k DISTINCT PRIME FACTORS

Abstract. We give some theoretical and computational results on "random" harmonic sums with prime numbers, and more generally, for integers with a fixed number of prime factors.

Keywords: Egyptian fractions; harmonic numbers; harmonic sums. 2010 Mathematics Subject Classification: Primary 11D75, Secondary 11B99.

#### 1. Introduction and general setting

It is well known that the harmonic series restricted to prime numbers diverges, as the harmonic series itself. This was first proved by Leonhard Euler in 1737 [7], and it is considered as a landmark in number theory. The proof relies on the fact that

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + O(1/N),$$

where  $\gamma \simeq 0.577215...$  is the Euler–Mascheroni constant. The corresponding result for primes is one of the formulae proved by Mertens, namely

$$\sum_{p \le N} \frac{1}{p} = \log \log N + A + O\left(\frac{1}{\log N}\right),$$

where  $A \simeq 0.2614972...$  is the Meissel–Mertens constant. It is also referred to as Hadamard–de la Vallée-Poussin constant that appears in Mertens' second theorem.

Recently, Bettin, Molteni and Sanna [2] studied the random harmonic series

(1) 
$$X := \sum_{n=1}^{\infty} \frac{s_n}{n}$$

where  $s_1, s_2,...$  are independent uniformly distributed random variables in  $\{-1, +1\}$ . Based on the previous work by Morrison [9, 10] and Schmuland [12], they proved the almost sure convergence of (1) to a density function g, getting lower and upper bounds of the minimum of the distance of a number  $\tau \in \mathbb{R}$  to a partial sum  $\sum_{n=1}^{N} s_n/n$ . In 1976 Worley studied the same problem in terms of upper bound of (1) both in the case  $\tau = 0$  (see [13]) and for a generic  $\tau \in \mathbb{R}$  (see [14]); his approach is not probabilistic but he has achieved an upper bound comparable to that of [2]. For further references, see also Bleicher and Erdős [3, 4], where the authors treated the number of distinct subsums

of  $\sum_{i=1}^{N} 1/n$ , which corresponds to taking  $s_i$  independent uniformly distributed random variables in  $\{0, 1\}$ . A more complete list of references can be found in [2].

The purpose of this paper is firstly to show that basically the same results hold for a general sequence of integers under some suitable, and not too restrictive, conditions; moreover, that a stronger result can be reached if we restrict to integers with exactly k distinct prime factors.

Although Bettin, Molteni and Sanna [2] treat both the lower bound and the upper bound, we are mainly interested in the upper bound using a probabilistic approach. As we will see, in the cases that we treat, we will not be able to say anything about the lower bound, except in terms of numerical computations.

We will use a consistent notation with the previous works by Bettin, Molteni and Sanna [1], [2], Crandall [6] and Schmuland [12].

#### 1.1. General setting of the problem

We denote by  $\mathbb{N}$  the set of positive integers. Let  $(a_n)_{n \in \mathbb{N}}$  be a strictly decreasing sequence of positive real numbers such that

(2) 
$$\lim_{n \to +\infty} a_n = 0$$
 and  $\sum_{n \ge 1} a_n = +\infty.$ 

Notice that

$$\sum_{n\geq 1}(-1)^n a_n$$

converges (not absolutely) by Leibniz's rule. Hence, by Riemann's theorem, given  $\lambda$ ,  $\Lambda \in [-\infty, +\infty]$  with  $\lambda \leq \Lambda$ , we can arrange the choice of the signs  $s_n = s_n(\lambda, \Lambda) \in \{-1, 1\}$ , in such a way that

$$\liminf_{N\to+\infty}\sum_{n\leq N}s_na_n=\lambda \qquad \text{and} \qquad \limsup_{N\to+\infty}\sum_{n\leq N}s_na_n=\Lambda.$$

As we said above, we are mainly interested in prime numbers, so we introduce some further reasonable hypotheses on the sequence  $a_n$ : we assume that  $b_n = a_n^{-1} \in \mathbb{N}$ , so that  $b_n$  is strictly increasing, and that

$$(3) n \le b_n \le nB(n),$$

where  $B(n) = n^{\beta(n)}$ , with  $\beta$  a real-valued decreasing function such that  $\beta(n) = o(1)$ . In order to prove Proposition 20 below, we will assume a more restrictive condition on  $\beta$ , that is

(4) 
$$\beta(n) \le \frac{1}{8\log\log n}$$
 for sufficiently large *n*.

Actually, this assumption is not strictly necessary and we will discuss this in Remark 25. Nevertheless, since the series  $\sum a_n$  must diverge, this condition is not too restrictive, and besides it is satisfied by most of the interesting sequences, like arithmetic progressions, the one of primes, and primes in arithmetic progressions.

Let us introduce some more notation: we consider the set

(5) 
$$\mathfrak{S}(N) = \left\{ \sum_{n \le N} s_n a_n \colon s_n \in \{-1, 1\} \text{ for } n \in \{1, \dots, N\} \right\},$$

and, for a given  $\tau \in \mathbb{R}$ , we set

$$\mathfrak{m}_N(\mathfrak{r}) = \min\{|S_N - \mathfrak{r}| \colon S_N \in \mathfrak{S}(N)\}.$$

In other words, for a given  $N \in \mathbb{N}$ , the goal is to find the choice of signs such that  $|S_N - \tau|$  attains its minimum value. Finally, we define the random variable

$$X_N := \sum_{n=1}^N s_n a_n,$$

where the signs  $s_n$  are taken uniformly and independently at random in  $\{-1,1\}$ . We will study its small scale distribution. With a slight abuse of notation, we denote by  $s_n$  both the signs in the definition (**b**) and the random variables in the definition above.

#### 1.2. Results

For ease of comparison with the results in Bettin, Molteni and Sanna [2], we now state our main results in the following form, even though more precise versions of them are to be found within the paper.

**Theorem 12.** Let  $\beta$  satisfy (A). Then there exists C > 0 such that for every  $\tau \in \mathbb{R}$  we have

$$\mathfrak{m}_N(\mathfrak{r}) < \exp(-C\log^2 N)$$

for all sufficiently large N depending on  $\tau$ .

**Theorem 13.** Let  $(b_n)_{n \in \mathbb{N}}$  be the sequence of integers having exactly k distinct prime factors. Then, for every  $\tau \in \mathbb{R}$  and for all sufficiently large N depending on  $\tau$ , we have

$$\mathfrak{m}_N(\tau) < \exp(-f(N))$$

where f is any function satisfying

$$f(N) = o\left(N^{1/(2k+1)-\varepsilon}\right).$$

*Remark* 14. We emphasize the fact that the estimate obtained in Theorem  $\square$  holds uniformly for every  $\tau \in \mathbb{R}$  in any fixed compact set.

**Corollary 15** (J. Benatar and A. Nishry). For any fixed  $\tau \in \mathbb{R}$ ,  $\varepsilon > 0$  and any sufficiently large N there exists a choice of signs  $(s_n)_{n \le N} \in \{-1, 1\}^N$ , such that

$$\sum_{n\leq N} \frac{s_n}{n} - \tau \bigg| \ll_{\tau,\varepsilon} \exp\left(-N^{1/3-\varepsilon}\right).$$

We collect some numerical results for k = 1 in Tables 11, 2 and 3. The sequence of Tables 11 and 2 appears in OEIS A 332399; see [5].

Acknowledgements. We thank Sandro Bettin and Giuseppe Molteni for many conversations on the subject, and Mattia Cafferata for his help in computing the tables at the end of the present paper. We also warmly thank Jacques Benatar and Alon Nishry for their fruitful suggestions which improved our paper, for providing us references and for letting us include their proof of Corollary 15 in this paper. R. Tonon and A. Zaccagnini are members of the INdAM group GNSAGA, which partially funded their participation to the Second Symposium on Analytic Number Theory in Cetraro, where some of this work was done.

