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Classical and Quantum Perturbations to the Primordial Universe

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Abstract

In this Ph.D. thesis we analyse both classical and quantum effects relevant for the study of cosmological perturbations. We choose this particular topic because, through the analysis of cosmological perturbations, it is possible to explore a wide range of different physical phenomena. Moreover, they are a central and important piece in the puzzle of the history of the universe.

The most obvious relevance of cosmological perturbations is the study of structure formation and the large scale structure of the universe. In this regard, such perturbations are related to primordial gravitational waves and primordial magnetic fields. Given their dependence on pre-recombination phenomena, they could give us some information on the universe before hydrogen recombination.

Classical perturbations have been widely studied in literature, with the main focus on isotropic cosmological models. While this is usually a good approximation, the presence of a primordial magnetic field causes a coupling between different algebraic modes of the usual decomposition, connecting density perturbations, primordial magnetic fields and primordial gravitational waves. Moreover, the presence of the magnetic field requires the use of an anisotropic cosmological model. While small, these relations are important in the evolution of anisotropic structures. Furthermore, such primordial seeds of the magnetic fields are widely believed to be the origin of the magnetic fields measured today in galaxies. In the first part of this thesis, we analyse these relations, together with the possible effects that a non ideal, i.e. viscous, cosmological fluid could have on the growth of perturbations. We focus our attention to a Bianchi I model, improving the results of some preceding papers.

The second part of the thesis focuses on the semiclassical approximation of quantum gravity. Quantum effects are believed to influence the birth and dynamics of perturbation seeds and, in general, the dynamics of the primordial universe. This way, the mathematical scheme used to represent these effects is a central point in the description of quantum gravity regarding such seeds.

Furthermore, even more care is required to split the WKB action between embedding variables and physical degrees of freedom, and in many models the quantum gravity corrections to the Schrödinger equation violate the unitarity of the system evolution. This decomposition shares some similarities with the Born-Oppenheimer approximation of molecular physics.

We perform a critical analysis of two different ways to apply this decomposition. In particular, we analyse limits and perspectives of the different proposals to solve the non unitarity problem, even comparing expansions in different fundamental physical constants (Planck constant and mass). We find the source of non-unitary effects in a common assumption in the definition of WKB time, and we propose an alternative formulation. Also, we show how the usual assumptions of classicality of the physical quantities must be handled with care, focusing our attention to the implementation of the classical background in the perturbation scheme.

Studies in this research field are very important because they could bind CMB measurements and primordial gravitational waves to quantum gravity, bringing us finally an experimental playground.

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Contents

Introduction and thesis outline

The formation of large scale structures across the universe is a fascinating and puzzling issue, still unanswered. Also, it could be related to the peculiarity of the matter distribution across the universe, the possibility for large scale filaments [99], as well as hypotheses for structure fractal dimension [82, 83].

Having the universe a Debye length of the order of 10 cm, it can be very well described by a fluid theory, like general relativistic magneto-hydrodynamics. This opens an excellent opportunity to study the effects of primordial magnetic fields [73] on the evolution of perturbations [89]. Surely, they are tightly constrained by the Cosmic Microwave Background Radiation up to a maximum allowed intensity of 10^{-9} G by a number of different studies [54–56, 85, 87, 95, 96]. This limits their effects, yet still they are able to trigger important anisotropies in the linear growth cosmological perturbations evolution [89, 104]. It is argued that such anisotropies could trigger the formation of large scale filaments.

A weak point in this description is the plasma nature of the universe, i.e. the small ionized component after recombination. However, it has been demonstrated that the coupling between neutral and ionized matter is still very strong at the scales of interest [72, 89, 104]. Thus, the dynamical features of the plasma component of the cosmological fluid are tightly to be attributed to the neutral component, too. Moreover, the large photon to baryon ratio is responsible for keeping active a strong Thomson scattering, even after recombination and up to $z \simeq 100$ [18, 72, 89].

Thus, we face the issue of the description of the cosmological perturbation seeds, in presence of a weak magnetic field, through the general relativistic magneto-hydrodynamics formulation.

Many authors already faced this subject, however nearly always neglecting the anisotropic effects, see [77] and references therein. However, the presence itself of the magnetic field causes strong anisotropic effects, already visible in the Newtonian approximation [89]. Thus, we study a Bianchi I model whose anisotropy is completely controlled by the primordial magnetic field. We use this model to study the cosmological perturbations in chapter 2, providing an accurate description of the perturbations equations both in radiation dominated and in matter dominated universe. We provide both analytical and numerical solutions, enhancing the results of the present studies on this topic.

A possible issue may be caused by the anisotropy itself, related to shear [89, 90]. If viscous effects were to occur at a relevant scale, they could stop the anisotropies from growing. In chapter 3, we investigate the non-ideal contributions due to viscosity. This time, we provide a limited analytical description and a more advanced numerical result, showing that viscosity plays almost no role at the scales of interest.

Moreover, we derive a threshold of about 5000 solar masses to its related scales. Thus, we clearly affirm that the magnetised perturbation are free from its effects, as their anisotropic behaviour.

In the second part of this thesis, we deal with the cosmological perturbations on a quantum gravity level. This is because cosmological perturbations are expected to start in the quantum gravity regime, and then evolve first when we can safely apply classical general relativity in the linear regime, and lately undergo a non-linear evolution until they form the structures that populate the present universe.

The most traditional attempt to derive a quantum theory of gravity is canonical quantization, and it is expected to lead at least to a good approximation of the real quantum theory [6]. The Hamiltonian formulation of GR leads to the concept of superspace, the configuration space of all the geometric and matter variables. Since, in general, the variables are fields defined over a curved spacetime, the full theory has a functional nature and requires some renormalization procedure to yield finite predictions. To avoid this kind of difficulties, a possibility is to concentrate on highly symmetric spacetimes, reducing the dynamics to a finite-dimensional scheme. The concept of superspace is then replaced by its finite-dimensional analogous, i.e. minisuperspace. This reduced theory finds its main applications in cosmology, where homogeneous spacetimes are considered.

The first thing we have to face is the problem of time. This is one of the most relevant open issues in canonical quantum gravity, and although there is a huge literature about this problem, a commonly accepted solution has not been found yet. Moreover, many tentative solutions clearly show non-unitary behaviours at the quantum gravity level [38, 53].

Many different approaches have been proposed to solve the issue of the lack of time in the quantum theory, trying to introduce it through some matter source [40, 51] or identifying it with an internal source-time variable [42]. All of them rely on the concept of *relational time* [41]: under the request of suitable conditions, each subsystem can be properly adopted as a clock for the remaining part of the quantum system. However, all of them seem qualitatively far from the idea of time of quantum mechanics, i.e. an external parameter. Moreover, there is still the issue of measurements performed by a classical observer, which is not possible in quantum gravity. Actually, it is not clear how to reproduce the proper limit of quantum field theory, starting from the Wheeler-DeWitt (WDW) equation [9] and on the base of a relational time approach.

A different proposal has been investigated in [36], where there is a decomposition of the system into "slow", classical, WKB variables and "fast", quantum ones. This resembles a Born-Oppenheimer approximation, with the peculiar feature that now time is introduced through the dependence of the quantum system on the classical variables, bringing to standard label time of the spacetime slicing. This approach allows to reconstruct quantum field theory on a classical, curved background. In principle, it can be applied to any set of variables [81]. It is probably the most natural way to introduce time in the minisuperspace [100].

In [36], the analysis is performed through an expansion in the Planck constant up to first order in $\hbar$. [38] implements the same idea is implemented expanding in the Planck mass and going up to the quantum gravity corrections. The results are similar to [36] up to quantum mechanics, but then a non-unitary character of the quantum dynamics emerges at the quantum gravity level. This prevents the predictivity of the theory.

There are two different proposals to solve this nasty issue in [53] and [103]. One relies on a finer separation between classical and quantum parts of the system, while the other tries to to reconstruct a posteriori a well-behaving Schrödinger evolution of the quantum subsystem, manually altering the WKB dynamics of the gravitational background.

We perform a deep critical analysis of all these proposals, showing that, unfortunately, none is free from the non-unitarity problem. Then, we make a new proposal to recover time through the introduction of the kinematical action defined in [25]. This way, we free the time definition from the ill-defined background Laplacian that appears at quantum gravity level, ad we are able to recover a unitary evolution for the quantum gravity Schrödinger equation.

Eventually, we present the application of the model of [38] to cosmological perturbations, as done in [98]. The calculations, however, rely on some assumptions that deserve a lot much care. We try to "clean" the procedure by using a more robust formalism instead and not relying on additional assumptions on the quantum variables. However, we found ourselves stuck at the quantum gravity level. This could mean that those assumptions were to be taken with greater care, as we suggested, and that this issue still deserves attention, being our quantum gravity order not compatible with the original one.

Thesis outline

Chapter 1 is a very short introduction to classical cosmology. We present here the Friedmann–Robertson–Walker model, together with the cosmological theory of small fluctuations. We then make a brief description of the first extensions to the FRW model, that is the Bianchi models. They keep the assumption of homogeneity, while non requiring isotropy: this leads to an increased freedom on the dynamics they are able to describe.

In chapter 2, based on [107, 114], we provide the general relativistic descriptions of the cosmological perturbations in presence of a magnetic field. We enhance the present descriptions of both the background model and the perturbations themselves, while always comparing our results with the present literature. We are able to add the anisotropic effects to both the large scale perturbations in radiation dominated universe, and the small scale ones in matter dominated era. Finally, we confirm the validity of present works in the Newtonian limit, and we provide a numerical description of the studied phenomena.

In chapter 3, we base our presentation on [111] and we analyse the influence of viscous effects, due to the non-ideal contributions to the background cosmological model. We focus our study in matter dominated universe, and we show that such influences appear only below a few thousand solar masses, showing that they do not modify the anisotropic effects due to the magnetic field presence.

Now, we move to the quantum effects related to cosmological perturbations. Chapter 4 has the aim to introduce to the formalism and basic results we need in the following ones. Here, we present a short overview of the ADM formalism of general relativity, canonical quantization and the Wheeler-DeWitt equation and, eventually, we present the minisuperspace formalism and the semiclassical approximation of quantum gravity, through which we recover a notion of time in the quantised model.

In chapter 5, we present a critical analysis of the current definitions of time through a WKB expansion, and we present a novel proposal based on the kinematical action, following [112]. We perform a critical analysis of the current proposals for a definition of time through a WKB expansion [36, 38]. We perform a deep comparison between them, and we apply the procedure of [103] to extend them to any perturbative order. Unfortunately, both of them show a non-unitary behaviour at the quantum gravity level. A solution is said to be found in [103], but we show it not to be acceptable. A different expansion is carried out in [53], based on a more robust formalism. Although it appears to solve the issue, we show that it lacks some important considerations and that, unfortunately, the non-unitarity is still there.

Eventually, we make a new proposal to recover time through the introduction of the kinematical action defined in [25]. This way, we are able to recover a unitary evolution for the quantum gravity Schrödinger equation.

In chapter 6, we present the results of [115]. We analyse some non-trivial assumptions of [98] and we try to circumvent them by making use of a more robust formalism. However, we end up stuck at the quantum gravity level. We are able to show that our final equation is not compatible with the original one of [98]. This could signal that the assumptions we mentioned were to be taken with greater care.

Notation

Throughout this thesis, Greek indices indicate 4-dimensional quantities and Latin indices refer to the 3-space variables, i. e. $\alpha, \beta, \ldots \in [0,3]$ while $a, b, \ldots \in [1,3]$. For simplicity, we often assume an adimensional light speed $c = 1$, or geometric units with $c = 1$ and $G = 1/(8\pi)$. The Einstein constant is $\mathcal{K} = 8\pi G/c^4 = 1$.

Chapter 1

Classical cosmology

We will introduce the FRW cosmological model in section 1.1, together with the theory of cosmological perturbations in the linear regime, both in Newtonian and general relativistic formalisms; we will make use of both these models and we will extend them in chapters 2 and 3.

Next, in section 1.2, we will make a short introduction to the Bianchi models, which are general homogeneous models and, as such, the first candidates for the regimes when the FRW models does not hold.

1.1 The FRW cosmological model

1.1.1 RW geometry

The Cosmological Principle states that each observer looks at the same universe. For this to be true, the universe must be both homogeneous and isotropic, i.e. there aren't either privileged space points or preferred space directions. Assuming these requirements, the most general metric left is that of a non-stationary homogeneous and isotopic three-geometry.

The homogeneity requires the spatial directions to evolve with the same time dependence, while the isotropy imposes a vanishing g_{0i}. Thus, any synchronous reference must be described by the Robertson-Walker line element

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}l^2_{\mathrm{RW}} = -\,\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\,\mathrm{d}\Omega^2\right). \tag{1.1}$$

The cosmic scale factor $a(t)$ is the only available degree of freedom, while K denotes the spatial curvature; r and Ω are the usual spherical coordinates. If $K \neq 0$, it is always possible to rescale a and r through

$$r \to r\sqrt{|K|}, \qquad a \to \frac{a}{\sqrt{|K|}} \tag{1.2}$$

such that the line element acquires the same form, with $|K| = 1$.

Redshift

Under this geometry, the trajectory of a test particle with 4-velocity u^μ, given by the geodesic equation, shows the phenomenon known as redshift. Focusing our attention on the zero component, we get

$$\frac{\mathrm{d}u^0}{\mathrm{d}s} + \frac{\dot{a}}{a}u^2 = 0, \tag{1.3}$$

Where $u^2 = u^i u_i = h_{ij}u^i u^j$. In a synchronous reference frame $u^\mu u_\mu = -(u^0)^2 + u^2 = -1$, so $u^0\,\mathrm{d}u^0 = u\,\mathrm{d}u$ and, using $u^0\,\mathrm{d}s = \mathrm{d}t$,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\dot{a}}{a}u = 0. \tag{1.4}$$

This shows that $u \propto 1/a$. In other words, if m_0 is the rest mass of a particle, its spatial momentum behaves as $p = m_0 u \propto 1/a$. This derivation does not rely on the non-vanishing of proper time, so it is also valid for massless particles, for which

$$E = p = \frac{2\pi}{\lambda} \propto \frac{1}{a}, \tag{1.5}$$

where E is the energy and λ the particle wavelength: in an expanding universe, a increases with time and the light spectrum is shifted towards the red. If we consider a photon emitted at t_e and observed at $t_0 = t_{\text{now}}$, the ratio of the wavelengths is

$$\frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} \equiv 1 + z, \tag{1.6}$$

where the quantity z is the amount of redshift and it is measurable, with $z_0 \equiv 0$. In practice, the role of the redshift is equivalent to the one of the scale factor. The physical distance between a pair of observers scales exactly in the same way, so any intrinsic comoving length is scaled with a.

Universe expansion

Let us now study the relative motion of two different points, assuming their position to be fixed, i.e. the relative motion due to the geometry. Looking far from us, we are actually looking backward in time. A Taylor expansion on the scale factor

$$a(t) = a_0 + \dot{a}|_{t=t_0}(t - t_0) + \dots, \qquad a_0 \equiv a(t_0), \tag{1.7}$$

together with (1.6), leads to

$$\frac{1}{1+z} \equiv \frac{a}{a_0} = 1 - H_0(t_0 - t) + \dots, \qquad H(t) \equiv \frac{\dot{a}}{a}. \tag{1.8}$$

H is called the Hubble parameter and it measures the (logarithmic) expansion rate of the universe at a given time; $H_0 \equiv H(t_0)$ is the Hubble constant. If we measure the distance between the points through photons, using $\mathrm{d}s^2 = 0$ we get

$$\int_t^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1 - Kr'^2}} = \begin{cases} r & \text{if } K = 0 \\ \sin r & \text{if } K = 1 \\ \sinh r & \text{if } K = -1 \end{cases} = r \text{ for } r \ll 1. \tag{1.9}$$

Thus, the spatial curvature can be neglected when the distance is much smaller than the curvature radius of the universe, and the space can be assumed to be flat. Inserting the Taylor series for a in the left hand side, up to first order we get

$$t_0 - t = d + \dots \,, \tag{1.10}$$

where $d = a_0 r$ is the present distance of the light source. Using this result and approximating it for $z \ll 1$, eq. (1.8) becomes the Hubble law

$$v = z = H_0 d + \dots \,, \tag{1.11}$$

where we have interpreted the geometrical redshift as Doppler effect due to a physical velocity v. The last equation describes the recession of the galaxies, which is due to the motion of the background. This is because, apart from local interaction and small proper motions, the galaxy flow can be described as the motion of pressure-less particles, i.e. a dust system, freely falling on the geometry. On the other hand, each single galaxy has enough binding energy to detach from the Hubble flow and be a non-expanding substructure, with an essentially flat internal spacetime. To derive the usual form of the Hubble law, we required $z \ll 1$, i.e. galaxies close to our own.

Hubble and particle horizons

Through the quantities defined above, it is possible to construct the characteristic times and scales of the expanding universe. The cosmological time is given by the inverse of the expansion rate H^{-1}. If $a \propto t^\alpha$ (as we will see in the next section for the Friedmann model), then $H^{-1} \propto t$. In the same way, the Hubble length is given by $L_H(t) = H^{-1}(t)$ and represents a real and measurable horizon for the microphysics of the universe: it roughly represents the distance a photon can travel in an expansion time and, for a phenomenon to operate coherently on a cosmological scale, its time scale must be much faster then the Hubble time, i.e. its spatial scale must be much smaller than the Hubble length.

The maximal causal distance at which physical signals can propagate is called *particle horizon* (or simply *horizon*) and, starting the signal from $t = 0$, can be easily obtained through

$$\mathrm{d}s^2 = 0 \implies \mathrm{d}t = a(t)\,\mathrm{d}l_{\mathrm{RW}} \implies l_{\mathrm{RW}} = \int_0^t \frac{\mathrm{d}t'}{a(t')}. \tag{1.12}$$

The last result must be rescaled to obtain the physical horizon:

$$d_{\mathrm{H}} = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')}. \tag{1.13}$$

Objects more distant than d_{H} have never been in causal contact and cannot have been affected by each other. This way, spatial regions farther than the cosmological horizon cannot be in thermal equilibrium. As we will see in the next section, in the Friedmann model the scale factor behaves as a power of time $a(t) \propto t^\alpha$, with $\alpha < 1$. This means that the cosmological horizon is a finite quantity. Moreover, the comoving horizon d_{H}/a is always increasing, and since comoving distances are constant by definition this means that things that are in causal contact today may not have been so in the past, i.e. a today causal connected region, at some time in the past, was a collection of many different independent regions.

1.1.2 FRW dynamics

We now analyse the dynamics of a homogeneous and isotropic universe, under the Einstein equations. The line element, as noted before, must be in the RW form (1.1).

The matter component of the universe is described by a cosmological perfect fluid, with energy momentum tensor

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{1.14}$$

where u_μ is the unit timelike 4-velocity of the fluid and ρ and P are respectively the energy density and pressure, as measured by an observer in a local inertial frame comoving with the fluid, and are related by an equation of state of the form $P = P(\rho)$. For the isothermal universe, the equation of state takes the form

$$P = (\gamma - 1)\rho = w\rho, \tag{1.15}$$

where γ is the polytropic index.

The equations of motion of the fluid can be derived from the conservation law of the energy-momentum tensor

$$\nabla_\nu T_\mu{}^\nu = 0 = u_\mu \nabla_\nu \left[(\rho + P)u^\nu\right] + (\rho + P)u^\nu \nabla_\nu u_\mu + \partial_\mu P. \tag{1.16}$$

Multiplying by u^μ we get

$$\nabla_\nu \left[(\rho + P)u^\nu\right] = u^\mu \partial_\mu P, \tag{1.17}$$

and substituting in the initial equation

$$u^\nu \nabla_\nu u_\mu = -\frac{1}{\rho + P}\left(\partial_\mu P + u_\mu u^\nu \partial_\nu P\right). \tag{1.18}$$

These are the equation of motion of the fluid flow. In general, the pressure elements prevent the geodesic motion and the comoving frame cannot be synchronous, unless $P = \text{const}$. However, for a homogeneous and isotropic space, P is function only of time and the right end side vanishes for $u^\mu = (1, 0, 0, 0)$: in this case, the comoving frame is also synchronous and the stress-energy tensor takes the form

$$T_\mu{}^\nu = \mathrm{diag}(-\rho, P, P, P). \tag{1.19}$$

From now on, for simplicity, we assume the cosmological fluid as comoving with the synchronous reference frame. This way, u^μ and $T_\mu{}^\nu$ take the form expressed before and the non-vanishing Einstein equations take the form

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{1}{3}\rho - \frac{K}{a^2} \tag{1.20}$$

$$2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2} = -P. \tag{1.21}$$

The first one is usually called the *Friedmann equation*. Combining them we get the equation for the universe acceleration

$$\frac{\ddot a}{a} = -\frac{1}{6}(\rho + 3P). \tag{1.22}$$

Usually, instead of solving them together, the Friedmann equation is accompanied by the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0, \tag{1.23}$$

which is derived from the stress energy tensor conservation law. Moreover, eqs. (1.20) and (1.22), together with the fluid equation of state (1.15) and calculated at $t = t_0$, provide simple expressions for the Hubble constant and the deceleration parameter.

Let us now describe the basic properties of the universe dynamics. From eq. (1.22), as far as the matter component of the universe has an equation of state with $P > -\rho/3$, the expansion has to decelerate. This way, the current evidences that the universe is accelerating contrast with the FRW dynamics, and require a revision of our understanding of the nature of the matter component of the universe or of the FRW model itself.

From eq. (1.20), we see that for $K = 0, -1$ there is no time in the universe evolution in which $H = 0$, i.e. $\dot{a}$ never vanishes. The Big Bang is thus followed by the radiation phase, an equilibrium era and finally a decelerating matter dominated phase, without any re-collapse. On the other hand, if $K = 1$ there exists a time when $\dot{a} = 0$. It is easy to see that, calling such time $t_{\rm tp}$, we have $a_{\rm tp} = \sqrt{(3/\rho_{\rm tp})}$. Both in a radiation and a matter dominated universe the second time-derivative (1.22) is negative, so $a_{\rm tp}$ is a maximum for the expansion and is followed by a collapse towards a singularity in which $a = 0$, i.e. a Big Crunch.

Finally, eq. (1.20), together with the definitions

$$\rho_{\rm crit} \equiv 3H^2 \tag{1.24}$$

$$\Omega \equiv \frac{\rho}{\rho_{\rm crit}}, \tag{1.25}$$

leads to

$$\Omega - 1 = \frac{1}{H^2 a_{\rm curv}^2}, \qquad a_{\rm curv} \equiv \frac{a}{\sqrt{K}}. \tag{1.26}$$

The definition of $\rho_{\rm crit}$ is simply the density the universe would have for $K = 0$, and is called *critical density*. The *density parameter* Ω is larger, equal or smaller than 1 respectively for closed, flat and negatively curved RW models. Presently, $\rho^0_{\rm crit} = 1.03 \times 10^{-29}\,{\rm g/cm^3} \approx 5.8 \times 10^{-6}\,{\rm GeV/cm^3}$, while Ω_0 is very closed to unity, with sign yet to be determined [86].

1.1.3 Cosmological perturbation theory

The FRW model is a good description of our universe at large scales, i.e. over $\sim 100\,{\rm Mpc}$ [75], but it clashes with the strong inhomogeneities that the universe presents at smaller scales. To explain such inhomogeneities, the general approach is to resort to the cosmological perturbation theory: we suppose that the present structures have formed starting from very small inhomogeneities that, according to the current model, have formed after an inflation driven expansion era and grew up to building the structures that we now observe [48, 69].

The formation of large scale structures across the Universe is one of the most fascinating and puzzling questions, still opened in theoretical cosmology. Among the long standing problems of this investigation area is the determination of the basic nature and dynamics of the cold dark matter [93], responsible for the gravitational skeleton in which the baryonic matter falls in, forming the radiative component of the present structures.

The standard cosmological perturbation theory, as explained in [18, 69, 80, 86], describes the perturbations both within the Hubble radius through the Newtonian approximation and outside of the Hubble horizon making use of General Relativity. Its main results are the following:

- outside of the Hubble horizon perturbations grow as $\sim t$ in the radiation dominated universe. This is important because, during the inflation era, only these modes are able to survive;
- inside the Hubble horizon and in matter dominated universe, the only perturbations able to grow are the ones with physical scale $\lambda \gg v_s\sqrt{\frac{\pi}{G\rho}}$, where v_s is the sound speed of the cosmological medium.

It should be noted that, at the time of recombination, density fluctuations are expected to be still very small, in relation to the very small temperature fluctuations of the CMB. This way, we can safely make use of a linear perturbation theory.

Newtonian theory of small fluctuations

As far as the scale of perturbations is much smaller than the Hubble radius and the fluid velocity is non-relativistic, we can neglect general relativistic corrections and use the Newtonian theory. This means that the Newtonian theory is valid from the onset of the matter dominated era, when $\rho \gg P \simeq 0$. In the following, $\boldsymbol{v}$ is the fluid velocity and ϕ the gravitational potential. The basic equations that we need are the continuity equation

$$\dot{\rho} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}), \tag{1.27}$$

Euler equation

$$\dot{\boldsymbol{v}} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\phi \tag{1.28}$$

and the gravitational field equation

$$\nabla^2\phi = 4\pi G\rho. \tag{1.29}$$

The background dynamics can be easily derived from eqs. (1.23) and (1.20) and it is

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3 \tag{1.30}$$

$$\boldsymbol{v} = H\boldsymbol{r} \tag{1.31}$$

$$\boldsymbol{\nabla}\phi = \frac{4}{3}\pi G\rho\boldsymbol{r}. \tag{1.32}$$

These are also solutions of the previous equations.

We now apply a linear perturbation to each quantity and we obtain

$$\dot{\rho} + 3H\,\delta\rho + H(\boldsymbol{r}\cdot\boldsymbol{\nabla})\,\delta\rho + \rho\boldsymbol{\nabla}\cdot\delta\boldsymbol{v} = 0 \tag{1.33}$$

$$\dot{\delta\boldsymbol{v}} + H\,\delta v + H(\boldsymbol{r}\cdot\boldsymbol{\nabla})\,\delta\boldsymbol{v} + \frac{1}{\rho}\boldsymbol{\nabla}\,\delta P + \boldsymbol{\nabla}\,\delta\phi = 0 \tag{1.34}$$

$$\nabla^2\,\delta\phi - 4\pi G\,\delta\rho = 0. \tag{1.35}$$

Assuming adiabatic perturbations, the pressure can be expressed through the adiabatic sound speed

$$\delta P = v_S^2\,\delta\rho\,. \tag{1.36}$$

The solutions are expected to be in form of plane waves, so we apply a Fourier transformation to the system

$$\delta x = \widetilde{\delta x}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{1.37}$$

where δx is any perturbed quantity, $\boldsymbol{k}$ is the physical wavenumber and $\boldsymbol{q} = a\boldsymbol{k}$ is the comoving wavenumber, constant during the expansion. From now on, every variable is intended as Fourier transformed. For simplicity of notation, we will drop the tilde over the variables. This leads to the simplified system

$$\dot{\delta\rho} + 3H\,\delta\rho + \mathrm{i}\rho\,(\boldsymbol{k}\cdot\delta\boldsymbol{v}) = 0 \tag{1.38}$$

$$\dot{\delta\boldsymbol{v}} + H\,\delta\boldsymbol{v} + \mathrm{i}\left(\frac{v_S^2}{\rho} - \frac{4\pi G}{k^2}\right)\delta\rho\,\boldsymbol{k} = 0. \tag{1.39}$$

To write the system in a simpler form, we decompose the fluid velocity in its components parallel and orthogonal to the direction of $\boldsymbol{k}$, respectively $\delta v^{\parallel}$ and $\delta\boldsymbol{v}^{\perp}$ (where $\delta\boldsymbol{v}^{\perp}\cdot\boldsymbol{k}\equiv 0$). We then define the variables

$$\delta \equiv \frac{\delta\rho}{\rho}, \qquad \theta \equiv \mathrm{i}(\boldsymbol{k}\cdot\delta\boldsymbol{v}) = \mathrm{i}k\,\delta v^{\parallel} \tag{1.40}$$

to obtain

$$\delta\boldsymbol{v} = \delta v^{\parallel}\,\hat{\boldsymbol{k}} + \delta\boldsymbol{v}^{\perp} \tag{1.41}$$

$$\dot{\delta} + \theta = 0 \tag{1.42}$$

$$\dot{\theta} + 2H\theta - \left(v_S^2 k^2 - 4\pi G\rho\right)\delta = 0. \tag{1.43}$$

From here, we obtain the final equation

$$\ddot{\delta} + 2H\dot{\delta} + \left(v_S^2 k^2 - 4\pi G\rho\right)\delta = 0. \tag{1.44}$$

From here, we need to make explicit the time dependence in the last equation. Restricting our analysis to the flat universe, i.e. assuming $K = 0$, we have

$$a \propto t^{2/3}, \qquad \rho = \frac{1}{6piGt^2}. \tag{1.45}$$

The assumption of flatness, however, is not a big limitation because, for old enough times or far inside the Hubble radius, K is negligible with respect to the other terms in eq. (1.20). On the other hand, for recent times, i.e. when we cannot neglect K, the Jeans mass is so small that its precise value is of little interest.

