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Globally diffeomorphic σ-harmonic mappings

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Abstract
Given a two-dimensional mapping U whose components solve a divergence structure elliptic equation,we give necessary and sufficient conditions on the boundary so that U is a global diffeomorphism.

Keywords Elliptic equations · Beltrami operators · Quasiconformal mappings

Mathematics Subject Classification 30C62 · 35J55

1 Introduction

Let $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ denote the unit disk. We denote by $\sigma = \sigma(x)$, $x \in B$, a possibly non-symmetric matrix having measurable entries and satisfying the ellipticity conditions

$$\begin{aligned} \sigma(x)\xi \cdot \xi \geq K^{-1}|\xi|^2, \text{ for every } \xi \in \mathbb{R}^2, x \in B, \\ \sigma^{-1}(x)\xi \cdot \xi \geq K^{-1}|\xi|^2, \text{ for every } \xi \in \mathbb{R}^2, x \in B, \end{aligned} \tag{1.1}$$

for a given constant $K \geq 1$.

Given a diffeomorphism $\Phi = (\varphi^1, \varphi^2)$ from the unit circle ∂B onto a simple closed curve $\gamma \subseteq \mathbb{R}^2$, we denote by D the bounded domain such that $\partial D = \gamma$. With no loss of generality, we may assume that Φ is orientation preserving.

Let us consider the mapping $U = (u^1, u^2) \in W^{1,2}(B;\mathbb{R}^2) \cap C(\overline{B};\mathbb{R}^2)$ whose components are the solutions to the following Dirichlet problems

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$$\begin{cases} \operatorname{div}(\sigma \nabla u^i) = 0, & \text{in } B, \\ u^i = \varphi^i, & \text{on } \partial B\,, i = 1, 2\,. \end{cases} \tag{1.2}$$

Loosely speaking, the question that we intend to address here is:

Under which conditions can we assure that U is an invertible mapping between B and D (or $\overline{B}$ and $\overline{D}$)?

The classical starting point for this issue is the celebrated Radó–Kneser–Choquet Theorem [13, 14, 18, 20] which asserts that assuming $\sigma = I$, the identity matrix, (that is: u^1, u^2 are harmonic) if D is convex then U is a homeomorphism. Generalizations to equations with variable coefficients have been obtained in [3 9] and to certain nonlinear systems in [8, 10, 16]. Counterxamples [4, 13] show that if D is not convex then the invertibility of U may fail, see also [7] for a counterexample when σ is variable.

In [4], the present authors investigated, in the case of harmonic mappings, which additional conditions are needed for invertibility in the case of a possibly non-convex target D. In particular, in [4, Theorem 1.3] it is proven that, assuming $\sigma = I$, U is a diffeomorphism if and only if $\det DU > 0$ everywhere on ∂B. An improvement to this result, still in the harmonic case, is due to Kalaj [17].

Here we intend to treat the case of equations with variable coefficients. The main result in this note is the following:

Theorem 1.1 *Assume that the entries of σ satisfy $\sigma_{ij} \in C^\alpha(\overline{B})$ for some $\alpha \in (0, 1)$ and for every $i, j = 1, 2$. Assume, in addition, that $U \in C^1(\overline{B}; \mathbb{R}^2)$.*

The mapping U is a diffeomorphism of $\overline{B}$ onto $\overline{D}$ if and only if

$$\det DU > 0 \ \text{ everywhere on } \ \partial B. \tag{1.3}$$

It is evident that, if U is a diffeomorphism on $\overline{B}$, then $\det DU \neq 0$ on ∂B. Thus, from now on, we shall focus on the reverse implication only.

New tools are required for this extension from the purely harmonic case. First we make use of an index calculus on the gradient of solutions of elliptic equations in two variables, first developed by R. Magnanini and the first named author [1]. A novel adaptation is however needed, because the theory in [1] requires Lipschitz continuity of the coefficients σ_{ij}. An approximation argument is then introduced to pass to the case $\sigma_{ij} \in C^\alpha(\overline{B})$, see Sect. 3. Furthermore, we make use of a recently obtained variant, Theorem 3.2, to the celebrated H. Lewy's Theorem [19], which was proven by the present authors in [6, Theorem 1.1].