#### 2. Lemmas

In this section we study some properties of the general sequence defined in (2), using the classical notation:  $\mathbb{E}[X]$  denotes the expected value of a random variable X,  $\mathbb{P}(E)$ the probability of an event E. For each continuous function with compact support  $\Phi \in C_c(\mathbb{R})$  we denote by  $\widehat{\Phi}$  its Fourier transform defined as follows:

$$\widehat{\Phi}(x) := \int_{\mathbb{R}} \Phi(y) e^{-2\pi i x y} dy.$$

We are actually interested in smooth functions, because the smoothness of the density of any random variable *X* is related to the decay at infinity of its characteristic function, defined precisely by its Fourier transform.

For each  $N \in \mathbb{N} \cup \{\infty\}$ , for any  $x \in \mathbb{R}$  and for any sequence satisfying (2), we also define the product

$$\rho_N(x) := \prod_{n=1}^N \cos(\pi x a_n) \quad \text{and} \quad \rho(x) := \rho_\infty(x).$$

We begin with the following lemma, which is a more general version of Lemma 2.4 from [2].

Lemma 16. We have

$$\mathbb{E}[\Phi(X_N)] = \int_{\mathbb{R}} \widehat{\Phi}(x) \rho_N(2x) \, \mathrm{d}x$$

for all  $\Phi \in \mathcal{C}^1_c(\mathbb{R})$ .

*Proof.* By the definition of expected value we have

$$\mathbb{E}[\Phi(X_N)] = \frac{1}{2^N} \sum_{s_1,\ldots,s_N \in \{-1,1\}} \Phi\left(\sum_{n=1}^N s_n a_n\right).$$

Using the inverse Fourier transform we get

$$\mathbb{E}[\Phi(X_N)] = \frac{1}{2^N} \sum_{s_1, \dots, s_N \in \{-1, 1\}} \int_{\mathbb{R}} \widehat{\Phi}(x) \exp\left(2\pi i x \sum_{n=1}^N s_n a_n\right) dx$$
$$= \int_{\mathbb{R}} \widehat{\Phi}(x) \frac{1}{2^N} \sum_{s_1, \dots, s_N \in \{-1, 1\}} \exp\left(2\pi i x \sum_{n=1}^N s_n a_n\right) dx.$$

Exploiting the fact that  $e^{i\alpha} + e^{-i\alpha} = 2\cos(\alpha)$ , we have

$$\sum_{s_1,\dots,s_N\in\{-1,1\}} \exp\left(2\pi i x \sum_{n=1}^N s_n a_n\right) = \frac{1}{2} \sum_{s_1,\dots,s_N\in\{-1,1\}} 2\cos\left(2\pi x \sum_{n=1}^N s_n a_n\right).$$

Finally, taking advantage of Werner's trigonometric identities, we obtain

$$\mathbb{E}[\Phi(X_N)] = \int_{\mathbb{R}} \widehat{\Phi}(x) \rho_N(2x) \, \mathrm{d}x.$$

We will need also a generalisation of Lemma 2.5 from [2], which is the following

**Lemma 17.** For all  $N \in \mathbb{N}$  and  $x \in [0, \sqrt{N}]$  we have

$$\rho_N(x) = \rho(x) \left( 1 + O\left( x^2/N \right) \right).$$

*Proof.* We recall that  $a_n$  is defined as in (2) and satisfies (3). In particular  $a_n = O(1/n)$ , so that the same argument in the proof of Lemma 2.5 of [2] holds.

Let us now define, for every positive integer N and any real  $\delta$  and x the set

 $\mathcal{S}(N,\delta,x,(a_n)_{n\geq 1}):=\{n\in\{1,\ldots,N\}\colon ||xa_n||\geq \delta\},\$ 

where  $\|\cdot\|$  denotes the distance from the nearest integer. For brevity, we sometimes drop the dependence on the sequence  $(a_n)_{n\geq 1}$ .

**Lemma 18.** *For all*  $N \in \mathbb{N}$  *and for all*  $x, \delta \ge 0$  *we have* 

$$|\mathbf{\rho}_N(x)| \leq \exp\left(-\frac{\pi^2\delta^2}{2} \cdot \#\mathcal{S}(N,\delta,x)\right).$$

Proof. It is a straightforward consequence of the inequality

$$|\cos(\pi x)| \le \exp\left(-\frac{\pi^2 ||x||^2}{2}\right).$$

**Lemma 19.** For any  $N \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $0 < \delta < 1/2$  we have

$$\frac{N}{2} - D(N, y(\delta), x) < \#\mathcal{S}(N, \delta, x) < N - D(N, y(\delta)/2, x),$$

where

$$D(N, y, x) = D(N, y, x, (b_n)_{n \ge 1}) := \sum_{\substack{x - y < m < x + y \\ N/2 \le n \le N}} \sum_{\substack{b_n \mid m \\ N/2 \le n \le N}} 1$$

and  $y(\delta) := \delta NB(N)$ .

Proof. As in Lemma 3.3 of [2], we observe that

$$\frac{N}{2} - T(N, \delta, x) < \#S(N, \delta, x) < N - T(N, \delta, x),$$

where

$$T(N,\delta,x) := \#\{n \in \mathbb{N} \cap [N/2,N] : \|xa_n\| < \delta\}.$$

Now, recalling that  $a_n = 1/b_n$ , we have

$$T(N,\delta,x) = \#\{n \in \mathbb{N} \cap [N/2,N] : \exists \ell \in \mathbb{N}, \ \ell - \delta < xa_n < \ell + \delta\}$$
$$= \#\{n \in \mathbb{N} \cap [N/2,N] : \exists \ell \in \mathbb{N}, \ x - \delta b_n < \ell b_n < x + \delta b_n\}.$$

From our hypothesis (3) we know that  $b_n \leq NB(N)$ ; then

$$T(N, \delta, x) < \#\{n \in \mathbb{N} \cap [N/2, N] \colon \exists \ell \in \mathbb{Z}, \ x - y(\delta) < \ell b_n < x + y(\delta)\}$$
  
=  $D(N, y(\delta), x).$ 

This proves the lower bound; the upper bound follows with the same argument.  $\Box$ 

Proposition 20. Let A be a fixed positive constant and, for N sufficiently large,

$$\beta(N) \leq \frac{1}{8\log \log N}.$$

Then there exists C' > 0 such that  $|\rho_N(x)| < x^{-A}$  for all sufficiently large positive integers N and for all  $x \in [N, \exp(C'(\log N)^2)]$ .

Proof. The proof follows along the same lines as Proposition 3.2 of [2]: we take

$$\overline{\delta} = \frac{2\sqrt{2A\log x}}{\pi} N^{-1/2}$$
 and  $x \in \left[N, \exp\left(\frac{\pi^2 N}{32A}\right)\right)$ ,

so that  $0 < \overline{\delta} < 1/2$  and  $y(\overline{\delta}) = \overline{\delta}NB(N) < x$ .

By Lemmas  $\mathbb{IS}$  and  $\mathbb{IS}$ , if we show that  $D(N, y(\overline{\delta}), x) < N/4$ , then we get  $|\rho_N(x)| < 1/x^4$ . Considering that  $b_n$  is a sequence of positive integers, we use Rankin's

trick with  $w \in (1/4, 1/2)$  and Ramanujan's result on  $\sigma_{-s}(n)$  [11] to obtain

$$\begin{split} D(N, y(\overline{\delta}), x) &< \frac{4}{\pi} \sqrt{2AN \log x} B(N) \cdot \max_{m \le 2x} \sum_{\substack{b_n \mid m \\ N/2 \le n \le N}} 1 \\ &< \frac{4}{\pi} \sqrt{2AN \log x} B(N) \cdot \max_{m \le 2x} \sum_{\substack{k \mid m \\ N/2 \le k \le NB(N)}} 1 \\ &\leq \frac{4}{\pi} \sqrt{2AN \log x} B(N) \cdot \max_{m \le 2x} \sum_{\substack{k \mid m \\ N/2 \le k \le NB(N)}} \left( \frac{NB(N)}{k} \right)^w \\ &= \frac{4}{\pi} N^{\frac{1}{2} + w} B(N)^{1 + w} \sqrt{2A \log x} \cdot \max_{m \le 2x} \sum_{\substack{k \mid m \\ N/2 \le k \le NB(N)}} k^{-w} \\ &\leq \frac{4}{\pi} N^{\frac{1}{2} + w} B(N)^{1 + w} \sqrt{2A \log x} \cdot \max_{m \le 2x} \sigma_{-w}(m) \\ &< \frac{4}{\pi} N^{\frac{1}{2} + w} B(N)^{1 + w} \sqrt{2A \log x} \cdot \exp\left(C_1 \frac{(\log 2x)^{1 - w}}{\log \log 2x}\right), \end{split}$$

where  $C_1$  is the constant of Ramanujan's theorem, as it is stated in Lemma 3.4 of [2].