To see the maximum allowed growth rate of a matter overdensity, we neglect the sound speed and eq. (1.44) becomes

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0. \tag{1.46}$$

This admits the solution

$$\delta(t) = \delta_+ t^{2/3} + \delta_- \frac{1}{t}, \tag{1.47}$$

where δ_+ and δ_- are integration constants.

The general solution involves the explicit time dependence of v_S^2. Given the heat ratio γ, the pressure is $P \propto \rho^\gamma$ and the adiabatic sound speed is

$$v_S^2 = \frac{\delta P}{\delta \rho} = \left(\frac{\partial P}{\partial \rho}\right)_{\text{adiabatic}} \propto \rho^{\gamma-1} \propto t^{2(1-\gamma)} = t^{-2\left(\nu+\frac{1}{3}\right)}, \tag{1.48}$$

where $\nu = \gamma - 4/3 \geq 0$. Eq. (1.44) can be expressed through the constant

$$\Lambda^2 = v_S^2 k^2 t^{2\gamma-2/3} \tag{1.49}$$

as

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} + \left(\frac{\Lambda}{t^{2+2\nu}} - \frac{2}{3t^2}\right)\delta = 0. \tag{1.50}$$

Right after recombination $\nu = 0$ and the independent solutions are

$$\delta_\pm(t) =\propto t^{-\frac{1}{6}\pm\sqrt{\frac{25}{36}-\Lambda^2}}. \tag{1.51}$$

Both solutions follow a damped oscillation for $\Lambda > 5/6$, decay for $5/6 > \Lambda > \sqrt{2/3}$ and it is present a growing mode only if

$$\Lambda^2 < \frac{2}{3} \iff k < k_J = \sqrt{\frac{4\pi G\rho}{v_S^2}}, \tag{1.52}$$

which is the Jeans condition in this regime. On the other hand, for more recent times $\nu > 0$ and the solutions are

$$\delta_\pm(t) \propto t^{-\frac{1}{6}} J_{\mp\frac{5}{6\nu}}\left(\frac{\Lambda}{\nu t^\nu}\right), \tag{1.53}$$

where J is the Bessel function. When $\Lambda t^{-\nu}/\nu \gg 1$ the solutions oscillate, while for $\Lambda t^{-\nu}/\nu \ll 1$ it holds [7]

$$\delta_\pm(t) \propto t^{-\frac{1}{6}\pm\frac{5}{6}}. \tag{1.54}$$

This can be translated in the Jeans condition

$$\frac{\Lambda}{\nu t^\nu} \ll 1 \iff k \ll k_J = \sqrt{\frac{6\pi G\nu^2\rho}{v_S^2}}. \tag{1.55}$$

General relativistic theory of small fluctuations

As before, we assume a flat universe with $K = 0$. The background metric is given by eq. (1.1) with Cartesian spatial coordinates, i.e.

$$g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2). \tag{1.56}$$

The only non-vanishing Christoffel symbols are[1]

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij}, \qquad \Gamma^k_{0i} = \frac{\dot{a}}{a}{g_i}^k \tag{1.57}$$

[1] δ_{ab} is the Kronecker delta. We remind that we use Latin characters for spatial only indices, and Greek characters for 4-dimensional ones.

and the non-vanishing components of the Ricci tensor are

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{ij} = \left(2\dot{a}^2 + a\ddot{a}\right)\delta_{ij}. \tag{1.58}$$

The stress-energy tensor is given by eq. (1.14). The background quantities are

$$a \propto t^{\frac{2}{3(1+w)}}, \qquad H = \frac{2}{3(1+w)}, \qquad \rho = 3H^2. \tag{1.59}$$

We are now going to apply a linear perturbation to every physical quantity. It will be important in the following to be able to distinguish between physical perturbations and mathematical artefacts due to the change in the coordinate system. Following the same scheme as of [86], we make a generic coordinate transformation of the form

$$x^\mu \to x^\mu + \epsilon^\mu \tag{1.60}$$

with small ϵ^μ and we keep terms up to $\mathcal{O}(\epsilon)$. The metric tensor becomes

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - g_{\mu\sigma}(x)\partial_\nu \epsilon^\sigma - g_{\rho\nu}(x)\partial_\mu \epsilon^\rho. \tag{1.61}$$

Here, x and x' correspond to the same physical point in the different reference frames. However, we are interested in the change in the value of $g_{\mu\nu}$ when evaluated at the same coordinate point x, which corresponds to two different physical points. We define

$$\begin{aligned} \Delta g_{\mu\nu} &= g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -g_{\mu\lambda}(x)\partial_\nu \epsilon^\lambda - g_{\lambda\nu}(x)\partial_\mu \epsilon^\lambda - \epsilon^\lambda \partial_\lambda g_{\mu\nu}(x) \\ &= -\nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu \end{aligned} \tag{1.62}$$

and, in the same way,

$$\begin{aligned} \Delta T_{\mu\nu} &= T'_{\mu\nu}(x) - T_{\mu\nu}(x) = -T_{\mu\lambda}\partial_\nu \epsilon^\lambda - T_{\lambda\nu}\partial_\mu \epsilon^\lambda - \epsilon^\lambda \partial_\lambda T_{\mu\nu} \\ &= -T_{\mu\lambda}\nabla_\nu \epsilon^\lambda - T_{\lambda\nu}\nabla_\mu \epsilon^\lambda - \epsilon^\lambda \nabla_\lambda T_{\mu\nu}. \end{aligned} \tag{1.63}$$

If we take the functions ϵ^μ of the same order of the perturbations, then the transformations given by eqs. (1.62) and (1.63) can be seen both as gauge transformations and as transformations of the perturbations $\delta g_{\mu\nu}$ and $\delta T_{\mu\nu}$ within fixed gauge: in the latter case those equations give the values of $\Delta\,\delta g_{\mu\nu}$ and $\Delta\,\delta T_{\mu\nu}$.

We will derive a linear system in the perturbed variables, so the gauge transformations will solve our equations. We call these solutions gauge perturbations or gauge modes: they are not physical because they correspond to a simple change in the reference frame. We will be looking for physical solutions for the time dependence of $\delta\rho$ so the most interesting gauge transformation is the one related to $\Delta\,\delta\rho$.

We can use the gauge freedom fo fix the synchronous gauge also at the perturbation order. The perturbed metric tensor is

$$g_{\mu\nu} = g^{\mathrm{B}}_{\mu\nu} + \delta g_{\mu\nu}\,, \tag{1.64}$$

where B means that it is the background value. With an appropriate choice of the functions ϵ^μ, we can impose the conditions

$$\delta g_{\mu 0} = 0. \tag{1.65}$$

If we fix this way the synchronous gauge and perform another gauge transformation, in order to preserve the conditions (1.65) we need $\Delta g_{0\mu} = 0$ and we are subject to the constraints

$$\epsilon^0 = \epsilon^0(x^j) \tag{1.66a}$$

$$\epsilon^i = \tilde{\epsilon}^i(x^j) + \partial^i \epsilon^0(x^j)\, a^2 \int \frac{\mathrm{d}t}{a^2}, \tag{1.66b}$$

where $\epsilon_0(x^j)$ and $\tilde{\epsilon}^i(x^j)$ are arbitrary functions of the spatial coordinates: we still have 4 unused degrees of freedom. Performing this second transformation and substituting the explicit expression of $T_{\mu\nu}$ in eq. (1.63) we find

$$\Delta\, \delta\rho = -\epsilon^0 \dot{\rho}^{\mathrm{B}} = 3H(\rho^{\mathrm{B}} + p^{\mathrm{B}})\epsilon^0 = 3H(1+w)\rho^{\mathrm{B}}\epsilon^0. \tag{1.67}$$

The last equation means that the gauge mode will be

$$\delta\rho^{\mathrm{gauge}} \propto \frac{1}{t}\rho^{\mathrm{B}}; \tag{1.68}$$

it should be present between the solutions and it should be discarded.

We are now ready to write the perturbed equations. We define

$$\gamma_{\mu\nu} = \delta g_{\mu\nu}\,, \qquad g_{\mu\rho} g^{\rho\nu} = \delta_\mu{}^\nu \implies \delta g^{\mu\nu} = -\gamma^{\mu\nu}, \tag{1.69}$$

where the indices of $\gamma_{\mu\nu}$ are raised and lowered with the unperturbed metric $g^{\mathrm{B}}_{\mu\nu}$. In the following we write the trace of $\gamma_{\mu\nu}$ as $\gamma = \gamma_k{}^k$. Accordingly to [18, 92] the perturbed Christoffel symbols are

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}_{\mathrm{B}} \left(\nabla^{\mathrm{B}}_\mu \gamma_{\nu\sigma} + \nabla^{\mathrm{B}}_\nu \gamma_{\mu\sigma} - \nabla^{\mathrm{B}}_\sigma \gamma_{\mu\nu} \right) \tag{1.70}$$

and the perturbed Ricci tensor is

$$\delta R_{\mu\nu} = \nabla^{\mathrm{B}}_\rho\, \delta\Gamma^\rho_{\mu\nu} - \nabla^{\mathrm{B}}_\nu\, \delta\Gamma^\rho_{\mu\rho}\,. \tag{1.71}$$

This means that the Christoffel symbols are

$$\delta\Gamma^0_{00} = 0, \qquad \delta\Gamma^0_{i0} = 0, \qquad \delta\Gamma^k_{00} = 0 \tag{1.72a}$$

$$\delta\Gamma^0_{ij} = \frac{1}{2}\dot{\gamma}_{ij} \tag{1.72b}$$

$$\delta\Gamma^k_{i0} = \frac{1}{2}\dot{\gamma}_i{}^k \tag{1.72c}$$

$$\delta\Gamma^k_{ij} = \frac{1}{2}\left(\partial_i \gamma_j{}^k + \partial_j \gamma_i{}^k - \partial^k \gamma_{ij} \right) \tag{1.72d}$$

and we define

$$\delta\Gamma_\mu \equiv \delta\Gamma^\nu_{\mu\nu} = \frac{1}{2}\partial_\mu \gamma. \tag{1.73}$$

We will need only one component of the variation of the Ricci tensor, namely

$$\delta R_{00} = \nabla^{\mathrm{B}}_\rho\, \delta\Gamma^\rho_{00} - \nabla^{\mathrm{B}}_0\, \delta\Gamma_0 = -\frac{1}{2}\ddot{\gamma} - \frac{\dot{a}}{a}\dot{\gamma}. \tag{1.74}$$

The fluid velocity perturbation is δu^μ, with

$$u_\mu u^\mu = -1 \implies \delta u^0 = 0. \tag{1.75}$$

The fluid energy perturbation is $\delta\rho$ and the fluid pressure perturbation is $\delta P = v_S^2\,\delta\rho$; from the equation of state (1.15) we have

$$\dot{w} = -3H(1+w)\left(v_S^2 - w\right) \tag{1.76a}$$

$$w = \text{const} \implies v_S^2 = w. \tag{1.76b}$$

If we write the perturbed Einstein equations in the form

$$\delta R_{\mu\nu} = \delta T_{\mu\nu} - \frac{1}{2}\,\delta(g_{\mu\nu}T)\,, \tag{1.77}$$

then the 00 component is

$$\ddot{\gamma} + 2H\dot{\gamma} + \delta\rho + 3\,\delta P = 0. \tag{1.78}$$

The perturbed Bianchi identities give the fluid energy conservation

$$\dot{\delta\rho} + 3H(\delta\rho + \delta P) + (\rho^{\mathrm{B}} + P^{\mathrm{B}})\left(\partial_i\,\delta u^i + \frac{1}{2}\dot{\gamma}\right) = 0 \tag{1.79}$$

and the conservation of the momentum

$$(\rho^{\mathrm{B}} + P^{\mathrm{B}})\left(\partial_0\partial_i\,\delta u^i + 2H\partial_i\,\delta u^i\right) + \partial_i\,\delta u^i\,\dot{P}^{\mathrm{B}} + \partial_P\partial^P\,\delta P = 0. \tag{1.80}$$

Defining the new variable to describe the perturbations

$$\Delta \equiv \frac{\delta\rho}{\rho^{\mathrm{B}} + P^{\mathrm{B}}}, \tag{1.81}$$

eq. (1.79) becomes

$$\dot{\Delta} + \frac{1}{2}\dot{\gamma} + \partial_i\,\delta u^i\,. \tag{1.82}$$

We now use it to replace $\dot{\gamma}$ in the other equations and we also express P through the equation of state, v_S^2 and $\delta\rho$. The other equations become

$$\ddot{\Delta} + 2H\dot{\Delta} - \frac{1}{2}(1+3v_S^2)(1+w)\rho\Delta + \partial_0\partial_i\,\delta u^i + 2H\partial_i\,\delta u^i = 0 \tag{1.83}$$

$$\partial_0\partial_i\,\delta u^i + (2-3w)H\partial_i\,\delta u^i + v_S^2\partial_i\partial^i\Delta = 0. \tag{1.84}$$

We solve this system in radiation dominated universe, with $w = v_S^2 = 1/3$. We perform a Fourier transformation of the functions through

$$\delta x = \widetilde{\delta x}(t)\mathrm{e}^{\mathrm{i}k_i x^i}, \qquad k_i = \text{const}, \tag{1.85}$$

and as before we drop the tilde for simplicity. The background variables are

$$a = \left(\frac{t}{t_0}\right)^{1/2}, \qquad H = \frac{1}{2t}, \qquad \rho = \frac{3}{4t^2} \tag{1.86}$$

and the wavenumber behaves as

$$k^2 = k_i k^i = \frac{q^2}{t}, \qquad q \equiv \sqrt{a^2 k^2} = \text{const}\,. \tag{1.87}$$

After some algebra, eqs. (1.83) and (1.84) combine to give

$$\dddot{\Delta} + \frac{5}{2}\frac{\ddot{\Delta}}{t} - \left(\frac{1}{2t^2} + \frac{q^2}{3t}\right)\dot{\Delta} + \left(\frac{1}{2t^3} - \frac{q^2}{3t^2}\right)\Delta = 0. \tag{1.88}$$

This last equation can be easily solved in the large scale limit, i.e. for $k^2 \to 0$, giving

$$\Delta(t) = \delta_1 t + \delta_{1/2}\sqrt{t} + \delta_{-1}\frac{1}{t}, \tag{1.89}$$

where δ_1, $\delta_{1/2}$ and δ_{-1} are arbitrary constants. We can easily identify $1/t$ as the gauge mode of eq. (1.68). It can be shown that the solution $\propto \sqrt{t}$ is proportional to the fluid velocity divergence, i.e. if $\partial_i\,\delta u^i \neq 0$, then its behaviour and the relative solution of Δ are

$$\partial_i\,\delta u^i = u_0\sqrt{\frac{t}{t_0}}, \qquad \Delta = \frac{2}{3}t_0 u_0\sqrt{\frac{t}{t_0}}; \tag{1.90}$$

we should note that this solution vanishes in the large scale limit because $\partial_i\,\delta u^i = k_i\,\delta u^i \propto \sqrt{k^2}$, yet we will need it in one of the next chapters. The only remaining solution gives the perturbation growth rate, that is $\Delta \propto t$.

1.2 Bianchi models

The first natural extension of the FRW model are the Bianchi models, where the homogeneity hypothesis is still valid, but we give up the assumption of isotropy [86, 92]. Homogeneity means invariance under spatial transformations, i.e.

$$\mathrm{d}l^2 = h_{ij}(t,x)\,\mathrm{d}x^i\,\mathrm{d}x^j = h_{ij}(t,x')\,\mathrm{d}x'^i\,\mathrm{d}x'^j\,, \tag{1.91}$$

where h_{ij} has the same form. Using the triadic representation we have

$$h_{ij} = \eta_{ab}e^a_i e^b_j, \qquad \eta_{ab} = \eta_{ab}(t). \tag{1.92}$$

Now we have the invariant line element $e^a_i\,\mathrm{d}x^i$, so

$$e^a_i(x)\,\mathrm{d}x^i = e^a_i(x')\,\mathrm{d}x'^i \tag{1.93}$$

and so

$$\frac{\partial x'^j}{\partial x^i} = e^j_a(x')e^a_i(x), \qquad e^a_i e^j_a = \delta^j_i, \qquad e^a_i e^i_b = \delta^a_b. \tag{1.94}$$

The Schwartz condition leads to

$$\left[\frac{\partial e^j_a(x')}{\partial x'^l}e^l_b(x') - \frac{\partial e^j_b(x')}{\partial x'^l}e^l_a(x')\right]e^b_k(x)e^a_i(x) = e^j_a(x')\left[\frac{\partial e^a_k(x)}{\partial x^i} - \frac{\partial e^a_i(x)}{\partial x^k}\right]. \tag{1.95}$$

Multiplying last eq. by $e^i_d(x)e^k_c(x)e^f_j(x')$ we find

$$e^f_j(x')\left[\frac{\partial e^j_d(x')}{\partial x'^l}e^l_c(x')-\frac{\partial e^j_c(x')}{\partial x'^l}e^l_d(x')\right]=e^j_c(x')e^l_d(x')\left[\frac{\partial e^f_j(x')}{\partial x'^l}-\frac{\partial e^f_l(x')}{\partial x'^k}\right].\tag{1.96}$$

Since both sides are equal and x and x' generic, both sides must be equal to a constant

$$\left(\frac{\partial e^c_i}{\partial x^j}-\frac{\partial e^c_j}{\partial x^i}\right)e^i_a e^j_b = C^c{}_{ab},\tag{1.97}$$

where the $C^c{}_{ab}$ are the structure constants of the group. Multiplying last equation by e^k_c we finally have

$$e^a_i\frac{\partial e^k_b}{\partial x^i}-e^b_j\frac{\partial e^k_a}{\partial x^j}=C^c{}_{ab}e^k_c.\tag{1.98}$$

This is a constraint that must be satisfied. Moreover, by construction we have

$$C^c{}_{ab}=-C^c{}_{ba}\tag{1.99}$$

and the Jacobi identity gives

$$C^f{}_{ab}C^d{}_{cf}+C^f{}_{bc}C^d{}_{af}+C^f{}_{ca}C^d{}_{bf}=0.\tag{1.100}$$

Given these relations, the structure constants can be written as

$$C^a{}_{bc}=\epsilon_{bcd}n^{da}+\delta^a_c a_b-\delta^a_b a_c,\tag{1.101}$$

where ϵ_{bcd} is the totally antisymmetric tensor, or equivalently as

$$C^{ab}=n^{ab}+\epsilon^{abc}a_c,\tag{1.102}$$

where $n^{ab}=n^{ba}$ and $a_a=C^b{}_{ba}$. The Jacobi identity reads

$$n^{ab}a_b=0.\tag{1.103}$$

A particular homogeneous model is identified by its a_c and n^{ab}. Without loss of generality we can write $a_c=(a,0,0)$ and $n^{ab}=\mathrm{diag}(n_1,n_2,n_3)$. This classification leads to the nine different Bianchi models [86, 92].

The Bianchi models are much more general than the FRW one. Bianchi I is similar to FRW, but it has 3 different scale factors, two of which increasing and one decreasing with time; this is the Kasner solution. Bianchi I describes two connected Kasner epochs, where a Kasner epoch is the time in which a Kasner dynamics take place: there is an exchange in the Kasner indices. The more general models are Bianchi VIII and Bianchi IX, which describe an oscillatory regime known as mixmaster [13]. The presence of matter, however, causes the isotropization of the models [68, 86] and recovers the FRW dynamics for long times, while towards the initial singularity the matter contribution is negligible.

It is possible to show that, through the Bianchi IX model, it is possible to build a general cosmological solution [86]. The Bianchi models overcome the problems related to the limits of FRW towards the initial singularity, while recovering the correct behaviour at present times: Bianchi I, V and IX tend towards FRW with the 3 possible curvatures.

Chapter 2

Anisotropic cosmological perturbations in presence of a magnetic field

In this chapter, we study the effect of a uniform magnetic field on the growth of cosmological perturbations.

Following [107, 114], we develop a mathematical consistent treatment in which a perfect fluid and a uniform magnetic field evolve together in a Bianchi I universe. Then, we study the energy density perturbations on this background with particular emphasis on the effect of the background magnetic field.

We develop a full relativistic solution which refines previous analysis in the relativistic limit [65, 77], recovers the known ones in the Newtonian treatment with adiabatic sound speed [89], and it adds anisotropic effects to the relativistic ones for perturbations with wavelength within the Hubble horizon. This represents a refined approach on the perturbation theory of an isotropic universe in GR, since most of the present studies deal with fully isotropic systems.

The chapter is structured as follows. In section 2.1 we introduce the importance of magnetic fields, together with the motivations and a summary of the contents of this analysis. In section 2.2 we give a brief motivation of the presence of such fields, while in section 2.3 we make a quick overview of the properties of the Bianchi models as found in present literature, with particular focus on the effects caused by the presence of magnetic fields. Then, section 2.4 presents the main formalism we use for the description of the magnetic field, in order to account properly for the ideal MHD assumption.

In section 2.5, we analyse the background cosmological model, improving the analysis carried out in [55]: this is the first achievement of this chapter. Then, in section 2.6, we present the main framework of our analysis and we derive the starting equations.

We start the main part of our analysis looking for the gauge modes, that would appear as fake solutions, in section 2.7. We are finally able to solve our system analytically in section 2.8 and numerically in section 2.9. Throughout all of this, we compare our results with the main relevant ones in literature. In short, we enhance the analysis of [77] by adding the anisotropic effects, and the one of [65] with a better handling of the coupling between density perturbations and background anisotropy; we finally confirm the validity of the results of [89] in matter dominated universe, when the perturbations have a characteristic scale much smaller than the Hubble horizon, without the addition of any general relativistic correction. Doing this, we present a coherent set of equations that can handle the evolution of magnetized perturbations since they become classical and until they enter non-linear regime.

Finally, we summarize our results in section 2.10, together with the future possible improvements.

2.1 Cosmological perturbations and magnetic fields

The formation of large scale structures across the universe is one of the most fascinating and puzzling questions, still opened in theoretical cosmology. Among the long standing problems of this investigation area is the determination of the basic nature and dynamics of the cold dark matter [93], responsible for the gravitational skeleton on which the baryonic matter falls in, forming the radiative component of the present structures.

However, also the peculiarity of the matter distribution across the universe, in particular the possibility for large scale filaments [99], as well as hypotheses for structure fractal dimension [82, 83] call attention for a deeper comprehension.

In this respect, we observe that the universe plasma nature, both before the hydrogen recombination and, for a part in 10^5 also in the later matter dominated era [72, 89], has to be taken into account.

At the recombination the universe Debye length is of the order of $10\,\mathrm{cm}$ and therefore the implementation of a fluid theory, like general relativistic magneto-hydrodynamics is to be regarded as a valid and viable approach to treat the influence of the primordial magnetic field [73] on the evolution of perturbations [89]. Nonetheless, the smallness of such magnetic field, as constrained by the Cosmic Microwave Background Radiation (CMBR) up to $10^{-9}\,\mathrm{G}$ [54–56, 85, 87, 95, 96], significantly limits the impact of the plasma nature of the cosmological fluid on the evolution of perturbations. As shown in [89, 104], the presence of the magnetic field is able to trigger anisotropy in the linear perturbations growth and it can be inferred that in the full non-linear regime, such anisotropy grows up to account for the formation of large scale filaments.

Apparently, a weak point in the perspective traced above consists of the small plasma component surviving when the hydrogen recombines and in the observation that the most relevant cosmological scales enter the non-linear regime in such a neutral universe. Instead, it can be surprisingly demonstrated [72, 89, 104] that the coupling between the neutral and ionized matter is very strong at spatial scale of cosmological interest (for overdensities of mass greater than 10^6 solar masses, the ambipolar Reynold number is much greater than unity for redshift $10 < z < 1000$). Thus, the dynamical features, for instance anisotropy, that we recover for the plasma component clearly concern the universe baryonic component too. This statement is not affected by the presence of dark matter gravitational skeleton in formation, simply because the radiation pressure prevents, up to $z \sim 100$ the real fall down of the baryonic fluid into the gravitational well. In fact, the large photon to baryon ratio, about 10^9 (also constant during the universe evolution), maintains active a strong Thomson scattering process, even after the hydrogen is recombined into atoms [18, 48, 72, 80, 89].

These considerations are meant to underline that a single fluid general relativistic magneto-hydrodynamics formulation is an appropriate tool to investigate the impact of the universe plasma features on structure formation, at least for a large range of the cosmological thermal history.

In this context many works have been developed, mainly assuming as negligible the backreaction of the magnetic field on the isotropic universe, see [77] and references therein. However, the presence of a magnetic field rigorously violates the isotropy of the space and the (essentially) flat Robertson-Walker geometry must be replaced by a Bianchi I model. This chapter faces the general question of how the linear perturbations evolve on a background Bianchi I cosmology, thought as a weak perturbation of the isotropic case.

We discuss in detail the structure of the perturbation equations in the synchronous gauge and the specific form of the spectrum time dependence in specific important limits, like the large scale limit, when the dependence on the wavenumber can be suppressed, and the sub-horizon limit, when the dependence on the wavenumber is dominant.

Furthermore, the change of the Jeans scale, when passing from the ionized to the (essentially) recombined universe, is determined for the small scales, shedding light on the role of the magnetic field and on the real nature of the gauge perturbations.

We recover the slowing-down of the growing mode in super-horizon scales, long known in FRW models. This effect is very small given the upper limits on the cosmological magnetic fields, of order $\mathcal{O}(v_A^2) \ll 1$. At sub-horizon scales, we generalise the solutions of [97] and [105], which in turn generalise the results of [77]. While they consider random (i.e. isotropic) magnetic fields to preserve the FRW model, we work in the anisotropic case and also consider a non-vanishing sound speed.

Finally, we stress that, along the whole analysis, we compare our results with previous achievements in literature, providing a significant contribution to the understanding of the different effects that the universe anisotropy, due to the magnetic field, induces on the perturbation evolution and stability.

We notice that there is another paper about this matter [65], which was the first analytical study to address this issue. There, the authors study the model in 3 different physical limits with specific anisotropies, while we completely relate the background anisotropy to the magnetic field.

2.2 Magnetic fields in cosmology

There are numerous and different measurements of magnetic fields in the universe, making use of Zeeman effect, synchrotron radiation and Faraday rotation [49, 57, 67, 73]. However, the origin of such magnetic fields in galaxies and galaxy clusters is still a mystery (see [70] for a review).

Ordered magnetic fields of about 1–10 kpc over galactic scales are fairly common in spiral galaxies disks and halos [49, 70], together with smaller scale fields. Moreover, magnetic fields appear to be a common property of the intra-cluster medium of clusters of galaxies, where in the central region they can reach the strength of about 40 µG with coherence scales of 10^1–10^2 kpc [47, 54, 67]. Probably the most interesting for the purposes of this thesis, microgauss magnetic fields have been observed in the intra-cluster medium of a number of rich clusters, with coherence length comparable to the scale of the cluster [49, 70] of a few Mpc. However, there are no detections of purely cosmological fields (i.e. fields not associated with gravitationally bound or collapsing structures).

In particular, galaxy cluster's magnetic fields are typically observed through Faraday rotation of cosmic ray electrons. Generally, Faraday rotation is the most versatile way of measuring cosmological magnetic fields at most scales, while the Zeeman effect is mainly used for molecular clouds and synchrotron radiation is an effective tool for nebulae and galaxies.

On the other hand, the isotropy observed in the present universe prohibits the presence of a strong large scale (over the Hubble horizon) magnetic field. However, a cosmological uniform magnetic field could still be present at cosmological scales if weak enough. Specifically, CMB observations constraint a maximum allowed intensity of 10^{-9} G for present magnetic fields, for both cosmological homogeneous magnetic fields and stochastic magnetic fields with a scale of 1 Mpc. Such limits derive from the expected effects of the primordial magnetic fields on the thermal spectrum of the CMB and its temperature anisotropies [55, 56, 87, 96] and on the CMB power spectra temperature polarization correlation [54, 60, 85, 95, 96], related to Faraday rotation. Other less strong limits of 10^{-7} G are related to the Helium-4 nucleosynthesis [55, 56]. A thorough review of these constraints is present in [96].

While weaker constraints are expected for smaller scale primordial fields, they are of far lesser physical interest and they are usually not taken into account.

The invocation of protogalactic dynamos to explain the magnitude of the observed fields involves many uncertain assumptions but still requires a small primordial (pregalactic) seed field [35]. Hence the possibility of a primordial field merits serious consideration.

A large scale magnetic field, comparable to the constraints and with an amplitude close to the upper limit, may well be of cosmological origin. A similar pregalactic (or protogalactic) field strength is inferred from the detection of fields of order 10^{-6} G in high redshift galaxies [43] and in damped Lyman alpha clouds [45], where the observed fields are likely to have been adiabatically amplified during protogalactic collapse.

