On proper applications of Galërkin's approach in structural mechanics courses

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ABSTRACT: An incautious use of the well-known Galërkin's technique to find approximate solutions of a differential problem may lead to apparently wrong results. Examples are based on an inverse approach to investigate buckling of compressed axisymmetric circular plates, a common subject in courses on mechanics of structures and stability of structural elements. We discuss how a mistake may originate and show how it is possible to recover the expected results, thus providing a means for the students to cross-check their outputs.

Keywords: Galërkin's technique; critical loads; axisymmetric plates.

1. INTRODUCTION

In many engineering courses it is often useful, if not necessary, to abandon the search of the exact solutions of a certain differential problem: this may be due, for instance, to its highly non-linear feature or to the excessive computational cost required for the resolution. The student will thus be advised to look for acceptable approximate solutions within the accuracy limits required by the applications. This can be done by passing from a continuous to a discrete domain, in order to reduce the differential problem to a system of algebraic equations, no matter how rich in equations and unknowns (finite differences and finite elements techniques). Alternatively, under suitable regularity assumptions, the solution can be expressed as the superposition of a certain combination of the (generally infinite) elements of a basis of functions. The desired approximation relies on the number of elements of this basis that are adopted.

Among these techniques, one of the best known, and traditionally widely used in engineering graduate courses, is Galërkin's method, proposed in 1915 [1]. This admits that the searched function has a finite pseudo-spectral decomposition in terms of a finite number of so-called comparison functions, rescaled by unknown amplitude coefficients. The comparison functions should at least be continuously derivable until the maximum order of the derivatives appearing in the considered differential problem, and are required to satisfy all the prescribed boundary conditions. Thus, students should be advised that comparison functions satisfy the constraints at the boundary (hence provide possible varied configurations of the considered mechanical system) but do not yield inner forces balanced with the external actions in general. Then, the differential equation(s) representing the physical behaviour of the system are not satisfied, i.e., their right-hand-side does not vanish identically and equals a residual. This provides the amount of error induced by replacing the actual solution with its pseudo-spectral approximation. Some developments of this technique, a century since its inception, were reviewed recently by Repin [2].

To make this residual as small as possible, Galërkin's technique requires that the error is projected onto each comparison function, and that all these projections vanish. The students can be told that, in the regular enough functional space commonly adopted, an orthogonal projection is represented by the finite integral of the product of the residual by each comparison function, extended over the domain of the considered mechanical system. Thus, a system of as many homogeneous algebraic equations as the number of unknown amplitude coefficients of the comparison functions is obtained. The students should be reminded that such a system always admits the trivial solution when the matrix of the coefficients of the system of algebraic equations expressing the projection of the residual onto the comparison functions is non-singular. Then, all the amplitude coefficients vanish, and the approximate solution searched for is trivially identically nil.

However, it is the non-trivial solutions that are of interest: thus, if the problem depends on a physical parameter, this shall assume the values making the matrix of the coefficients of the above said system of homogeneous algebraic equations singular. The students should be advised that this is, for instance, one way to find the approximate values of natural angular frequencies of elastic systems, or, analogously, of their bifurcation loads (either in a static or dynamic setting).

The authors of this paper usually present Galërkin's technique in their post-graduate course in structural mechanics and stability, where students are encouraged to apply it for either conventional and novel problems. They also show them how they have applied Galërkin's technique in their investigations on the dynamic stability of pipes that convey fluid and rest on an elastic foundation [3, 4] and on the search for unusual closed-form solutions for the buckling of circular FGM plates [5] via the adoption of the semi-inverse technique developed in [6].

It is known (the proof is not difficult and might be interesting for students) that the approximate solution *à la* Galërkin approaches the exact one from above. As a rule of thumb to be given to the class, the pseudo-spectral representation comprehends only a limited set of comparison functions. Thus, the approximate configurations, however respectful of the prescribed constraints at the boundary, are actually stiffer than the exact solution. Indeed, the latter is represented by an infinite sum of admissible functions, each contributing with an elastic energetic contribution, however small. Neglecting some of them, then, makes the approximated mechanical system stiffer with respect to the actual one. This means that the approximate solutions that are found are given a technical meaning in all applications, since the designer can dominate the error in approximation.

However, we realized that an incautious application of Galërkin's technique might lead to manifestly wrong results, approximating the correct result from below. This emerged in a post-graduate task for the final exam in structural mechanics, and was corrected by the authors in a recent investigation on unusual closed-form buckling solutions of circular FGM plates [7]. This could be of interest in the engineering courses where Galërkin's technique is used. Thus, here we sketch one possible mechanical problem in which the phenomenon occurs and present the results obtained by a non-critical application of Galërkin's technique. Finally, we present our interpretation of the causes inducing the unexpected outcomes. We also show how it is actually possible to overcome the errors and recover the expected behaviour, providing some didactic comments as well.