Let  $w = w(x) := 1/2 - \varphi(x)$ , where  $\varphi$  is a positive decreasing function that we will choose later. Then we have

$$B(N)^{1+w} = \exp\left(\left(\frac{3}{2} - \varphi(x)\right)\beta(N)\log N\right),$$

and so we would be done if we showed that

$$N^{1-\varphi(x)+(3/2-\varphi(x))\beta(N)}\sqrt{\log x} \cdot \exp\left(C_1\frac{(\log 2x)^{1/2+\varphi(x)}}{\log \log 2x}\right) = o(N),$$

that is

$$\sqrt{\log x} \cdot \exp\left(C_1 \frac{(\log 2x)^{1/2 + \varphi(x)}}{\log \log 2x}\right) = o(N^{\varphi(x) + (\varphi(x) - 3/2)\beta(N)}).$$

Hence we must have

$$\varphi(x) + (\varphi(x) - 3/2)\beta(N) > 0,$$

that is

$$\beta(N) < \frac{\varphi(x)}{3/2 - \varphi(x)} \approx \frac{2}{3}\varphi(x).$$

Since  $\varphi$  is decreasing and we want to maintain the same range for x as in [2], that is  $x \in [N, \exp(C'(\log N)^2)]$ , we need to have

$$\beta(N) \lesssim \frac{2}{3} \varphi\left(\exp\left(C'(\log N)^2\right)\right).$$

Let us take  $\varphi(x) = (\log \log 2x)^{-1}$  and  $\beta(N)$  such that for  $x \in [N, \exp(C'(\log N)^2)]$  it holds

(6) 
$$\beta(N) \le \frac{2}{3J}\varphi(x) = \frac{2}{3J}\frac{1}{\log\log 2x}$$

where  $J \in \mathbb{R}$ , J > 1. Then we would achieve our goal if we showed that

$$\sqrt{\log x} \cdot \exp\left(C_1 e \frac{(\log 2x)^{1/2}}{\log \log 2x}\right) = o\left(\exp\left(\left(1 - \frac{1}{J} + o(1)\right) \frac{\log N}{\log \log 2x}\right)\right),$$

that is

$$\exp\left(C_1 e \frac{(\log 2x)^{1/2}}{\log \log 2x} - \left(1 - \frac{1}{J} + o(1)\right) \frac{\log N}{\log \log 2x} + \frac{1}{2} \log \log x\right) = o(1).$$

This condition is equivalent to

$$C_1 \operatorname{e} \frac{(\log 2x)^{1/2}}{\log \log 2x} - \left(1 - \frac{1}{J} + o(1)\right) \frac{\log N}{\log \log 2x} + \frac{1}{2} \log \log x \to -\infty.$$

Taking into account the ranges for *x*, we see that it is sufficient to have

$$\frac{1}{\log \log N} \left[ C_1 \sqrt{C'} \operatorname{e} \log N(1+o(1)) - \left(1 - \frac{1}{J}\right) \log N + O\left( (\log \log N)^2 \right) \right] \to -\infty.$$

We recall that, by our choice of x and N, we have  $\log \log x \approx \log \log N$ . Hence, we just need to take C' sufficiently small, in a way that

(7) 
$$C' < \left(\frac{J-1}{C_1 e J}\right)^2,$$

to guarantee that  $D(N, y(\overline{\delta}), x) < N/4$  for large *N*. For the sake of simplicity, we take J = 2 and the proposition is proved as stated.

*Remark* 21. We remark here that condition ( $\underline{A}$ ) on  $\beta$ , which we assumed to prove the proposition, was necessary to ensure the existence of the function  $\phi$  satisfying all the properties we needed, and in particular ( $\underline{B}$ ).

**Corollary 22.** Let A be a fixed positive constant and  $\beta$  satisfy (4). Then  $|\rho(x)| < x^{-A}$  for all sufficiently large  $x \in \mathbb{R}$ .

Proof. It holds

$$|\rho(x)| = \left| \rho_{\lfloor x \rfloor + 1}(x) \prod_{n > \lfloor x \rfloor + 1} \cos(\pi x a_n) \right| < x^{-A} . \Box$$

**Theorem 23.** Let C' > 0 satisfy (**D**) and  $\beta$  satisfy (**D**). Then for all intervals  $I \subseteq \mathbb{R}$  of length  $|I| > \exp(-C'(\log N)^2)$  one has

133

$$\mathbb{P}[X_N \in I] = \int_I g(x) \,\mathrm{d}x + o(|I|),$$

as  $N \rightarrow \infty$ , where

$$g(x) := 2\int_0^\infty \cos(2\pi ux) \prod_{n=1}^\infty \cos\left(\frac{2\pi u}{b_n}\right) du = 2\int_0^\infty \cos(2\pi ux)\rho(2u) du.$$

The proof follows along the same lines as Theorem 2.1 in [2] and we omit the details for brevity.

**Corollary 24.** Let  $\beta$  satisfy ((4)). For all  $\tau \in \mathbb{R}$  and C' > 0 satisfying ((1)), we have

$$#\left\{ (s_1,\ldots,s_N) \in \{-1,+1\}^N : \left| \tau - \sum_{n=1}^N \frac{s_n}{b_n} \right| < \delta \right\} \sim 2^{N+1} g(\tau) \delta(1 + o_{C',\tau}(1))$$

as  $N \to \infty$  and  $\delta \to 0$ , uniformly in  $\delta \ge \exp(-C'(\log N)^2)$ . In particular, for large enough N, one has  $\mathfrak{m}_N(\tau) < \exp(-C'(\log N)^2)$ .

*Remark* 25. We have imposed condition (2) for  $\beta$  to keep the same range of validity for x as in [2]. We remark that the hypotheses on  $\beta$  could be relaxed at the price of restricting this range: for example, we could take

$$\beta(N) = \frac{\log \log \log N}{\log \log N},$$

and obtain the result of Proposition 20 for  $x \in [N, \exp(\log^a N)]$ , where  $a \in (1, 2)$  is a suitable constant. In fact, this would weaken directly the estimates that we have just found in Theorem 23 and Corollary 24, where  $\exp(-C'(\log N)^2)$  would be replaced by  $\exp(-\log^a N)$ .

#### 3. Products of *k* primes

We now leave the general case and concentrate on primes and products of k distinct primes. Hence, we define

$$\mathcal{P}_k := \{ n \in \mathbb{N} \mid n \text{ is the product of } k \text{ distinct primes } \};$$

we will denote by  $b_n^{(k)}$  the *n*-th element of the ordered set  $\mathcal{P}_k$ . Let us recall the definition of  $\mathcal{S}(N, \delta, x)$  in the case  $a_n = 1/b_n^{(k)}$ :

$$S(N,\delta,x) := \{ n \in \{1,\ldots,N\} : ||x/b_n^{(k)}|| \ge \delta \}.$$

We remark that, since we left the general case, we can now take  $B(n) = b_n^{(k)}/n$ , and denote it by  $B_k(n)$ . In 1900, Landau [8] proved that

$$\pi_k(t) := |\mathcal{P}_k \cap \{n \in \mathbb{N} \mid n \le t\}| = \frac{t}{\log t} \frac{(\log \log t)^{k-1}}{(k-1)!} + O\left(\frac{t(\log \log t)^{k-2}}{\log t}\right),$$

which implies that

(8) 
$$B_k(n) \sim \log n \frac{(k-1)!}{(\log \log n)^{k-1}}$$

We can now start with a refinement of Proposition 20, where we extend the interval of validity for x in the case  $b_n = b_n^{(k)}$ .