Moreover, the potential existence of a primordial magnetic field is also consistent with observations of clusters of galaxies. As already pointed out, Faraday rotation measurements of radio sources inside and behind clusters indicate strong magnetic fields in many of them [39, 47]. The detected cluster fields have a typical magnitude of a few µG and a coherence lengths from 10^1–10^2 kpc up to a few Mpc. The cores of several clusters contain tangled magnetic fields with amplitudes as high as 10^1–10^2 µG [31, 46, 47]. In the outer halos of clusters, lower limits $\gtrsim 0.1$ µG were set on the field amplitude, by combining measurements of synchrotron radio-emission from relativistic electrons in these halos together with lower limits on the associated hard X-ray emission due to Comptonization of the microwave background [24, 33].

Therefore, we must consider such primordial magnetic fields when dealing with the cosmological perturbation theory.

2.3 General properties of the Bianchi models in presence of a magnetic field

As we already said, it is impossible to accommodate a magnetic field in an isotropic model. Moreover, although present observations show that the isotropic FRW model describes very well the present universe, it is only a very special description of the universe towards the initial singularity, while the general one should incorporate anisotropy [27, 92].

In the first stage of the evolution of the universe the matter contribution is negligible, while it is necessary to have an isotropic matter field to achieve the isotropization of the model [28, 86]. The general solution is constructed through the Bianchi VIII and IX models [27, 86, 92], but we will focus for simplicity on a single Kasner era and so we will use a Bianchi I model.

The Bianchi I model is similar to the FRW one, but with three different scale factors. It is intrinsically anisotropic in vacuum, i.e. the three cosmic scale factors are never all equal; moreover, in vacuum one of the three scale factor always decreases with time, meaning that one of the spatial direction is contracting.

Near enough to the cosmological singularity, any matter source in the form of perfect fluid energy density, having equation of state $p = w\rho$ always behaves as a test fluid, i.e. it induces negligible backreaction, as far a $0 < w < 1$. Since the background magnetic field energy density is a radiation-like term in the Universe and it is associated to an equation of state $p = \rho/3$, near enough to the singularity, we can expect a typical vacuum solution of the Kasner form [86, 92].

The more general Bianchi IX model can be described as a succession of Kasner epochs, in which the different directions exchange time evolutions, alternating moments of growing and decreasing [86]. For more detailed informations regarding the Bianchi models we recommend [23].

Clearly, as soon as the Universe expands enough, the matter source can no longer be negligible and, if the pressure term is isotropic, the solution must correspondingly isotropize, i.e. the three scale factors tend to be equivalent. This process of isotropization is particularly efficient in the case of an inflationary paradigm [68, 86], when a vacuum energy, having an equation of state $p = -\rho$ is dominating the Universe dynamics.

The relevance of our study for the structure formation takes place when the isotropization process reduced the Bianchi I cosmology to a flat Robertson-Walker Universe, except for the residual intrinsic anisotropy due to the presence of a background magnetic field.

There exist already a large number of studies regarding Bianchi I models, analysing cases with different values for the barotropic index w of the matter source in addition to the magnetic field. [58] was probably the first to address their stability. [28] studies the effect of a pure magnetic matter component, [10] contains analytic solutions for dust $w = 0$ and radiation $w = 1/3$, [12] contains solutions for $w = 1$ and $1/3 \leq w \leq 1$ and for the pure magnetic case, [78] analyses the case of vacuum energy $w = -1$. The nature of the solutions depends on the values of various constants, it can collapse isotropically or anisotropically, only in the longitudinal or in the transverse direction towards the Big Bang. In general the magnetic fields accelerates expansion (or decelerates collapse) in the transverse direction of the magnetic pressure and it decelerates expansion (or accelerates collapse) in the direction of the magnetic tension. For general properties of the solutions, see [71].

Some interesting cases are analysed in [10]: if $B^2/\rho \to 0$ towards the singularity then the magnetic field effects are negligible; if B^2/ρ does not approach 0, then it is constant and both fluids determine the dynamics, or the magnetic field causes a rapid expansion in the transverse direction and this change of the dynamics causes $B^2/\rho \to 0$. Moreover, [78] shows that in presence of a cosmological constant the magnetic field has a strong effect at early times, decelerating the collapse in the transverse direction and accelerating it in the longitudinal one, and it is negligible at later times, when the vacuum energy causes accelerated expansion in both directions; the authors also describe the shape of the singularity.

It should be noted that in general the presence of the magnetic field causes a slowing down in the process of isotropization, making the shear more important; this way the CMB gives a strong constraint on primordial homogeneous magnetic fields [55, 56].

2.4 1+3 covariant formalism

The correct handling of magnetic fields in cosmology is something both necessary and very difficult from a mathematical point of view. The ease of change of reference frame in general relativity could cause a transformation of electric and magnetic fields through the Maxwell equations that could lead to unintuitive or wrong results, if underestimated. We think the 1+3 covariant formalism to be the best and safest way to decompose the electromagnetic field in its electric and magnetic components, and to correctly define their respective equations. Here, we recap the main features and equations of the formalism, mainly following [77, 79]. A more detailed description can be found in a number of review articles [4, 15, 62]

The covariant approach to general relativity dates back to the 1950s [2, 3], and since then it has been used in numerous applications by many authors [15, 19, 62]. The basic idea is to use the kinematic quantities of the cosmological fluid, its energy-momentum tensor and the gravito-electromagnetic parts of the Weyl tensor instead of the metric. The key equations are the Ricci and Bianchi identities applied to the fluid 4-velocity, while Einstein's equations are incorporated via algebraic relations between the Ricci and the energy-momentum tensors. From the next section, we will use the fundamental definitions and equations of this formalism, but not the formalism itself for reasons we will explain there. Instead, we will still follow the traditional approach of solving the system through the Einstein's equations and the stress-energy tensor conservation laws. For this reason, we will present only the concepts we will need later leaving a more complete review to the before mentioned papers.

2.4.1 1+3 covariant decomposition and covariant kinematics

The basis of the 1+3 decomposition is to locally split the spacetime in a way comoving with the cosmological fluid. Consider a general spacetime with Lorentzian metric $g_\mu\nu$ of signature $(-,+,+,+)$ and introduce a family of fundamental observers along a timelike congruence of worldlines tangent to the 4-velocity of the fluid

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}, \qquad u^\mu u_\mu = -1, \tag{2.1}$$

where t is the observers' proper time. Such velocity introduces naturally the required splitting, where the vector u^μ determines the time direction and the tensor

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \qquad h_{\mu\nu}u^\mu = 0 \tag{2.2}$$

projects orthogonally to the 4-velocity into the observer's instantaneous rest space. Moreover, in absence of rotation the vector field u^μ is hypersurface-orthogonal and $h_{\mu\nu}$ is the 3-metric of the spatial surfaces.

Through u^μ and $h_{\mu\nu}$ it is possible to define the time derivative

$$\dot{S}_{\mu\nu\dots}{}^{\rho\sigma\dots} \equiv u^\alpha \nabla_\alpha S_{\mu\nu\dots}{}^{\rho\sigma\dots} \tag{2.3}$$

and the orthogonally projected gradient

$$\mathrm{D}_\lambda \, S_{\mu\nu\dots}{}^{\rho\sigma\dots} \equiv h_\lambda{}^\alpha h_\mu{}^\beta h_\nu{}^\gamma h^\rho{}_\delta h^\sigma{}_\epsilon \dots \nabla_\alpha S_{\beta\gamma\dots}{}^{\delta\epsilon\dots} \tag{2.4}$$

of any given tensor $S_{\mu\nu\dots}{}^{\rho\sigma\dots}$. Moreover, u^μ and $h_{\mu\nu}$ allow a unique decomposition of every spacetime quantity into its irreducible timelike and spacelike components. There hold the important relations

$$\mathrm{D}_\rho \, h_{\mu\nu} = 0 \tag{2.5}$$

$$\dot{h}_{\mu\nu} = 2u_{(\mu}A_{\nu)}, \tag{2.6}$$

where A_μ is the fluid 4-acceleration defined as

$$A_\mu \equiv \dot{u}_\mu = u^\alpha \nabla_\alpha u_\mu. \tag{2.7}$$

The spacetime volume element is given by the covariantly constant tensor

$$\eta_{\mu\nu\rho\sigma}, \qquad \eta^{0123} = \frac{1}{\sqrt{-g}}, \qquad \eta_{\mu\nu\rho\sigma}\eta^{\alpha\beta\gamma\delta} = -4!\delta_{[\mu}{}^\alpha \delta_\nu{}^\beta \delta_\rho{}^\gamma \delta_{\sigma]}{}^\delta. \tag{2.8}$$

Contracting it along the time direction it is possible to construct the effective volume element in the observers' instantaneous rest space

$$\varepsilon_{\mu\nu\rho} = \eta_{\mu\nu\rho\alpha} u^\alpha, \tag{2.9}$$

for which there hold the relations

$$\varepsilon_{\mu\nu\rho} u^\mu = 0 \tag{2.10a}$$

$$\eta_{\mu\nu\rho\sigma} = 2u_{[\mu}\varepsilon_{\nu\rho\sigma]} - 2\varepsilon_{\mu\nu[\rho}u_{\sigma]} \tag{2.10b}$$

$$\varepsilon_{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} = 3!\delta_{[\mu}{}^\alpha \delta_\nu{}^\beta \delta_{\rho]}{}^\gamma \tag{2.10c}$$

$$\mathrm{D}_\sigma \, \varepsilon_{\mu\nu\rho} = 0. \tag{2.10d}$$

The time derivative of the spatial antisymmetric tensor is [64]

$$\dot{\varepsilon}_{\mu\nu\rho} = 3u_{[\mu}\varepsilon_{\nu\rho]\sigma}A^{\sigma}. \tag{2.11}$$

Here comes the crucial point of the 1+3 decomposition. The skew part of a projected rank-2 tensor is spatially dual to the projected vector

$$S_{\mu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho}S^{[\nu\rho]}, \tag{2.12}$$

and any projected rank-2 tensor has the irreducible covariant decomposition[1]

$$S_{\mu\nu} = \frac{1}{3}Sh_{\mu\nu} + S_{\langle\mu\nu\rangle} + \varepsilon_{\mu\nu\rho}S^{\rho}, \qquad S = S_{\mu\nu}h^{\mu\nu}. \tag{2.14}$$

This decomposition is readily applied to a general imperfect fluid which, with respect to the fundamental observers, decomposes as

$$T_{\mu\nu} = \rho u_{\mu}u_{\nu} + Ph_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \pi_{\mu\nu} \tag{2.15a}$$
$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} \tag{2.15b}$$
$$P = \frac{1}{3}T_{\mu\nu}h^{\mu\nu} \tag{2.15c}$$
$$q_{\mu} = -h_{\mu}{}^{\nu}T_{\nu\rho}u^{\rho} \tag{2.15d}$$
$$\pi_{\mu\nu} = T_{\langle\mu\nu\rangle}, \tag{2.15e}$$

where ρ is the matter energy density, P is the effective isotropic pressure, i.e. the sum between equilibrium pressure and bulk viscosity, q_{μ} is the total energy-flux vector and $\pi_{\mu\nu}$ is the symmetric and trace-free tensor that describes the anisotropic pressure. Eqs. (2.15) can describe any type of matter.

Through the identity $R = 4\Lambda - T$, Einstein's equations can be recast as

$$R_{\mu\nu}u^{\mu}u^{\nu} = \frac{1}{2}(\rho + 3P) - \Lambda \tag{2.16a}$$
$$h_{\mu}{}^{\nu}R_{\nu\rho}u^{\rho} = -q_{\mu} \tag{2.16b}$$
$$h_{\mu}{}^{\rho}h_{\nu}{}^{\sigma}R_{\rho\sigma} = \frac{1}{2}(\rho - P)h_{\mu\nu} + \Lambda h_{\mu\nu} + \pi_{\mu\nu}. \tag{2.16c}$$

The irreducible decomposition of the 4-velocity gradient is

$$\nabla_{\nu}u_{\mu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - A_{\mu}u_{\nu} \tag{2.17a}$$
$$\sigma_{\mu\nu} \equiv \mathrm{D}_{\langle\nu}\, u_{\mu\rangle} \tag{2.17b}$$
$$\omega_{\mu\nu} \equiv \mathrm{D}_{[\nu}\, u_{\mu]} \tag{2.17c}$$
$$\theta \equiv \nabla^{\mu}u_{\mu} = \mathrm{D}^{\mu}\, u_{\mu}. \tag{2.17d}$$

[1] In the following, angled brackets denote the symmetric and trace-free part of spatially projected tensors, i.e.

$$v_{\langle\mu\rangle} = h_{\mu}{}^{\alpha}v_{\alpha} \quad \text{and} \quad S_{\langle\mu\nu\rangle} = h_{\langle\mu}{}^{\alpha}h_{\nu\rangle}{}^{\beta}S_{\alpha\beta} = \left(h_{(\mu}{}^{\alpha}h_{\nu)}{}^{\beta} - \frac{1}{3}h_{\mu\nu}h^{\alpha\beta}\right)S_{\alpha\beta}. \tag{2.13}$$

Each of the terms has an important physical interpretation: $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are respectively the shear and vorticity tensors, θ is the volume expansion (or contraction) scalar and A_μ is the 4-acceleration vector due to non-gravitational forces, i.e. it vanishes when the dynamics is only due to gravity [62]. By construction, there hold the relations

$$\sigma_{\mu\nu}u^\mu = 0 \tag{2.18a}$$
$$\omega_{\mu\nu}u^\mu = 0 \tag{2.18b}$$
$$A_\mu u^\mu = 0 \tag{2.18c}$$

and

$$v_{\mu\nu} \equiv \mathrm{D}_\nu\, u_\mu = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} = \nabla_\nu u_\mu + A_\mu u_\nu \tag{2.19}$$

describes the relative motion of neighbouring observers (with the same 4-velocity). In particular, if χ^μ is their relative position, then $v_{\mu\nu}\chi^\nu$ is the relative velocity of the observers' worldlines [15, 19]. The volume scalar represents the average separation between neighbouring observers and is also used to introduce a representative length scale a, i.e. the cosmological scale factor, through

$$\frac{\dot{a}}{a} = \frac{1}{3}\theta. \tag{2.20}$$

The effect of vorticity is to change the orientation of a given fluid element without any deformation, while the shear changes the shapes while leaving the volume unaffected. The vorticity can also be described through the vector

$$\omega_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}\omega^{\nu\rho}. \tag{2.21}$$

The non-linear covariant kinematics is determined by a set of propagation and constraint equations, which are purely geometric and essentially independent of the Einstein's equations, and emerge applying the Ricci identities to the fundamental 4-velocity

$$2\nabla_{[\mu}\nabla_{\nu]}u_\rho = R_{\mu\nu\rho\sigma}u^\sigma. \tag{2.22}$$

We will need only one of them, which is known as Raychaudhuri's formula

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \frac{1}{2}(\rho + 3P) - 2(\sigma^2 - \omega^2) + D^\mu A_\mu + A_\mu A^\mu + \Lambda, \tag{2.23}$$

Where $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}/2$ and $\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}/2 = \omega_\mu\omega^\mu$. This is the key formula to study the gravitational collapse.

The conservation laws derived from the Bianchi identities

$$\nabla^\nu T_{\mu\nu} = 0 \tag{2.24}$$

split into a timelike part, responsible of the energy conservation, and a spacelike one for the momentum conservation

$$\dot{\rho} = -\theta(\rho + P) - D^\mu q_\mu - 2A^\mu q_\mu - \sigma^{\mu\nu}\pi_{\mu\nu} \tag{2.25a}$$
$$(\rho + P)A_\mu = -D_\mu P - \dot{q}_{\langle\mu\rangle} - \frac{4}{3}\theta q_\mu - (\sigma_{\mu\nu} + \omega_{\mu\nu})q^\nu - D^\nu\pi_{\mu\nu} - \pi_{\mu\nu}A^\nu. \tag{2.25b}$$

2.4.2 Electromagnetic fields

The 1+3 formalism provides an intuitive description of the electromagnetic field [76], which is characterized by the antisymmetric Faraday tensor. Relative to a fundamental observer, the Faraday tensor decomposes as [19, 61]

$$F_{\mu\nu} = 2u_{[\mu}E_{\nu]} + \varepsilon_{\mu\nu\rho}B^{\rho} \tag{2.26a}$$

$$E_{\mu} = F_{\mu\nu}u^{\nu} \tag{2.26b}$$

$$B_{\mu} = \frac{1}{2}\varepsilon_{\mu\nu\rho}F^{\nu\rho}, \tag{2.26c}$$

where E_μ and B_μ are respectively the electric and magnetic field measured by the observer. This guarantees that

$$E_{\mu}u^{\mu} = 0 = B_{\mu}u^{\mu}, \tag{2.27}$$

i.e. E_μ and B_μ are spacelike vectors.

The Faraday tensor determines the stress-energy tensor of the electromagnetic field in the Heaviside-Lorentz units.

$$T_{\mu\nu} = -F_{\mu\rho}F^{\rho}{}_{\nu} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}. \tag{2.28}$$

this has the irreducible decomposition

$$T_{\mu\nu} = \frac{1}{2}\left(E^2 + B^2\right)u_{\mu}u_{\nu} + \frac{1}{6}\left(E^2 + B^2\right)h_{\mu\nu} + 2\mathcal{P}_{(\mu}u_{\nu)} + \Pi_{\mu\nu} \tag{2.29a}$$

$$\mathcal{P}_{\mu} = \varepsilon_{\mu\nu\rho}E^{\nu}B^{\rho} \tag{2.29b}$$

$$\Pi_{\mu\nu} = -E_{\langle\mu}E_{\nu\rangle}B_{\langle\mu}B_{\nu\rangle}. \tag{2.29c}$$

Here $E^2 = E_\mu E^\mu$ and $B^2 = B_\mu B^\mu$ are the square magnitudes of the fields and $\mathcal{P}_\mu$ is the electromagnetic Poynting vector. This expression provides a fluid description of the electromagnetic field and ensures that $T_{\mu}{}^{\mu} = 0$.

The Maxwell equations

$$\nabla_{[\rho}F_{\mu\nu]} = 0 \tag{2.30a}$$

$$\nabla^{\nu}F_{\mu\nu} = J_{\mu} \tag{2.30b}$$

contain the 4-current J_μ, which splits according to

$$J_{\mu} = \rho_{\mathrm{e}}u_{\mu} + \mathcal{J}_{\mu} \tag{2.31a}$$

$$\rho_{\mathrm{e}} = -J_{\mu}u^{\mu} \tag{2.31b}$$

$$\mathcal{J}_{\mu} = h_{\mu}{}^{\nu}J_{\nu}, \tag{2.31c}$$

with ρ_{e} representing the measurable charge density and $\mathcal{J}_\mu$ the orthogonally projected current. Relative to the fundamental observer, the Maxwell equations decompose as

$$\dot{E}_{\langle\mu\rangle} = \left(\sigma_{\mu\nu} + \varepsilon_{\mu\nu\rho}\omega^{\rho} - \frac{2}{3}\theta h_{\mu\nu}\right)E^{\nu} + \varepsilon_{\mu\nu\rho}A^{n}uB^{\rho} + \varepsilon_{\mu\nu\rho}\,\mathrm{D}^{\nu}\,B^{\rho} - \mathcal{J}_{\mu} \tag{2.32a}$$

$$\dot{B}_{\langle\mu\rangle} = \left(\sigma_{\mu\nu} + \varepsilon_{\mu\nu\rho}\omega^{\rho} - \frac{2}{3}\theta h_{\mu\nu}\right)B^{\nu} + \varepsilon_{\mu\nu\rho}A^{n}uE^{\rho} - \varepsilon_{\mu\nu\rho}\,\mathrm{D}^{\nu}\,E^{\rho} \tag{2.32b}$$

$$\mathrm{D}^{\mu}\,E_{\mu} = \rho_{\mathrm{e}} - 2\omega^{\mu}B_{\mu} \tag{2.32c}$$

$$D^{\mu}B_{\mu} = 2\omega^{\mu}E_{\mu}. \tag{2.32d}$$

These equations, in addition to the usual terms, also contain the effect of motion of relative observers: $-2\omega^\mu B_\mu$ is the effective charge caused by a moving magnetic field and $2\omega^\mu E_\mu$ the effective magnetic charge caused by a moving electric field.

The electromagnetic stress energy tensor obeys the constraint

$$\nabla^\nu T_{\mu\nu} = -F_{\mu\nu} J^\nu, \tag{2.33}$$

where the right-hand side represents the Lorentz 4-force. Through this, the conservation laws (2.25) become

$$\dot{\rho} = -\theta(\rho + P) - D^\mu q_\mu - 2A^\mu q_\mu - \sigma^{\mu\nu}\pi_{\mu\nu} + E_\mu \mathcal{J}^\mu \tag{2.34a}$$

$$\begin{aligned}(\rho + P)A_\mu = &- D_\mu P - \dot{q}_{\langle\mu\rangle} - \frac{4}{3}\theta q_\mu - (\sigma_{\mu\nu} + \omega_{\mu\nu})q^\nu \\ &- D^\nu \pi_{\mu\nu} - \pi_{\mu\nu}A^\nu + \rho_\mathrm{e} E_\mu + \varepsilon_{\mu\nu\rho}\mathcal{J}^\nu B^\rho,\end{aligned} \tag{2.34b}$$

where now ρ, p, q_μ and $\pi_{\mu\nu}$ represents only the charged matter variables and the electromagnetic fields are expressed through E_μ and B_μ. The antisymmetry of the Faraday tensor and the Maxwell equations imply

$$\nabla^\nu J_\mu = 0, \tag{2.35}$$

that is the conservation of the 4-current. This leads to the charge density conservation law

$$\dot{\rho}_\mathrm{e} = -\theta\rho_\mathrm{e} - \mathrm{D}^\mu \mathcal{J}_\mu - A^\mu \mathcal{J}_\mu. \tag{2.36}$$

In absence of spatial currents, the charge density evolution depends only on the expansion (or contraction) of the fluid element.

2.4.3 Ideal MHD approximation

The universe, along his lifetime, has been a good conductor, with the exception of any inflationary period or reheating phase. This means that cosmological magnetic fields have remained frozen in an expanding medium during most of its evolution. This allows them to be described through ideal magnetohydrodynamics, which is described through Ohm's law.

Following [16, 63], Ohm's law takes the form

$$J_\mu = \rho_\mathrm{e} u_\mu + \zeta E_\mu, \tag{2.37}$$

where ζ is the scalar conductivity of the medium. This way, the spacelike part of the 4-current, due to conduction, is

$$\mathcal{J}_\mu = \zeta E_\mu. \tag{2.38}$$

This form is valid for single-fluid resistive MHD approximation. thanks to the 1+3 decomposition, induced fields have already been taken care of and are not present in the equations, because they are always expressed in the correct reference frame.

The ideal MHD limit consist in assuming an infinite conductivity $\zeta \to \infty$, i.e. a vanishing electric field even in the presence of spatial currents. The matter component of the universe is assumed to be a perfect fluid, that is eqs. (2.15) with $q_m u = 0$ and $\pi_{\mu\nu} = 0$. Maxwell's eqs. (2.32) now are

$$\varepsilon_{\mu\nu\rho}\,\mathrm{D}^{\nu}\,B^{\rho} = \mathcal{J}_{\mu} - \varepsilon_{\mu\nu\rho}A^{\nu}B^{\rho} \tag{2.39a}$$

$$\dot{B}_{\langle\mu\rangle} = \left(\sigma_{\mu\nu} + \varepsilon_{\mu\nu\rho}\omega^{\rho} - \frac{2}{3}\theta h_{\mu\nu}\right)B^{\nu} \tag{2.39b}$$

$$\omega^{\mu}B_{\mu} = \frac{1}{2}\rho_{\mathrm{e}} \tag{2.39c}$$

$$D^{\mu}B_{\mu} = 0. \tag{2.39d}$$

Eq. (2.39b) guarantees that the magnetic field is frozen within the fluid [61]. Moreover, contracting it with B^{μ} we get the magnetic energy conservation

$$\dot{(B^2)} = -\frac{4}{3}\theta B^2 - 2\sigma_{\mu\nu}\Pi^{\mu\nu}. \tag{2.40}$$

This allows to split the total energy conservation into two different equations: the last one and the perfect fluid energy conservation. The energies are separately conserved. Conservation laws (2.25), or equivalently (2.34), now become

$$\dot{\rho} = -(\rho + P)\theta \tag{2.41a}$$

$$\left(\rho + P + \frac{2}{3}B^2\right)A_{\mu} = -\,\mathrm{D}_{\mu}\,P - \varepsilon_{\mu\nu\rho}B^{\nu}\varepsilon^{\rho\alpha\beta}D_{\alpha}B_{\beta} - \Pi_{\mu\nu}A^{\nu}. \tag{2.41b}$$

The last one can be simplified through

$$\varepsilon_{\mu\nu\rho}B^{\nu}\varepsilon^{\rho\alpha\beta}D_{\alpha}B_{\beta} = \frac{1}{2}D_{\mu}B^2 - B^{\nu}\,\mathrm{D}_{\nu}\,B_{\mu}, \tag{2.42}$$

giving us

$$\left(\rho + p + \frac{2}{3}B^2\right)A_{\mu} = -\,\mathrm{D}_{\mu}\,p - \frac{1}{2}D_{\mu}B^2 + B^{\nu}\,\mathrm{D}_{\nu}\,B_{\mu} - \Pi_{\mu\nu}A^{\nu}. \tag{2.43}$$

2.5 Background model

We assume that our system is homogeneous and perturbed at first order by weak inhomogeneous perturbations. At the background level we have a homogeneous universe with an isotropic perfect fluid and a uniform magnetic field: such field cannot live with an isotropic metric, such as FRW, but it can be accommodated in an anisotropic model. We must use one of the Bianchi models because of the homogeneity and our model fits best in a Bianchi I universe, which is the simplest anisotropic generalization of FRW, so our metric in synchronous gauge is

$$g_{\mu\nu} = \mathrm{diag}\left(-1, a_1^2(t), a_2^2(t), a_3^2(t)\right). \tag{2.44}$$

These types of models were widely studied in literature in different assumptions and physical limits (see for example [8, 10, 12, 28, 78]); here we are interested mainly in their behaviour after the matter-radiation equivalence, where the magnetic field can be reasonably small compared to the matter component. This regime was already studied in different works, for example by [28] in radiation dominated universe; here we will recap [55], which accounts for different types of anisotropic stresses in both radiation an matter dominated universe. We will, however, amend for their time behaviour in matter dominated universe and we will not neglect higher order corrections in the isotropic components.

We assume that the magnetic field is oriented along the 3 axis, so the system is axisymmetric and $a_1 = a_2$; for simplicity we call $a = a_1 = a_2$ and $c = a_3$. We have $u^\mu = (1, 0, 0, 0)$.

It is now straightforward to write the Einstein equations (2.16)

$$2\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} = -\frac{1}{2}\left(\rho + 3P + B^2\right) \tag{2.45a}$$

$$\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right) = \frac{1}{2}\left(\rho - P + B^2\right) \tag{2.45b}$$

$$\frac{\ddot{c}}{c} + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} = \frac{1}{2}\left(\rho - P - B^2\right) \tag{2.45c}$$

and the energy conservation laws for the system (2.41a) and (2.40)

$$\dot{\rho} + \left(2\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right)(\rho + P) = 0 \tag{2.46}$$

$$(\dot{B^2}) + 4\frac{\dot{a}}{a}B^2 = 0. \tag{2.47}$$

We define the Alfvén velocity, which is the energy ratio between magnetic field and fluid

$$v_A^2 = \frac{B^2}{\rho}, \tag{2.48}$$

witch is responsible for the intensity of the anisotropies, the isotropic expansion H and the anisotropy parameter S

$$3H = 2\frac{\dot{a}}{a} + \frac{\dot{c}}{c}, \qquad S = \frac{1}{H}\left(\frac{\dot{a}}{a} - \frac{\dot{c}}{c}\right). \tag{2.49}$$

If we now assume a barotropic fluid with equation of state $P = w\rho$ and $w = \mathrm{const}$ the Einstein equation (2.45a) becomes

$$3\dot{H} + H^2\left(3 + \frac{2}{3}S^2\right) = -\frac{1}{2}\left(1 + 3w + v_A^2\right)\rho, \tag{2.50}$$

subtracting equation (2.45c) from equation (2.45b) we get

$$H\dot{S} + \dot{H}S + 3H^2S = v_a^2\rho \tag{2.51}$$

and summing 2 times equation (2.45b) to equation (2.45c) we eventually have

$$3\dot{H} + 9H^2 = \frac{1}{2}\left[3(1 - w) + v_A^2\right]\rho. \tag{2.52}$$

From the definition of v_A^2 (2.48) and from the energy conservations (2.46) and (2.47) we have

$$(\dot{v_A^2}) = \frac{(\dot{B^2}) - \dot{\rho}v_A^2}{\rho} = v_A^2H\left(3w - 1 - \frac{4}{3}S\right). \tag{2.53}$$

If we now assume that the magnetic field energy is small compared to the fluid energy we have $v_A^2 \ll 1$ and if we write

$$H = H_{(0)} + H_{(1)}, \qquad \rho = \rho_{(0)} + \rho_{(1)} \tag{2.54}$$

with $H_{(1)}, \rho_{(1)} = \mathcal{O}(v_A^2)$ it is easy to see from equations (2.50) and (2.52) that at 0-order in v_A^2 we recover FRW and we have

$$H_{(0)} = \frac{2}{3(1+w)t}, \qquad \rho_{(0)} = 3H_{(0)}^2, \qquad S_{(0)} = 0. \tag{2.55}$$

The anisotropy is described by S and equation (2.51) becomes at first order in v_A^2

$$\dot{S} + \frac{1-w}{1+w}\frac{S}{t} = \frac{2}{1+w}\frac{v_A^2}{t}, \tag{2.56}$$

while equation (2.53) gives

$$\dot{(v_A^2)} = -\frac{2}{3}\frac{1-3w}{1+w}\frac{v_A^2}{t}. \tag{2.57}$$

The isotropic part is contained in equations (2.50) and (2.52), which form a system whose solution is

$$\rho_{(1)} = \frac{4}{1+w}\frac{H_{(1)}}{t} - \frac{2}{3(1+w)^2}\frac{v_A^2}{t^2} \tag{2.58}$$

$$\dot{H}_{(1)} + 2\frac{H_{(1)}}{t} = -\frac{1-3w}{9(1+w)^2}\frac{v_A^2}{t^2}. \tag{2.59}$$

We are interested only in anisotropies caused by the magnetic field so we will put to 0 the homogeneous solution of each equation, with the exception of (2.57).