The plan of the paper is as follows.

In Sect. 2, we begin by proving Theorem 2.1, that is, a version of Theorem 1.1 which requires stronger regularity on σ and on Φ.

Section 3 contains the completion of the proof of Theorem 1.1; let us mention that, as an intermediate step, we also prove Theorem 3.4, which treats the case when the Dirichlet data Φ is merely a homeomorphism, extending to the case of variable coefficients the result proved in [4, Theorem 1.7] for the case of $\sigma = I$.

In the final Sect. 4, we sketch the arguments for an improvement, Theorem 4.2 to Theorem 1.1, in analogy with [4, Theorem 5.2].

2 A smoother case

Theorem 2.1 *In addition to the hypotheses of Theorem 1.1, let us assume that the entries of σ satisfy $\sigma_{ij} \in C^{0,1}(\overline{B})$ and that $\Phi = (\varphi^1, \varphi^2) \in C^{1,\alpha}(\partial B, \mathbb{R}^2)$ for some $\alpha \in (0, 1)$. If*

$$\det DU > 0 \text{ everywhere on } \partial B\,, \tag{2.1}$$

then the mapping U is a diffeomorphism of $\overline{B}$ onto $\overline{D}$.

We observe that, assuming that σ_{ij} are Lipschitz continuous in $\overline{B}$, it is a straightforward matter to rewrite equation

$$\mathrm{div}(\sigma\nabla u) = 0$$

in the form

$$\mathrm{div}(A\nabla u) + b\cdot\nabla u = 0\,, \tag{2.2}$$

where $b = (b^1, b^2)$ is in L^∞ and A is a uniformly elliptic symmetric matrix in the sense of (1.1), with Lipschitz entries, and it satisfies $\det A = 1$ everywhere.

The calculation is as follows. Denote

$$\widehat{\sigma} = \frac{1}{2}(\sigma + \sigma^T)\,, \check{\sigma} = \frac{1}{2}(\sigma - \sigma^T)\,,$$

where $(\cdot)^T$ denotes the transposition. Writing the equation in weak form and using smooth test functions, we obtain

$$0 = \mathrm{div}(\sigma\nabla u) = \mathrm{div}(\widehat{\sigma}\nabla u) + \partial_{x_i}\check{\sigma}_{ij}\partial_{x_j}u\,,$$

next we pose $\gamma = \sqrt{\det\widehat{\sigma}}$ and $A = \frac{1}{\gamma}\widehat{\sigma}$ and we compute

$$0 = \gamma\,\mathrm{div}(A\nabla u) + \partial_{x_i}(\gamma\delta_{ij} + \check{\sigma}_{ij})\partial_{x_j}u\,,$$

hence $b^j = \frac{1}{\gamma}\partial_{x_i}(\gamma\delta_{ij} + \check{\sigma}_{ij})$.

We recall that local weak solutions u to (2.2) are indeed $C^{1,\alpha}$; their critical points are isolated and have finite integral multiplicity. This theory has been developed in [1]. As a consequence of such a theory, we can state the following auxiliary result. Let us start with some notation.

We denote

$$u_\alpha = \cos\alpha\, u^1 + \sin\alpha\, u^2\,, \alpha \in \mathbb{R}\,, \tag{2.3}$$

where u^1, u^2 are the components of the mapping U appearing in Theorem 1.1. Next we define

$$M_\alpha = \text{number of critical points of } u_\alpha \text{ in } B, \text{ counted with their multiplicities}\,. \tag{2.4}$$

Note that, in view of (1.3), M_α is finite for all α.

Proposition 2.2 *Under the assumptions of Theorem 2.1, we have*

$$M_\alpha = \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\arg(\partial_z u_\alpha)\ , \tag{2.5}$$

moreover $M_\alpha = M$ is constant with respect to α.