**Proposition 26.** Let A be a fixed positive constant,  $k \in \mathbb{N}$  be fixed and  $a_n = 1/b_n^{(k)}$ , where  $b_n^{(k)}$  is the n-th element of the ordered set  $\mathcal{P}_k$ . Then  $|\rho_N(x)| < x^{-A}$  for all sufficiently large positive integers N and for all  $x \in [U, \exp(f(N))]$ , where  $\log N = o(f(N))$  and

$$f(N) = o\left(\left(\frac{N}{B_k^2(N)}\right)^{1/(2k+1)}\right)$$

and U > 1 is a constant depending on f.

*Proof.* Let  $x \in [N, \exp(f(N))]$ . As in the proof of Proposition 20, we need to show that  $D(N, y(\overline{\delta}), x) < N/4$ , where  $\overline{\delta}$  is chosen in the same way and  $y(\overline{\delta}) = \overline{\delta}NB_k(N)$ . Since now we are considering  $x \ge N$ , it is easy to see that for sufficiently large N we have  $y(\overline{\delta}) \le x$ . We recall here that the prime omega function  $\omega(n)$  is defined as the number of different prime factors of n, and that

$$\omega(n) \ll \frac{\log n}{\log \log n},$$

as a consequence of the prime number theorem. In this case, we have

$$D(N, y(\overline{\delta}), x) := \sum_{x-y(\overline{\delta}) < m < x+y(\overline{\delta})} \sum_{\substack{b_n^{(k)} \mid m \\ N/2 \le n \le N}} 1 \le \sum_{x-y(\overline{\delta}) < m < x+y(\overline{\delta})} \sum_{\substack{p_1 \dots p_k \mid m \\ p_i \text{ distinct primes}}} 1 \le \sum_{x-y(\overline{\delta}) < m < x+y(\overline{\delta})} \omega(m)^k \le (2y(\overline{\delta})+1) \max_{m < x+y(\overline{\delta})} \omega(m)^k \le (N\log x)^{1/2} B_k(N) \left(\frac{\log 2x}{\log \log 2x}\right)^k \ll N^{1/2} B_k(N) (\log x)^{k+1/2} g_k(N) (\log x)^{k+1/2}$$

where we used the trivial bound for the prime omega function. If we show that this quantity is o(N), we are done. So we need

$$\log x = o\left(\left(\frac{N}{B_k^2(N)}\right)^{1/(2k+1)}\right).$$

Hence we can take any f that satisfies

$$f(N) = o\left(\left(\frac{N}{B_k^2(N)}\right)^{1/(2k+1)}\right),$$

where we recall that  $B_k$  satisfies (**N**). The theorem is then proved for  $x \in [N, \exp(f(N))]$ . If x < N, it holds

$$|\rho_N(x)| \le |\rho_{|x|}(x)|,$$

hence the result we have just proved holds also whenever  $x \le \exp(f(\lfloor x \rfloor))$ . But there must exist U > 0 such that this holds for any x > U, since  $\log x = o(f(x))$ .

We are now ready to prove a more general version of Theorem 2.1 of [2] for the sequence  $(b_n^{(k)})_{n \in \mathbb{N}}$ .

**Theorem 27.** Let f and  $a_n$  be defined as in Proposition 26. Then for all intervals  $I \subseteq \mathbb{R}$  of length  $|I| > \exp(-f(N))$  one has

$$\mathbb{P}[X_N \in I] = \int_I g(x) \,\mathrm{d}x + o(|I|),$$

as  $N \rightarrow \infty$ , where

$$g(x) := 2\int_0^\infty \cos(2\pi ux) \prod_{n=1}^\infty \cos\left(\frac{2\pi u}{b_n^{(k)}}\right) du = 2\int_0^\infty \cos(2\pi ux)\rho(2u) du$$

*Proof.* The proof follows the one of Theorem 2.1 of [2]. Let  $\varepsilon > 0$  be fixed. We define

$$\begin{aligned} \xi &= \xi_{N,-\varepsilon} := \exp(-(1-\varepsilon)f(N)), \\ \xi_+ &= \xi_{N,+\varepsilon} := \exp(-(1+\varepsilon)f(N)), \\ \xi_0 &:= \xi_{N,0} = \exp(-f(N)), \end{aligned}$$

so that  $\xi^{-1} < \xi_0^{-1}$  and Proposition 26 holds for  $x \in [N, \xi_0^{-1}]$ . For an interval I = [a, b] with  $b - a > 2\xi_0$ , let us define  $I^+ := [a - \xi, b + \xi]$  and  $I^- := [a + \xi_+, b - \xi_+]$ . Then one can construct two smooth functions  $\Phi_{N,\varepsilon,I}^{\pm}(x) : \mathbb{R} \to [0, 1]$  (from now on, we will drop the subscripts when they are clear by the context) such that

$$\begin{cases} \sup \Phi^+ \subseteq I^+ \\ \Phi^+(x) = 1 & \text{for } x \in I, \\ \sup \Phi^- \subseteq I \\ \Phi^-(x) = 1 & \text{for } x \in I^-, \\ (\Phi^\pm)^{(j)}(x) \ll_j \xi^{-j} & \text{for all } j \ge 0. \end{cases}$$

By the last equation, we know that the Fourier transforms of  $\Phi^{\pm}$  satisfy

(9) 
$$\widehat{\Phi^{\pm}}(x) \ll_B (1+|x|\xi)^{-B} \text{ for any } B > 0 \text{ and } x \in \mathbb{R}.$$

Since

$$\mathbb{E}[\Phi^{-}(X_{N})] \leq \mathbb{P}[X_{N} \in I] \leq \mathbb{E}[\Phi^{+}(X_{N})],$$

we just need to show that

$$\mathbb{E}[\Phi^{\pm}(X_N)] = \int_{\mathbb{R}} \Phi^{\pm}(x)g(x)\,\mathrm{d}x + o_{\varepsilon}(|I|).$$

From now on,  $\Phi$  will indicate either  $\Phi^+$  or  $\Phi^-$ . By Lemma 16 we have

$$\mathbb{E}[\Phi(X_N)] = \frac{1}{2} \int_{\mathbb{R}} \widehat{\Phi}(x/2) \rho_N(x) \,\mathrm{d}x = I_1 + I_2 + I_3,$$

where  $I_1$ ,  $I_2$  and  $I_3$  are the integrals supported respectively in  $|x| < N^{\varepsilon}$ ,  $|x| \in [N^{\varepsilon}, \xi^{-(1+\varepsilon)}]$ and  $|x| > \xi^{-(1+\varepsilon)}$ . Note that  $\xi^{-(1+\varepsilon)} = \exp((1-\varepsilon^2)f(N)) > \exp(\varepsilon \log N) = N^{\varepsilon}$ , that  $\xi^{-(1+\varepsilon)} = \xi_0^{-(1-\varepsilon^2)} < \xi_0^{-1}$ , and that  $\xi^{-(1+\varepsilon)} \cdot \xi = \xi^{-\varepsilon} = \xi_0^{-\varepsilon(1-\varepsilon)} \to +\infty$  as  $N \to +\infty$ . By Lemma [7] and Corollary [22], we have

$$\begin{split} I_{1} &= \frac{1}{2} \int_{-N^{\varepsilon}}^{N^{\varepsilon}} \widehat{\Phi}(x/2) \rho_{N}(x) \, \mathrm{d}x = \frac{1}{2} \int_{-N^{\varepsilon}}^{N^{\varepsilon}} \widehat{\Phi}(x/2) \rho(x) \, \mathrm{d}x + O\left( \|\widehat{\Phi}\|_{\infty} N^{-1+3\varepsilon} \right) \\ &= \frac{1}{2} \int_{\mathbb{R}} \widehat{\Phi}(x/2) \rho(x) \, \mathrm{d}x + O_{A}\left( \|\widehat{\Phi}\|_{\infty} N^{-(A-1)\varepsilon} \right) + O\left( \|\widehat{\Phi}\|_{\infty} N^{-1+3\varepsilon} \right) \\ &= \int_{\mathbb{R}} \widehat{\Phi}(x) \rho(2x) \, \mathrm{d}x + O_{\varepsilon}\left( \|\Phi\|_{1} N^{-1+3\varepsilon} \right), \end{split}$$

where to conclude we chose  $A = A(\varepsilon)$  sufficiently large. For the second integral, we use Proposition 26 and obtain