2.5.1 Radiation dominated universe

For radiation dominated universe $w = 1/3$ and equation (2.57) gives

$$v_A^2 = v_{A0}^2 = \text{const}\,. \tag{2.60}$$

Equation (2.56) then gives

$$S = 3v_A^2 = 3v_{A0}^2. \tag{2.61}$$

From equation (2.58) we get ρ.

From the definitions (2.49) we can get the values of a and c. Finally we have

$$v_A^2 = v_{A0}^2 = \text{const}, \quad t_0 = \text{const} \tag{2.62}$$

$$a \sim \left(\frac{t}{t_0}\right)^{1/2}\left(1 + \frac{1}{2}v_{A0}^2 \ln\left(\frac{t}{t_0}\right)\right) \tag{2.63}$$

$$c \sim \left(\frac{t}{t_0}\right)^{1/2}\left(1 - v_{A0}^2 \ln\left(\frac{t}{t_0}\right)\right) \tag{2.64}$$

$$H = \frac{1}{2t} \tag{2.65}$$

$$\rho = \frac{3}{4t^2}\left(1 - \frac{1}{2}v_{A0}^2\right). \tag{2.66}$$

2.5.2 Matter dominated universe

For matter dominated universe $w = 0$ and equation (2.57) gives

$$v_A^2 = v_{A0}^2 \left(\frac{t}{t_0}\right)^{-2/3}, \quad v_{A0}^2, t_0 = \text{const}\,. \tag{2.67}$$

From equation (2.56) we get

$$S(t) = 6v_A^2(t). \tag{2.68}$$

For the isotropic part we proceed as before: equation (2.59) gives

$$H_{(1)} = -\frac{v_A^2(t)}{3t} \tag{2.69}$$

From equation (2.58) we get ρ.

From the definitions (2.49)we can get the values of a and c. Finally we have

$$v_A^2 = v_{A0}^2 \left(\frac{t}{t_0}\right)^{-2/3} \tag{2.70}$$

$$a \sim \left(\frac{t}{t_0}\right)^{2/3} - \frac{3}{2}v_{A0}^2 \tag{2.71}$$

$$c \sim \left(\frac{t}{t_0}\right)^{2/3} + \frac{9}{2}v_{A0}^2 \tag{2.72}$$

$$H = \frac{2}{3t}\left(1 - \frac{1}{2}v_A^2(t)\right) \tag{2.73}$$

$$\rho = \frac{4}{3t^2}\left(1 - \frac{3}{2}v_A^2(t)\right). \tag{2.74}$$

2.6 Perturbed equations

We now perform an extension of the relativistic calculation presented in sec. 1.1.3. We perturb all the quantities that govern our system while keeping synchronous gauge. We will show in sec. 2.7 that we will use a variable that, for large times, can be considered gauge invariant. This way, we will be sure to not incur in issue related to the gauge choice, even if the synchronous one is the most common to handle this kind of study.

The perturbed metric is

$$g_{\mu\nu} = g_{\mu\nu}^{\mathrm{B}} + \delta g_{\mu\nu} \tag{2.75a}$$

$$\delta g_{\mu 0} = 0, \tag{2.75b}$$

where B means the background value; we can define

$$\gamma_{\mu\nu} = \delta g_{\mu\nu} \tag{2.76a}$$

$$g_{\mu\rho}g^{\rho\nu} = \delta_\mu^{\ \nu} \implies \delta g^{\mu\nu} = -\gamma^{\mu\nu}, \tag{2.76b}$$

where the indices of $\gamma_{\mu\nu}$ are raised and lowered with the unperturbed metric $g_{\mu\nu}^{\mathrm{B}}$. In the following we write the trace of $\gamma_{\mu\nu}$ as $\gamma = \gamma_k^{\ k}$. The fluid velocity perturbation is δu^μ, with

$$u_\mu u^\mu = -1 \implies \delta u^0 = 0. \tag{2.77}$$

The fluid energy perturbation is $\delta\rho$ and the fluid pressure perturbation is $\delta P = v_S^2\,\delta\rho$; it holds

$$\dot{w} = -3H(1+w)\left(v_S^2 - w\right) \tag{2.78a}$$

$$w = \text{const} \implies v_S^2 = w, \tag{2.78b}$$

but we keep v_S^2 as an arbitrary function and possibly different from w; the reason of this choice will be clear in section 2.8.2.

The perturbed magnetic field must remain pure spatial at all orders[2], so the condition $B_\mu u^\mu = 0$ holds at all perturbative orders and the perturbation to the magnetic field satisfies

$$\delta(B_\mu B^\mu) = \delta(B^2) = \gamma_{33} B^3 B^3 + 2c^2\,\delta B^3\,B^3 \tag{2.79a}$$

$$B_\mu u^\mu = 0 \implies \delta B^0 = c^2 B^3\,\delta u^3\,. \tag{2.79b}$$

Accordingly to [18, 92] the perturbed Christoffel symbols are

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2} g_{\mathrm{B}}^{\rho\sigma}\left(\nabla^{\mathrm{B}}_\mu \gamma_{\nu\sigma} + \nabla^{\mathrm{B}}_\nu \gamma_{\mu\sigma} - \nabla^{\mathrm{B}}_\sigma \gamma_{\mu\nu}\right) \tag{2.80}$$

and the perturbed Ricci tensor is

$$\delta R_{\mu\nu} = \nabla^{\mathrm{B}}_\rho\,\delta\Gamma^\rho_{\mu\nu} - \nabla^{\mathrm{B}}_\nu\,\delta\Gamma^\rho_{\mu\rho}\,. \tag{2.81}$$

We are now ready to perturb the exact equations of section 2.4. We notice that, because of the homogeneity of the background model, when applied to the perturbation of a scalar quantity the comoving time derivative $\dot{s}$ is the same as the synchronous time derivative $\partial_0 s$, so we make no difference between them in the following. The fluid energy conservation (2.41a) becomes

$$\dot{\delta\rho} + \left(2\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right)(\delta\rho + \delta P) + (\rho^{\mathrm{B}} + P^{\mathrm{B}})\left(\partial_i\,\delta u^i + \frac{1}{2}\dot{\gamma}\right) = 0 \tag{2.82}$$

and the magnetic field energy conservation (2.40) gives

$$(\delta(\dot{B^2})) + 4\frac{\dot{a}}{a}\,\delta(B^2) + 2B_{\mathrm{B}}^2\left(\partial_i\,\delta u^i - \partial_3\,\delta u^3 + \frac{1}{2}\dot{\gamma} - \frac{1}{2}\dot{\gamma_3}^3\right) = 0. \tag{2.83}$$

[2] In literature there are different definitions of the magnetic field at a perturbative level, but it is easy to recognize that not all of them satisfy the required properties. After a careful analysis we concluded that the correct one, at least with respect to the physical phenomenon we study here, it the one of [77] made through the 1+3 formalism. This way, the magnetic field is defined as the spatial projected part of the Faraday tensor $F_{\mu\nu}$, while the electric field as the temporal one (see eqs.) (2.26)) and we have $B_\mu u^\mu = 0$ at all orders.

There are two important reasons for this requirement. The first one is that the electromagnetic field is decomposed in electric and magnetic components by the observer and we are interested in its interaction with the cosmological fluid, so the natural observer is the fluid itself. Beside that, we force a vanishing electric field $E_\mu = 0$ through the assumption of infinite conductivity of the medium, thus we work in the limit of ideal MHD. To do this we need these fields to be defined with respect to the fluid. Using this definition there are no induced fields, reflecting the fact that the covariant form of Maxwell's formulae and of the electric and magnetic field definitions already incorporates the effects of relative motion [77].

The second reason is that with different definitions we would have a nonvanishing trace for the perturbed magnetic stress energy tensor, while this way all goes well and it is traceless. This is easy to check using the definition of perturbations from sec. 2.6.

The Einstein 00 equation is (we will always use Einstein equations with a lower and an upper index)

$$\frac{1}{2}\ddot{\gamma} + \frac{\dot{a}}{a}\dot{\gamma} - \left(\frac{\dot{a}}{a} - \frac{\dot{c}}{c}\right)\dot{\gamma_3}^3 + \frac{1}{2}\left(\delta\rho + 3\,\delta P\right) + \frac{1}{2}\,\delta(B^2) = 0, \tag{2.84}$$

while the 33 equation reads

$$\begin{aligned}
&\partial_k\partial^3{\gamma_3}^k - \frac{1}{2}\left(\partial_k\partial^k{\gamma_3}^3 + \partial_3\partial^3\gamma\right) + \frac{1}{2}\ddot{\gamma_3}^3 \\
&\quad + \frac{1}{2}\left(2\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right)\dot{\gamma_3}^3 + \frac{1}{2}\frac{\dot{c}}{c}\dot{\gamma} - \frac{1}{2}\left(\delta\rho - \delta P\right) + \frac{1}{2}\,\delta(B^2) = 0;
\end{aligned} \tag{2.85}$$

to remove $\partial_3\partial^k{\gamma_k}^3$ from the last equation we need to use the derivative of the 03 equation with respect to the 3 index

$$\begin{aligned}
&\partial_0\left(\partial_3\partial^k{\gamma_k}^3\right) - \partial_3\partial^3\dot{\gamma} + 2\frac{\dot{a}}{a}\partial_3\partial^k{\gamma_k}^3 - \left(\frac{\dot{a}}{a} - \frac{\dot{c}}{c}\right)\partial_3\partial^3\gamma - \left(\frac{\dot{a}}{a} - \frac{\dot{c}}{c}\right)\partial_3\partial^3{\gamma_3}^3 \\
&\quad = -2(\rho^{\mathrm{B}} + P^{\mathrm{B}})\partial_3\,\delta u^3\,.
\end{aligned} \tag{2.86}$$

If we had used eqs. (2.16) we would have found the same results.

By imposing the null divergence of the magnetic field (2.39d) we get

$$\partial_i\,\delta B^i + \frac{1}{2}B^3_{\mathrm{B}}\partial_3\gamma = 0. \tag{2.87}$$

The last equation we need is the conservation of the momentum (2.43) (note that A^μ has only the first order component): we define an index $P \in \{1, 2\}$ that lies on the plane orthogonal to the background magnetic field and we write the divergence of the momentum conservation on the 12-plane $(\partial_1()^1 + \partial_2()^2)$

$$\begin{aligned}
&(\rho^{\mathrm{B}} + P^{\mathrm{B}})\left(\partial_0\partial_P\,\delta u^P + 2\frac{\dot{a}}{a}\partial_P\,\delta u^P\right) + B^2_{\mathrm{B}}\left[\partial_0\partial_P\,\delta u^P + \left(2\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right)\partial_P\,\delta u^P\right] \\
&\quad + \partial_P\,\delta u^P\,\partial_0\left(P^{\mathrm{B}} + \frac{1}{2}B^2_{\mathrm{B}}\right) + \partial_P\partial^P\left(\delta P + \frac{1}{2}\,\delta(B^2)\right) \\
&\quad - B^3_{\mathrm{B}}\partial_3\partial_P\,\delta B^P + B^2_{\mathrm{B}}\left(\frac{1}{2}\partial_P\partial^P{\gamma_3}^3 - \partial_P\partial^3{\gamma_3}^P\right) = 0
\end{aligned} \tag{2.88}$$

and the derivative of the 3 component along the 3 axis

$$\begin{aligned}
&(\rho^{\mathrm{B}} + P^{\mathrm{B}})\left(\partial_0\partial_3\,\delta u^3 + 2\frac{\dot{c}}{c}\partial_3\,\delta u^3\right) + \partial_3\,\delta u^3\,\partial_0\left(P^{\mathrm{B}} + \frac{1}{2}B^2_{\mathrm{B}}\right) \\
&\quad + \partial_3\partial^3\left(\delta P + \frac{1}{2}\,\delta(B^2)\right) + 2\frac{\dot{a}}{a}B^2_{\mathrm{B}}\partial_3\,\delta u^3 \\
&\quad - B^3_{\mathrm{B}}\partial_3\partial_3\,\delta B^3 - \frac{1}{2}B^2_{\mathrm{B}}\partial_3\partial^3{\gamma_3}^3 = 0.
\end{aligned} \tag{2.89}$$

The system (2.82)-(2.89) fully characterizes the evolution of the perturbed quantities and it is the ground of the following analysis. Compared to [65] we fully related the background anisotropy to the magnetic field, without the need of additional hypothesis.

2.7 Gauge Modes

Fixing the synchronous gauge does not end the freedom of coordinate choice: we can still make a gauge transformation preserving the synchronous gauge.

We follow the same scheme as of [86]: we make a generic coordinate transformation of the form

$$x^\mu \to x^\mu + \epsilon^\mu \tag{2.90}$$

with small ϵ^μ and we keep terms up to $\mathcal{O}(\epsilon)$.

The metric tensor becomes

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - g_{\mu\sigma}(x)\partial_\nu\epsilon^\sigma - g_{\rho\nu}(x)\partial_\mu\epsilon^\rho. \tag{2.91}$$

If we define

$$\begin{aligned}\Delta g_{\mu\nu} &= g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -g_{\mu\lambda}(x)\partial_\nu\epsilon^\lambda - g_{\lambda\nu}(x)\partial_\mu\epsilon^\lambda - \epsilon^\lambda\partial_\lambda g_{\mu\nu}(x)\\ &= -\nabla_\mu\epsilon_\nu - \nabla_\nu\epsilon_\mu\end{aligned} \tag{2.92}$$

to preserve the synchronous gauge we need $\Delta g_{0\mu} = 0$ which gives $\epsilon^0 = \epsilon^0(x^j)$ and

$$\epsilon^P = \tilde{\epsilon}^P(x^j) + \partial^P\epsilon^0(x^j)\, a^2 \int \frac{\mathrm{d}t}{a^2}, \tag{2.93a}$$

$$\epsilon^3 = \tilde{\epsilon}^3(x^j) + \partial^3\epsilon^0(x^j)\, c^2 \int \frac{\mathrm{d}t}{c^2}, \tag{2.93b}$$

where $\epsilon_0(x^j)$ and $\tilde{\epsilon}^i(x^j)$ are arbitrary functions of the spatial coordinates: we still have 4 unused degrees of freedom represented by the functions ϵ^0 and $\tilde{\epsilon}^i$.

If we take the functions ϵ^0 and $\tilde{\epsilon}^i$ of the same order of the perturbations then the transformation given by equation (2.92) can be seen both as a gauge transformation and as a transformation of the functions $\gamma_{\mu\nu}$ within fixed synchronous gauge: in the latter case equation (2.92) gives the value of $\Delta\gamma_{\mu\nu}$. In the same way the stress-energy tensor transforms as

$$\begin{aligned}\Delta T_{\mu\nu} &= -T_{\mu\lambda}\partial_\nu\epsilon^\lambda - T_{\lambda\nu}\partial_\mu\epsilon^\lambda - \epsilon^\lambda\partial_\lambda T_{\mu\nu}\\ &= -T_{\mu\lambda}\nabla_\nu\epsilon^\lambda - T_{\lambda\nu}\nabla_\mu\epsilon^\lambda - \epsilon^\lambda\nabla_\lambda T_{\mu\nu}\end{aligned} \tag{2.94}$$

and if we see these as transformations on the physical variables instead of the coordinates we obtain the gauge modes for $\delta T_{\mu\nu}$. Substituting the explicit expression of $T_{\mu\nu}$ as the sum of the fluid and the magnetic field components we see that the transformation acts separately on the two components and we get for the fluid density perturbation

$$\Delta\,\delta\rho = -\epsilon^0\dot{\rho}^{\mathrm{B}} = 3H(\rho^{\mathrm{B}} + P^{\mathrm{B}})\epsilon^0 = 3H(1+w)\rho^{\mathrm{B}}\epsilon^0. \tag{2.95}$$

In section 2.6 we linearised the equations and so the gauge transformations solve our equations and we call them gauge perturbations or gauge modes: these solutions are not physical because they correspond to a simple change in the reference frame. We are looking for a physical solution for the time dependence of $\delta\rho$ so the most interesting gauge transformation is given by equation (2.95).

Having the knowledge of gauge modes it is possible to construct gauge invariant variables, in a similar way as done in [52]. We have

$$\Delta\, \delta u^i = \partial^i \epsilon^0 \tag{2.96}$$

so our main scalar variable should be

$$\delta\rho^{\mathrm{GI}} = \partial^i \partial_i\, \delta\rho - 3H(1+w)\rho^{\mathrm{B}} \partial_i\, \delta u^i \tag{2.97a}$$

$$\Delta\, \delta\rho^{\mathrm{GI}} = 0. \tag{2.97b}$$

It is easy to check that it is exactly the variable used in [77], expressed in synchronous gauge. However, we will not need it because the vorticity part $H\partial_i\, \delta u^i$ decays in time with respect to $\partial_i \partial^i\, \delta\rho\,/\rho^{\mathrm{B}}$ and we are interested in late time dynamics. We will also not need the Laplacian, because we will use Fourier expansions so it will reduce to a multiplicative term: for late times we can assume $\delta\rho$ to be gauge invariant.

Moreover, it is possible to watch this approximation from another perspective. We will analyse now the FRW case, to clarify the meaning of $\delta\rho$ becoming gauge invariant for late times. Following [18] and using the Newtonian approximation we showed in eq. (1.53) that the solutions after recombination are

$$\delta_\pm \propto t^{-1/6} J_{\mp\frac{5}{6\nu}}\left(\frac{\Lambda t^{-\nu}}{\nu}\right), \tag{2.98}$$

where $\gamma = \nu + 4/3 > 4/3$ is the heat ratio of the fluid (after recombination $\gamma \simeq 5/3$), $\delta = \delta\rho\,/\rho$,

$$\Lambda = t^{2\gamma-2/3} v_S^2 k^2 \tag{2.99}$$

is a constant, v_S^2 is the squared sound speed and k the wavenumber. The functions $J_a(z)$ are the Bessel functions: when their argument is large they oscillate, but when the argument is small they behave like

$$\delta_\pm \propto t^{(-1\pm 5)/6}. \tag{2.100}$$

The growing mode is the physical solution we are looking for, while the other one decays to zero.

We cannot speak of gauge modes in Newtonian theory, but the decaying mode corresponds exactly to the relativistic gauge mode, and as expected it decays in time with respect to the growing one. This means that, for large times, gauge modes naturally decay to zero and we can neglect them as long as we are looking only for the growing ones.

It should be noted that in our calculations we are in the same situation: we cannot have a relativistic sound speed different from w in a single fluid model, but we make this “approximation” in section 2.8.2 because from a physical point of view we need a non-vanishing sound speed (we refer to sec. 2.8.2 for the details, and for the reasons that allow us to do so). This way we “break” the gauge invariance, but the gauge modes manifest themselves in one of the decaying solutions[3]. We are only looking for growing modes, so we can safely neglect them.

[3] This happens because, as shown in this section, the gauge modes appear from a link between background quantities and perturbed variables: applying a small change of reference frame, if the

2.8 Analytical Solutions

If we write the perturbations as Fourier transforms we see that the system imposes different evolution to the perturbations that propagates along the background magnetic field, with $\partial_P(\dots) = 0$, and the perturbations that propagates orthogonally to the background magnetic field, with $\partial_3(\dots) = 0$. These different modes are however coupled by the magnetic stress-energy tensor tensorial nature.

To simplify the equations we use the barotropic state equation for the fluid, so $P^{\mathrm{B}} = w\rho^{\mathrm{B}}$ with $w = \text{const}$ and $\delta P = v_S^2\,\delta\rho$, and the Fourier expansion for the spatial part of the perturbations, so the spatial dependence is of the form $\mathrm{e}^{\mathrm{i}k_j x^j}$. We define the new variables

$$\Delta = \frac{\delta\rho}{(1+w)\rho^{\mathrm{B}}} \tag{2.101}$$

$$G = \frac{1}{2}\gamma \tag{2.102}$$

$$T = \frac{1}{2}{\gamma_3}^3 \tag{2.103}$$

$$M = \frac{\delta(B^2)}{B_{\mathrm{B}}^2}. \tag{2.104}$$

Our differential equation system is not simple but we can solve it for small magnetic fields by keeping only terms up to first order in v_A^2: we shall remember that S is already at first order while Δ, G, T and δu^i have also a 0-order (FRW) part; M has only the 0-order part because it is always multiplied by v_A^2 because $\delta(B^2) = B_{\mathrm{B}}^2 M = 2\rho^{\mathrm{B}} v_A^2 M$. In the same way, looking at our system also T is always multiplied by v_A^2: this is because it does not affect density perturbations unless some anisotropy is present.

We also use eq. (2.78b) to discard terms proportional to $w - v_S^2$ or to $(\dot{v_S^2})$, unless multiplied by $k_i k^i$ or $k_3 k^3$. This is because, while they are equal to 0 for $w = \text{const}$, we will need them in sec. 2.8.2.

The fluid energy conservation equation (2.82) in the new variables reads

$$\dot{G} = -\dot{\Delta} - \partial_i\,\delta u^i\,. \tag{2.105}$$

Similarly we rewrite the magnetic energy conservation (2.83)

$$\dot{M} = -2\left(\partial_P\,\delta u^P + \dot{G} - \dot{T}\right) = 2\left(\dot{\Delta} + \dot{T} + \partial_3\,\delta u^3\right), \tag{2.106}$$

---

intensity of such change is of the same order of the perturbed variables, than the solutions mix together and we cannot distinguish between real, physical solutions and mathematical artefacts.

Allowing a non-vanishing sound speed, while keeping $w = 0$, violates eqs. (2.78). However, these equations are due to the equation of state (in fact, eq. (2.78a) is exactly the perturbed equation of state for the fluid, plus the definition of sound speed). The EOS of the system is a delicate topic and, only for this special equation, we are allowed to perform this "approximation". We refer to sec. 2.8.2 for more details.

Moreover, the limit $v_S^2 \to 0$ recovers the perturbed EOS. This means that, in this limit, we are able to identify the gauge mode. For example, in eq. (2.100) we clearly identify the gauge mode with δ_-. Indeed, we would have found this solution in a general relativistic framework if using $v_S^2 = w = 0$. This allows us to identify the solution δ_- in eq. (2.98) as the one related to the gauge mode.

where we found the last equality by using the fluid energy conservation.

Combining Einstein 33 equation (2.85) with its derivative with respect to time and using the derivative of Einstein 03 equation (2.86) with respect to the 3-index in order to take care of $\partial_i \partial^3 {\gamma_3}^i$ terms we get an equation for T. Because T only appears in the system in terms that are multiplied by v_A^2, we will only need this equation at 0-order:

$$\begin{aligned} &3(1+w)\dddot{T} + 10\frac{\ddot{T}}{t} + 2\frac{1-3w}{1+w}\frac{\dot{T}}{t^2} - 8\frac{\partial_3\,\delta u^3}{t^2} + 2\frac{\ddot{G}}{t} + \frac{2}{3}\frac{1-3w}{1+w}\frac{\dot{G}}{t^2} \\ &- 2\left(1-v_S^2\right)\frac{\dot{\Delta}}{t^2} + \frac{4}{3}(1-v_S^2)\frac{1+3w}{1+w}\frac{\Delta}{t^3} + 3(1+w)(k_i k^i \dot{T} - k_3 k^3 \dot{G}) = 0 \end{aligned} \tag{2.107}$$

We can use the fluid energy conservation equation (2.105) to eliminate G from the other equations. This way the Einstein 00-equation (2.84) reads

$$\begin{aligned} &\ddot{\Delta} + 2H\left(1+\frac{1}{3}S\right)\dot{\Delta} - \frac{1}{2}(1+3v_S^2)(1+w)\rho\Delta + \partial_0\partial_i\,\delta u^i \\ &\quad + 2H\left(1+\frac{1}{3}S\right)\partial_i\,\delta u^i + \frac{4}{3(1+w)}S\frac{\dot{T}}{t} - \frac{2}{3(1+w)^2}v_A^2\frac{M}{t^2} = 0. \end{aligned} \tag{2.108}$$

We obtain the evolution equation for the divergence of the 4-velocity by summing equations (2.88) an (2.89); we then use equation (2.85) to remove the $\partial_i \partial^3 {\gamma_3}^i$ term and equation (2.87) to remove the divergence of the magnetic field. Doing so we find

$$\begin{aligned} &\left(1+\frac{1}{1+w}v_A^2\right)\partial_0\partial_i\,\delta u^i + \\ &\quad + \left[(2-3w)H + \left(\frac{v_A^2}{2(1+w)} + \frac{1}{3}S\right)\frac{4}{3(1+w)}\frac{1}{t}\right]\partial_i\,\delta u^i = \\ &\quad = -v_S^2\partial_i\partial^i\Delta - \frac{v_A^2}{2(1+w)}\partial_i\partial^i M \\ &\quad + \frac{1}{1+w}v_A^2\partial_0\partial_3\,\delta u^3 + \left(\frac{v_A^2}{1+w} + 2S\right)\frac{2}{3(1+w)}\frac{\partial_3\,\delta u^3}{t} \\ &\quad - \frac{1}{1+w}v_A^2\left[\ddot{T} + \frac{2}{1+w}\frac{\dot{T}}{t} + \frac{2}{3(1+w)}\frac{\dot{G}}{t}\right] + \frac{2}{3(1+w)}(1-v_S^2)v_A^2\frac{\Delta}{t^2}. \end{aligned} \tag{2.109}$$

We will need also equation (2.89) which reads, using equation (2.79a) to remove $\partial_3\,\delta B^3$,

$$\partial_0\partial_3\,\delta u^3 + \left(2-3w-\frac{4}{3}S\right)H\partial_3\,\delta u^3 + \partial_3\partial^3(v_S^2\Delta) = 0. \tag{2.110}$$

Thus we restated the dynamical system (2.82)-(2.89) in a more suitable form which is more appropriate for the following analysis.

2.8.1 Radiation dominated universe at large scales

In radiation dominated universe we have $w = v_S^2 = 1/3$ and at large scales we can set $k^2 \approx k_3 k^3 \approx 0$. It is easy to check that, once we get rid of the scale dependent terms, eq. (2.107), (2.108) and (2.109) reduces respectively to

$$2\dddot{T} + 5\frac{\ddot{T}}{t} - 4\frac{\partial_3\,\delta u^3}{t^2} - \frac{\partial_0\partial_i\,\delta u^i}{t} - \frac{\ddot{\Delta}}{t} - \frac{2}{3}\frac{\dot{\Delta}}{t^2} + \frac{2}{3}\frac{\Delta}{t^3} = 0 \tag{2.111}$$

$$\begin{aligned}\ddot{\Delta} &+ \left(1+v_A^2\right)\frac{\dot{\Delta}}{t} - \left(1-\frac{1}{2}v_{A0}^2\right)\frac{\Delta}{t^2} + 3v_A^2\frac{\dot{T}}{t} - \frac{3}{8}v_A^2\frac{M}{t^2}\\ &+ \partial_0\partial_i\,\delta u^i + \left(1+v_A^2\right)\frac{\partial_i\,\delta u^i}{t} = 0\end{aligned} \tag{2.112}$$

$$\begin{aligned}&\left(1+\frac{3}{4}v_A^2\right)\partial_0\partial_i\,\delta u^i + \frac{1+2v_A^2}{2}\frac{\partial_i\,\delta u^i}{t} =\\ &\quad = \frac{3}{4}v_A^2\partial_0\partial_3\,\delta u^3 + \frac{27}{8}v_A^2\frac{\partial_3\,\delta u^3}{t} - \frac{3}{4}v_A^2\ddot{T} - \frac{9}{8}v_A^2\frac{\dot{T}}{t} + \frac{3}{8}v_A^2\frac{\dot{\Delta}}{t} + \frac{1}{4}v_A^2\frac{\Delta}{t^2}.\end{aligned} \tag{2.113}$$

This system, together with (2.106) and (2.110), is satisfied by a power law solution and could be reduced to a pure algebraic problem, but we found simpler to solve it for $v_A^2 = 0$ and then look perturbatively for the corrections in v_A^2. We found

$$\Delta = \frac{\Delta_{\text{gauge}}}{t} + \Delta_{\text{grow}}t^{1-v_{A0}^2} + \Delta_1 t^{1/2-v_{A0}^2} + \Delta_2 t^{1/2+2v_{A0}^2}. \tag{2.114}$$

It can be shown that the $t^{1/2}$ modes are related to a non-vanishing divergence of the background velocity $\partial_i\,\delta u^i = ik_i\,\delta u^i$: strictly speaking, we should have imposed the $k_i \approx 0$ condition, thus finding only the t and $1/t$ modes:

$$\Delta = \frac{\Delta_{\text{gauge}}}{t} + \Delta_{\text{grow}}t^{1-v_{A0}^2} \tag{2.115}$$

and recovering the usual FRW solution in the limit $v_A^2 \to 0$.