Here ∂_z denotes the usual complex derivative, where it is understood $z = x_1 + ix_2$.

Proof Formula (2.5) is a manifestation of the argument principle. A proof, with some changes in notation, can be found in [1, Proof of Theorem 2.1]. Also, a special case of Theorem 2.1 in [1] tells us that if ξ is a C^1 unitary vector field on ∂B such that $\nabla u_\alpha \cdot \xi > 0$ everywhere on ∂B; then, we have

$$M_\alpha = \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\arg(\xi)\ . \tag{2.6}$$

Let us denote

$$\xi = \frac{1}{|\nabla u_1|} J\nabla u_1\ . \tag{2.7}$$

where the matrix J represents the counterclockwise 90° rotation

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.8}$$

, and we compute

$$\nabla u_\alpha \cdot \xi = \frac{\sin\alpha}{|\nabla u_1|}\nabla u_2 \cdot J\nabla u_1 = \frac{\sin\alpha}{|\nabla u_1|}\det DU$$

which is positive for all $\alpha \in (0, \pi)$. Hence M_α is constant for all $\alpha \in (0, \pi)$; by continuity the same is true for all $\alpha \in [0, \pi]$. The proof is complete, by noticing that $u_{\alpha+\pi} = -u_\alpha$. □

Our next goal being to prove that $M = M_\alpha = 0$, we return to the equation in pure divergence form. Denoting $u = u_\alpha$ for any fixed α, we have that equation

$$\mathrm{div}(\sigma\nabla u) = 0$$

holds in B. It is well-known that there exists $v \in W^{1,2}(B)$, called the *stream function* of u such that

$$\nabla v = J\sigma\nabla u\ , \tag{2.9}$$

where, again, the matrix J denotes the counterclockwise 90° rotation (2.8), see, for instance, [2]. Denoting

$$f = u + iv\ , \tag{2.10}$$

it is well-known that f solves the Beltrami type equation

$$f_{\bar z} = \mu f_z + \nu\overline{f_z}\ \text{ in } B\ , \tag{2.11}$$

where the so-called complex dilatations μ, ν are given by

$$\mu = \frac{\sigma_{22}-\sigma_{11}-i(\sigma_{12}+\sigma_{21})}{1+\mathrm{Tr}\,\sigma+\det\sigma} \quad , \; \nu = \frac{1-\det\sigma+i(\sigma_{12}-\sigma_{21})}{1+\mathrm{Tr}\,\sigma+\det\sigma} \; , \tag{2.12}$$

and satisfy the following ellipticity condition

$$|\mu| + |\nu| \le k < 1, \tag{2.13}$$

where the constant k only depends on K, see [5, Proposition 1.8] and the notation $\mathrm{Tr}A$ is used for the trace of a square matrix A.

Furthermore, it is also well-known, Bers and Nirenberg [11], Bojarski [12], that a $W^{1,2}$ solution to (2.11) fulfills the so-called Stoilow representation

$$f = F \circ \chi, \tag{2.14}$$

where F is holomorphic and χ is a quasiconformal homeomorphism, which can be chosen to map B into itself. Moreover, χ solves the Beltrami equation

$$\chi_{\bar{z}} = \widetilde{\mu}\chi_z \text{ in } B, \tag{2.15}$$

where $\widetilde{\mu}$ is defined almost everywhere by

$$\widetilde{\mu} = \mu + \frac{\overline{f_z}}{f_z}\nu \; ,$$

Note that, under the present assumptions, μ, ν are Lipschitz continuous in $\overline{B}$ and f is in $C^{1,\alpha}(\overline{B}, \mathbb{C})$.

From now on, for simplicity, we denote by B_ρ be the disk of radius $\rho > 0$ concentric to B.

In view of (1.3), there exists $0 < \rho < 1$ such that $\partial_z f \neq 0$ on $\overline{B} \setminus B_\rho$. As a consequence, $\widetilde{\mu}$ is C^α on $\overline{B} \setminus B_\rho$, and the following Lemma holds.