$$\begin{aligned} |I_2| &\leq \|\widehat{\Phi}\|_{\infty} \int_{N^{\varepsilon}}^{\xi^{-(1+\varepsilon)}} |\rho_N(x)| \, \mathrm{d}x \leq \|\Phi\|_1 \int_{N^{\varepsilon}}^{\xi^{-(1+\varepsilon)}} x^{-A} \, \mathrm{d}x \leq \|\Phi\|_1 \int_{N^{\varepsilon}}^{+\infty} x^{-A} \, \mathrm{d}x \\ &\ll_{\varepsilon} \|\Phi\|_1 N^{-A\varepsilon+\varepsilon} \ll_{\varepsilon} \|\Phi\|_1 N^{-1}, \end{aligned}$$

where, as before, to conclude we took  $A = A(\varepsilon)$  sufficiently large. For the last integral, we recall that trivially  $|\rho_N(x)| \le 1$ ; using the bound (9), we obtain

$$\begin{aligned} |I_3| &\leq \int_{|x| > \xi^{-(1+\varepsilon)}} |\widehat{\Phi}(x/2)| \, \mathrm{d}x \ll_B \int_{\xi^{-(1+\varepsilon)}}^{+\infty} (1+x\xi)^{-B} \, \mathrm{d}x = (B-1)(\xi^{-1} + \xi^{-(1+\varepsilon)})^{1-B} \\ &\ll_B \xi_0^{B-1} = o_\varepsilon(\xi_0) = o_\varepsilon(|I|), \end{aligned}$$

where to conclude we chose  $B = B(\varepsilon)$  sufficiently large. We can now put these results together: using Parseval's theorem and the fact that  $\|\Phi\|_1 = O_{\varepsilon}(|I|)$ , we get

$$\mathbb{E}[\Phi(X_N)] = \int_{\mathbb{R}} \widehat{\Phi}(x) \rho(2x) \, \mathrm{d}x + O_{\varepsilon} \left( \|\Phi\|_1 N^{-1+3\varepsilon} \right) + o_{\varepsilon}(|I|) = \int_{\mathbb{R}} \Phi(x) g(x) \, \mathrm{d}x + o_{\varepsilon}(|I|)$$

and the theorem is then proved.

*Remark* 28. By Corollary 22, for any  $n \in \mathbb{N}$  it holds

$$\int_{-\infty}^{+\infty} |t^n \rho(t)| \, \mathrm{d}t < \infty,$$

which implies by standard arguments (see e.g. §5 of [12]) that the density g is a smooth strictly positive function. Besides, by the same corollary,  $g(x) \ll_D x^{-D}$  for any D > 0.

**Corollary 29.** For all  $\tau \in \mathbb{R}$ , we have

$$\#\left\{ (s_1,\ldots,s_N) \in \{-1,1\}^N : \left| \tau - \sum_{n=1}^N \frac{s_n}{b_n^{(k)}} \right| < \delta \right\} \sim 2^{N+1} g(\tau) \delta(1 + o_\tau(1))$$

as  $N \to \infty$  and  $\delta \to 0$ , uniformly in  $\delta \ge \exp(-f(N))$ , where f is defined as in Proposition 2d. In particular, for N large enough, one has  $\mathfrak{m}_N(\tau) < \exp(-f(N))$ .

#### 4. Addendum (by J. Benatar and A. Nishry): proof of Corollary **IIS**

*Proof.* Let  $c_m$  denote the *m*-th non-prime integer, so that  $c_1 = 1$ ,  $c_2 = 4$ ,  $c_3 = 6$ , ... We first approximate  $\tau$  with a restricted harmonic sum of the form  $\sum_{m \le M} s_m c_m$ , where  $M = M(N) = N - \pi(N)$ . Since  $C_m := c_m/m \sim 1$ , we may apply Theorem 12 to obtain a sequence of signs  $(s_n)_{n \le M} \in \{-1, 1\}^M$  such that

$$-1 \leq \tau' := \sum_{m \leq M} s_m c_m - \tau \leq 1.$$

Moreover, taking  $(p_n)_{n \in \mathbb{N}}$  to be the sequence of primes, we have that  $B(n) \sim \log n$  and hence we may apply Theorem 13 to get a choice of signs  $(\sigma_n)_{n \leq \pi(N)} \in \{-1, 1\}^{\pi(N)}$  such that

$$\left| \tau' - \sum_{n \leq \pi(N)} \frac{\sigma_n}{p_n} \right| \ll_{\tau, \varepsilon} \exp\left( -N^{1/3 - \varepsilon} \right) . \Box$$

#### References

- [1] S. Bettin, G. Molteni, and C. Sanna, *Greedy approximations by signed harmonic sums and the Thue–Morse sequence*, Adv. Math. 366, no. 3 (2020), 1–42.
- [2] S. Bettin, G. Molteni, and C. Sanna, *Small values of signed harmonic sums*, C. R. Math. Acad. Sci. Paris 356 (2018), no. 11-12, 1062–1074.
- [3] M. N. Bleicher and P. Erdős, *The number of distinct subsums of*  $\sum_{i=1}^{N} 1/i$ , Math. Comp. **29** (1975), 29–42.
- [4] M. N. Bleicher and P. Erdős, *Denominators of Egyptian fractions*. II, Illinois J. Math. 20 (1976), no. 4, 598–613.

- 138 A. Gambini, R. Tonon, A. Zaccagnini, with an addendum by J. Benatar and A. Nishry
- [5] M. Cafferata, A. Gambini, R. Tonon, and A. Zaccagnini, Sequence A332399 in The On-Line Encyclopedia of Integer Sequences (2020), published electronically at https://oeis.org/A332399.
- [6] R. E. Crandall, *Theory of ROOF walks*, Unpublished. Available at http://www. reed.edu/physics/faculty/crandall/papers/ROOF11.pdf, 2008.
- [7] L. Euler, *Variae observationes circa series infinitas*, Commentarii academiae scientiarum imperialis Petropolitanae **9** (1737), 160–188.
- [8] E. Landau, Sur quelques problèmes relatifs à la distribution des nombres premiers, Bull. Soc. Math. France **28** (1900), 25–38.
- [9] K. E. Morrison, Cosine products, Fourier transforms, and random sums, Amer. Math. Monthly 102 (1995), no. 8, 716–724.
- [10] K. E. Morrison, *Random walks with decreasing steps*, Unpublished manuscript, California Polytechnic State University, 1998.
- [11] S. Ramanujan, *Highly composite numbers*, Proc. London Math. Soc. **14** (1915), 347–409.
- [12] B. Schmuland, *Random harmonic series*, Amer. Math. Monthly **110** (2003), no. 5, 407–416.
- [13] R. T. Worley, Signed sums of reciprocals. I, J. Austral. Math. Soc. Ser. A 21 (1976), no. 4, 410–413.
- [14] R. T. Worley, Signed sums of reciprocals. II, J. Austral. Math. Soc. Ser. A 21 (1976), no. 4, 414–417.

Alessandro Gambini Dipartimento di Matematica Guido Castelnuovo Sapienza Università di Roma Piazzale Aldo Moro, 5 00185 Roma, Italia email (AG): alessandro.gambini@uniroma1.it

Remis Tonon, Alessandro Zaccagnini Dipartimento di Scienze, Matematiche, Fisiche e Informatiche Università di Parma Parco Area delle Scienze, 53/a 43124 Parma, Italia email (RT): remis.tonon@unimore.it email (AZ): alessandro.zaccagnini@unipr.it

Lavoro pervenuto in redazione il 02.10.2019.