Using (2.95) and (2.65) we find that $1/t$ is a gauge mode, while $t^{1-v_{A0}^2}$ is the physical growing mode, with the correction due to the magnetic field.

We see in (2.115) that the magnetic field reduces the growing rate of density perturbations, but by an amount of order $\mathcal{O}(v_A^2) \ll 1$. This effect has long been known, and it is due to the extra magnetic pressure. A similar behaviour was found in [77] and [65], although with the differences stated below.

We find that our physical growing mode follows a slightly different temporal law with respect to [77]. We also note another difference between our solution (2.114) and the one of [77]: the non dominant mode is $t^{1/2}$ in our formalism, while it is $t^{-1/2}$ in their. At a more careful analysis, our equations tend correctly to the ones of [18] for $v_A^2 \to 0$ and we obtain in such limit the same solutions of [48, 86], including the $t^{1/2}$ mode. Thus, we argue such discrepancy is therefore between the synchronous and covariant formalisms, and that is besides the purposes of our analysis. However, we point out that we correctly recover the well known solution in absence of magnetic fields, and we showed in sec. 2.7 that, for large times, our solutions should match with the ones of [77].

Moreover, at some point in [77] the dependence on the direction of the magnetic field is lost, as it says "the magnetic anisotropic stress $\Pi_{\mu\nu}$ is treated as a first-order perturbation and the only zero-order magnetic variable is B^2". On the other side, we kept both of them, because both proportional to v_A^2. This, however, does not explain the results because the $t^{1/2}$ mode is not caused by the magnetic field, as shown in eq. (1.89), and because [77] faces gauge invariant perturbations while using a synchronous FRW background, so there is no reason why this mode should not be found. Thus, we suggest that the issue could be in their solution.

2.8.2 Matter dominated universe at small scales

In this section we analyse the perturbations in a matter dominated universe ($w = 0$), in the regime in which the anisotropies are small with respect to the background. We expand in Fourier the spatial part of each quantity like $\mathrm{e}^{\mathrm{i}k_j x^j}$, with $k_j = \text{const}$, and we define $k^2 = k_i k^i$.

Being at small scales means $k^2 \gg H^2$ and assuming $v_S^2, v_A^2 \ll 1$ we can greatly simplify our equations, keeping only terms in v_S^2 or v_A^2 that are multiplied by k^2 and dropping terms of order v_S^2 and v_A^2. This means that the effect of the sound speed and the Alfvén speed is relevant only at very small scales, as we will see from the solutions of our equations. This approximation, although still relativistic and so comparable to other result in literature, for example [77], will give the non-relativistic limit, as shown below.

Sound speed and Alfvén speed

First we need some considerations regarding the sound speed. From a formal point of view, the sound speed is related to the barotropic index w by (2.78a) and $w = \text{const}$ implies $v_S^2 = w$, so it should vanish. From a physical point of view we need a non-vanishing sound speed and we can also estimate its value. While formally the best solution to this problem would be using a two fluid model, with a different equation of state for perturbations, here we will simply drop the relation between v_S^2 and w and assume that the perturbed fluid follows a different equation of state with respect to the background fluid. This is correct in the Newtonian approximation and it is in fact the standard way of handling things[4] [18, 89], while putting $v_S^2 = 0$ at the end will recover the full covariant value of our calculations for studying pure magnetic effects.

[4]Eq. (2.78a) is the perturbed equation of state of the fluid. From a formal point of view, the perturbed fluid must follow the same EOS. On the other hand, from a physical point of view, the perturbed fluid presents a non-vanishing sound speed (eq. (2.117)). Moreover, the sound speed is necessary for the Jeans mechanism, i.e. without the sound speed every small perturbations should grow as $t^{2/3}$, as in eq. (1.47).

The properties of the perturbations do not have to be the same of the background fluid, however treating them formally as a single fluid causes this fictitious behaviour. The alternative, i.e. working with a multi-component fluid, would introduce other issues because background fluid and perturbations are not different fluids. Moreover, it would make the equations much more complex. Thus, the best solution is to "break" the EOS for the perturbations. Indeed, even if we had used the same EOS, v_S^2 is so small that is nearly irrelevant if not multiplied by k^2, so we could simply choose the perturbations EOS and w would have disappeared from the system anyway. However, this is not a clean way to face this issue.

Formally, the only solution would be to resort to statistical mechanics. But this would make everything much more complex. Instead, this "breaking" of the EOS for perturbations is accepted unquestioned in classical mechanics, see for example [18, 89], as well as many other papers. However, knowing this limit of our procedure, we will also solve our equations assuming a vanishing sound speed in sec. 2.8.3.

We proceed as in [18]: we use an adiabatic sound speed

$$v_S^2 = \frac{\delta P}{\delta \rho} \sim \frac{\gamma p}{\rho} \sim \rho^{\gamma-1} \sim t^{2(1-\gamma)} \tag{2.116}$$

where γ is the heat ratio. We write $\nu = \gamma - 4/3 \geq 0$ so

$$v_S^2 = v_{S0}^2 \left(\frac{t}{t_0}\right)^{-2\left(\nu+\frac{1}{3}\right)}. \tag{2.117}$$

We can estimate more precisely the sound speed value, and it is possible to show that the adiabatic sound speed is [18, 89]

$$v_S^2\Big|_{z<z_{\mathrm{rec}}} = \frac{1}{3}\frac{k_B T_b \sigma}{m_p + k_B T_b \sigma}, \qquad v_S^2\Big|_{z>z_{\mathrm{rec}}} = \frac{5}{3}\frac{k_B T_b}{m_p}, \tag{2.118}$$

where z_{rec} is the redshift value at recombination, k_b is the Boltzmann constant, T_b is the baryons temperature, m_p is the proton mass and σ is the specific entropy, whose value is $\sigma = 4a_{\mathrm{SB}}T^3/3n_b k_B \approx 1.5 \cdot 10^9$, being a_{SB} the Stefan–Boltzmann constant and T the gas temperature. We neglected any anisotropic effects in temperature, because they would be related to the next order corrections. The baryons temperature is the same of the photons until $z \approx 100$, due to residual Thomson scattering, and decreases faster thereafter:

$$T_b\big|_{z>100} = T_\gamma = T_\gamma\big|_{z=0}(1+z), \;\; T_\gamma\big|_{z=0} \approx 2.7\,\mathrm{K} \tag{2.119a}$$

$$T_b\big|_{z<100} \propto (1+z)^2. \tag{2.119b}$$

Comparing the two expressions we see that right after recombination and until complete decoupling, so for $z_{\mathrm{rec}} = 1100 > z > 100 = z_{\mathrm{rec}}$, we have $\nu = 0$ and the cosmic medium behaves like a non-relativistic fluid with $\gamma = 4/3$: the total energy density is dominated by hydrogen rest mass but the pressure is dominated by radiation. After the end of Thomson scattering effects and until reionization, for $100 > z > 10$, $\nu \simeq 1/3$ and the cosmic medium behave like a relativistic fluid with $\gamma \simeq 5/3$. The plot of the sound speed and of the Alfvén speed is in fig. 2.1.

We define two constants addressing the effect of sound speed and Alfvén speed after recombination. Taking the time dependence of k^2 depending only on the 0-order part of the background metric because it always appears multiplied by v_S^2 or v_A^2, we have respectively

$$\Lambda_S^2 = v_S^2 k^2 t^{2\gamma-2/3}, \qquad \Lambda_A^2 = v_A^2 k^2 t^2. \tag{2.120}$$

For a more detailed discussion about the sound speed see [22].

Analytical solutions

Using the assumptions of section 2.8.2 we can greatly simplify our equations. The energy conservation (2.105) and the magnetic field energy conservation (2.106) retain the same form. The Einstein 00-equation (2.108) now reads

$$\ddot{\Delta} + \frac{4}{3t}\dot{\Delta} - \frac{2}{3t^2}\Delta + \partial_0\partial_i\,\delta u^i + \frac{4}{3t}\partial_i\,\delta u^i = 0. \tag{2.121}$$

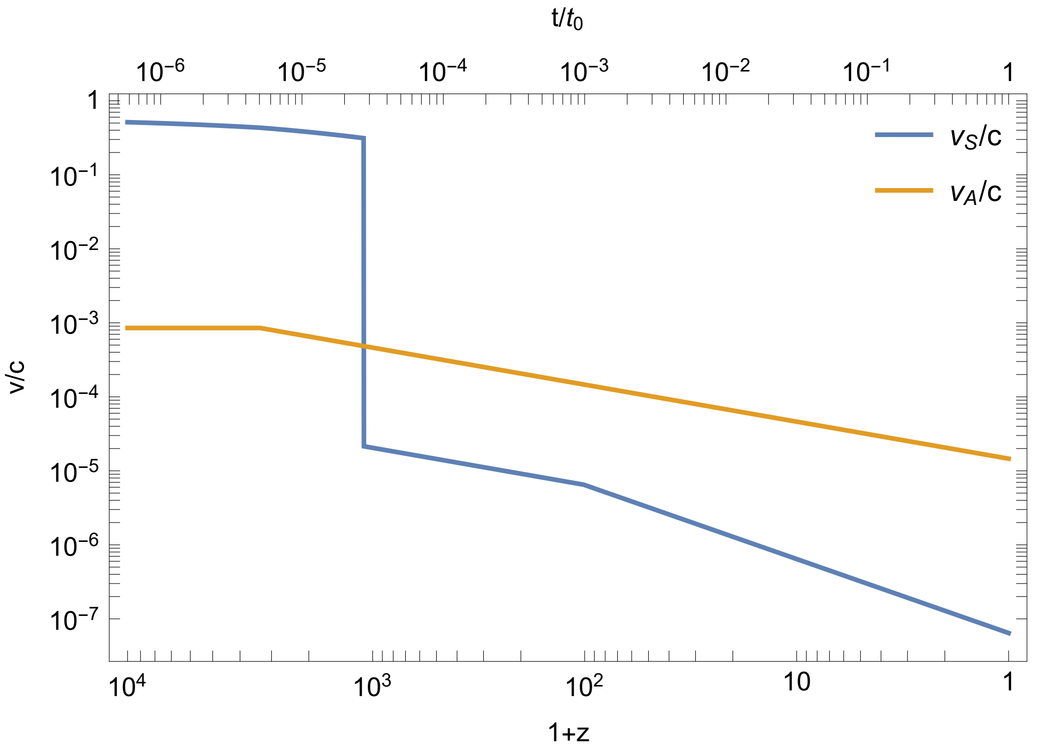


Figure 2.1. Plot of the sound speed and the Alfvén speed. We see that sound speed dominates until recombination, where suddenly the Alfvén velocity becomes important.

The momentum conservation 2.109 becomes

$$\partial_0 \partial_i \, \delta u^i + \frac{4}{3t} \partial_i \, \delta u^i = -v_S^2 \partial_i \partial^i \Delta - \frac{1}{2} v_A^2 \partial_i \partial^i M \tag{2.122}$$

and its counterpart along the z-axis remains (2.110):

$$\partial_0 \partial_3 \, \delta u^3 + \frac{4}{3t} \partial_3 \, \delta u^3 + v_S^2 \partial_3 \partial^3 \Delta = 0. \tag{2.123}$$

We need the Einstein 33-equation only at 0-order in the magnetic field, after being multiplied by v_A^2, so equation (2.107) in our limit reads

$$v_A^2 \partial_i \partial^i \dot{T} + v_A^2 \partial_3 \partial^3 (\partial_i \, \delta u^i + \dot{\Delta}) = 0. \tag{2.124}$$

With some algebra it is possible to reduce this system to a single equation. Expanding the spatial part in Fourier, defining the anisotropy parameter μ of the solution as

$$k_3 k^3 = \mu^2 k^2 \tag{2.125}$$

and using the constants (2.120) we find, after some algebra,

$$\begin{aligned} 9t^4 \Delta^{(4)} &+ 60 t^3 \Delta^{(3)} + \left[76 + 9 \Lambda_S^2 t^{-2\nu} + 9 \Lambda_A^2 \right] t^2 \Delta^{(2)} \\ &+ \left[8 + 12 \Lambda_S^2 (1 - 3\nu) t^{-2\nu} + 12 \Lambda_A^2 \right] t \Delta^{(1)} \\ &+ \left[6 \Lambda_S^2 \left(-\nu + 6\nu^2 + \frac{3}{2} \mu^2 \Lambda_A^2 \right) t^{-2\nu} - 6 \mu^2 \Lambda_A^2 \right] \Delta = 0, \end{aligned} \tag{2.126}$$

where $\Delta^{(i)}$ is the i-th derivative of Δ. This corresponds exactly to equation (29) of [89], except for a difference in the definition of v_A^2 and so in Λ_A.

We believe interesting to analyse separately the two cases of $\nu = 0$ and $\nu = 1/3$, instead of studying them together as in [89].

Post recombination evolution

For $1100 > z > 100$ we have $\nu = 0$. The solution of (2.126) is

$$\Delta = \Delta_i t^{x_i}, \tag{2.127}$$

where Δ_i are arbitrary constants and

$$x_1 = \left(-1 + \sqrt{\delta_-}\right)/6 \qquad x_2 = \left(-1 - \sqrt{\delta_-}\right)/6 \tag{2.128a}$$

$$x_3 = \left(-1 + \sqrt{\delta_+}\right)/6 \qquad x_4 = \left(-1 - \sqrt{\delta_+}\right)/6 \tag{2.128b}$$

$$\delta_\pm = \delta_1 \pm 6\sqrt{\delta_2} \tag{2.128c}$$

$$\delta_1 = 13 - 18\Lambda_S^2 - 18\Lambda_A^2 \tag{2.128d}$$

$$\delta_2 = \left(-2 + 3\Lambda_S^2 + 3\Lambda_A^2\right)^2 - 12\mu^2\Lambda_A^2\left(-2 + 3\Lambda_S^2\right). \tag{2.128e}$$

The only possible growing solution is x_3, and the requirement is that it holds one of the conditions

$$\mu > 0 \text{ and } \Lambda_S^2 < \frac{2}{3} \tag{2.129a}$$

$$\mu = 0 \text{ and } \Lambda_S^2 + \Lambda_A^2 < \frac{2}{3}; \tag{2.129b}$$

using (2.120) and (2.54), making explicit the presence of Newton's constant we get $\rho = 1/6\pi G t^2$, conditions (2.129) become

$$\mu > 0 \text{ and } k < k_J = \sqrt{\frac{4\pi G\rho}{v_S^2}} \tag{2.130a}$$

$$\mu = 0 \text{ and } k < \sqrt{\frac{4\pi G\rho}{v_S^2 + v_A^2}} < k_J. \tag{2.130b}$$

While the first one is the standard Jeans condition, the second one means that, orthogonally to the background magnetic field, there is a heavier requirement dependent on the strength of the magnetic field: some modes could grow in every direction but the one of the field. The presence of the magnetic field also imposes a slowing down of the growing mode:

$$x_3 \leq x_3|_{\Lambda_A=0} = \frac{1}{6}\left(-1 + \sqrt{25 - 36\Lambda_S^2}\right), \tag{2.131}$$

where the equal sign holds only in absence of a magnetic field, that is only if $\Lambda_A = 0$.

Late times evolution

This is exactly the case analysed in [89]. For $z < 100$ we have $\nu > 0$ and the solution of (2.126) is

$$\Delta = \Delta_i t^{x_i} {}_2F_3 \left[\begin{matrix} a_{i1}, a_{i2} \\ b_{i1}, b_{i2}, b_{i3} \end{matrix}; -\frac{\Lambda_S^2 t^{-2\nu}}{4\nu^2} \right], \tag{2.132}$$

where Δ_i are arbitrary constants, ${}_2F_3$ is a generalized hypergeometric function with

$$x_1 = \left(-1 + \sqrt{\delta_-}\right)/6 \qquad x_2 = \left(-1 - \sqrt{\delta_-}\right)/6 \tag{2.133a}$$

$$x_3 = \left(-1 + \sqrt{\delta_+}\right)/6 \qquad x_4 = \left(-1 - \sqrt{\delta_+}\right)/6 \tag{2.133b}$$

$$\delta_\pm = 13 - 18\Lambda_A^2 \pm 6\sqrt{\left(2 - 3\Lambda_A^2\right)^2 + 24\mu^2\Lambda_A^2} \tag{2.133c}$$

and constant coefficients $a_{ij},\ b_{ij}$ depending only on the constants[5] $\nu,\ \Lambda_S,\ \Lambda_A$

$$a_{({}^1{}_2)1} = 1 \mp \sqrt{\delta_-}/12\nu - \sqrt{1 - 36\mu^2\Lambda_A^2}/12\nu \tag{2.134a}$$

$$a_{({}^1{}_2)2} = 1 \mp \sqrt{\delta_-}/12\nu + \sqrt{1 - 36\mu^2\Lambda_A^2}/12\nu \tag{2.134b}$$

$$a_{({}^3{}_4)1} = 1 \mp \sqrt{\delta_+}/12\nu - \sqrt{1 - 36\mu^2\Lambda_A^2}/12\nu \tag{2.134c}$$

$$a_{({}^3{}_4)2} = 1 \mp \sqrt{\delta_+}/12\nu + \sqrt{1 - 36\mu^2\Lambda_A^2}/12\nu \tag{2.134d}$$

$$b_{({}^1{}_2)1} = 1 \mp \sqrt{\delta_-}/6\nu \tag{2.134e}$$

$$b_{({}^1{}_2)2} = 1 \mp \sqrt{\delta_-}/12\nu - \sqrt{\delta_+}/12\nu \tag{2.134f}$$

$$b_{({}^1{}_2)3} = 1 \mp \sqrt{\delta_-}/12\nu + \sqrt{\delta_+}/12\nu \tag{2.134g}$$

$$b_{({}^3{}_4)1} = 1 \mp \sqrt{\delta_+}/6\nu \tag{2.134h}$$

$$b_{({}^3{}_4)2} = 1 \mp \sqrt{\delta_+}/12\nu - \sqrt{\delta_-}/12\nu \tag{2.134i}$$

$$b_{({}^3{}_4)3} = 1 \mp \sqrt{\delta_+}/12\nu + \sqrt{\delta_-}/12\nu. \tag{2.134j}$$

The solutions can grow only if the argument of the hypergeometric functions is small, i.e. if

$$\Lambda_S^2/4\nu^2 t^{2\nu} \ll 1: \tag{2.135}$$

this way we have

$$\Delta = \Delta_i t^{x_i} \left(1 + \mathcal{O}\left(\frac{\Lambda_S^2 t^{-2\nu}}{4\nu^2}\right)\right). \tag{2.136}$$

Condition (2.135) is the standard Jeans condition [18]: using (2.120) and (2.54), eq. (2.135) translates in [89]

$$k \ll k_J = \sqrt{\frac{24\nu^2\pi G\rho}{v_S^2}}. \tag{2.137}$$

[5]For brevity of notation we couple the indices in pairs. Thus, in eq. (2.134a), the them a_{11} has to be taken with the sign $-$, while a_{21} should have the sign $+$ in its right end side.

The only solution in (2.136) that can grow is 3: $x_3 > 0$ only if it holds one of

$$0 < \mu \leq 1 \tag{2.138a}$$

$$\mu = 0 \text{ and } \Lambda_A^2 < \frac{2}{3}. \tag{2.138b}$$

The first one means that, in any direction but orthogonal to the background magnetic field, the only necessary condition is the standard one. The second one is an additional condition that must hold for perturbations propagating orthogonally to the background magnetic field, and using (2.120) and (2.54) it reads [89]

$$k < k_A = \sqrt{\frac{4\pi G\rho}{v_A^2}}. \tag{2.139}$$

The presence of this new condition makes possible the existence of Jeans unstable modes, that orthogonally to the background magnetic field are stabilized by the magnetic pressure if $k_A < k_J$ and $k_A < k < k_J$ [89].

Studying the growing rate of this solution with more care, we see that x_3 satisfies

$$\mu = 1 \quad \Longrightarrow \quad x_3 = x_3|_{\Lambda_A = 0} = \frac{2}{3} \tag{2.140a}$$

$$\mu \neq 0 \quad \Longrightarrow \quad x_3 < x_3|_{\Lambda_A = 0}: \tag{2.140b}$$

orthogonally to the background magnetic field the growing rate is unchanged, while in other directions it is slowed down, depending on the field strength.

2.8.3 Full relativistic case

If we put $v_S^2 = 0$ we recover the exact relativistic solution. As we can see from the previous solutions, the growing condition is

$$\mu > 0 \tag{2.141a}$$

$$\mu = 0 \text{ and } k < k_A = \sqrt{\frac{4\pi G\rho}{v_A^2}}. \tag{2.141b}$$

Moreover, the solution is

$$\Delta = \Delta_i t^{x_i} \tag{2.142}$$

with x_i given by (2.128) with $\Lambda_S = 0$, or equivalently by (2.133).

If we compare our result with [77], we identify the anisotropic behaviour and we obtain the correct Newtonian limit of [89]. However, our solutions are different. We argue they may have found some sort of average effect because, as already said at the end of section 2.8.1, they neglect anisotropic effects on perturbations. Moreover, given the strong anisotropy of the model, the magnetic Jeans wavenumber is present only in one direction, the one with $\mu = 0$. This is not true in their solution, and we guess that they could have neglected some important contribution. This, however, claims for further investigation.

In case $\Lambda_A^2 \ll 1$ we have

$$x_{\binom{1}{2}} = \frac{1}{6}\left(-1 \pm \sqrt{1 - 36\mu^2\Lambda_A^2}\right) \tag{2.143a}$$

$$x_{\binom{3}{4}} = \frac{1}{6}\left(-1 \pm \sqrt{25 - 36(1-\mu^2)\Lambda_A^2}\right) \tag{2.143b}$$

and setting $\mu^2 = 1/3$ the solutions x_3 and x_4 recover eq. (31) of [97] and eq. (31) of [105], so our small scales solution of sec. 2.8.2 is a generalization of their work, while including a non-vanishing sound speed and pressure.

2.9 Numerical integration

To better show our results, we numerically integrated the system (2.82)–(2.89), using estimates from [113] to set the numerical values for the background functions. We followed the same procedure of [89] to determine the initial conditions: we started the integration from a very early time and we verified that the initial perturbations were outside the Hubble horizon and we used the large scale solution to match the initial conditions to the growing mode; in our case such conditions come from eq. (2.115).

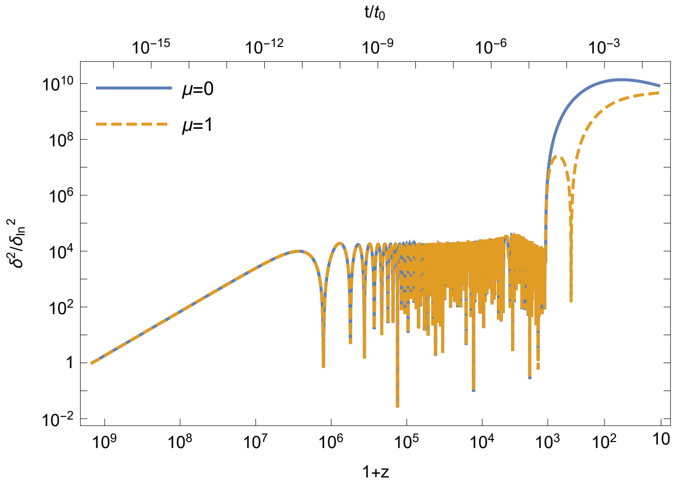
We assumed to perturb only the baryon component of the universe, while leaving the CDM component unperturbed; a rigorous treatment should rely on a multi–fluid model, but we ague that we can still extract meaningful information within our approximation. Practically speaking, this assumption means that every quantity present in our equations at perturbative level must be replaced by its baryonic component, while the background model still depends on CDM. Our equations are still correct, because the background interaction is only due to energy density, while at perturbative level every dependence on CDM disappears, except from background quantities.

We chose to study the same scales of [89], i.e. $k \approx (17, 1.7, 0.37)$ Mpc^{-1} normalized at present time, corresponding to baryonic masses of $M \approx (1.5 \times 10^8, 1.5 \times 10^{11}, 1.5 \times 10^{13})$ M$_\odot$ and roughly equivalent respectively to a dwarf galaxy, a galaxy and a galaxy cluster. The results of the numerical integration are shown in figure 2.2.

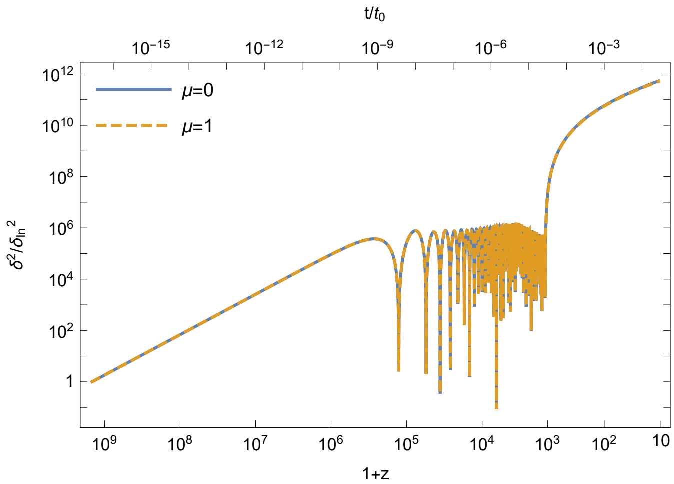
Our results must be compared to the ones of [89]. Until equivalence ($z \approx 3400$) we are in radiation dominated universe and the comparison is obvious: our solutions grow, while theirs decay; this is because in [89] the authors always consider matter dominated universe.

After equivalence, in both cases we are subject to a decaying period, followed by a new growth after recombination, but in our case this happens for a shorter time; most of the anisotropic effects comes in this era, because before equivalence the thermal pressure is much stronger than the magnetic one and most of the anisotropy is suppressed, so they are less relevant in our simulations. This is clear in fig. 2.2b, where we see almost no anisotropy. As a further confirmation, it can be shown that $\Delta(z \approx 10)/\Delta(z \approx 1100)$ has the same value in both the analysed cases, so the main anisotropic contribution comes from the region $3400 \lesssim z \lesssim 1100$.

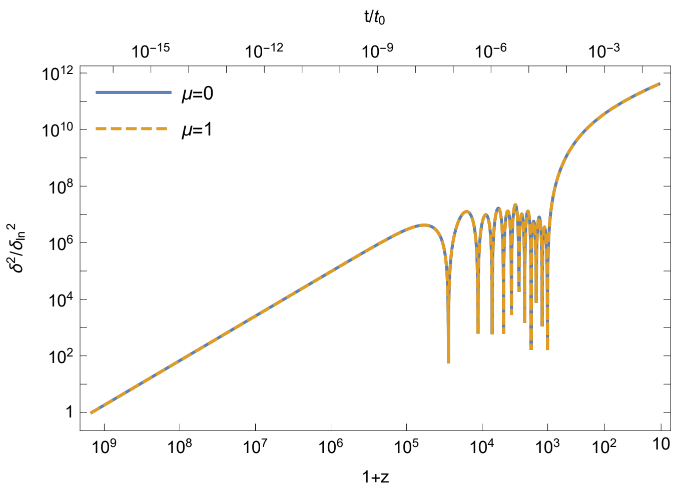
After recombination we have a behaviour similar to [89], because here we are at scales were the Newtonian approximation is correct. The apparent discrepancy



(a) Perturbations at dwarf galaxy scale: $k \simeq 17\,\mathrm{Mpc}^{-1}$, $M \approx 1.5 \times 10^{8}\ \mathrm{M}_{\odot}$.



(b) Perturbations at galactic scale: $k \simeq 1.7\,\mathrm{Mpc}^{-1}$, $M \approx 1.5 \times 10^{11}\ \mathrm{M}_{\odot}$.



(c) Perturbations at galaxy cluster scale: $k \simeq 0.37\,\mathrm{Mpc}^{-1}$, $M \approx 1.5 \times 10^{13}\ \mathrm{M}_{\odot}$.