2. A BENCHMARK PROBLEM: THE BUCKLING OF CIRCULAR PLATES

The deflection of a linearly elastic, homogeneous and isotropic circular plate of radius R, subjected to axial symmetrical constrains and external loads, turns its middle plane into a surface of revolution. The authors rely on the well-known handbooks [8, 9, 10, 11] for the presentation of the subject to students. By axial symmetry, any diametral section of the plate behaves in the same way; cylindrical coordinates r, ϕ, z (radial, angular, and transverse, respectively) label a material point. The origin of the coordinate system is at the centre of the middle plane of the plate, characterised by $z=0$. All the mechanical fields of interest are defined on the middle plane, and their dependance on z is at most linear. By axial symmetry, their dependance on ϕ ceases, and the partial derivatives with respect to r turn into ordinary derivatives. This is very interesting for didactic purposes, since it may be dealt with using ordinary calculus, plus some information on special solutions for particular differential equations.

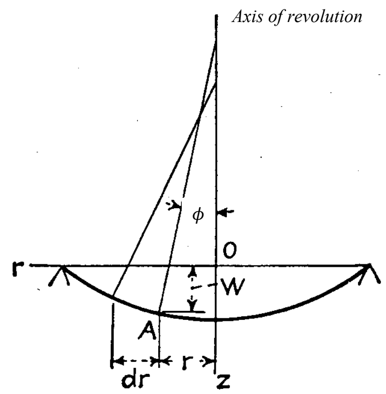

Figure 1: Sketch of the Kirchhoff-Love hypothesis for circular plates (inspired by [8]).

In thin plates, the transverse material fibres practically remain normal to the deflected surface (Kirchhoff-Love hypothesis, or assumption of the 'classical plate theory'). This assumption can be easily described during course work, comparing it with the Bernoulli-Euler hypothesis for the ideal radial beams of which the plate might be imagined as a sort of weave. Let $w(r)$ be the deflection of the middle plane of the plate and $\theta(r)$ be the angle of the axis of revolution of the deflected surface with the orthogonal segment to this at any of its points. Then, by the Kirchhoff-Love hypothesis,

$\theta(r)$ equals the rotation of a transverse material fibre in the plane rz [8, 10, 11], see Fig. 1,

$$\theta(r) = -\frac{dw(r)}{dr} \, . \tag{1}$$

The radial and hoop curvatures χ_r, χ_ϕ of the deflected surface are given by [8, 10, 11]

$$\chi_r = \frac{d\theta}{dr} = -\frac{d^2w}{dr^2}, \qquad \chi_\phi = \frac{\theta}{r} = -\frac{1}{r}\frac{dw}{dr} \tag{2}$$

where, for simplicity of notation, the dependance on r is understood and omitted. By axial symmetry, the curvatures in Eq. (2) are principal, and no twisting (or mixed) curvature appears.

A plate with uniform thickness h, composed of material with Young's modulus E and Poisson's ratio ν, has flexural rigidity D given by [8, 10, 11]

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{3}$$

The linear elastic homogeneous constitutive relations for the radial and hoop inner bending couples per unit length, denoted by M_r, M_ϕ respectively, are [8, 9, 10, 11]

$$\begin{aligned} M_r &= -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) = D\left(\frac{d\theta}{dr} + \frac{\nu}{r}\theta\right) \\ M_\phi &= -D\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) = D\left(\frac{\theta}{r} + \nu\frac{d\theta}{dr}\right) \end{aligned} \tag{4}$$

i.e., they can be expressed either in terms of the transverse displacement, or of the rotation of the transverse material fibres. Here and in the following, as we also did above, the dependance on r is omitted for simplicity of notation for all the mechanical fields of interest.

A volume element of the circular plate subjected to transverse external loads and inner actions is sketched in Fig. 2. In our case, the transverse load vanishes and, since the curvatures given by Eq. (2) are principal, no torsion couples per unit length $M_{r\phi}$ arise. Axial symmetry implies that M_ϕ must be uniform along ϕ, and no shearing transverse force $Q_{\phi z}$ may arise. On the other hand,

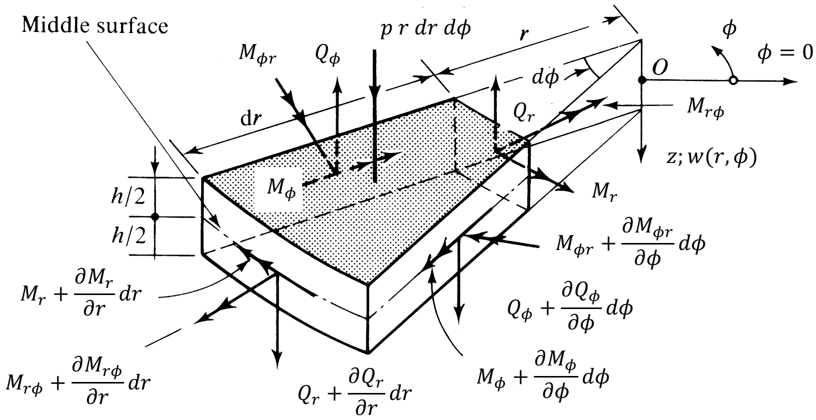

Figure 2: Free body of a volume element of the plate (inspired by [10]).

M_r may vary along the radial coordinate, and it must be balanced by a shearing transverse force per unit length Q_{rz}. This will be the only transverse contact force present, hence its subscripts will be dropped from notation for simplicity of notation.