<u>a</u> . 1	1 .		· · .	.1 1	1	
Signad	harmonic	cume of	intanarc	with b	dictinct	nrima tactore
Signed	narmonic	sums or	miceus	WILLIN	usunci	Diffic factors

N	$\mathfrak{m}_N(0) \cdot p_1 \cdots p_N$
1	1
2	1
3	1
4	23
5	43
6	251
7	263
8	21013
9	1407079
10	4919311
11	818778281
12	2402234557
13	379757743297
14	3325743954311
15	54237719914087
16	903944329576111
17	46919460458733911
18	367421942920402841
19	17148430651130576323
20	1236225057834436760243
21	4190310920096832376289
22	535482916756698482410061
23	29119155169912957197310753
24	443284248908491516288671253
25	28438781483496930396689638231
26	10196503226925713726754541885481
27	137512198125317766267968137765087
28	5572821202475305606211985553786081
29	77833992457426020006787481021085581
30	24244850423688161715955346535954790877
31	2030349334778419995324119439659994086131
32	76860130392109667765387079377871685276909
33	51919/0624445/608828445331682/0184721318637
34	3296432092/1348431895096550/92159132283920307
35	191/1590315567357340242017182966253037383120953
36	58192378490977430486851365332352874578233287403
37	83/477642920747839191618216897250374978659503996169
38	130665466261033919414441892800025408642432364448372023
39	7541550169407232608689149525984967898398947805296216009
40	23868339955752715692132986729285170427530832996153507207

### 1. Numerical data

Table 1: The values, multiplied by  $p_1 \cdots p_N$ , of the smallest signed harmonic sums with the first *N* primes, with *N* up to 40. See also <u>OEIS A332399</u>.