Lemma 2.3 *Under the assumptions of Theorem 2.1, there exists $0 < \rho < 1$ such that the mapping χ, appearing in (2.14), belongs to $C^{1,\alpha}$, for some $0 < \alpha < 1$, when restricted to $\overline{B} \setminus B_\rho$.*

Proof For ρ sufficiently close to 1, we may represent $\chi = \exp(\omega)$ in the annulus $\overline{B} \setminus B_\rho$. Also, for every determination of ω, we have

$$\omega_{\bar{z}} = \widetilde{\mu}\omega_z \; . \tag{2.16}$$

Now, posing $w = \Re e(\omega) = \log|\chi|$, it is well-known that we have

$$\mathrm{div}(\widetilde{\sigma}\nabla w) = 0, \text{ in } B \setminus \overline{B_\rho}$$

where $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma} = \begin{pmatrix} \dfrac{|1-\widetilde{\mu}|^2}{1-|\widetilde{\mu}|^2} & -\dfrac{2\Im m(\widetilde{\mu})}{1-|\widetilde{\mu}|^2} \\ -\dfrac{2\Im m(\widetilde{\mu})}{1-|\widetilde{\mu}|^2} & \dfrac{|1+\widetilde{\mu}|^2}{1-|\widetilde{\mu}|^2} \end{pmatrix}, \tag{2.17}$$

and satisfies uniform ellipticity conditions of the form (1.1), see, for instance, [5]. Moreover, $\widetilde{\sigma}$ has Hölder continuous entries in $\overline{B} \setminus B_{\rho}$. Now, since, trivially, $w = 0$ on ∂B, then, by standard regularity at the boundary, w is $C^{1,\alpha}$ near ∂B. Such a regularity extends to ω and then to χ, because (2.16) can be rewritten as $\nabla \Im m(\omega) = J\widetilde{\sigma}\nabla w$. □

Next we recall the following classical notion, see for instance [22].

Definition 2.4 Given a closed curve γ, parametrized by $\Phi \in C^1([0,2\pi];\mathbb{R}^2)$ and such that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\vartheta} \neq 0, \text{ for every } \vartheta \in [0,2\pi],$$

we define the *winding number* of γ as the following integer

$$\mathrm{WN}(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\arg\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\vartheta}\right).$$

Proposition 2.5 *Under the previously stated assumptions*

$$\mathrm{WN}(f(\partial B)) = M + 1\ ,$$

with M as in Proposition 2.2.

Proof With no loss of generality, we may assume $\chi(1) = 1$.

We have that for every $\vartheta \in \mathbb{R}$,

$$f(e^{i\vartheta}) = F(e^{i\varphi(\vartheta)})$$

where

$$e^{i\varphi(\vartheta)} = \chi(e^{i\vartheta})$$

hence φ is a strictly increasing function from $[0,2\pi]$ into itself, with $C^{1,\alpha}$ regularity. Consequently

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\arg\left(\frac{\mathrm{d}f(e^{i\vartheta})}{\mathrm{d}\vartheta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\arg\big(F'(e^{i\varphi(\vartheta)})\big) + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\arg\big(e^{i\varphi(\vartheta)}\varphi'(\vartheta)\big)\ .$$

For the second integral, we trivially have

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\arg\big(e^{i\varphi(\vartheta)}\varphi'(\vartheta)\big) = 1\ ,$$

whereas, by the argument principle, the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\arg\big(F'(e^{i\varphi(\vartheta)})\big) = \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\arg\big(F'(z)\big)$$

equals the number of zeroes of F' when counted with their multiplicities, which coincides with the number of critical points of u, again counted with their multiplicities, that is, M. This is a consequence of the notions of *geometrical critical points* and *geometric index* introduced in [2, Definition 2.4], which in the present circumstances, coincide with the usual concepts of critical points and multiplicity, respectively. □