The projection along r of the balance of moments about the circumferential direction for an element of the plate, the original shape of which is a portion of a circular crown, yields [8, 10, 11]

$$\frac{1}{r}\frac{d}{dr}\left(rM_r\right) - \frac{M_\phi}{r} + Q = 0. \tag{5}$$

By the Kirchhoff-Love hypothesis, the transverse shearing force Q has reactive nature, and can be determined by balance only. Under a uniform axisymmetric distribution of compressive forces of magnitude N_r per unit length at the boundary $r = R$ of the plate, in any deflected configuration the balance of the transverse force in any circular crown element of the plate yields [8, 10, 11]

$$\frac{1}{r}\frac{d}{dr}\left[r\left(Q + N_r\theta\right)\right] = 0. \tag{6}$$

Inserting Eqs. (4), (6) into Eq. (5) yields the governing equations for buckling [8, 9]: the first

$$D\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} + N_r\frac{d}{dr}\left(r\frac{dw}{dr}\right) = 0 \tag{7}$$

is in terms of the deflection; another, equivalent to Eq. (7),

$$D\left(\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}-\frac{\theta}{r^2}\right)+N_r\theta=0 \tag{8}$$

is in terms of the rotation of transverse fibres. Indeed, it can be easily seen that Eq. (8) is provided by the direct integration of Eq. (7) and the condition that the rotation of the transverse fibre at the origin vanishes due to axial symmetry, in keeping with the Kirchhoff-Love hypothesis Eq. (1).

Eq. (8) easily turns into an equation of first order, labelled with the name of the German astronomer Bessel. There is no need to provide details on such equations in class other than that they have special solutions, called Bessel functions: this is similar to the harmonic functions being solutions of harmonic differential equations. The solutions of Eq. (8) depend on the constraints at $r=R$, providing the relevant boundary conditions.

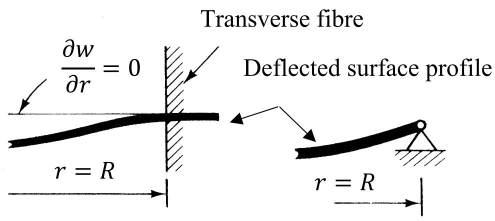

Figure 3: Boundary conditions for the plate: clamp (left) and simple support (right) (inspired by [10]).

Asking for non-trivial solutions (different from the uniformly compressed flat reference shape) provides the critical buckling loads [8, 10, 11, 5]. For simply supported and clamped plates the boundary conditions are sketched in Fig. 3; one gets, if $\nu=1/3$,

$$(\bar{N}_r)_{ss}\approx\frac{4.282D}{R^2},\qquad(\bar{N}_r)_{cl}\approx\frac{14.682D}{R^2} \tag{9}$$

where the bar indicates the critical magnitude of the compressive load, and the subscripts suggest the simple support, or the clamp, at the boundary. In class we point out that the approximation symbols in Eq. (9) refer to the meaningful digits in the roots of the transcendental equation yielding the non-triviality condition of the searched solution, not to a non-exact analytical approach.

3. GALËRKIN'S APPROACH FOR APPROXIMATE SOLUTIONS

In class we point out that the singularity condition leading to the critical values of the load parameter leads to a highly non-linear algebrai equation. Then, we stress that it is not always possible to find an exact expression for the buckling load, and that for application purposes it is appropriate to find approximate results. To this aim, Galërkin's technique fits: we choose a finite set of admissible functions (also named comparison functions), and let the amount of the compressive load at the boundary $r = R$ of the plate be the physical control parameter.

In this section, we present at first the approximate values for the fundamental buckling load of uniformly compressed homogenous uniform circular plates that can be attained by adopting a single comparison function in Galërkin's approach. These functions are, for both cases of the clamped and the simply supported plate, the static deflections of the corresponding transversely uniformly loaded plates. These are easily found in closed form by well-known methods, found for instance in [8, 10, 11], and are thus a fruitful exercise for students. Hence, we will operate Galërkin's technique with reference to the governing equation for buckling provided by Eq. (7). Then, we evaluate the approximate buckling load by resorting to the governing equation for buckling provided by Eq. (8), adopting the comparison function given by the rotation of the transverse material fibres of the corresponding transversely uniformly loaded plates. These functions are also easily found in closed form. Somehow unexpectedly, a manifest mistake appears in this case: this is a good example for students of an incautious application.

We then present a two-term approximation: one term of the pseudo-spectral decomposition is the static deflection of the single-term approximation, the other comparison function is the static deflection of the corresponding plates under a transverse suitably distributed load. In doing so, we resort to the governing equation for buckling provided by Eq. (7). As above, we perform the same operations with reference to Eq. (8). In this case we adopt the rotation of the transverse material fibres of the corresponding transversely loaded plates, once uniformly, once distributed, as in the case of the deflection. In both cases of the deflection and the rotation of the transverse material fibres, the expressions for the buckling load are found in closed form by methods described in the literature. This represents a further positive exercise for students; again, an apparent error is present when resorting to Eq. (8).