139

537867238 17916375626 12331033593	44537867238057 17916375626793 22331033593181 6891955087197 31209864412247 88005845271177	4537867238057 17916375626793 2331033593181 6891955087197 51209864412247 58005845271177 6904086039541 5381936073433	24537867238057 77916375626793 22331033593181 6891955087197 6891955087197 68905845271177 5381936073433 5274894064063 5274894064063	4537867238057 77916375626793 22331033593181 6891955087197 6891955087197 68905845271177 6904086039541 5381936073433 52274894064063 60827415289259	4537867238057 77916375626793 72331033593181 6891955087197 68905845271177 6904086039541 55381936073433 52274894064063 60827415289259 60827415289259 7104653817109	4537867238057 77916375626793 22331033593181 6891955087197 68905845271177 6904086039541 55381936073433 52274894064063 60827415289259 4364040946261 77104653817109 0946387592789	4537867238057 4537867238057 77916375626793 6891955087197 68905845271177 6904086039541 6904086039541 6904086039541 65381936073433 55381936073433 60827415289259 63877415289259 638740946261 77104653817109 60946387592789 60946387592789	4537867238057 77916375626793 77916375626793 6891955087197 68905845271177 88005845271177 76904086039541 5381936073433 55381936073433 55381936073433 60827415289259 4364040946261 77104653817109 0946387592789 6384458406969 3481014623817109 6684458406969 5684458406969	4537867238057 77916375626793 77916375626793 6891955087197 68905845271177 6904086039541 5381936073433 55381936073433 55381936073433 55381936073433 60827415289259 60827415289259 608244584064063 7104653817109 77104653817109 6084458406969 50684458406969 66884458406969 4195964225281	4537867238057 7916375626793 7916375626793 6891955087197 68905845271177 6904086039541 5381936073433 55381936073433 55381936073433 60827415289259 60827415289259 638749640946261 77104653817109 6946387592789 6084458406969 4195964225281 80682245858603
75668001545378672 19861279479163756 82605691723310335	0221474220015479163756 75668001545378672 19861279479163756 82605691723310335 06447930168919550 03688338812098644 16545780680058452	75668001545378672 19861279479163756 82605691723310335 06447930168919550 03688338812098644 16545780680058452 19240184469040860 15350069753819360	75668001545378672 19861279479163756 82605691723310335 06447930168919550 03688338812098644 16545780680058452 19240184469040860 15350069753819360 15350069753819360	7566800154537667 7566800154537667 19861279479163756 82605691723310335 06447930168919550 03688338812098644 16545780680058452 19240184469040860 15350069753819360 15350069753819360 153500294808274152 25202364808274152	75668001545376672 19861279479163756 82605691723310335 66447930168919550 03688338812098644 16545780680058452 19240184469040860 15350069753819360 15350069753819360 15350069753819360 15350069753819360 15350069753819360 11253204971046538	75668001545378672 19861279479163756 82605691723310335 82605691723310335 06447930168919550 03688338812098644 16545780680058452 16545780680058452 1624040860058452 192408469040860 15350069753819360 15350069753819360 12629109243640409 1125320497104653875 109047122094653875	75668001545376672 19861279479163756 82605691723310335 66447930168919550 03688338812098644 16545780680058452 19240184469040860 15350069753819360 15350069753819360 470218788227448940 15350049715209463875 10204712209463875 77869882870753747 77869882870753747	75668001545378672 19861279479163756 82605691723310335 82605691723310335 06447930168919550 03688338812098644 16545780680058452 19240184469040860 15350069753819360 15350069753819360 15350069753819360 15350069753819360 15350069753819360 10253204971046538 1094712209463875 77869882870753741 63915736034810146 184692622266844584	7566800154537672 19861279479163756 82605691723310335 82605691723310335 06447930168919550 03688338812098644 16545780680058452 19240184469040860 15350069753819360 470218788227448940 770809109243640409 11253204971046538 10904712209463875 77869882870753747 63915736034409 11253204971046538 10904712209463875 77869882870753747 639157360344584 18469262266844584 18469262266844584	7566800154537672 19861279479163756 82605691723310335 66447930168919550 03688338812098644 16545780680058452 19240184469040860 15350069753819360 47021878822748940 47021878822748940 47021878822748940 15350049136409 11253204971046538 10904712209463875 77869882870753747 63915736034810146 63915736034810146 63915736034810146 63915736034810146 63915736034810146 63915736034810146 6391573603482870753747 5792922141959642 66570600806822458
193601517566800154 141755151986127947 127987318260569172	193601517566800154 141755151986127947 127987318260569172 564737720644793016 116464350368833881 548880741654578068	193601517566800154 141755151986127947 127987318260569172 564737720644793016 116464350368833881 116464350368833881 116464350368833881 135741161924018446 135741161924018446 255197731535006975	193601517566800154 141755151986127947 127987318260569172 564737720644793016 116464350368833881 548880741654578068 135741161924018446 135741161924018446 255197731535006975 047366774702187882	193601517566800154 141755151986127947 127987318260569172 564737720644793016 116464350368833881 548880741654578068 135741161924018446 135741161924018446 255197731535006975 255197731535006975 255197731535006975 1497366774702187882	<pre>193601517566800154 141755151986127947 127987318260569172 564737720644793016 116464350368833881 54880741654578068 135741161924018446 135741161924018446 135741161924018446 255197731535006975 255197731535006975 25519773153506975 25531131629910924 499473941125320497 </pre>	193601517566800154 141755151986127947 127987318260569172 564737720644793016 164737720644793016 1564737720644793016 1554941654578068 135741161924018446 135741161924018446 255197731555006975 255197731555006975 149178582620236480 1491785882620236480 1491785882620236480 1491785882620236480 1491785882620236480 1491785882620236480 1491785882620236480 1491785882620236480 1491785882620236480 1491785882620236480 1491785882620236480 149178588260 1491785882620236480 1491785882620236480 149178588262023647722878582620236477220 149178588260 149178588260 149178588260 149178588260 149178588260 149178588260 149178588260 149178588260 149178588260 149178588260 149178588260 149178588260 149178588260 149178588588578585857785885778585857858585857857	193601517566800154 141755151986127947 127987318260569172 564737720644793016 116464350368833881 54880741654578068 135741161924018446 135741161924018446 135741161924018446 135741161924018446 149178582620236480 892531131629910924 892531131629910924 149473941125320497 784365401090471220 569512737786988287	<pre>193601517566800154 141755151986127947 127987318260569172 564737720644793016 564737720644793016 564737720644793016 135741161924018446 135741161924018446 135741161924018446 135741161924018448 135741161924018482 14917858202036480 14917858202036480 14917858202036480 14917858202036480 14917858202036480 14917858202036480 14917858202036480 1491785820203688 14917858200375 1491785820203688 14917858200375 14917858200375 140129876391573603 153298137184692622 153298137184692622 1401297730 14012876391573603 153298137184692622 140129876391573603 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 153298137184692622 15329813718 152 153298137184692622 15329813718 152 15 15 15 15 15 15 15 15 15 15 15 15 15</pre>	<pre>193601517566800154 141755151986127947 127987318260569172 564737720644793016 1664737720644793016 1564737720644793016 15647375068833881 135741161924018446 135741161924018446 135741161924018486 135741161924018486 149178582620336480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236480 149178582620236226 1491220 14917858262023648 15213761579292222 1491287639157730 1521376157929222 1521376157929222 1521376157929222 1521376157929222 1521376157929222 1521376157929222 1521376157929222 1521376157929222 1521376157929222 1521376157929222 152137615792922 152137615792922 152137615792922 152137615792922 152137615792922 152137615792922 15213761579 1521376 1521376 1521376 1521376 152137 1521 152 152137 152 1521 15 15 15 15 15 15 15 15 15 15 15 15 15</pre>	<pre>193601517566800154 141755151986127947 127987318260569172 564737720644793016 116464350368833881 54880741654578068 135741161924018446 135741161924018446 135741161924018446 135741161924018446 135741161924018446 135741161924018446 135741161924018446 14947391131629910924 149473941125320497 149473941125320497 149473941125320497 149473941125320497 149473941125320497 149473941125320497 149473941125320497 149473941125320497 149473941125320497 149473941125320497 149473941125320497 149473941125320497 149473911125320497 149473941125320497 149473941125320497 184365401090471220 16012987639157786988287 16012987639157786988287 16012987639137786988287 16012987639137786988287 1522137615792922214 143356657060080 125137615792922214 12522137615792922214 125222137615792922214 12522137615792922214 12522137615792922214 125222137615792922214 12522137615792922214 125222137615792922214 125222137615792922214 1252222137615792922214 1252222137615792922214 1252222137615792922214 125222214 125222214 1252222214 125222224 1252222224 1252222224 125222224 12522224 125222224 125222224 12522224 125222224 12522224 125222224 12522224 12522224 12522224 125222224 12522224 12522224 12522224 12522224 12522224 125222224 12522224 125222224 1252 1252</pre>
805290441755151986 641150127987318260	805290441755151986 641150127987318260: 246782664737720644 581772116464350368: 806288648880741654	805290441755151986 641150127987318260: 246782664737720644 581772116464350368 806288648880741654 841680135741161924 483564255197731535	805290441755151986 641150127987318260 246782664737720644 581772116464350368 880741654 881680135741161924 4835642551977315351 167655047366774702	805290441755151986 541150127987318260 546782664737720644 581772116464350368 806288648880741654 841680135741161924 483564255197731535 167655047366774702 0813231491785825020	805290441755151986 641150127987318260 246782664737720644 581772116464350368 806288648880741654 841680135741161924 483564255197731535 167655047366774702 081323149178582620 484621892531131629 336245449473941125 336245449473941125	805290441755151986 541150127987318260 541150127987318260 581772116464350368 881628864880741619244 881680135741161924 483564255197731535 167655047366774702 081323149178582620 081323149178582620 081323149178582620 081323149178582620 081323149178582620 081323149178582620 081323149178582620 081323149178582620 081323149178582620 081323149178582620 081325178436541125 294051784365410900	805290441755151986 641150127987318260 246782664737720644 581772116464350368 806288648880741654 841680135741161924 841680135741161924 883564255197731535 167655047366774702 081323149178582620 081323149178582620 336245449473941125 294051784365401090 163881669512737786	805290441755151986 541150127987318260 541150127987318260 581772116464350368 8628880716348 8628880716348 8806444 58177211646435038 880535197731535 1676550473657731535 167655047367731535 167655047367731535 167655047367731535 16765504736773151629 336245449473941125 2381669512737786 163881669512737786 430193460129876391 579866232981371846	805290441755151986 541150127987318260 581772116464350368 881772116464350368 881772116464350368 8807441654 881680135741161924 483564255197731535 167655047366774702 081323149178582620 081323149178582620 081323149178582620 881323149178582620 081323149178582620 836245449473941125 336245449473941125 336245449473941125 336245449473941125 336245449473941125 379866232981371846 379555222137615792	805290441755151986 541150127987318260 581772116464350368 881772116464350368 880741654 880628864880741654 483564255197731535 167655047366774702 167655047366774702 167655047366774702 336245449473941125 336245449473941125 336245449473941125 336245449473941125 336245449473941125 3795522137615798 37955522137615792 177591943696826657 177591943696826657