Next we compute:

$$\mathrm{WN}(f(\partial B)) = \mathrm{WN}(\Phi(\partial B)) = 1.$$

Proposition 2.6

Proof We may fix $\alpha = 0$, that is, $u = u^1$, and let v^1 be its stream function. For every $t \in [0, 1]$ let us consider $U_t = (u^1, (1-t)v^1 + tu^2)$. Trivially

$$U_0 \approx u^1 + iv^1 = f \,, U_1 = U \,.$$

We compute

$$\det DU_t = (1-t)\sigma \nabla u \cdot \nabla u + t \det DU > 0 \,, \text{ on } \partial B \,, \text{ for every } t \in [0, 1] \,,$$

consequently

$$\beta_t(\vartheta) = \frac{d}{d\vartheta} U_t(e^{i\vartheta}) \,, \text{ for every } t \in [0, 1] \,, \vartheta \in [0, 2\pi] \,.$$

never vanishes. By homotopic invariance of the winding number, [22, Theorem 1], the thesis follows. □

Proof of Theorem 2.1 Combining Propositions 2.2, 2.5 and 2.6 we deduce that, for all α, ∇u_α nowhere vanishes. Hence $\det DU > 0$ everywhere in $\overline{B}$. Hence it is a local diffeomorphism which is one-to-one on the boundary, by the Monodromy Theorem, see for instance [21, p.175]; the thesis follows. □

3 Proof of Theorem 1.1

We start by removing the hypothesis of Lipschitz continuity on σ and obtain an intermediate weaker result.

Lemma 3.1 *In addition to the hypotheses of Theorem 1.1, let us assume $\Phi = (\varphi^1, \varphi^2) \in C^{1,\alpha}(\partial B, \mathbb{R}^2)$, for some $\alpha \in (0, 1)$. Then U is locally a homeomorphism in B.*

Proof Let σ_ε be a family of C^∞ mollifications of σ, which satisfy ellipticity and Hölder regularity uniformly with respect to ε. Let U_ε be the solution to

$$\begin{cases} \mathrm{div}(\sigma_\varepsilon \nabla u^i_\varepsilon) = 0, & \text{in } B, \\ u^i_\varepsilon = \varphi^i, & \text{on } \partial B \,, i = 1, 2 \,. \end{cases} \tag{3.1}$$

By regularity theory, $U_\varepsilon \in C^{1,\alpha}(\overline{B}, \mathbb{R}^2)$ uniformly with respect to ε; hence, by the Ascoli–Arzelà Theorem, $U_{\varepsilon_n} \to U$ in $C^1(\overline{B}, \mathbb{R}^2)$ for some sequence $\varepsilon_n \to 0$. Therefore, for n large enough

$$\det DU_{\varepsilon_n} > 0 \text{ everywhere on } \partial B$$

thus, by Theorem 2.1, U_{ε_n} is a diffeomorphism of $\overline{B}$ onto $\overline{D}$. In particular, the number $(M_{\varepsilon_n})_\alpha$, associated to U_{ε_n} according to definition (2.4), equals zero for all α and for n large enough. In view of the stability of the geometric index, established in [2, Proposition 2.6], we have that $u_\alpha = \cos\alpha\, u^1 + \sin\alpha\, u^2$ has no (geometrical) critical point in B for any α. We may invoke now [3, Theorem 3] to obtain that U is locally a homeomorphism in B. □

We now recall a variant to the celebrated H. Lewy's Theorem [19], recently obtained in [6, Theorem 1.1]. Here $\Omega \subset \mathbb{R}^2$ is any open set.

Theorem 3.2 *Assume that the entries of σ satisfy $\sigma_{ij} \in C^\alpha_{loc}(\Omega)$ for some $\alpha \in (0,1)$ and for every $i,j = 1,2$. Let $U = (u^1, u^2) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^2)$ be such that*

$$\mathrm{div}(\sigma \nabla u^i) = 0\ ,\ i = 1,2, \tag{3.2}$$

weakly in Ω. If U is locally a homeomorphism, then it is, locally, a diffeomorphism, that is

$$\det DU \neq 0 \text{ for every } x \in \Omega\ . \tag{3.3}$$

Before introducing the next Theorem, we recall the following definition.