3.1 Single-term approximation

An admissible functionfor performing Galërkin's approach is taken here as the static deflection of a uniform homogenous circular plate under a uniform transverse load. Since the shape only of the admissible function matters, with no loss of generality, we suppose that the magnitude of this fictitious load equals N_r/R. Then, according to [8, 10, 11] the searched deflection $\tilde{w}$ is given by

$$\tilde{w} = \frac{1}{D}\int \frac{1}{r}\left\{\int \left[\frac{1}{r}\left(\int \frac{N_r}{R} r\, dr\right)\right] r\, dr\right\} dr \tag{10}$$

supplemented by the relevant boundary conditions.

For a clamped plate, these are, in keeping with the Kirchhoff-Love hypothesis Eq. (1),

$$w(R) = 0, \qquad \theta(R) = -\left.\frac{dw}{dr}\right|_{r=R} = 0 \tag{11}$$

and lead to the admissible function

$$w_{1,cl} = \frac{N_r}{64RD}(R^2 - r^2)^2\,, \tag{12}$$

where the superscripts denote the one-term approximation and the clamp at the boundary.

For a simply supported plate, the boundary conditions to complete Eq. (10) are

$$w(R) = 0, \qquad M_r(R) = -\,D\left.\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)\right|_{r=R} = 0\,, \tag{13}$$

where Eq. (4) was accounted for, and lead to the admissible function (subscripts are as above)

$$w_{1,ss} = \frac{N_r}{64RD(1+\nu)}(R^2 - r^2)\left[(5+\nu)\,R^2 - (1+\nu)\,r^2\right] \tag{14}$$

Performing Galërkin's technique with reference to Eq. (7), always supposing $\nu = 1/3$, leads to

$$(\bar{N}_r)_{1,ss} = \frac{176D}{41R^2} \approx \frac{4.293D}{R^2}, \qquad (\bar{N}_r)_{1,cl} = \frac{16D}{R^2}\,, \tag{15}$$

where the subscript 1 indicates the single-term expansion, and the other signs are as in Eq. (9). As expected, the values in Eq. (15) approximate the actual ones in Eq. (9) from above; the error is 0.25% and 8.98% for the simply supported and the clamped plate, respectively. That is, the comparison function for the clamped plate returns a stiffer behaviour with respect to that provided by the comparison function for the simply supported plate, which is physically justified: this is a key point to encourage a critical attitude in students.

We might ask whether Galërkin's technique works with comparable efficiency also resorting to the governing equation for buckling in terms of the rotation of the transverse material fibres, Eq. (8). Then, comparison functions for θ must be looked for, and we adopt the same strategy as for the comparison functions for w. The primitives of the searched functions are given in closed form

$$\tilde{\theta} = -\frac{1}{Dr}\int\left\{\int\left[\frac{1}{r}\left(\int\frac{N_r}{R}r\,dr\right)\right]dr\right\}r\,dr \tag{16}$$

supplemented by the relevant boundary conditions; the obvious similitude with Eq. (10) is worth emphasising. For a clamped plate, the boundary condition to complete Eq. (16) is

$$\theta(R) = 0 \tag{17}$$

and leads to the admissible function

$$\theta_{1,cl} = \frac{N_r r}{16RD}(R^2 - r^2)\,, \tag{18}$$

where the subscripts suggest as in Eq. (12). It is interesting to remark that the expression in Eq. (18) is nothing but the derivative of $w_{1,cl}$ (see Eq. (12)) with respect to the radial coordinate.

For a simply supported plate, the boundary condition to complete Eq. (16) is

$$M_r(R) = D \left. \frac{d\theta}{dr} + \frac{\nu}{r}\theta \right|_{r=R} = 0\,, \tag{19}$$

(see Eq. (4)), and leads to the admissible function, where the subscripts are as above,

$$\theta_{1,ss} = \frac{N_r}{16RD(1+\nu)} \left[(3+\nu)\, R^2 - (1+\nu)\, r^2 \right] \tag{20}$$

Again, we remark that $\theta_{1,ss}$ in Eq. (20) is the r-derivative of the expression in Eq. (14).

Performing Galërkin's technique with reference to Eq. (8), always supposing $\nu = 1/3$, leads to

$$(\bar{N}_r)_{1,ss} = \frac{2128D}{515R^2} \approx \frac{4.132D}{R^2}, \qquad (\bar{N}_r)_{1,cl} = \frac{14D}{R^2}\,, \tag{21}$$

with notation as in Eq. (15). Note that both values are *below* the exact ones, with a percentagewise error of -3.50% and -4.65%, respectively. Thus, resorting to Eq. (8) in terms of the rotation of the transverse material fibres causes a paradox: all the passages in the two approaches are equivalent, even the comparison functions are exactly differentially related. Thus, the non-critical student could simply accept the results as correct within the limits of the chosen approximation. Yet the approximate values approach the exact values from below, which is not expected from the theory. This should be stressed in order for the students to form a critical examination of the obtained results: a seemingly paradoxical result implies a non-critical resolution. Thus, we operate a two-terms Galërkin's approach to corroborate the results and give a different insight.