51015564115012798	51015564115012798 01458424678266473 46862758177211640 01346780628864888	5101556411501279 5186275817721164 4686275817721164 1134678062886488 1134678062886488 1521888416801357 52942048356425519	5101556411501279 3145842467826647 4686275817721164 3134678062886488 1521888416801357 5294204835642551 3368741676550473	5101556411501279 5101556411501279 1145862758177211644 1134678062886488 1134678062886488 1521888416801357 5294204835642551 53368741676550473 404660813231491 404660813231491	5101556411501279 3101556411501279 4686275817721164 4686275817721164 134678062886488 1521888416801357 5294204835642551 3368741676550473 4046660813231491 56123048462189255 5291503362454494	5101556411501279 3145842467826647 4686275817721164 1134678062886488 1521888416801357 1521888416801357 5294204835642551 3368741676550473 4046660813231491 5612304846218925 55291503362454494 3756112940517843 55291503362454494 5529150336244494 552915033624494 55291503362454494 552915033624494 5529150336244494 55291503362454494 5529150336244494 55291503362454494 55291503362454494 55291503362454494 55291503362454494 552915033654494 5529150336544494 5529150336544494 5529150336544494 55291503365454494 55291503365454494 55291503365454494 55291503365454494 55291503365554494 55291503365454494 55291503365454494 55291503365444957 552915035555 55291503555555 552915035555 552915035555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 55291555 552915555 55291555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 552915555 5529155555 5529155555 5529155555 5529155555 552915555555555555555555555555555555555	5101556411501279 3145842467826647 4686275817721164 4686275817721164 1521888416801357 5294204835642551 3368741676550473 3368741676550473 4046660813231491 566123048462189255 5291503362454494 5661223048462189255 5291503362454494 5661223048462189255 5291503362454494 5661223048462189255 5291281638816695 5241281638816695 5245043010346017 528375043010346017 528375043010346017 528375043010346017 528375043010346017 528375043010346017 528375043010346017 528375043010346017 528375043010346017 528375043010346017 528576047010046605 528504300000000000000000000000000000000000	5101556411501279 145842467826647 46862758177211644 1346780628864888 15218884168013574 52942048356425519 53687416765504733 542048365031377 4046660813231491 5612304846218925 5691503362454494 56123048462189255 591503362454494 56123048462189255 591503362454494 56123048465189255 59126123048462189255 59126123048462189255 50245757986623298 40243757986623298	5101556411501279 145842467826647 4686275817721164 134678062886488 15218884168013575 529420483564253157 529420483564253157 5358741676550473 4046660813231491 56123048462189255 5291503362454494 56123048462189255 5291503362454494 56123048462189255 5291503362454494 56123048462189255 5291503362454494 56123048462189255 5291503362454494 56123048462189255 5291503362454494 56123048462189255 5291503362454494 56123048462189255 5291503362454494 56123048462189255 5291503362454494 517473757986623298 517473379555522211 517473379555522211 517473379555522211 517473379555522211 517473379555522212 517473379555522211 517473379555522212 517473379555522211 51747377986623298 517473379555522212 5174737757986623298 517473379555522212 5174737757986623298 517473379555522212 5174737757986623298 517473379555522212 5174737757986623298 517473379555522212 5174737757986623298 517473379555522212 5174737757986623298 5174737757986623298 5174737757986623298 51747377986623298 51747377986623298 51747377986623298 51747377986623298 51747377986623298 51747377986623298 51747377986623298 51747377986623298 51747377986623298 51747377986623298 51747377986623298 517478778 517478778 5174787878 51747878 51747878 5174788	5101556411501279 3145842467826647 4686275817721164 1134678062886488 152188416801357 5294204835642551 5294204835642551 5294204835642531491 5612304846218925 5291503362454494 56123048462189255 5291503362454494 56123048462189255 5291503362454494 56123048462189255 529150336245494 5174733795552221 56694217759194366 5174733795552221 56694217759194366 5174733795552221 56694217759194366 5174733795552221 56694217759194366 57201 56694217759194366 57201 56694217759194366 57201 56694217759194366 56694217759194366 56694217759194366 567201 56694217759194366 567201 56694217759194366 567201 56694217759194366 567201 567201 567201 567201 567201 567201 567201 567201 567201 567201 567201 567201 567201 577201 5
CIUI0/2118C43	C1010/2112/01012/ 145575301458 162487746862 3303054101346	C1010/211201012 145575301458 162487746862 162487746862 16248774682 386200315218 1344537262942	45575301458 145575301458 162487746862 162487746862 162487746862 165647133687 165647133687	455112/012/012/012/012/012/012/01458 145575301458 162487746862 303054101346 386200315218 344557262942 3445547133687 465647133687 329912040466	4381120121845 145575301458 162487746862 303054101346 386200315218 3845537262942 13445537262942 445647133687 465647133687 3329912040466 338078666123 412673852915	45575301458 145575301458 162487746862 303054101346 386200315218 386200315218 344537262942 4455647133687 465647133687 465647133687 398078666123 3412673852915 9662552237561	4551120121012110121101211012110121101211	455112/01012 145575301458 162487746862 303054101346 386200315218 386200315218 386200315218 33627466 3398078666123 3398078666123 39807866123 39807866123 39807866123 39807866123 39807866123 3007874888375 009874888375 009874888375	455112/011012 145575301458 162487746862 303054101346 386200315218 3345537262942 3345537262942 3398078666123 3398078666123 3398078666123 3412673852915 96513412673852915 96513412673852915 0962552237561 0651341240243 107401340243	4551120121201458 145575301458 162487746862 303054101346 336200315218 3345537262942 3345537262942 33687 3329912040466 3329912040466 3329912040466 3329912040466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 33291204466 3329120466 34291200000000000000000000000000000000000
1484163245	1484163245 2461132114 1780440916 4026505330	1484163245 2461132114 1780440916 4026505330 9880636738 9679954034	1484163245( 1484163245( 1780440916( 4026505330) 98026505330 9679954034 9283194746	1484163245 2461132114 1780440916 4026505330 9880636738 9679954034 9283194746 1285627032	1484163245 1484163245 2461132114 1780440916 4026505330 98806505330 98806505330 98806505330 9679954034 9283194746 1285627032 3451830339 1989346241	1484163245( 1484163245( 1780440916( 1780440916) 9679954034 9283194746( 1285627032) 1285627032 1989346241 1989346241 1989346241 1989346241	1484163245 1484163245 2461132114 1780440916 98806505330 98806505330 98806505330 988065033 94746 1285627032 1983346241 1989346241 1989346241 1989346241 3196330096 0721675665	$\begin{array}{c} 1484163245(\\ 1484163245(\\ 2461132114(\\ 1780440916(\\ 17804555330)(\\ 64026555330)(\\ 9679954054333)(\\ 9679954034\\ 9283194746(\\ 1285627032)\\ 1285627032\\ 1285627032\\ 1285627032)(\\ 12856770)(\\ 12856770$	1484163245( 1484163245( 1780440916( 1780440916) 9679954330 96799543333 9283194746( 1285627032) 13853294746( 1285627032) 1389346241 1983346241 1983346255555555555555555555555555555555555	1484163245( 1484163245( 1780440916( 1780440916) 98806505330( 9283194746( 1285627032) 1989346241( 1989346241( 1989346241( 1989346241( 1989346241( 1989346241( 1989346241( 1989346241( 1989346241( 1989346241( 1989346241( 1989346241( 1983373) 196183273 9618327613
	3570378824( 68861589177 2854952040	35703788246 68861589178 28549520400 99680710988 5533632496	3570378824( 68861589178 58861589178 58549520402 99680710988 5533632496 51332441923	3570378824( 6886158917( 2854952040) 9968071098( 5533632496 5133244192( 5133244192)	357037824( 6886158917( 5886158917( 2854952040) 9968071098( 5533632496 5133244192( 5133244192) 2133244192( 5133244192) 2761969934( 9761969934( 9761969934( 9761969934( 9761969934( 9761969934( 9761969934( 9761969934( 9761969934( 976196933( 976196933( 976196933( 976196933( 976196933( 976196933( 976196933( 976196933( 976196933( 976196933( 976196) 976196( 9761960000000000000000000000000000000000	35703788246 68861589178 568861589178 99680710988 5533632496 51332441928 513324119 51332411928 513324 513324 513324 513324 513324 513324 513325 513324 513324 513525 513525 513525 513525 513525 513525 513525 513525 513525 513525 513555 513555 513555 513555 5135555 5135555 5135555 51355555 51355555555	357037824( 58861589178 58861589178 28549520402 99680710988 5533632496 5533632496 5533632496 51332441928 51332441928 5613318128 29537081198 566392617319 56713803657 5713803657	35703788246 86861589178 86861589178 99680710988 55336324966 51332441928 561332441928 97619699344 97619699344 29537081198 29537081193 38742475072 5713893655 74980553934	35703788246 35703788246 558951590178 99680710988 55336324966 51332441928 553363318123 97619699349 56537081198 66993441928 5713893657 5713893655 5713893655 57138936553 74980553939	357037824( 68861589178 58861589178 58549520402 99680710988 5533632496 51332441928 51332441928 5633318128 9761969934 5713893653 5713893653 5713893653 5056214956 6869447496
000	203 19376 466412	203 19376 466412 25229409 164152705	203 19376 466412 25229409 164152705 642450545	203 19376 466412 25229409 164152705 164152705 642450545 642450545	203 19376 466412 466412 25229409 164152705 642450545 642450545 407776132 375146609 317665922 317665922	203 19376 466412 25229409 164152705 642450545 642450545 407776132 375146609 317665922 3166158296	203 19376 466412 25229409 164152705 642450545 642450545 407776132 375146609 317665922 166158296 166158296 876335793	203 19376 466412 25229409 164152705 642450545 642450545 407776132 375146609 317665922 1661582963 1661582923 166158293 876335793 876335793 002768497 002768497	203 19376 466412 25229409 164152705 642450545 642450545 407776132 375146609 317665922 166158296 876335793 876335793 876335793 876335793 002768497 002768497	203 19376 466412 25229409 164152705 164152705 1641552545 642450545 407776132 375146609 317665922 166158296 166158296 876335793 7774262135 777258925 335809366
			1 25436	1 25436 1780024	1 25436 1780024 115533643 644520573	1 25436 1780024 115533643 644520573	$\begin{array}{c} 1\\ 25436\\ 1780024\\ 115533643\\ 644520573\\ 668963051\\ 582248468\\ 033653\\ 03365\\ 0356\\$	1 25436 1780024 115533643 644520573 668963051 668963051 582248468 933638637 698188450 698188450	1 25436 1780024 115533643 644520573 668963051 582248468 582248468 933638637 933638637 698188450 698188450	1 25436 1780024 115533643 644520573 644520573 668963051 68963051 582248468 933638637 674905123 674905123
					346	1 3466 73696	1 346 73696 96325	1 346 96325 19819 19819	1 346 73696 73696 <u>96325</u> 96325 19026	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 6 1 1 1 6 9 9 1 1 1 1
							1199	199	19995 11351 12724	73 73 19996 13519 227249 54326 54326
							15125	199 15135 1530272	151351 151351 15302722 6269085432	73 19996 1513519 153027249 153027249 62690854326 4299187908371

Table 2: The values, multiplied by  $p_1 \cdots p_N$ , of the smallest signed harmonic sums with the first N primes, with N between 41 and 60. See also <u>OEIS A332399</u>.

140 A. Gambini, R. Tonon, A. Zaccagnini, with an addendum by J. Benatar and A. Nishry

N	$\Delta_N \cdot p_1 \cdots p_N$
1	1
2	1
3	1
4	2
5	22
6	35
7	263
8	4675
9	24871
10	104006
11	2356081
12	6221080
13	141769355
14	6096082265
15	6928889495
16	367231143235
17	1283811918935
18	78312527055035
19	5246939312687345
20	372532691200801495
21	8815359347599933286
22	223849990729887044174
23	6148176498383067879445
24	179847837287937160817963
25	663024394602752425373130

Table 3: The values, multiplied by  $p_1 \cdots p_N$ , of the shortest distances  $\Delta_N$  between different signed harmonic sums with the first *N* primes, with *N* up to 25.