Definition 3.3 Given $P \in \overline{B}$, a mapping $U \in C(\overline{B};\mathbb{R}^2)$ is a *local homeomorphism* at P if there exists a neighborhood G of P such that U is one-to-one on $G \cap \overline{B}$.

Theorem 3.4 *Let $\Phi : \partial B \to \gamma \subset \mathbb{R}^2$ be a homeomorphism onto a simple closed curve γ . Let D be the bounded domain such that $\partial D = \gamma$. Let $U \in W^{1,2}_{\mathrm{loc}}(B;\mathbb{R}^2) \cap C(\overline{B};\mathbb{R}^2)$ be the solution to (1.2). Assume that the entries of σ satisfy $\sigma_{ij} \in C^\alpha_{loc}(B)$ for some $\alpha \in (0,1)$ and for every $i,j = 1,2$. If, for every $P \in \partial B$, the mapping U is a local homeomorphism at P, then it is a homeomorphism of $\overline{B}$ onto $\overline{D}$ and it is a diffeomorphism of B onto D .*

We first need the following Lemma. Let us recall that B_ρ denotes the disk of radius $\rho > 0$ concentric to B.

Lemma 3.5 *Assume $\Phi : \partial B \to \gamma \subset \mathbb{R}^2$ is a homeomorphism onto a simple closed curve γ . Let $U \in W^{1,2}_{loc}(B;\mathbb{R}^2) \cap C(\overline{B};\mathbb{R}^2)$ be the solution to (1.2). Assume that the entries of σ satisfy $\sigma_{ij} \in C^\alpha_{loc}(B)$ for some $\alpha \in (0,1)$ and for every $i,j = 1,2$. If, in addition, for every $P \in \partial B$ the mapping U is a local homeomorphism near P, then there exists $\rho \in (0,1)$ such that U is a diffeomorphism of $B \setminus \overline{B_\rho}$ onto $U\left(B \setminus \overline{B_\rho}\right)$.*

Proof For every $P \in \partial B$ let

$$s(P) = \sup\left\{ s > 0 | U \text{ is a homeomorphism in } B_s(P) \cap \overline{B} \right\},$$

the function $s(P)$ is positive valued and lower semicontinuous; hence, by the compactness of ∂B, there exists $\delta > 0$ such that $s(P) > 2\delta$ for all $P \in \partial B$. Again by compactness, there exist finitely many points $P_1, \dots, P_K \in \partial B$ such that

$$\partial B \subset \bigcup_{k=1}^{K} B_\delta(P_k),$$

and U is one-to-one on $B_{2\delta}(P_k) \cap \overline{B}$ for every k. Note that there exists $\rho_0 \in (0,1)$ such that

$$\overline{B} \setminus B_{\rho_0} \subset \bigcup_{k=1}^{K} B_\delta(P_k).$$

Let P, Q be two distinct points in $\overline{B} \setminus B_{\rho_0}$. If $|P - Q| < \delta$, then there exists $k = 1, \dots, K$ such that $P, Q \in B_{2\delta}(P_k)$ and, hence, $U(P) \neq U(Q)$. Assume now $|P - Q| \geq \delta$. Let

$$P' = \frac{P}{|P|} \quad , \quad Q' = \frac{Q}{|Q|}.$$

We have $|P - P'| < 1 - \rho$, $|Q - Q'| < 1 - \rho$, and thus

$$|P' - Q'| > |P - Q| - 2(1 - \rho) \geq \delta - 2(1 - \rho).$$

Choosing ρ_1, $\rho_0 \leq \rho_1 < 1$ such that $(1 - \rho_1) < \frac{\delta}{4}$, we have $|P' - Q'| > \frac{\delta}{2}$. Now we use the fact that P' and Q' belong to ∂B and Φ is one-to-one to deduce that there exists $c > 0$ such that

$$|\Phi(P') - \Phi(Q')| \geq c.$$

Recall that U is uniformly continuous on $\overline{B}$. Denoting by ω its modulus of continuity, we have

$$\begin{aligned} |U(P) - U(Q)| &\geq |U(P') - U(Q')| - 2\omega(1 - \rho) \\ &= |\Phi(P') - \Phi(Q')| - 2\omega(1 - \rho) \geq c - 2\omega(1 - \rho). \end{aligned}$$