3.2 Two-term approximation

We add one term to the set of the comparison functions and resort to both the governing equation for buckling, Eqs. (7), (8), starting from the first one. The admissible functions are closed-form solutions of elastic static problems of the corresponding plates under an axisymmetric suitably distributed transverse loads (always in the spirit of letting the students do exercises). One of such functions is due to a uniform load, as before; other such functions are due to

$$p_z = \frac{N_r}{R}\left(1 - \frac{r}{R}\right) , \tag{22}$$

which is among the simplest conceivable, one order higher than the uniform load.

If we resort to the governing equation for buckling in terms of the deflection, Eq. (10) yields

$$w_{2,cl} = \frac{N_r\,(64r^5 - 225Rr^4 + 290R^3r^2 - 129R^5)}{14400R^2D} \tag{23}$$

for the clamped plate, using the relevant boundary conditions Eqs. (11). Analogously, we obtain

$$w_{2,ss} = \frac{N_r\,[64r^5(1+\nu) - 225Rr^4(1+\nu)10R^3r^2(71+29\nu) - 3R^5(183+43\nu)]}{14400R^2D(1+\nu)} \tag{24}$$

for the simply supported plate, using the relevant boundary conditions Eqs. (13).

Performing Galërkin's procedure with reference to Eq. (7), posing $\nu = 1/3$ as usual, yields

$$(\bar{N}_r)_{2,ss} \approx \frac{4.309D}{R^2}, \qquad (\bar{N}_r)_{2,cl} \approx \frac{15.059D}{R^2} \tag{25}$$

Here the subscript 2 indicates the two-term approximation, the others are as in Eq. (9). As expected, analogously to Eq. (15), the values in Eq. (25) approximate the actual ones in Eq. (9) from above; the error is 0.63% for the simply supported plate, 2.57% for the clamped one. The two-term approach for the clamped plate provides a better approximation than that for the simply supported one. It is interesting for the students to stress that for the simply supported plate the single-term approximation is so satisfactory that any refinement risks being almost meaningless. On the other hand, for the clamped plate the two-term approximation provides much better results than the single term. The physical interpretation of this is an interesting open question for the students, always with the aim of inspiring them towards a critical analysis.

We turn to a two-term approximation in Galërkin's technique applied to Eq. (8). The comparison functions will be the rotations of the transverse material fibres of the corresponding plates, transversely loaded by a uniform distribution, and by that in Eq. (22). The first expressions are given by Eqs. (18), (20); the second are obtained using Eq. (16), supplemented by the boundary conditions in Eqs. (17), (19) for the clamped and the simply supported plate, respectively. For a clamped plate one gets

$$\theta_{2,cl} = \frac{N_r r\,(45Rr^2 - 16R^3 - 29r^3)}{720R^2 D} \tag{26}$$

and for the simply supported plate

$$\theta_{2,ss} = \frac{N_r r\,[45Rr^2(1+\nu) - R^3(71+29\nu) - 16r^3(1+\nu)]}{720R^2 D(1+\nu)} \tag{27}$$

It should be noted that, as for the one-term approximation, the comparison functions in Eqs. (26), (27) can be derived from the expressions in Eqs. (23), (24) by applying the Kirchhoff-Love hypothesis Eq. (1).

Galërkin's procedure provides, with a Poisson's ratio of $\nu = 1/3$,

$$(\bar{N}_r)_{2,ss} \approx \frac{4.281D}{R^2}, \qquad (\bar{N}_r)_{2,cl} \approx \frac{14.707D}{R^2}\,, \tag{28}$$

the notation being as in Eq. (25). The values in Eq. (28) approximate the actual ones in Eq. (9) once from above, then from below; the error is -0.02% for the simply supported plate, and is 0.17% for the clamped one. Again, the approximation for the clamped plate returns a stiffer behaviour with respect to that for the simply supported plate; but, even though almost imperceptible, the approximation from below appears to lead to wrong results.

4. EXPLANATION OF THE ERRONEOUS PROCEDURE

The apparent wrong results evidenced above are not a bug in the theory, nor in Galërkin's method itself, but it is due to an erroneous application of it. Such possible errors were underlined in the monograph by Vol'mir [12] when dealing with Galërkin's approach for evaluating the critical loads in purely flexible (Euler-Bernoulli) beams. Vol'mir remarks that the stationarity condition for the total potential energy of such beams, pre-stressed in compression by a load P, is [13]

$$\delta\mathcal{E}(v(x)) = \delta\left[\int_0^l \frac{1}{2}B\left(\frac{d^2v(x)}{dx^2}\right)^2 dx - \int_0^l \frac{1}{2}P\left(\frac{dv(x)}{dx}\right)^2 dx\right] = 0 \tag{29}$$

where $B = EI$ is the bending stiffness of the beam, E is Young's modulus of the material, I the second moment of area of the cross-section with respect to an axis normal to the plane containing the beam, and $v(x)$ the x-axis deflection. After two integrations by parts, Eq. (29) leads to

$$\begin{aligned} \left|B\frac{d^2v(x)}{dx^2}\delta\left(\frac{dv(x)}{dx}\right)\right|_0^l - \left|\left(B\frac{d^3v(x)}{dx^3} + P\frac{dv(x)}{dx}\right)\delta v(x)\right|_0^l + \\ + \int_0^l \left(B\frac{d^4v(x)}{dx^4} + P\frac{d^2v(x)}{dx^2}\right)\delta v(x)\,dx = 0 \end{aligned} \tag{30}$$