Figure 2.2. Density perturbations evolution in time, relative to their initial value. While some anisotropy is present in (a) because of the magnetic Jeans length (see sec. 2.9 and [89]), most of the anisotropic effects of [89] here are suppressed because of thermal pressure in the radiation dominated era.

in the oscillating behaviour of fig. 2.2a is mainly due to the (small) difference in the numerical values of the background functions, because the oscillating behaviour is very sensible to such numbers; however, the qualitative evolution is the same, with the $\mu = 0$ case beginning to decay because of the magnetic Jeans length [89] (eq. (2.130b) and (2.139)).

Moreover, in the last region we should be outside of the linear regime, so we would need a full nonlinear treatment.

2.10 Results

We developed a self-consistent scheme for the analysis of cosmological perturbations in the presence of a magnetic field. We set up in the synchronous gauge a dynamical scheme which accounts for the effects induced by the magnetic field both on the background and the first order formulation. To this end, we considered a Bianchi I model, whose anisotropy with respect to the flat Robertson-Walker geometry is due to the privileged direction defined by the magnetic field.

We first solved in detail the equations describing the anisotropic background and then we analysed the perturbation dynamics, having awareness of the gauge contribution analytical form.

We amended for the previous analysis in [77] in the case of a super-horizon wavelength of the perturbation. In particular, our matches the non magnetic one in the correct limit, differently from theirs. We correctly recovered the slowing-down of the growing mode caused by the magnetic pressure, and so of order $\mathcal{O}(v_A^2) \ll 1$. This effect has long been known in FRW models and has been analysed in Bianchi I models with particular anisotropies by [65], while we worked always relating the background anisotropy to the magnetic field without additional assumptions.

We refined the results of [77] for the sub-horizon wavelength of the perturbations, showing that an anisotropic treatment is required. We also generalised the results of [97] and [105], while including a nonvanishing sound speed and considering the anisotropic case.

We finally enforced the Newtonian limit obtained in [89], completing it with the relativistic analysis, also facing a numerical treatment. We showed that the relativistic regime limits the anisotropy induced by the magnetic field.

Overall, despite the assumption of a Bianchi-I background, most of our solutions reproduce those obtained on an FRW background. At a closer look, the Bianchi I anisotropy enters the system via the S function defined in (2.49). At small scales the relevant terms are the ones with k^2, and none of those are related to such anisotropy. However, when the condition $H^2 \ll k^2$ does not hold, such terms become important; unfortunately, in this case the system would be much more complicated that the one of sec. 2.8.2. On the other hand, at large scales the background anisotropy survives, and we argue that it is mainly related to the perturbed fluid velocity. In particular, it can be shown that the solutions proportional to $t^{1/2}$ in (2.114) are related to δu^i, and more precisely in $\Delta_2 t^{1/2+2v_{A0}^2}$ we have both $k_i\,\delta u^i \neq 0$ and $k_3\,\delta u^3 \neq 0$, while in $\Delta_1 t^{1/2-v_{A0}^2}$ it holds $k_3\,\delta u^3 = 0$; the solutions $\Delta_{\rm grow} t^{1-v_{A0}^2}$ and $\Delta_{\rm gauge}/t$, on the other hand, both have $k_i\,\delta u^i = k_3\,\delta u^3 = 0$.

We stress that, in order to solve the equations, we assumed a small magnetic field and so all the effects we studied are related to $v_A^2 \ll 1$, and they become relevant only at small scales, due to the large wavenumber $k^2 \gg H^2$ and to the also small sound speed v_S^2. This is clear by looking at fig. 2.1.

The solutions we found, other then directly describing the behaviour of cosmological perturbations, are the starting point of the studies on the CMB or in non-linear regimes. These subsequent analyses should answer the questions on the observability of such a small magnetic field on the CMBR, or of the anisotropies is causes on the perturbations.

Chapter 3

Influence of viscosity on the anisotropic dynamics of magnetized cosmological perturbations

Following [111], we analyse the influence that viscous effects can induce on the evolution of primordial perturbations to the isotropic universe in the presence of a weak magnetic field. Previous analyses have shown that the presence of the magnetic field induces an intrinsic anisotropy in the perturbations dynamics, essentially because of the anisotropic character of the perturbed magnetic pressure. This anisotropic effect is of order unity in the perturbation amplitude, although it remains small in the linear theory when the density constraints are considered.

The aim of this study is to determine the impact of viscosity, surely present in the early universe, on the growth of the perturbation anisotropy. The main merit of this study consists of demonstrating that a tiny overlapping exists in the parameter space to deal simultaneously with anisotropic features due to the magnetic field and the viscous damping of such density fluctuation. Actually, we demonstrate that the viscosity affects the value of the anisotropy, by smoothing the growing rate of the instability only when structure smaller than about 5000 solar masses are concerned. This result allows us to guarantee that the intrinsic anisotropy of the magnetized universe perturbations is not affected by the viscosity due to friction among inhomogeneous layers or compressive-like effects, and therefore, they remain good candidates for being seeds for filament formation across the universe.

In section 3.1 we give the main motivations for this analysis. Sections 3.2 presents the viscous coefficients. In sec. 3.3 we derive the equations that describe our physical problem, and we solve them analytically in sec 3.4, in some particular limits, and numerically in sec. 3.5.

3.1 Introduction and motivation

One of the most interesting open questions in early cosmology is the generation and dynamics of primordial magnetic fields [77, 79]. The intensity of such magnetic field across the universe is fixed by the observational data on the Cosmic Microwave Background Radiation (CMBR) [87, 96] and the maximum value of about 10^{-9} G is naturally inferred. In [72, 89, 104], it has been argued that the presence of a non-zero magnetic field can significantly influence the dynamics of cosmological perturbations during their linear regime of evolution. In fact, before the re-combination era, the universe is in the state of a plasma and even when the neutral hydrogen is formed a part to 10^5 of the baryonic matter remains in the form of a plasma, since a perfect re-combination is inhibited by the universe expansion rate. The crucial point consist of the strong coupling of the remaining plasma portion of the baryonic matter to the neutrals, via a non-negligible ambipolar diffusion. Thus, properties derived for the plasma universe component can be extrapolated to the dynamic of the all baryonic perturbations. In the present analysis we deal with a single fluid representation just due to this strong coupling of the neutrals and the plasma for a significant period after the re-combination. On a different footing is treated the presence of the dark matter universe component, which does not directly interact with both neutrals and plasma, but influences the perturbation evolution via its gravitational field. However, since the dark matter perturbation dynamics can be neglected during the neutral-plasma interaction, we include dark matter in the present analysis only via its contribution to the total universe mass density.

In particular in [89], see also [90], it has been demonstrated that the magnetic field presence induces a significant degree of anisotropy in the perturbation evolution and hence shear. We investigate in the present chapter if and how this effect is influenced by the viscous properties of the cosmological plasma. In fact, it is naturally expected a possible damping phenomenon of the perturbation anisotropy when the non-ideal contribution due to the viscosity is taken into account into the perturbation dynamics.

We construct the dynamical equations governing the perturbation dynamics in magneto-hydrodynamics and determine at which scale the damping effect is visible. After analysing the dispersion relation, we describe the evolution of a spherical overdensity, characterizing its evolution toward an anisotropic configuration. As fundamental result of this study, we clearly show how the viscosity damping starts to affect the perturbation dynamics only for a perturbation mass below few thousand of solar masses.

Since the obtained mass scale threshold is very small with respect to the typical cosmological scales, i.e. of the typical masses characterizing structures across the universe, like cluster of galaxies and super-cluster of galaxies, we can firmly argue, on the base of the present analysis, that the anisotropic effect that the cosmological perturbation growth feels from the presence of a magnetic field is not removed by the presence of viscosity in the plasma universe component. The physical relevance of this claim consists of the possibility that the linear perturbation anisotropy can be enhanced during the non-linear regime and therefore the primordial cosmological field can be responsible for the formation of matter filaments across the universe, as it emerges in some surveys of the local universe. By mean of this study, we can exclude that the viscosity can affect the formation of the scale filaments in the proposed scenario, while its presence could reduce anisotropy in smaller scale structures, whose formation could not significantly deviate from the isotropy even in the presence of a primordial magnetic field.

3.2 Modelization of viscosity

We now modelise the viscosity of the cosmological medium. For simplicity, we work in a special-relativistic framework, with metric tensor $\eta^{\mu\nu}$. The dynamics of a viscous fluid can be encoded in a stress energy tensor of the form

$$T^{\mu\nu} = \rho u^\mu u^\nu + P(\eta^{\mu\nu} + u^\mu u^\nu) + \Delta T^{\mu\nu}. \tag{3.1}$$

We are interested only in effects of first order in $\Delta T^{\mu\nu}$ and we require $\Delta T^{\mu\nu}$ to be a linear combination of the spacetime derivatives of the fluid variables. It can be shown that, in a locally comoving frame, i.e with $u^0 = 1$ and $u^i = 0$, the more general form allowed for $\Delta T^{\mu\nu}$ is [17, 32]

$$\Delta T_{00} = 0 \tag{3.2a}$$
$$\Delta T_{0i} = -\chi \partial_i T - \xi \partial_t u_i \tag{3.2b}$$
$$\Delta T_{ij} = -\eta \left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u_k \right) - \zeta \delta_{ij} \partial_k u_k, \tag{3.2c}$$

where T is the fluid temperature. While η, ζ and χ are simply recognized as the usual coefficients of shear viscosity, bulk viscosity and heat conduction, ξ is a pure (special)relativistic contribution. Imposing the entropy generation to be positive, one finds $\xi = T\chi$.

Finding the exact expression for the coefficients is a much harder work, involving relativistic kinematics. The results are found in [1, 17], and considering the viscosity due to the Thomson scattering between the photons and the cosmological fluid, the coefficients read

$$\chi = \frac{4}{3} \mathrm{a_{SB}} T^4 \tau \tag{3.3a}$$
$$\eta = \frac{4}{15} \mathrm{a_{SB}} T^4 \tau \tag{3.3b}$$
$$\zeta = 4 \mathrm{a_{SB}} T^4 \tau \left(\frac{1}{3} - \frac{\partial p}{\partial \rho} \right)^2 , \tag{3.3c}$$

where $\mathrm{a_{SB}}$ is the Stefan-Boltzmann constant, T is the photon temperature, τ is the photon mean free time and is given by [89] $\tau \approx (n_\gamma \sigma_T v)^{-1}$, where n_γ is the photons density. Using $v \approx c$, $\sigma_T \approx 6.6 \times 10^{-29}\,\mathrm{m}^2$ the cross section for the Thomson scattering and $T = T_\gamma^0 (1+z)$ with $T_\gamma^0 \approx 2.73\,\mathrm{K}$, we find

$$n_\gamma \approx 2.0 \times 10^7\,\mathrm{m}^{-3} (T/\mathrm{K})^3 \approx 1.3 \times 10^4\,\mathrm{m}^2\mathrm{s}^{-}1 \left(\frac{1+z}{1+z_{rec}} \right)^{-3/2} . \tag{3.4}$$

3.3 Perturbation scheme

We focus our attention to matter dominated universe. Thus, we work in the Newtonian approximation, following the same perturbation scheme of [89], that in turn is an enhancement of the Newtonian model of sec. 1.1.3, while replacing the Euler equation with the Navier-Stokes equation (3.5b). The physical system, in the background synchronous and comoving reference frame, is described by[1]

$$\dot{\rho} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \tag{3.5a}$$

$$\begin{aligned} &\rho \dot{\boldsymbol{v}} + \rho (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} + \boldsymbol{\nabla} P + \rho \boldsymbol{\nabla} \phi \\ &\quad - \eta \nabla^2 \boldsymbol{v} - \left(\zeta + \frac{1}{3} \eta \right) \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{v}) - \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi} = 0 \end{aligned} \tag{3.5b}$$

$$\dot{\boldsymbol{B}} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) = 0 \tag{3.5c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{3.5d}$$

$$\nabla^2 \phi - 4\pi G \rho = 0, \tag{3.5e}$$

where the dot expresses the derivative with respect to the synchronous time. Here, G is the gravitational constant, ρ is the matter density, $\boldsymbol{v}$ its velocity, P its pressure, ϕ the gravitational potential, B the magnetic field and ζ and η are respectively the bulk and shear viscosity coefficients.

We chose to neglect heat conduction, because it is expected to give much smaller effects. Moreover, we stress that we must take account of the relativistic effects because, even if the cosmological fluid is considered classical, the single particles can still (and usually do) behave in a relativistic regime [18]. Thus, the relativistic kinematics is used to derive the properties of the classical cosmological fluid.

For the homogeneous background the viscous terms vanish and we are left with the usual solution

$$\rho \sim \frac{1}{a^3} \tag{3.6a}$$

$$\boldsymbol{v} = H \boldsymbol{r} \tag{3.6b}$$

$$\boldsymbol{B} \sim \frac{1}{a^2} \tag{3.6c}$$

$$\boldsymbol{\nabla} \phi = \frac{4}{3} \pi G \rho \boldsymbol{r}, \tag{3.6d}$$

where $\boldsymbol{r}$ are the spatial coordinates, a is the universe scale factor and $H = \dot{a}/a$.

Perturbing the system at first order we find the equations that describe the

[1] Throughout this chapter, we consider the light speed as $c = 1$.

inhomogeneities

$$\dot{\delta\rho} + 3H\,\delta\rho + H(\boldsymbol{r}\cdot\boldsymbol{\nabla})\,\delta\rho + \rho\boldsymbol{\nabla}\cdot\delta\boldsymbol{v} = 0 \tag{3.7a}$$

$$\nabla^2\,\delta\phi - 4\pi G\,\delta\rho = 0 \tag{3.7b}$$

$$\begin{aligned}\dot{\delta\boldsymbol{v}} + H\,\delta\boldsymbol{v} + H(\boldsymbol{r}\cdot\boldsymbol{\nabla})\,\delta\boldsymbol{v} + \frac{\boldsymbol{\nabla}\,\delta P}{\rho} + \boldsymbol{\nabla}\,\delta\phi\\ - \frac{\eta}{\rho}\nabla^2\,\delta\boldsymbol{v} - \frac{\zeta + \frac{1}{3}\eta}{\rho}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\delta\boldsymbol{v}) - (\boldsymbol{\nabla}\times\delta\boldsymbol{B})\times\frac{\boldsymbol{B}}{4\pi\rho} = 0\end{aligned} \tag{3.7c}$$

$$\boldsymbol{\nabla}\cdot\delta\boldsymbol{B} = 0 \tag{3.7d}$$

$$\dot{\delta\boldsymbol{B}} + 2H\,\delta\boldsymbol{B} + H(\boldsymbol{r}\cdot\boldsymbol{\nabla})\,\delta\boldsymbol{B} + \boldsymbol{B}(\boldsymbol{\nabla}\cdot\delta\boldsymbol{v}) - (\boldsymbol{B}\cdot\boldsymbol{\nabla})\,\delta\boldsymbol{v} = 0. \tag{3.7e}$$

Defining the dimensionless magnetic perturbation as

$$\boldsymbol{\mathcal{B}} = \frac{\delta\boldsymbol{B}}{B} \tag{3.8}$$

we can expand the inhomogeneities in plane waves

$$\delta x = \widetilde{\delta x}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{3.9}$$

where δx is any perturbed quantity, $\boldsymbol{k}$ is the physical wavenumber and $\boldsymbol{q} = a\boldsymbol{k}$ is the comoving wavenumber, constant during the expansion. Practically speaking, this implies the substitutions $\boldsymbol{\nabla} \to \mathrm{i}\boldsymbol{k}$ and $\partial_t \to \partial_t - \mathrm{i}H(\boldsymbol{k}\cdot\boldsymbol{r})$. From now on, every variable is intended as Fourier transformed. For simplicity of notation, we will drop the tilde over the variables. Our Fourier transformed system now reads

$$\dot{\delta\rho} + 3H\,\delta\rho + \mathrm{i}\rho(\boldsymbol{k}\cdot\delta\boldsymbol{v}) = 0 \tag{3.10a}$$

$$\begin{aligned}\dot{\delta\boldsymbol{v}} + H\,\delta\boldsymbol{v} + \mathrm{i}\boldsymbol{k}\,\delta\rho\left(\frac{v_S^2}{\rho} - \frac{4\pi G}{k^2}\right) + \frac{\eta}{\rho}k^2\,\delta\boldsymbol{v}\\ + \frac{\zeta + \frac{1}{3}\eta}{\rho}\boldsymbol{k}(\boldsymbol{k}\cdot\delta\boldsymbol{v}) + \mathrm{i}v_A^2\hat{\boldsymbol{B}}\times(\boldsymbol{k}\times\boldsymbol{\mathcal{B}}) = 0\end{aligned} \tag{3.10b}$$

$$\dot{\boldsymbol{\mathcal{B}}} + \mathrm{i}\hat{\boldsymbol{B}}(\boldsymbol{k}\cdot\delta\boldsymbol{v}) - \mathrm{i}(\hat{\boldsymbol{B}}\cdot\boldsymbol{k})\,\delta\boldsymbol{v} = 0 \tag{3.10c}$$

where the sound speed v_S and the Alfvén speed v_A are defined through

$$\delta P = v_S^2\,\delta\rho \tag{3.11}$$

$$v_A^2 = \frac{B^2}{4\pi\rho} \ll 1 \tag{3.12}$$

and we have removed $\delta\phi$ through eq. (3.7b).

We can finally write the system in a fashion similar to the final one of [89]. Decomposing $\delta\boldsymbol{v}$ in its components $\delta\boldsymbol{v}^{\parallel}$ and $\delta\boldsymbol{v}^{\perp}$ parallel and orthogonal to $\boldsymbol{k}$ and introducing the same scalar variables of [89], plus ω_η and ω_ζ, we have

$$\delta \boldsymbol{v} = \delta v^{\parallel}\, \hat{\boldsymbol{k}} + \delta \boldsymbol{v}^{\perp}\,, \quad \delta \boldsymbol{v}^{\perp} \cdot \hat{\boldsymbol{k}} = 0 \tag{3.13a}$$

$$\delta = \frac{\delta\rho}{\rho} \tag{3.13b}$$

$$\bar{b} = \boldsymbol{B} \cdot \hat{\boldsymbol{B}} \tag{3.13c}$$

$$\theta = \mathrm{i}\boldsymbol{k} \cdot \delta\boldsymbol{v} = \mathrm{i}k\, \delta v^{\parallel}\,, \quad \bar{v} = \mathrm{i}k\, \delta\boldsymbol{v}^{\perp} \cdot \hat{\boldsymbol{B}} \tag{3.13d}$$

$$\mu = \hat{\boldsymbol{B}} \cdot \hat{\boldsymbol{k}}, \quad 0 \le \mu \le 1 \tag{3.13e}$$

$$\omega_S^2 = v_S^2 k^2, \quad \omega_0^2 = \omega_S^2 - 4\pi G\rho, \quad \omega_A^2 = v_A^2 k^2 \tag{3.13f}$$

$$\omega_\eta = \frac{\eta}{\rho} k^2, \quad \omega_\zeta = \frac{\zeta}{\rho} k^2 \tag{3.13g}$$

and the final system reads

$$\dot{\delta} + \theta = 0 \tag{3.14a}$$

$$\dot{\theta} + \left(2H + \frac{4}{3}\omega_\eta + \omega_\zeta\right)\theta - \omega_0^2 \delta - \omega_A^2 \bar{b} = 0 \tag{3.14b}$$

$$\dot{\bar{b}} + (1-\mu^2)\theta - \mu\bar{v} = 0 \tag{3.14c}$$

$$\dot{\bar{v}} + 2H\bar{v} + \omega_\eta \bar{v} + \mu\omega_A^2 \bar{b} = 0. \tag{3.14d}$$

As we have seen in sec. 2.8.2, eq. (2.120), we can express the effect of the sound speed and the Alfvén speed using the constants Λ_S and Λ_A. In a similar way, using eqs. (3.3) and eq. (3.4) we see that $\omega_\eta = \text{const}$ is constant and also $\omega_\zeta = \text{const}$ is constant as long as we can neglect v_S^2, that is after recombination. Moreover, after recombination $v_s^2 = \frac{\partial p}{\partial \rho} \ll 1$ so $\zeta = \frac{5}{3}\eta$, $\omega_\zeta = \frac{5}{3}\omega_\eta$. This way, we can express the system (3.14) in a more suitable form:

$$\dot{\delta} + \theta = 0 \tag{3.15a}$$

$$\dot{\theta} + \left(2H + \frac{4}{3}\omega_\eta + \omega_\zeta\right)\theta - \left(\frac{\Lambda_S^2}{t^{2+2\nu}} - 4\pi G\rho\right)\delta - \frac{\Lambda_A^2}{t^2}\bar{b} = 0 \tag{3.15b}$$

$$\dot{\bar{v}} + (2H + \omega_\eta)\bar{v} + \mu\frac{\Lambda_A^2}{t^2}\bar{b} = 0 \tag{3.15c}$$

$$\dot{\bar{b}} + (1-\mu^2)\theta - \mu\bar{v} = 0. \tag{3.15d}$$

3.4 Analytical solutions in the main physical limits

The system (3.14) has no simple analytical solution, but we can get a hint on its evolution by looking at the most important limits for the quantities involved. The main parameters we are interested in are the viscous frequencies ω_η and ω_ζ, the magnetic field frequency ω_A and the anisotropy parameter μ. We consider the period after recombination, where $\omega_\zeta \simeq \frac{5}{3}\omega_\eta$.

The first obvious regimes to consider are $\omega_A = \omega_\eta = \omega_\zeta = 0$, which recovers the standard solution in the form of Bessel functions [18], and $\omega_\zeta = \omega_\eta = 0$, which is the magnetic case studied in chap. 2 and by [89, 107]. As we already know these solutions, we will firstly analyse the pure viscous case and then the general evolution under some interesting limits of the parameter μ.

3.4.1 Pure viscous limit

If we set $\omega_A = 0$ the system (3.15) reduces to its first two equations and can be easily solved. The viscosity enters the evolution through the coefficient $2H + \frac{4}{3}\omega_\eta + \omega_\zeta$. Given that after recombination $\omega_\zeta = \frac{5}{3}\omega_\eta$, this becomes $2H + 3\omega_\eta$: using eq. (2.117) we have

$$\ddot{\delta} + \left(\frac{4}{3t} + 3\omega_\eta\right)\dot{\delta} + \left(\frac{\Lambda_S^2}{t^{2\nu}} - \frac{2}{3}\right)\frac{\delta}{t^2} = 0, \tag{3.16}$$

where $\nu = \gamma - 4/3 \geq 0$.

For $1100 > z > 100$ we have $\nu = 0$. The density perturbations then evolve as

$$\delta = t^{\frac{-1+\Delta}{6}} \mathrm{e}^{-3\omega_\eta t}\left[\delta_1 U\left(\tfrac{7+\Delta}{6}, 1 + \tfrac{\Delta}{3}, \omega_\eta t\right) + \delta_2 L\left(-\tfrac{7+\Delta}{6}, \tfrac{\Delta}{3}, 3\omega_\eta t\right)\right] \tag{3.17a}$$

$$\Delta = \sqrt{25 - 36\Lambda_S^2}, \tag{3.17b}$$

where δ_1 and δ_2 are the two integration constants, U is the confluent hypergeometric U function and $L(n, a, x)$ is the Laguerre polynomial $L_n^a(x)$.

When $z < 100$ there is no analytical solution, unless we assume the reasonable condition that $\Lambda_S^2 t^{-2\nu} \to 0$: this is surely true enough inside the Jeans scale. This way we get

$$\delta = \delta_1\left(1 - \frac{2}{9\omega_\eta t}\right) + \delta_2\left[\frac{3^{5/3}\mathrm{e}^{-3\omega_\eta t}}{t^{1/3}} + \frac{2 - 9\omega_\eta t}{\omega_\eta^{2/3} t}\Gamma\left(\frac{2}{3}, 3\omega_\eta t\right)\right], \tag{3.18}$$

where, as before, δ_1 and δ_2 are the two integration constants and $\Gamma(a, z)$ is the incomplete gamma function.

3.4.2 Magnetic viscous case: parallel and orthogonal modes

In presence of a magnetic field, there is no general analytical solution. However, if we look only at modes parallel or orthogonal to the background magnetic field, we can find one.

When $\mu = 1$, by definition of the perturbed variables it holds $\bar{b} = 0$, and the solution reduces to the previous case, with no effects deriving from the magnetic field.

When $\mu = 0$, we solve eqs. (3.17b) and (3.14c) through

$$\bar{b} = \delta + \Delta_{\delta b}, \qquad \Delta_{\delta b} = \text{const}, \tag{3.19}$$

and we find a general solution of eqs. (3.14) of the form

$$\delta = \delta_b + \delta_{LU}(t), \tag{3.20}$$

where $\delta_b = \text{const}$ and δ_{LU} is given by eq. (3.17a), but with $\Delta = \Delta_b = \text{const}$. When $\nu = 0$ we have

$$\delta_b = \frac{3\Delta_{\delta b}\Lambda_A^2}{2 - 3\Lambda_S^2 - 3\Lambda_A^2} \tag{3.21a}$$

$$\Delta_b = \sqrt{25 - 36\Lambda_S^2 - 36\Lambda_A^2}, \tag{3.21b}$$

while for $\nu > 0$ we assume as before $\Lambda_S^2 t^{-2\nu} \to 0$ and we have

$$\delta_b = \frac{3\Delta_{\delta b}\Lambda_A^2}{2 - 3\Lambda_A^2} \tag{3.22a}$$

$$\Delta_b = \sqrt{25 - 36\Lambda_A^2}. \tag{3.22b}$$

3.4.3 Mathematical limits of the solutions

We will now analyse the most interesting limits for the solutions of eq. (3.17a) and eq. (3.18), i.e. either towards the non viscous case $\omega_\eta \to 0$ and for large times $t \to \infty$. This way, we will consider also the viscous-magnetic cases, because when $\mu = 1$ there are no magnetic effects, while for $\mu = 0$ the solution expressed by eq. (3.20) has the same form of eq. (3.17a), except for the value of the constant Δ.

For eq. (3.17a), for simplicity we will limit our study to the solutions with $\Delta \in \mathbb{R}$, i.e. when it exists a growing mode in the non-viscous case.

In the non-viscous limit, it holds

$$\omega_\eta^{\Delta/3} U \xrightarrow[\omega_\eta \to 0]{} c_U \cdot t^{(-1-\Delta)/6} \tag{3.23}$$

$$\omega_\eta^{\Delta/3} L \xrightarrow[\omega_\eta \to 0]{} c_L \cdot t^{(-1+\Delta)/6}. \tag{3.24}$$

The rescaling $\omega_\eta^{\Delta/3}$ is needed to avoid non-physical divergent terms, and it should be included in the constants δ_1 and δ_2, when fixed by a meaningful Cauchy problem, while c_U and c_L are numerical constants. We have recovered the usual growing and decaying modes.

For large enough times the limits are

$$U \xrightarrow[t->\infty]{} c_U \tag{3.25}$$

$$L \xrightarrow[t->\infty]{} \lim_{t->\infty} c_L \cdot t^{(-1+\Delta)/6} \mathrm{e}^{-3\omega_\eta t} = 0, \tag{3.26}$$

where c_U and c_L are again numerical constants: there is no growing mode.

For eq. (3.18) the situation is a little more complex. If we define the functions δ_1^t and δ_2^t as the independent solutions that compose $\delta(t)$

$$\delta_1^t(t) \equiv \delta_1 \left(1 - \frac{2}{9\omega_\eta t}\right) \tag{3.27}$$

$$\delta_2^t(t) \equiv \delta_2 \left[\frac{3^{5/3}\mathrm{e}^{-3\omega_\eta t}}{t^{1/3}} + \frac{2 - 9\omega_\eta t}{\omega_\eta^{2/3} t}\Gamma\left(\frac{2}{3}, 3\omega_\eta t\right)\right], \tag{3.28}$$

then we have

$$\omega_\eta \delta_1^t(t) \xrightarrow[\omega_\eta \to 0]{} -\frac{2\delta_1}{9t} \propto \frac{1}{t} \tag{3.29}$$

$$\omega_\eta^{2/3} \delta_2^t(t) \xrightarrow[\omega_\eta \to 0]{} \frac{2\Gamma(2/3)\delta_2}{t} \propto \frac{1}{t}, \tag{3.30}$$

where the powers of ω_η, as before, are necessary to avoid divergence of the limits. We already know the correct limits [18]: t^{-1} and $t^{2/3}$. Apparently, we miss the growing mode. The explanation is that it is present, but not at the dominant order, in δ_2^t. Fortunately, the dominant order is proportional to the other solution, so a linear combination of them should give the correct limit. After some calculations, we choose an appropriate value of δ_1 to find

$$\delta_1 = \frac{-2+9\omega_\eta}{9\omega_\eta^{5/3}} 3^{1/3} \left(\frac{1}{18} + \omega_\eta \frac{1}{4}\right) \Gamma\left(\frac{2}{3}\right) \delta_2 \tag{3.31a}$$

$$\frac{1}{\omega_\eta}\delta(t) \xrightarrow[\omega_\eta \to 0]{} 3^{2/3} \frac{81}{10} \delta_2 t^{2/3} : \tag{3.31b}$$

we restored the correct independent solutions.