Choosing ρ, $\rho_1 \leq \rho < 1$, such that $1 - \rho < \omega^{-1}\left(\frac{c}{4}\right)$ we obtain

$$|U(P) - U(Q)| \geq \frac{c}{2} > 0,$$

which implies the injectivity of U in $\overline{B} \setminus B_\rho$. Consequently, by Theorem 3.2, $\det DU \neq 0$ in $B \setminus \overline{B_\rho}$ and the thesis follows. □

Proof of Theorem 3.4 In view of the already quoted Monodromy Theorem, it suffices to show that $\det DU \neq 0$ everywhere in B.

For every $r \in (0, 1)$, let us write $\Phi^r : \partial B_r \to \mathbb{R}^2$ to denote the application given by

$$\Phi^r = U|_{\partial B_r}.$$

It is obvious, by interior regularity of U, that Φ^r belongs to $C^{1,\alpha}$. On the other hand, by Lemma 3.5, there exists $\rho \in (0, 1)$ such that for every $r \in (\rho, 1)$ the mapping $\Phi^r : \partial B_r \to \underline{\gamma_r} \subset \mathbb{R}^2$ is a diffeomorphism of ∂B_r onto a simple closed curve γ_r. Now, when restricted to B_r, U solves (1.2) with Φ replaced by Φ^r, and B by B_r . Then, up to a rescaling of coordinates, Lemma 3.1 is applicable, and we obtain, in combination with Theorem 3.2,

$$\det DU \neq 0, \quad \text{everywhere in } B_r.$$

Finally, by Lemma 3.5 we have $\det DU \neq 0$ in $B \setminus \overline{B_\rho(0)}$ so that $\det DU \neq 0$ everywhere in B. □

We now conclude the proof of the main Theorem 1.1.

Proof of Theorem 1.1 Having assumed $\det DU > 0$ on ∂B, by continuity, one can find $0 < \rho < 1$, sufficiently close to 1 such that $\det DU > 0$ on $\overline{B} \setminus B_\rho$. By Theorem 3.4, we have that U is a global homeomorphism and that $\det DU > 0$ in B. Consequently, $\det DU > 0$ on all of $\overline{B}$ and the thesis follows. □

4 An improvement

Finally, we prove a variation of Theorem 1.1. First, we recall the following:

Definition 4.1 Given a Jordan domain D, let us denote by co(D) its convex hull. We define the *convex part* of ∂D as the closed set $\gamma_c = \partial D \cap \partial(\mathrm{co}(D))$. Consequently, we define the *non-convex part* of ∂D as the open subset $\gamma_{nc} = \partial D \setminus \partial(\mathrm{co}(D))$.

Theorem 4.2 *Under the assumptions of Theorem 1.1, if*

$$\det DU > 0 \quad \text{everywhere on } \Phi^{-1}(\gamma_{nc}), \tag{4.1}$$

where γ_{nc} is the set introduced in Definition 4.1 above, then the mapping U is a diffeomorphism of $\overline{B}$ onto $\overline{D}$.

It is worth noticing that, if D is convex, then the condition (4.1) is void, which agrees with the known adaptations [3, 9] of the well-known Radó–Kneser–Choquet [18] to equation (1.2).

Proof The proof follows the same line of [4, Theorem 5.2], the only change is that the classical Zaremba–Hopf Lemma for harmonic functions must be replaced by its appropriate adaptation to divergence structure equations with Hölder coefficients, which is due to Finn and Gilbarg [15]. We omit the details. □

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