Eq. (30) may be interpreted as a version of the virtual work equality, accounting for linear elastic constitutive equations, as Vol'mir states in [12], if one identifies the inner couple M and the generalised shearing force Q in the boundary terms, respectively given by

$$B\frac{d^2v(x)}{dx^2} = M(x), \qquad B\frac{d^3v(x)}{dx^3} + P\frac{dv(x)}{dx} = Q(x) \tag{31}$$

Galërkin's technique considers an approximate expression of the transverse displacement field

$$v(x) \approx \sum_{i=1}^{n} f_i\eta_i(x) \tag{32}$$

and inserts it in Eq. (30). However, in order to discard the boundary terms in Eq. (30) and operate only on the integral term therein, the admissible functions $\eta_i(x)$ in eq. (32) shall satisfy *all* boundary conditions, be they geometric or natural. Thus, replacing Eq. (32) into Eq. (30) yields

$$\sum_{i=1}^{n} \int_0^l \left(B\frac{d^4 v(x)}{dx^4} + P\frac{d^2 v(x)}{dx^2} \right) \eta_i(x)\, dx = 0 \quad \forall\, i = 1, 2, \ldots, n \tag{33}$$

where the transverse displacement $v(x)$ among parentheses in the integrand is always expressed as a finite sum of admissible functions. The relations provided by Eq. (33) constitute a homogeneous linear algebraic system in the amplitude coefficients f_i that admits non-trivial solutions only imposing the singularity condition on the determinant of the matrix of the coefficients of the system. Such a condition provides the searched approximate values of the critical values of the compressive force P. Vol'mir [12] remarks that one cannot pretend to replace the expression between parentheses in Eq. (33) with another form of the field equation for the considered problem, keeping the same admissible functions in order to find correct solutions. Indeed, the variational problem leading to Eq. (33) is uniquely determined: for instance, one knows that

$$B\frac{d^2 v(x)}{dx^2} + Pv(x) = 0\,, \tag{34}$$

supplemented by suitable boundary conditions (only geometrical in this case), correctly provides the buckling loads for the considered problem. However, one cannot pretend to use the same admissible functions for both Eq. (33) and the analogous one obtained replacing Eq. (34) in the term between parentheses of Eq. (33). As a matter of fact, the boundary conditions to be satisfied in the integration by parts following the variational condition of Eq. (30) are different, thus the admissible functions are inevitably different. This leads to apparent incorrect results, as Vol'mir stresses in [12]: as a simple example, one could suggest the students to check what might occur for clamped beams, which are a common benchmark.

A similar phenomenon takes place when dealing with axisymmetric plates, and leads to the wrong results evidenced above. Let us recapitulate all passages in full detail, in order to catch the possible bugs in performing Galërkin's technique. The total potential energy for an axisymmetric plate, pre-stressed in compression by a uniform load per unit length N_r, is [8, 10, 11]

$$\int_0^R \left\{ D \left[\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} \right) \delta\left(\frac{d^2w}{dr^2}\right) + \left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2} \right) \frac{1}{r}\delta\left(\frac{dw}{dr}\right) \right] - N_r \frac{dw}{dr}\delta\left(\frac{dw}{dr}\right) \right\} r\, dr = 0 \quad (35)$$

where the dependance of the transverse displacement w on the radial coordinate r has been understood, hence omitted, for the sake of simplicity of notation. Galërkin's method foresees integration by parts of Eq. (35): a first step provides

$$\left| rD\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} \right) \delta\left(\frac{dw}{dr}\right) \right|_0^R + \int_0^R r \left[D\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} \right) + N_r \frac{dw}{dr} \right] \delta\left(\frac{dw}{dr}\right) dr = 0 \quad (36)$$

Accounting for the Kirchhoff-Love hypothesis, the transverse material fibres remain orthogonal to the deflected middle surface, hence their rotation θ equals the slope of the deflected middle surface in the radial direction. Then, inserting the linear elastic relation for the bending couple in the radial direction M_r, Eq. $(4)_1$, into Eq. (36), the latter becomes

$$|rM_r\, \delta\theta|_0^R - \int_0^R r \left[D\left(\frac{d^2\theta}{d\theta^2} + \frac{1}{r}\frac{d\theta}{dr} - \frac{\theta}{r^2} \right) + N_r\theta \right] \delta\theta\, dr = 0 \quad (37)$$

For the usual hypotheses on the generality of the variation of the test field $\delta\theta$, both the boundary term and the integral term shall vanish, which leads to the boundary conditions and to the bulk equations to be verified. This implies that Galërkin's method with comparison functions chosen from among transverse displacements that are solutions to some elastic problems for axisymmetric plates can, in principle, be unsatisfactory.