For long times, the solutions tend to

$$\delta_1^t(t) \xrightarrow[t->\infty]{} \delta_1 \tag{3.32}$$

$$\delta_2^t(t) \xrightarrow[t->\infty]{} \lim_{t->\infty} \frac{3^{5/3} \mathrm{e}^{-3\omega_\eta t}}{t^{4/3}\omega_\eta} = 0. \tag{3.33}$$

We have again a constant and a decaying mode, but no growing solution.

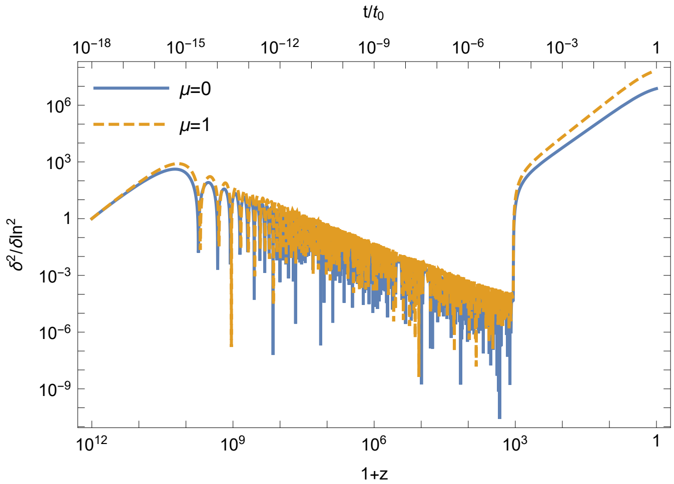
Given the preceding results, we can conclude that, when relevant in the physical system, viscosity prevents any growing mode. This, however, is not easily the case.

While it is not easy to derive analytically a limiting scale, looking at eq. (3.14b), the viscosity is important if $2H \simeq 9\omega_\eta$: for $z \simeq 100$ that means a wavenumber $q \simeq 0.97\,\mathrm{kpc}^{-1}$, corresponding to roughly 5000 solar masses. A similar condition appears from eq. (3.14d). On the other hand, the magnetic Jeans length, calculated in [107], is about $10\,\mathrm{Mpc}^{-1}$ and is a much more stringent constraint.

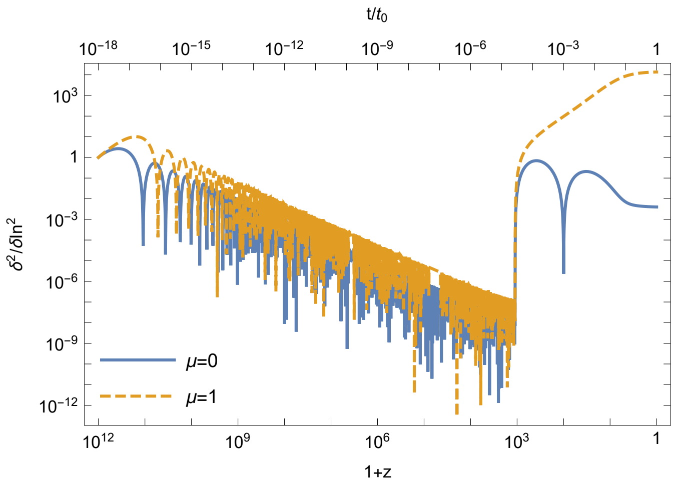
3.5 Numerical analysis

To better show the behaviour of our system, we numerically integrated the equations. The effect of the magnetic field alone is showed in [89, 107] and, when the wavenumber is below the magnetic scale, it consists in a slight splitting between perturbations parallel and orthogonal to the magnetic field; on the other hand, over the magnetic limiting wavenumber, perturbations with $\mu = 0$ are damped.

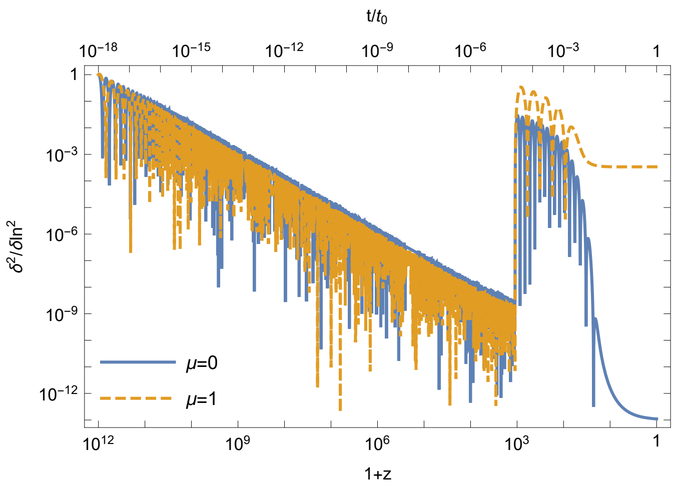
To show an interesting regime, i.e. one in which both viscosity and magnetic field play an important role, we supposed a magnetic field 10 times weaker than the maximum allowed by the constraints, i.e. $B(z = 0) = 10^{-10}\,\mathrm{G}$. The universe main variables magnitudes are taken from [113]. The results of the integration in fig. 3.1 clearly show the behaviour described by the analytical limits. In particular, while in fig. 3.1a we only see a small viscous damping, fig. 3.1c follows exactly the limits of eqs. (3.25) and (3.26) and shows both the viscous and the magnetic damping. Fig. 3.1b is in the middle between the other two, with a small viscous damping and a smaller presence of the one due to the magnetic field. We can thus state that inside the Jeans scale viscosity plays a role only at the very end of the evolution, if any at all, but at such a time nonlinear effects are expected to dominate. On the other end, between the standard Jeans length and the magnetic one and in presence a fast oscillating mode, the viscosity causes a fast damping of the oscillating mode, enhancing the anisotropic effects of the magnetic field. This is because, at



(a) Perturbations at a scale of $q \simeq 40\,\mathrm{Mpc}^{-1}$. This scale is nearly not affected by viscosity, but shows the splitting due to the magnetic field presence.



(b) Perturbations at a scale of $q \simeq 140\,\mathrm{Mpc}^{-1}$. Here, viscosity starts to damp the perturbations, while its effects combine with the magnetic ones for $\mu = 0$.



(c) Perturbations at a scale of $q \simeq 1\,\mathrm{kpc}^{-1}$. This scale suffers both the damping due to viscosity and the one due to the magnetic field, which becomes very strong.

Figure 3.1. Effect of combined viscosity and magnetic field on the cosmological perturbations, when $B(z = 0) = 10^{-10}\,\mathrm{G}$.

such a scale, the viscous effects are stronger and come into play at a smaller time (fig. 3.1c).

While the integration at a fixed scale shows the behaviour of the solution of eqs. (3.14), the effect on the final anisotropy is better shown in the coordinate space. Then, we integrated a Gaussian distribution of initial shape

$$\delta(\boldsymbol{x}, t_{\text{rec}}) = \delta(\boldsymbol{x} = 0, t_{\text{rec}})\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2}}, \quad \sigma = 0.05\,\text{kpc}^{-1}, \tag{3.34}$$

where $\boldsymbol{x}$ are the comoving coordinates, from recombination to $z = 10$, with the maximum allowed magnetic field of $10^{-9}\,$G. This Gaussian perturbation encloses in the 3σ region roughly the mass of a dwarf galaxy. The integration was performed as follows: we applied a Fourier transformation to the initial distribution, then we evolved each harmonic through eqs. (3.14), and finally we transformed back in the coordinate space at the preceding time. Thus, we are in the spatial coordinates domain, differently from both the analytical limits and the other numerical integrations. The results are shown in fig. 3.2: the effects of viscosity on the final anisotropy are negligible at this scale.

3.6 Concluding remarks

The observation of filament-like distribution of mass across the universe large scale calls attention for possible mechanisms for their generation. In [89], it was argued that the seeds of these filaments could be recognized in the anisotropy that the linear growth of cosmological perturbation takes in the presence of a magnetic field.

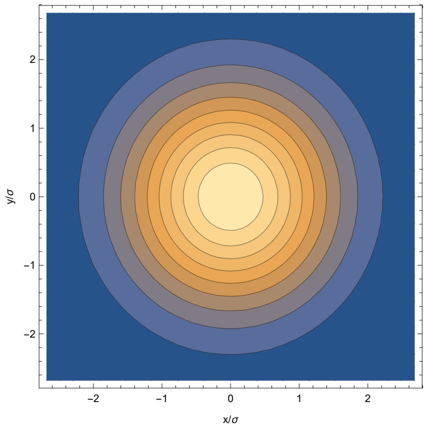
Clearly, while on the perturbation scale such an anisotropy of the perturbations is of order unity, when the density contrasts are calculated, such effect becomes not greater than few percent. However, the idea is that the amplitude of the density contrast anisotropy is amplified in the non-linear regime of the perturbation growth, towards the formation of large scale structures.

A crucial point to understand the reliability of such a proposal to explain the formation of filaments, consists of the possibility to exclude that the anisotropy induced by the magnetic field be suppressed by other physical effects also present in the perturbation dynamics, with particular reference to the dissipation phenomena. Here we consider the role played by both shear and bulk viscosity on the evolution of the Jeans instability for a magnetized universe.

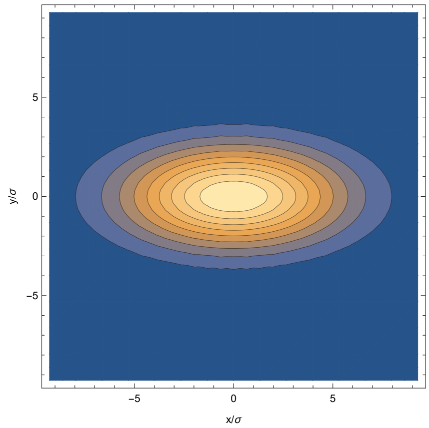
We treated the expanding universe via a fluid-like representation, according to the analysis in [18], whose background morphology are fixed by the universe averaged density and the expansion rate as unperturbed fluid velocity.

Once constructed the dispersion relation associated to the Jeans instability, we evolve the linear growth of a overdensity, which starts as a spherical blob and it is deformed via the anisotropic effect of the magnetic field.

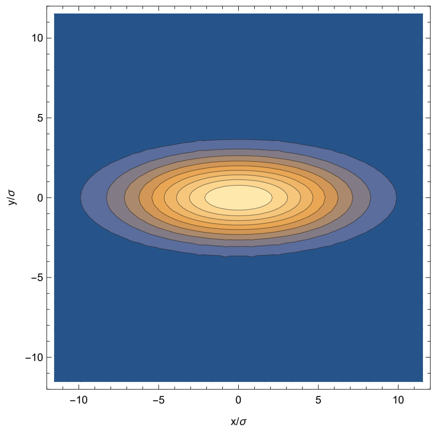
We provide the important evidence that the range of model parameters where the magnetic field effect and the viscosity damping simultaneously survive, is very tiny. As a consequence, we can fix the scale of about 5000 solar masses as that one below which the viscosity effects really attenuate the magnetized perturbation anisotropy. Since such a scale has a limited cosmological significance in view of the



(a) Distibution at $z = 1000$, with eccentricity of $\epsilon \approx 0.27$. There is no measurable difference between viscous and non viscous case.



(b) Distibution at $z = 100$, with eccentricity of $\epsilon \approx 0.89$ and a difference between viscous and non viscous case of order 10^{-6}.



(c) Distibution at $z = 10$, with eccentricity of $\epsilon \approx 0.93$ and a difference between viscous and non viscous case of order 10^{-4}.

Figure 3.2. Integration of an initial Gaussian density distribution with eq. (3.34), from $z = 1100$ to $z = 10$. Viscosity plays almost no role in the anisotropy. The figures show equal density contours, corresponding to $(0.1, 0.2, \ldots 0.9)$ times the central density. The background magnetic field with $B(z = 0) = 10^{-9}$ G is directed along the y axis.

structure formation across the universe, we can conclude that the dissipation due to the shear and bulk viscosity is unable to suppress the seed of filaments, at least at cosmologically relevant scales. Furthermore, the scale of 5000 solar masses fixes a cut-off on the filaments scale if they are actually due to magnetic properties of the primordial universe. This issue can give some hints on the phenomenological investigation of the proposed idea in terms of the very early universe observation at very small scales.

Finally, we recall how the reliability of the proposed model relies on the strong ambipolar diffusion which exist between the baryonic universe component and the weak plasma remnant after the recombination has taken place: the effects that the magnetic field induces on the plasma distribution are transferred by the Thomson scattering to the baryons, which are falling in the dark matter skeleton.

Chapter 4

Overview of quantum gravity and quantum cosmology

Here we present an overview of the quantization framework for General Relativity, which we will need in the following chapters, and its application to Cosmology. It can be found in a number of textbooks; in particular, we follow [5, 21, 30, 86, 94] and refer to those for more specific topics. This is not intended to be an exhaustive explanation of the topic or to present it with all of its subtleties, for which we refer to the already cited textbooks. Instead, the aim of this chapter is to present the general context of the analyses of the following ones.

In sec. 4.1 we present the Hamiltonian formulation of general relativity, obtained through the ADM formalism. In sec. 4.2 we apply the canonical quantization to general relativity, obtaining the Wheeler-DeWitt equation, and we present its main issues. We end the section presenting the framework of quantum cosmology and the minisuperspace models. In sec. 4.3 we face the semiclassical approximation of a minisuperspace model, in order to find both a meaningful definition of time at a quantum level ad a viable interpretation for the wave function of the universe.

4.1 Hamiltonian formulation of general relativity

The first step towards the quantization of general relativity is to express the Einstein equations in a Lagrangian formulation. From there, we will be able to move to the Hamiltonian description and to apply the quantization process. It must be noted that these formulations offer other perspectives of GR and are important by themselves, and not only in view of the quantization of the theory. Nevertheless, they are an essential step toward the quantum theory of GR.

4.1.1 Lagrangian formulation of general relativity

The dynamical content of general relativity is fully expressed in Einstein's field equations. The main field variable is the spacetime metric $g_{\mu\nu}$, defined over a manifold $\mathcal{M}$. The awkwardness of this theory relies in the dependence of the integration volume element on the metric, i.e. on the field variable.

We choose to present here the most common formulation, that is the Einstein-Hilbert action. Except for some boundary terms, the vacuum Einstein's equations can be derived from the Lagrangian density

$$\mathcal{L}_{\mathrm{G}} = \sqrt{-g}R. \tag{4.1}$$

To obtain the vacuum field equations, we must require the variation of the corresponding action with respect to the inverse metric to vanish. Moreover, the standard boundary conditions are not enough, and we must require also the first derivatives of the field to be fixed on the boundary

$$\delta g^{\mu\nu}|_{\partial\mathcal{M}} = 0, \qquad \delta\partial_\rho g^{\mu\nu}|_{\partial\mathcal{M}} = 0. \tag{4.2}$$

Because of this, there exists a number of different Lagrangian densities to solve this issue and require only the metric tensor to be fixed on the boundary; not all of them, however, have the same symmetries of the Einstein-Hilbert action. For example the $\Gamma\Gamma$ formulation is not scalar nor covariant, but it requires only the first condition of eq. (4.2).

Adding the matter Lagrangian density $\mathcal{L}_{\mathrm{M}}$, which takes account of the contribution from all matter sources, we may write the Einstein-Hilbert action functional as

$$S[g_{\mu\nu}] = \int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}\left[\frac{1}{2\mathcal{K}}R - \mathcal{L}_{\mathrm{M}}\right], \tag{4.3}$$

where $\mathcal{K} = 8\pi G/c^4$. The variation of this action gives Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \mathcal{K}T_{\mu\nu}, \tag{4.4}$$

where the stress-energy tensor is

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\left[\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{M}})}{\delta g^{\mu\nu}} - \frac{\partial}{\partial x^\rho}\frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta(\partial_\rho g^{\mu\nu})}\right]. \tag{4.5}$$

Since $\mathcal{L}_{\mathrm{G}}$ does not depend on the matter fields, variation of the action with respect to them will yield the same equations as variation of the matter action alone.

4.1.2 Spacetime foliation

The Lagrangian formulation of a field theory is spacetime covariant, with the meaning that once the action functional of the fields has been specified on the spacetime manifold, its extremization yields the field equations. On the other hand, the Hamiltonian formulation requires the breakup of spacetime into space and time.

The problem in the definition of the GR Hamiltonian lies exactly in the absence of a suitable time parameter: eqs. (4.4), obtained through the Euler-Lagrange equations, are not dynamical because they do not involve any time parameter. Rather, their solution describes the intensity, and not the evolution, of the gravitational field $g_{\mu\nu}$ at any point in spacetime, given the energy-matter stress-energy tensor. The problem we have to face is then the separation of space and time, and this is addressed through the ADM foliation.

The canonical formulation of general relativity assumes a global hyperbolic topology for the physical spacetime $\mathcal{M}$. This ensures that the Cauchy problem for the gravitational field is well posed and it allows the splitting [14]

$$\mathcal{M} = \mathbb{R} \otimes \Sigma, \tag{4.6}$$

where Σ is a compact spacelike 3-dimensional manifold representing the 3-space.

The hypersurfaces Σ are equal time surfaces, in the sense that they share the same parameter x^0. If we now call x^i the spatial coordinates of the surfaces Σ, the equations

$$u^\mu = u^\mu(x^i; x^0) \tag{4.7}$$

are a parametric set of equations for the family of hypersurfaces. Given a basis $\{\boldsymbol{f}_\mu\}$ on $\mathcal{M}$, we can define a basis tangent to the surface Σ_{x^0} with fixed x^0 through

$$\boldsymbol{b}_i = \frac{\partial u^\mu}{\partial x^i} \boldsymbol{f}_\mu. \tag{4.8}$$

We can complete this basis with a vector orthogonal by definition to all $\boldsymbol{b}_i$

$$\boldsymbol{\eta} = \eta^\mu \boldsymbol{f}_\mu, \qquad \boldsymbol{\eta} \cdot \boldsymbol{b}_i \equiv 0. \tag{4.9}$$

Since Σ_{x^0} is spacelike, $\boldsymbol{\eta}$ must be timelike. We can finally impose the normalization of $\boldsymbol{\eta}$ as $\boldsymbol{\eta} \cdot \boldsymbol{\eta} = -1$.

We can define the *deformation vector*, which connects points with the same coordinates x^i of two infinitesimally close surfaces Σ_{x^0} and $\Sigma_{x^0+\delta x^0}$, as

$$\boldsymbol{b}_0 = \frac{\partial u^\mu}{\partial x^0} \boldsymbol{f}_\mu = N\boldsymbol{\eta} + N^i \boldsymbol{b}_i. \tag{4.10}$$

Its projections on the basis $\{\boldsymbol{b}_i, \boldsymbol{\eta}\}$ are the *lapse function* N, which measures the proper time separation between the surfaces Σ_{x^0} and $\Sigma_{x^0+\delta x^0}$, and the *shift vector* N^i, which measures the displacement of the point x^i on $\Sigma_{x^0+\delta x^0}$ from the intersection of $\Sigma_{x^0+\delta x^0}$ with the normal geodesic drawn from x^i on Σ_{x^0}, i.e. the shift between points with the same coordinates. Their role is better explained in fig. 4.1. We can use the new basis $\{\boldsymbol{b}_i, \boldsymbol{b}_0\}$ to write the metric tensor as

$$g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j = \frac{\partial u^\mu}{\partial x^i}\frac{\partial u^\nu}{\partial x^j} \boldsymbol{f}_\mu \cdot \boldsymbol{f}_\nu = \frac{\partial u^\mu}{\partial x^i}\frac{\partial u^\nu}{\partial x^j} g_{\mu\nu} = h_{ij} \tag{4.11}$$

$$g_{0i} = \boldsymbol{b}_0 \cdot \boldsymbol{b}_i = N^j h_{ij}, \tag{4.12}$$

$$g_{00} = \boldsymbol{b}_0 \cdot \boldsymbol{b}_0 = -N^2 + N^i N^j h_{ij}. \tag{4.13}$$

Thus, the metric tensor and its inverse are

$$g_{\mu\nu} = \begin{bmatrix} -N^2 + N^i N^j h_{ij} & N^j h_{ij} \\ N^j h_{ij} & h_{ij} \end{bmatrix}, \qquad g^{\mu\nu} = \begin{bmatrix} -1/N^2 & N^i/N^2 \\ N^i/N^2 & h^{ij} - N^i N^j/N^2 \end{bmatrix}, \tag{4.14}$$

where h_{ij} is the induced metric on the foliation hypersurfaces and $h^{ij}h_{jk} = \delta^i{}_k$. The ADM variables are related to the metric tensor through

$$N = \frac{1}{\sqrt{-g^{00}}}, \qquad N^i = -\frac{g^{0i}}{g^{00}}, \qquad h_{ij} = g_{ij}. \tag{4.15}$$

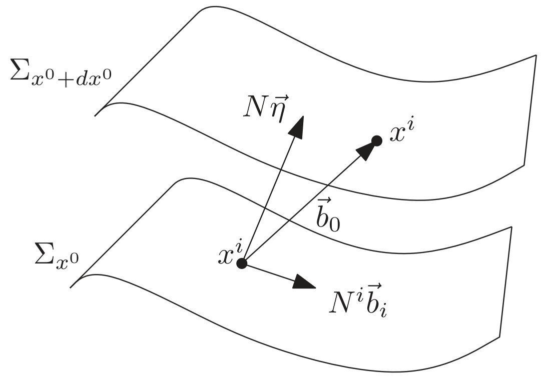


Figure 4.1. Foliation of the spacetime due to the ADM formulation.

It is still necessary to fix the sign of N: we choose it to be positive, so that the foliation evolves towards the future. It is easy to see that

$$\sqrt{-g} = N\sqrt{h}. \tag{4.16}$$

The information contained in (N, N^i, h_{ij}) is equivalent to the one contained in the metric tensor $g_{\mu\nu}$.

4.1.3 Gauss-Codazzi equation and ADM Lagrangian density

At this point, we need to express the gravitational action in terms of (N, N^i, h_{ij}). This is done through the Gauss-Codazzi equation, which states that

$$R = {}^{(3)}R + K_{ij}K^{ij} - K^2 - 2\nabla_\mu(\eta^\nu\nabla_\nu\eta^\mu - \eta^\mu K), \tag{4.17}$$

where

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - {}^{(3)}\nabla_i N_j - {}^{(3)}\nabla_j N_i\right) \tag{4.18}$$

is the *extrinsic curvature* and $K = K_i{}^i$. For its formal proof we refer to [94]. Besides the boundary terms, it allows to express the spacetime curvature R through the ADM variables. We must note, however, that the elimination of the boundary terms is non-trivial and requires again both the conditions expressed in eq. (4.2) to be imposed.

Neglecting the boundary terms we can rewrite the Einstein-Hilbert Lagrangian from eq. (4.1) as

$$\begin{aligned} 2\mathcal{K}\mathcal{L}_{\mathrm{EH}} &= \sqrt{-g}R = N\sqrt{h}\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) \\ &= N\sqrt{h}\left[{}^{(3)}R + (h^{ik}h^{jl} - h^{ij}h^{kl})K_{ij}K_{kl}\right] = 2\mathcal{K}\mathcal{L}_{\mathrm{ADM}}. \end{aligned} \tag{4.19}$$

The Einstein-Hilbert action is now an integral over the manifold $\mathbb{R}\otimes\Sigma_{x^0}$:

$$S = \int_{\mathcal{M}} \mathrm{d}^4x\,\mathcal{L}_{\mathrm{EH}} = \int_{\mathbb{R}\otimes\Sigma_{x^0}} \mathrm{d}x^0\,\mathrm{d}^3x\,\mathcal{L}_{\mathrm{ADM}}. \tag{4.20}$$

4.1.4 Hamiltonian formulation of general relativity

We are ready to derive the Hamiltonian formulation of general relativity. From eq. (4.19) we can find the conjugate momenta

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0 \tag{4.21}$$

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{N}^i} = 0 \tag{4.22}$$

$$\Pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \frac{1}{2\mathcal{K}}\sqrt{h}\left(K^{ij} - h^{ij}K\right). \tag{4.23}$$

Not all conjugate momenta are independent and eqs. (4.21) and (4.22) are not invertible with respect to $\dot{N}$ and $\dot{N}^i$: we are dealing with a constrained system and these are the primary constraints of the theory.

Inverting eq. (4.23) we get

$$K_{ij} = \frac{2\mathcal{K}}{\sqrt{h}} G_{ijkl}\Pi^{ij}, \tag{4.24}$$

where the tensor G_{ijkl} is called *supermetric* and it represents a metric tensor on the configuration space reduced to the h_{ij} coordinates. It is given by

$$G_{ijkl} = \frac{1}{2}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right). \tag{4.25}$$

The ADM Lagrangian density, expressed in the momenta, is

$$\mathcal{L}_{ADM} = 2\mathcal{K}\frac{N}{\sqrt{h}} G_{ijkl}\Pi^{ij}\Pi^{kl} + \frac{1}{2\mathcal{K}} N\sqrt{h}\,{}^{(3)}R. \tag{4.26}$$

In order to make the Legendre transformation invertible, we introduce the Lagrange multipliers λ and λ^i for the primary constraints. Thus the Hamiltonian density is

$$\begin{aligned} H_{ADM} &= \Pi\lambda + \Pi_i\lambda^i + \Pi^{ij}\dot{h}_{ij} - \mathcal{L}_{ADM} \\ &= \Pi\lambda + \Pi_i\lambda^i + N\mathcal{H} + N^i\mathcal{H}_i + 2\,{}^{(3)}\nabla_i\left(h_{kj}N^k\Pi^{ij}\right), \end{aligned} \tag{4.27}$$

where the *super-Hamiltonian* $\mathcal{H}$ and the *supermomentum* $\mathcal{H}_i$ are

$$\mathcal{H} = \frac{2\mathcal{K}}{\sqrt{h}} G_{ijkl}\Pi^{ij}\Pi^{kl} - \frac{1}{2\mathcal{K}}\sqrt{h}\,{}^{(3)}R \tag{4.28}$$

$$\mathcal{H}_i = -2h_{ij}\,{}^{(3)}\nabla_k\Pi^{kj}. \tag{4.29}$$

The last term in eq. (4.27) is a boundary term and from now on we will neglect it.

We can now see the secondary constraints of the theory:

$$[\Pi, H_{ADM}] = -\mathcal{H} = 0, \qquad [\Pi_i, H_{ADM}] = -\mathcal{H}_i = 0: \tag{4.30}$$

the super-Hamiltonian and the supermomentum are exactly the secondary constraints of the theory. The whole Hamiltonian of the theory is a combination of constraints, and therefore vanishes

$$H_{ADM} = 0. \tag{4.31}$$

The equations of motion of the lapse function and the shift vector are

$$\dot{N} = [N, H_{ADM}] = \lambda, \qquad \dot{N^i} = [N^i, H_{ADM}] = \lambda^i : \tag{4.32}$$

the dynamics of N and N^i is completely arbitrary, but the condition $N > 0$ must always hold.

We could show that the constraints form a closed algebra [86, 94].

The primary constraints tell us that N and N^i should not be viewed as dynamical variables, so that the configuration space should be that of the Riemannian metrics h_{ij} on Σ_{x^0}. Moreover, the presence of constraints tells us that the configuration space is still "too large", i.e. we have not yet isolated the true dynamical degrees of freedom. This is related to the gauge freedom present in h_{ij}. Following the analogy with the electromagnetic case (see [30]), we should take the configuration space of general relativity to be the set of equivalence classes of Riemannian metrics on Σ_{x^0}, where two metrics are considered equivalent if they can be carried into each other by a diffeomorphism. This configuration space is called *superspace* [11].

Using the superspace as configuration space, the supermomentum constraint in eq. (4.30) is automatically satisfied, i.e. (4.29) vanishes [30]. However, the super-Hamiltonian constraint remains. It can be viewed as resulting from the gauge arbitrariness in the choice of how to slice the spacetime. It is very similar to the constraint which arises when one parametrizes an originally unconstrained theory in a fixed, background spacetime [20, 25], but the non-linearity of (4.28) makes impossible to "de-parametrize" the constraint. Thus, it is not possible to find a configuration space for general relativity such that only the "true dynamical degrees of freedom" are present in its phase space, and the presence of the super-Hamiltonian constraint appears as an unavoidable feature of the Hamiltonian formulation of general relativity.

4.2 Quantum geometrodynamics

4.2.1 Canonical quantization and the Wheeler-DeWitt equation

As we have seen, the configuration space of canonical quantum gravity is the space of all the Riemannian 3-metrics modulo the spatial diffeomorphism group, i.e. the equivalence class of 3-metrics connected by spatial diffeomorphisms

$$\{h_{ij}\} = \frac{\mathrm{Riemm}(\Sigma)}{\mathrm{Diff}(\Sigma)}. \tag{4.33}$$

The approach one usually follows in quantizing the gravitational field is the *Dirac scheme*, that is the quantization of the constrained system[1].