Twice repeated integration by parts of Eq. (35) provides, after some simplifications,

$$\left| rM_r\, \delta\theta + \left[\frac{d(rM_r)}{dr} + M_\phi + rN_r\theta \right] \delta w \right|_0^R - \int_0^R r(D\Delta\Delta w + N_r \Delta w)\, \delta w\, dr = 0 \quad (38)$$

where: M_ϕ is the hoop bending couple, according to Eq. $(4)_2$; the term multiplying the variation δw in the boundary term is easily proved to correspond to the shearing transverse force Q per unit length *in the adjacent, buckled, configuration pulled back to the referential one*, i.e.,

$$Q = \frac{d(rM_r)}{dr} + M_\phi + rN_r\theta\,; \tag{39}$$

and Δ is Laplace operator in cylindrical coordinates. It is quite apparent that the admissible, or comparison, functions to be used in Galërkin's method for finding the buckling load of uniformly compressed axisymmetric plates cannot be arbitrarily chosen if one considers Eq. (37) or (38). The bulk equations they provide are actually *equivalent* from the viewpoint of analytical solutions (when at ease). However, the boundary conditions for Eq. (37) are not sufficient to find an expression for the transverse displacement w; so, one shall consider Eq. (38). Thus, it seems that a correct Galërkin's approach should consider this, and revert to the relevant admissible functions.

5. OVERCOMING THE ERRONEOUS APPLICATION

The correct implementation of Galërkin's approach for the cases, already examined, of the simply supported and the clamped axisymmetric plates are shown by a right choice of admissible functions. This depends on the step of integration by parts taken starting from the stationarity condition Eq. (35) and the boundary conditions to respect, and eliminates all the evidenced incorrect results.

5.1. Simply supported plates

The boundary conditions are partly geometrical: the transverse displacement vanishes at $r = R$ and the rotation of the transverse material fibres vanishes, due to axial symmetry, at the centre of the middle plane. The boundary conditions are also partly natural, i.e. the radial bending couple vanishes at $r = R$. As usual, we start with a single term approximation, and use polynomials that are solutions of linear elastic problems for the same structural element. If we wish to adopt Eq. (37), a good admissible function is the field of rotation of transverse fibres due to a uniform shearing force,

$$\theta_1(r) = \frac{r\left[r\left(1+\nu\right) - R\left(2+\nu\right)\right]}{3D\left(1+\nu\right)} \tag{40}$$

which, introduced in the bulk part of (37) and assuming the usual value $\nu = 1/3$, leads to

$$\left(\bar{N}_r\right)_1 = \frac{15D\left(1+\nu\right)\left(5+\nu\right)}{R^2\left(22+8\nu+\nu^2\right)} \approx \frac{4.305D}{R^2} \;\Rightarrow\; \Delta\left(\bar{N}_r\right) = \frac{\left(\bar{N}_r\right)_1 - \left(\bar{N}_r\right)_e}{\left(\bar{N}_r\right)_e} \approx 0.54\%\,, \tag{41}$$

where the subscript 1 stands for the single-term Galërkin's procedure, and the subscript e stands for the exact value provided by the theory.

A second admissible function is the field of rotation of transverse fibres due to a shearing force varying linearly with respect to the radial coordinate,

$$\theta_2(r) = \frac{r\left[r^2\left(1+\nu\right) - R^2(3+\nu)\right]}{8RD\left(1+\nu\right)} \;\Rightarrow\; \theta(r) \approx f_1\theta_1(r) + f_2\theta_2(r) \tag{42}$$

which, introduced in the bulk part of (37) and assuming the usual value $\nu = 1/3$, leads to

$$\left(\bar{N}_r\right)_2 \approx \frac{4.284D}{R^2} \;\Rightarrow\; \Delta\left(\bar{N}_r\right) = \frac{\left(\bar{N}_r\right)_2 - \left(\bar{N}_r\right)_e}{\left(\bar{N}_r\right)_e} \approx 0.04\% \tag{43}$$

with the same notation of Eq. (42). In both cases, the approximation is very good, and the values approach the actual ones from above, as the theory predicts: the error is overcome because the *correct* admissible functions were chosen, contrary to what was done above, naively. This is a good point to be stressed for students to practice good self-checking and show that they have a good knowledge of the theory.

If we wish to adopt Eq. (38), a satisfactory admissible function is the field of transverse displacement due to a uniform load,

$$\eta_1(r) = \frac{\left(R^2 - r^2\right)\left[-r^2\left(1+\nu\right) + R^2\left(5+\nu\right)\right]}{64D\left(1+\nu\right)} \tag{44}$$

which, introduced in the bulk part of (38) and assuming the usual value $\nu = 1/3$, leads to

$$\left(\bar{N}_r\right)_1 = \frac{16D\left(1+\nu\right)\left(7+\nu\right)}{R^2\left(33+10\nu+\nu^2\right)} \approx \frac{4.293D}{R^2} \;\Rightarrow\; \Delta\left(\bar{N}_r\right)_{cr} = \frac{\left(\bar{N}_r\right)_1 - \left(\bar{N}_r\right)_e}{\left(\bar{N}_r\right)_e} \approx 0.25\% \tag{45}$$

with the same notation of Eqs. (40), (42). A second admissible function is the field of transverse displacement due to a load varying linearly with respect to the radial coordinate,

$$\eta_2(r) = \frac{\left[2r^5\left(1+\nu\right) - 5R^3r^2\left(4+\nu\right) + 3R^5\left(6+\nu\right)\right]}{450RD\left(1+\nu\right)} \;\Rightarrow\; w(r) \approx f_1\eta_1(r) + f_2\eta_2(r) \tag{46}$$

which, introduced in the bulk part of (37) and assuming the usual value $\nu = 1/3$, leads to

$$\left(\bar{N}_r\right)_2 \approx \frac{4.282D}{R^2} \;\Rightarrow\; \Delta\left(\bar{N}_r\right) = \frac{\left(\bar{N}_r\right)_2 - \left(\bar{N}_r\right)_e}{\left(\bar{N}_r\right)_e} \approx 0.01\% \tag{47}$$

with the same notation of Eqs. (40), (42). It is apparent that this version of Galërkin's procedure provides better approximated results than that related to the bulk equation in terms of the rotation of the transverse material fibres. This is to be expected, since the admissible functions must verify more boundary conditions in the case of a bulk equation in terms of the deflection. In any case, the values approach the actual ones from above, as expected, and no error exists.

5.2. Clamped plates

The boundary conditions to be considered for the admissible functions of Galërkin's method are purely geometrical in this case. We start with a single polynomial, solution of the linear elastic problem for the same structural element. If we adopt Eq. (37), an admissible function is the field of rotation of transverse fibres due to a uniform shearing force,

$$\theta_1(r) = \frac{r\left(r-R\right)}{3D} \tag{48}$$

which, introduced in the bulk part of (38), leads to

$$\left(\bar{N}_r\right)_1 = \frac{15D}{R^2} \;\Rightarrow\; \Delta\left(\bar{N}_r\right) = \frac{\left(\bar{N}_r\right)_1 - \left(\bar{N}_r\right)_e}{\left(\bar{N}_r\right)_e} \approx 2.17\% \tag{49}$$

with the usual notation. A second admissible function is the field of rotation of transverse fibres due to a shearing force varying linearly with respect to the radial coordinate,

$$\theta_2(r) = \frac{r\left(r^2 - R^2\right)}{8RD} \;\Rightarrow\; \theta(r) \approx f_1\theta_1(r) + f_2\theta_2(r) \tag{50}$$

which, introduced in the bulk part of (38), leads to

$$\left(\bar{N}_r\right)_2 = \frac{7\left(29 - \sqrt{265}\right)D}{6R^2} \approx \frac{14.841D}{R^2} \;\Rightarrow\; \Delta\left(\bar{N}_r\right) = \frac{\left(\bar{N}_r\right)_2 - \left(\bar{N}_r\right)_e}{\left(\bar{N}_r\right)_e} \approx 1.09\% \tag{51}$$

The results provide a very good approximation of the actual ones, again, and no error exists.

If we wish to adopt Eq. (38), a satisfactory admissible function is the field of transverse displacement due to a uniform load,

$$\eta_1(r) = \frac{\left(R^2 - r^2\right)^2}{64D} \tag{52}$$

which, introduced in the bulk part of (38), leads to

$$\left(\bar{N}_r\right)_1 = \frac{16D}{R^2} \;\Rightarrow\; \Delta\left(\bar{N}_r\right) = \frac{\left(\bar{N}_r\right)_1 - \left(\bar{N}_r\right)_e}{\left(\bar{N}_r\right)_e} \approx 8.98\% \tag{53}$$

A second comparison function is the field of transverse displacement due to a load varying linearly with respect to the radial coordinate,

$$\eta_2(r) = \frac{6R^5 - 10R^3r^2 + 4r^5}{900Rd} \;\Rightarrow\; w(r) \approx f_1\eta_1(r) + f_2\eta_2(r) \tag{54}$$

which, introduced in the bulk part of (38), leads to

$$\left(\bar{N}_r\right)_2 = \frac{9\left(183 - \sqrt{12369}\right)D}{44R^2} \approx \frac{14.683D}{R^2} \;\Rightarrow\; \Delta\left(\bar{N}_r\right) = \frac{\left(\bar{N}_r\right)_2 - \left(\bar{N}_r\right)_e}{\left(\bar{N}_r\right)_e} \approx 0.01\% \tag{55}$$

For a single-term approximation, Galërkin's procedure applied to the bulk equation in terms of the rotation of the material fibres is good; for a two-term approximation, Galërkin's procedure applied to the bulk equation in terms of the deflection provides almost exact results. In addition, once again the approximate values approach the actual ones from above, and no error occurs.

6. CONCLUSIONS

We have presented a benchmark example where an incautious application of Galërkin's method may lead to erroneous results, approximating the actual ones from below, which is not justified by the physical interpretation of the approximation technique itself. Simply by re-considering with due accuracy the passages that must be operated to perform Galërkin's method, we have shown that there can be different expressions that have to be weighted by the so-called comparison functions in order to provide the relation yielding the approximate values searched for. We have shown that it is apparent that adopting comparison functions for Eq. (38) and inserting them into Eq. (37), or *vice-versa*, may lead to erroneous results, approximating the actual values from below, which is not correct. On the other hand, a rigorous search of the correct comparison functions immediately provides very good approximate values, with no errors arising. We are of the opinion that this is a benchmark example that can be particularly helpful in instructing students not only to use a well-known approximating method, but also to develop a critical attitude in the search for approximate results and their interpretation in terms of theoretical expectations and physical interpretations.

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