Proceeding in a formal way, the space of states is that of proper functional of the configurations variables

$$\Psi = \Psi\left(N, N^i, h_{ij}\right). \tag{4.34}$$

[1]The alternative approach is the *reduced quantization*, that consists in solving the constraints at a classical level before the quantization procedure. However, it has several faults, as for example in quantum electrodynamics it is consistent only in the non-interacting case.

The configuration variables and momenta are promoted to operators

$$N(x) \to \hat{N}(x) \equiv N(x), \qquad \Pi(x) \to \hat{\Pi}(x) \equiv -\mathrm{i}\hbar \frac{\delta}{\delta N(x)} \tag{4.35a}$$

$$N^i(x) \to \hat{N}^i(x) \equiv N^i(x), \qquad \Pi_i(x) \to \hat{\Pi}_i(x) \equiv -\mathrm{i}\hbar \frac{\delta}{\delta N^i(x)} \tag{4.35b}$$

$$h_{ij}(x) \to \hat{h}_{ij}(x) \equiv h_{ij}(x), \qquad \Pi^{ij}(x) \to \hat{\Pi}^{ij}(x) \equiv -\mathrm{i}\hbar \frac{\delta}{\delta h_{ij}(x)} \tag{4.35c}$$

and the Poisson brackets are the implemented as commutators

$$\left[\hat{N}(x), \hat{\Pi}(y)\right] = \mathrm{i}\hbar \delta^{(3)}(x-y) \tag{4.36a}$$

$$\left[\hat{N}^i(x), \hat{\Pi}_j(y)\right] = \mathrm{i}\hbar \delta^i_j \delta^{(3)}(x-y) \tag{4.36b}$$

$$\left[\hat{h}_{ij}(x), \hat{\Pi}^{kl}(y)\right] = \mathrm{i}\hbar \frac{1}{2}\left(\delta^k_i \delta^l_j + \delta^k_j \delta^l_i\right) \delta^{(3)}(x-y), \tag{4.36c}$$

where the other commutators vanish.

The primary constraints simply state the independence of the wave functional from the lapse function and shift vector, so we have

$$\Psi = \Psi\left(h_{ij}\right), \tag{4.37}$$

hence the physical states are independent from the variables defining the spacetime slicing. The supermomentum constraint

$$\hat{\mathcal{H}}_i = -2h_{ij}\, {}^{(3)}\nabla_k \left[\frac{\delta \Psi}{\delta h_{kj}(x)}\right] = 0 \tag{4.38}$$

means that the wave functional Ψ depends on the 3-geometry $\{h_{ij}\}$ only, rather than on any specific representation. The last constraint, the super-Hamiltonian, is the famous Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = -\frac{2\hbar^2 \mathcal{K}}{\sqrt{h}} G_{ijkl} \frac{\delta^2 \Psi}{\delta h_{ij}(x)\, \delta h_{kl}(x)} - \frac{1}{2\mathcal{K}} \sqrt{h}\, {}^{(3)}R \Psi = 0, \tag{4.39}$$

which is the fundamental equation for the quantum dynamics of the gravitational field. We stress that we did not approach the issue of factor order ambiguity, and instead we chose the simplest one, with the momenta on the right.

The WDW equation (4.39) carries many problems that prevent a successful quantization of the gravitational field. It still has to be found a Hilbert space for the solutions of the constraints. Moreover, we still miss both a suitable scalar product and the capacity to solve the WDW equation. The 3-dimensional Ricci scalar ${}^{(3)}R$ behaves as a potential in eq. (4.39), however the absence of a restriction on its sign means that tachyon-like objects are expected to appear in a rigorous quantum framework. The theory is not compatible with the requirement of a positive definite spectrum for the operator $\hat{h}_{ij}(x)$; its classical counterpart, however, is a Riemannian metric tensor,l so it is positive definite and we still need to find a physical meaning for these negative values. Moreover, the physical meaning of the functional Ψ is not clear, and it requires a notion of time that is not trivial at a quantum level.

4.2.2 The problem of time

The Hamiltonian is the generator of the time displacements in phase-space, so the Wheeler-DeWitt equation (4.39) and the supermomentum constraint (4.30) lead to a Schrödinger equation for a quantum state not dependent on time

$$\hat{H} = \int_{\Sigma} \mathrm{d}^3x \left(N\hat{\mathcal{H}} + N^i \hat{\mathcal{H}}_i \right) \tag{4.40}$$

$$\mathrm{i}\frac{\partial}{\partial t}\Psi_t = \hat{H}\Psi_t = 0. \tag{4.41}$$

This is called *frozen formalism* because it apparently implies that nothing evolves in the quantum theory, i.e. there is no quantum dynamics. The problem here is that we are identifying two different notions of time. In Newtonian theory and in quantum mechanics, time is a fixed external parameter, which lies at the basis of the quantum commutation relations, as well as the notion of Hilbert space with conserved scalar product. On the other hand, in general relativity time is merely a coordinate, since the theory is invariant under spacetime diffeomorphisms, hence it is not observable.

The notion of time plays a fundamental role in the formulation of the quantum theory, and the conventional Copenhagen interpretation of QFT, as well as its whole framework, breaks down as soon as the metric is no longer fixed. Moreover, even the notion of a clock (in the sense of a quantum observable whose values monotonically grow with an abstract time t) is not compatible with the physical requirement of an energy positive spectrum.

Keeping in mind these considerations, there are essentially three ways to face this serious issue: introducing time before the quantization, after the quantization or dealing with a timeless framework.

In the first approach, time is regarded as a fundamental quantity and time is extracted from the set of the dynamical variables, and not as a functional of the canonical ones. This can be done finding an internal time with respect to the other gravitational degrees of freedom, for example in the ADM reduction of the dynamics, or looking for an external time defined with respect to the matter fields. In this last scenario, we speak of matter clocks. In the first case, the constraints must be solved classically before the quantization of the system, and this violated the geometrical nature of the gravitational field, in favour of the real degrees of freedom. This approach, however, leads to some issues [86].

The time after quantization paradigm is carried out by applying the standard Dirac quantization, i.e. the one described in the previous section, and then recovering a time notion after the quantization. This can be done, for example, in the semiclassical approach, as we will see in sec. 4.3. the main idea is that spacetime does not exists at a fundamental level, but emerges as an approximate feature only under suitable conditions. This approach is very useful in quantum cosmology and it is the standard way to solve the time issue.

The last approach, i.e. that of a timeless framework, is based on the idea that there is no need of time at a quantum level. The quantum theory of gravity can be constructed without the notion of time, which may arise only in some special situations, i.e. in some specific approximation of the theory. This approach is the closest to the principles of general relativity, although it does not come without issues [86].

4.2.3 Quantum cosmology and the minisuperspace

The application of the quantum gravity framework at the cosmological models comes with a huge simplification. The cosmological models arise with the requirement of spatial homogeneity (and, in some cases, also isotropy). For each point $x^i \in \Sigma$ there is a finite number of degrees of freedom in the superspace. This way, the homogeneity causes the fields to be restricted to a finite dimensional subspace of the Wheeler superspace, while all but a finite number of degrees of freedom are frozen out by the symmetries. The resulting finite dimensional configuration space is known as *minisuperspace*.

In a minisuperspace, the diffeomorphism constraint is automatically satisfied and one deals with a purely constrained quantum mechanical system, instead of a field theory, described by a single WDW equation for all the spatial points. However, it has not yet been demonstrated that the truncation to minisuperspace can be regarded as a rigorous approximation of the full superspace. Strictly speaking, setting most of the field modes and their conjugate momenta to zero violates the uncertainty principle. On the other hand, classical cosmology is based on these symmetries and their quantization should answer its fundamental questions. Moreover, a minisuperspace model can be relevant to the description of a generic universe toward the classical singularity then restricted to each cosmological horizon.

A generic n/dimensional minisuperspace model involves the following assumptions:

- the lapse function is time independent, i.e. $N = N(t)$;
- the shift vector is zero, i.e. $N^i = 0$;
- the 3-metric is described by a finite number n of homogeneous coordinates $q^A(t)$, where $A = 1, \dots, n$.

This way, the line element is

$$\mathrm{d}s^2 = N^2(t)\,\mathrm{d}t^2 - h_{ij}(x,t)\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \tag{4.42}$$

The momenta conjugate to the field q^A are $p_B(t)$, with $B = 1, \dots, n$: we deal with a n-dimensional system.

We describe now the vacuum case. In presence of matter, the variables q^A should include also the matter degrees of freedom. The action of the model is

$$S = \int \mathrm{d}t\,\left(p_A\dot{q}^A - N\mathcal{H}\right) = \int \mathrm{d}t\,\left[p_A\dot{q}^A - N\left(\mathcal{G}^{AB}p_Ap_B + U(q)\right)\right], \tag{4.43}$$

where the *minisupermetric* $\mathcal{G}_{AB}$ is the reduced version of G_{ijkl}, where $A, B = \{ij\}, \{kl\}$ run over the independent components of h_{ij}. It has Lorentzian signature $(+,-,-,-,-,-)$ and it is explicitly defined as

$$\mathcal{G}_{AB}\,\mathrm{d}q^A\,\mathrm{d}q^B = \int \mathrm{d}^3x\,\frac{2\mathcal{K}}{\sqrt{h}}G^{ijkl}\,\delta h_{ij}\,\delta h_{kl}\,. \tag{4.44}$$

The potential U is given by the 3-dimensional Ricci scalar

$$U = -\frac{1}{2\mathcal{K}}\int \mathrm{d}^3x\,\sqrt{h}\,{}^{(3)}\!R. \tag{4.45}$$

The variation of eq. (4.43) with respect to the lapse function leads to the scalar constraint

$$\mathcal{H}(q^A, p_A) = \mathcal{G}^{AB} p_A p_B + U(q) = 0, \tag{4.46}$$

and the equations of motions are

$$\dot{q^A} = N\{q^A, \mathcal{H}\}, \qquad \dot{p_A} = N\{p_A, \mathcal{H}\}. \tag{4.47}$$

The system resembles a relativistic particle moving in a n-dimensional curved spacetime, with metric $\mathcal{G}_{AB}$ and subjected to the potential $U(q)$. The Hamiltonian constraint (4.46) reflects the parametrization invariance of the theory, which is the residual of the 4-dimensional diffeomorphism invariance of the full theory.

The canonical quantization of this model is straightforward, with the WDW equation that reads

$$\hat{\mathcal{H}}\Psi = \left(-\nabla^2 + U\right)\Psi = 0, \tag{4.48}$$

where $\Psi = \Psi(q)$ is the wave function of the universe. Here, ∇_A is the covariant derivative constructed from the metric $\mathcal{G}_{AB}$ and the Laplacian is

$$\nabla^2 = \nabla_A \nabla^A = \frac{1}{\sqrt{\mathcal{G}}}\partial_A \left(\sqrt{\mathcal{G}}\mathcal{G}^{AB}\partial_B\right), \tag{4.49}$$

with $\mathcal{G} = |\det \mathcal{G}_{AB}|$. the factor ordering is fixed by the last equation, and this choice is peculiar because the WDW equation has the same form in any minisuperspace coordinate system and it is invariant under the redefinition of the 3-metric fields $q^A \to q'^A(q^A)$.

This theory, however, does not come without its own issues. The system is closed and isolated, and there is no a priori splitting between classical and quantum observers. This way, the concept of measurement is problematic, because there is no external classical observer. Moreover, the probabilistic interpretation of quantum measurement is problematic, too, because the universe is unique and it is not possible to perform many measurements. Finally, the concept of time has the same problems expressed before.

The most accepted idea to face these issues is that a meaningful interpretation of the wave function of the universe can be reached only at a semiclassical level. A quantum mechanical interpretation is possible only for a small subsystem of the entire universe, i.e. when at least some of the minisuperspace variables can be considered semiclassical in the sense of the WKB approximation.

4.3 The problem of time: semiclassical expansion and Vilenkin proposal

Following the discussion of last section, we present here the proposal of [36]. It consist of a semiclassical approximation of the wave function of the universe, through which it is possible to achieve both the definition of a time parameter and a probabilistic interpretation for the wave function of the universe, in a sense similar to quantum mechanics.

To simplify the discussion, we concentrate on minisuperspace models. The action of the model takes the form (4.43), where the variables q^A span over both gravitational and matter degrees of freedom, i.e. h_{ij} and the matter fields ϕ_α. The superpotential is given by eq. (4.45), with the addition of matter

$$U = \int \mathrm{d}^3x\,\sqrt{h}\left(V(\phi) - \frac{1}{2\mathcal{K}}\,{}^{(3)}R\right), \qquad h = \det h_{ij} \tag{4.50}$$

instead of (4.45), where $V(\phi)$ is the potential energy of the fields ϕ_α. The Wheeler-DeWitt equation is (4.48). Even if we chose the same factor ordering of last section, the results of this section are independent from the ordering; this is shown by [38] for a similar calculation, which we will show in the next chapter to be (almost) analogue to the one we are going to present. For brevity of notation, in the following we will drop the hat over the operators.

4.3.1 Classical universes

To distinguish between classical variables and quantum ones (which we will need later), we will rename the classical ones as c^a, and keep q^ν for the quantum ones, where a and ν span the entire original set of the index A, i.e. if $A = 1, \dots, n$, then $a = 1, \dots, n-m$ and $\nu = 1, \dots, m$, with $n > m \geq 0$. Let us start considering the case when all the variables q^A are classical. Then Ψ is a superposition of terms of the form

$$\Psi(c) = A(c)\mathrm{e}^{\mathrm{i}S(c)}, \qquad S(c) \in \mathbb{R}. \tag{4.51}$$

For now, we focus on a single one of such terms. We make explicit the dependence of the quantities by $\hbar^2$ writing the WDW equation as[3]

$$\left(\hbar^2\nabla^2 - U\right)\Psi = 0 \tag{4.52}$$

and requiring that $S(h^a) = \mathcal{O}(\hbar^{-1})$. Replacing eq. (4.51) into the WDW equation we find at the lowest order the Hamilton-Jacobi equation

$$\mathcal{G}^{ab}(\nabla_a S)(\nabla_b S) + U = 0, \tag{4.53}$$

and at the next order

$$2\nabla A \cdot \nabla S + A\nabla^2 S = 0, \tag{4.54}$$

which expresses the conservation of the current

$$j^a = -\frac{\mathrm{i}}{2}\mathcal{G}^{ab}(\Psi^*\nabla_b\Psi - \Psi\nabla_b\Psi^*) = \left|A^2\right|\nabla^a S, \tag{4.55}$$

i.e. it means

$$\nabla_a j^a = 0. \tag{4.56}$$

The action $S(c^a)$ describes a congruence of classical trajectories. The momentum on the trajectory at the point c^a is $p_b = \nabla_b S(c^a)$, and the "velocity" is

$$\dot{c}^a = 2N\nabla^a S. \tag{4.57}$$

[2]The original paper [36] uses a "small dimensionless parameter $\lambda \propto \hbar$". Although this ensures a simpler treatment of the physical dimensions of the various terms of the expansion, for simplicity of the expansion itself we use directly $\hbar$.

[3]In the original paper, instead of making explicit the proportionality of the Laplacian to $\hbar^2$, it was present the assumptions that $U(h^a) = \mathcal{O}\left(\hbar^{-2}\right)$. This is equivalent to our procedure, although we consider our approach more intuitive.

There is a trajectory through each point of the superspace, except forbidden regions when $S \notin \mathbb{R}$. The trajectories can begin or end at the boundaries of the superspace, which represent singular configurations, or at points where $\nabla S = 0$ and the semi-classical approximation breaks down, but closed trajectories are not possible if $S(c)$ is a single valued function on superspace (we assume that the superspace variables are chosen such that this is true).

We define probability distributions on $(n-1)$-dimensional surfaces, which play the role of equal time surfaces. We can choose such surface in a way that they are crossed exactly once by all the trajectories of the congruence, and this happens always in the same direction (we refer to the original paper [36] for more details). This means that

$$\dot{c}^a \, \mathrm{d}\Sigma_a > 0; \tag{4.58}$$

the choice of the sign in the last inequality is arbitrary, and we chose that one. Thus, we can define the probability density

$$\mathrm{d}P = j^a \, \mathrm{d}\Sigma_a \,, \tag{4.59}$$

which is semidefinite. The normalization of Ψ should be chosen such that

$$\int \mathrm{d}P = \int j^a \, \mathrm{d}\Sigma_a = 1. \tag{4.60}$$

An example of a possible choice of such surfaces is given by the surfaces of constant S which are orthogonal to the congruence o trajectories.

4.3.2 Small quantum subsystems

We now introduce some quantum variables q^ν. We assume that the effect of q^ν on the dynamics of c^a is negligible, in the sense that the variables q^ν correspond to a small subsystem of the universe. The WDW equation ca be written as

$$\left(\hbar^2 \nabla_c^2 - U_c(c) - H_q\right) \Psi = 0, \tag{4.61}$$

where $H_c = -\hbar^2 \nabla_c^2 + U_c(c)$ is the part of the WDW obtained by neglecting all quantum variables and their momenta, and we mark with c the terms related to the classical part and with q the ones related to the quantum part. A critical assumption in the following is that[4]

$$\frac{H_q \Psi}{H_c \Psi} = \mathcal{O}(\hbar). \tag{4.62}$$

This is the heart of the next expansion, because it means that the quantum subsystem is small with respect to the whole universe, and both the semiclassical character of the universe and the smallness of the quantum subsystem are due to the fact that the universe is large. It implies that $H_q \Psi = \mathcal{O}(\hbar \cdot H_c \Psi) = \mathcal{O}(\hbar)$.

We assume that c^a and q^ν are normalized such that the leading order of the metric tensor components is of order $\hbar^0 = 1$. We can achieve a separation between classical and quantum subspaces through the assumptions

$$\mathcal{G}^c_{ab}(c) = \mathcal{O}\left(\hbar^0\right), \qquad \mathcal{G}_{ab}(c,q) = \mathcal{G}^c_{ab}(c) + \mathcal{O}(\hbar), \qquad \mathcal{G}_{a\nu}(c,q) = \mathcal{O}(\hbar), \tag{4.63}$$

[4] Like before, the original paper makes use of a dimensionless parameter proportional to $\hbar$.

i.e. the subspaces defined by c^a and q^ν are approximately orthogonal. The classical Laplacian is contracted using the classical part of the metric, that is

$$\nabla_c^2 = \mathcal{G}_c^{ab}(c)\nabla_a\nabla_b. \tag{4.64}$$

The wave function of the universe can be written as

$$\Psi(c,q) = \sum_k \psi_k(c)\chi_k(c,q), \tag{4.65}$$

where $\psi_k(c)$ is the classical part an is given by eq. (4.51), while $\chi_k(c,q)$ is the quantum component.

Let us start with a single one of such terms, i.e.

$$\Psi(c,q) = \psi(c)\chi(c,q) = A(c)\mathrm{e}^{\mathrm{i}S(c)}\chi(c,q). \tag{4.66}$$

At the highest order, that is $\mathcal{O}(\hbar^0)$, we get as before

$$H_c\psi = 0 \implies \nabla_c^2 S(c) + U_c = 0. \tag{4.67}$$

At order $\mathcal{O}(\hbar)$ we have

$$\mathrm{i}\hbar(\nabla_c^2 S)\chi + 2\mathrm{i}\hbar\mathcal{G}_c^{ab}(\nabla_a \ln A)(\nabla_b S)\chi + 2\mathrm{i}\hbar\mathcal{G}_c^{ab}(\nabla_a S)(\nabla_b\chi) - H_q\chi = 0. \tag{4.68}$$

We make the assumption that eq. (4.54) is still satisfied by the classical part, that is that $H_c\psi = 0$ even at order $\mathcal{O}(\hbar)$, and we obtain

$$2\mathrm{i}\hbar\mathcal{G}_c^{ab}(\nabla_a S)(\nabla_b\chi) - H_q\chi = 0. \tag{4.69}$$

Using eq. (4.57) eq finally get

$$\mathrm{i}\hbar\frac{\partial\chi}{\partial t} = NH_q\chi; \tag{4.70}$$

this corresponds to defining the time derivative through

$$\partial_t = 2\mathcal{G}_c^{ab}(\nabla_a S)\nabla_b. \tag{4.71}$$

We found the Schrödinger equation for the subsystem in the background defined by $c^a(t)$. The presence of the lapse function is due to the time reparametrization invariance, so that time can appear only in the combination $N(t)\,\mathrm{d}t$.

The nice property of the definition (4.71) is that the $(n-1)$-dimensional sections of the minisuperspace labelled by the time parameter (i.e. the equal-time hypersurfaces) are orthogonal to the hypersurfaces $S_0 = \text{const}$. In other words, the equal-time hypersurfaces are crossed once and only once by each element of the congruence of the classical trajectories. This property is essential to define positive probabilities starting from the Klein-Gordon-like scalar product and to recover the standard interpretation of quantum mechanics for the small subsystem of the Universe.

We can obtain the probability distribution defined by the wave function through

$$j^a = |\chi|^2|A|^2\nabla_c^a S = j_c^a\rho_\chi \tag{4.72a}$$

$$j^\nu = -\frac{\mathrm{i}}{2}|A|^2(\chi^*\nabla^\nu\chi - \chi\nabla^\nu\chi^*) = \frac{1}{2}|A|^2 j_\chi^\nu, \tag{4.72b}$$

with $j_c^a \equiv |A|^2 \nabla_c^a S$ and $\rho_\chi \equiv |\chi|^2$. Using the conservation of the total current, we get

$$\frac{\partial \rho_\chi}{\partial t} + N \nabla_\nu j_\chi^\nu = 0. \tag{4.73}$$

The probability distribution associated with j^a is

$$\rho(c, q, t) = \rho_c(c, t) |\chi|^2, \tag{4.74}$$

where ρ_c is the classical probability distribution for the variables c^a and ρ_χ is the probability distribution for the quantum variables q^ν on the classical trajectories $c^a(t)$. If we write the surface element on the equal-time surfaces as $\mathrm{d}\Sigma = \mathrm{d}\Sigma_c \, \mathrm{d}\Omega_q$, where $\mathrm{d}\Sigma_c$ is the surface element in the subspace defined by c^a, then we have the normalization

$$\int \rho_c \, \mathrm{d}\Sigma_c = 1 \tag{4.75}$$

and χ can be normalised by

$$\int |\chi|^2 \, \mathrm{d}\Omega_q = 1, \tag{4.76}$$

where $\mathrm{d}\Omega_q = \sqrt{|\det \mathcal{G}_{\mu\nu}|} \, \mathrm{d}^m q$. hence, we recovered the standard interpretation of the wave function for a small subsystem of the universe.

Finally, we stress that the presence of classical variables is needed to define a time label that ensures the positive semidefiniteness of the Klein-Gordon-like scalar product induced by the WDW equation and finds a conceptual justification in the role played by classical devices in the interpretation of quantum measurement.

Chapter 5

Semiclassical expansion and the problem of time

The problem of time is one of the most relevant open issues in canonical quantum gravity. Although there is a huge literature about this problem, a commonly accepted solution has not been found yet. Here, following [112], we focus on the Semiclassical Approach to the problem of time, that has the main goal of reproducing quantum field theory on a fixed WKB background.

We analyse the different choices of the expansion parameter, in order to include matter in the background equations in a clean way and without any need of manually rescaling the matter fields. We also discuss the problem of the non-unitary evolution at the order of the expansion where quantum gravity corrections to quantum field theory appear: we claim that the proposed solutions are non viable and that either the problem may need for new theoretical paradigms to be solved or one has to relax some of the fundamental hypotheses of the Semiclassical Approach, at least at the quantum gravity order.

Having showed that the current proposals suffer form non-unitarity at the quantum gravity level, we formulate a new proposal free from this issue, using the kinematical action to define time.

The structure of the manuscript is as follows. In sec. 5.1 we present the main motivations for this work, highlighting the different proposals and their merits and drawbacks. In sec. 5.2, after we already discussed the main concepts of the semiclassical approach to the problem of time in sec. 4.3, we present the semiclassical expansions of [38] in the Planck mass. in sec. 5.3 we critically review both the $\hbar$ and Plank mass expansions, while also making a deep comparison between them, allowed by the extension of the $\hbar$ expansion to arbitrary orders. Sec. 5.4 deals with the solution to the non-unitarity problem proposed in [103], showing that is no acceptable.

In sec. 5.5 we present the expansion of [53], based on an exact decomposition of the wave function. We complete such expansion in sec. 5.6, in order not to break the gauge invariance of this approach, and we show that also this expansion is deemed to find non-unitary corrections.

Eventually, in sec. 5.7 we make some useful considerations on the definition of the WKB time. Learning from the already presented approaches, we formulate a new proposal based on the kinematical action [25] in sec. 5.8, and we show that it is free from that non-unitarity issue.

We make the point of this chapter in sec. 5.9.

5.1 Motivation

One of the long standing problems of canonical quantum gravity is the so-called *frozen formalism*, i.e. the absence of an evolution of the quantum gravitational field with respect to an external clock [94]. Over the years, many approaches have been proposed to address this question, based both on introducing time through some matter source [40, 51] or identifying it with an internal source-time variable [42]. These approaches differ among each other also for considering time proposals before or after the canonical quantization procedure has been performed, but they all rely on the concept of *relational time* [41]: under the request of suitable conditions, each subsystem can be properly adopted as a clock for the remaining part of the quantum system. However, these approaches seem qualitatively far from the idea that in quantum mechanics time is an external parameter and measurements are performed by a classical observer. Actually, on the base of a relational time approach it is not clear how to reproduce the proper limit of quantum field theory on a curved background, starting from the Wheeler-DeWitt (WDW) equation [9].

In this respect, a different proposal has been investigated in [36], where the situation considered is that in which the quantum system can be separated into a set of semiclassical WKB variables and a "small", fast, purely quantum component. This scenario is the quantum gravity version of a Born-Oppenheimer (BO) approximation, with the peculiar feature that now the dependence of the quantum system on the classical variables allows to re-introduce the notion of an external time for the fast system component, essentially coinciding with the standard label time of the spacetime slicing. Although this approach can be applied to any set of variables (see for instance [81]), it is particularly appropriate to reconstruct the limit of quantum field theory on a classical curved background. For an application of the Vilenkin proposal to the minisuperspace, which clarifies under which conditions the BO approximation can be adopted, see [100].

In [36], the analysis is performed by using the Planck constant as the natural expansion parameter and cutting the dynamics up to first order in $\hbar$. In [38], the same idea is implemented by using as expansion parameter the Planck mass (de facto the Newton constant) and the expansion of the dynamics is considered up to, in principle, any order of approximation. This study has the merit to arrive to similar results than those proposed in [36], but without requiring the rapid variation of the wave function with respect to the small, quantum subsystem variables. The emerging problem is here that, as far as the next order of approximation is considered, corresponding to quantum gravity corrections to quantum field theory, a non-unitary character of the quantum dynamics emerges. This fact prevents the predictivity of the approach at this level. Nonetheless in [98] the study of the cosmological perturbations on a classical isotropic Robertson-Walker background is developed in the framework of quantum gravity corrections. The results of this analysis calculate the modification of the inflationary spectrum of perturbations, due to non-classical effects of the gravitational field and show the smallness of the non-unitary contributions. Despite such interesting issues, the basic conceptual problem remains open and calls attention to validate the viability of the basic idea of a BO approximation.

Two different proposals to solve the non-unitary problem of the WKB theory, at the order of quantum gravity corrections, have been developed in [53] and [103]. The proposed solutions rely on two different points of view: one aims to define a conserved probability density, disregarding the details of the evolution quantum operator; the other aims to reconstruct a posteriori a well-behaving Schrödinger evolution of the quantum subsystem, by altering the pure WKB dynamics of the gravitational background.

This study offers a critical analysis of all this field of investigation and outlines how the fundamental problem of dealing with non-unitary contributions in the quantum dynamics has not yet been properly addressed. This problem remains an open non-trivial issue of the BO approximation applied to the semiclassical limit of quantum gravity.

As a first step, we compare the approach in [38] with an expansion in terms of the natural parameter $\hbar$, upgrading the analysis in [36], up to any order of approximation, and always requiring the rapid variation of the wave function on the small quantum subset. Via this analysis, we clarify that the two approaches are essentially equivalent, but for the classical limit. In fact, the approach in [38] is associated to a classical limit which corresponds to gravity in vacuum. The reason of this feature is simply that, by using the Planck mass as expansion parameter, the gravity-matter coupling is naturally lost in the Hamilton-Jacobi (HJ) equation: matter is ruled out at the zeroth order of the expansion. Differently, the expansion in $\hbar$ has no problem in reproducing the classical Einstein equations in the presence of a matter source. A very important example where this feature is relevant can be found in the behaviour of the scalar field in a cosmological setting. In this case the scalar field is able to be the matter source of the isotropic universe expansion, and, simultaneously, is responsible for the generation of a fluctuation spectrum via its quantum dynamics on such a classical background. Actually, this scenario has been considered in [98], but the discussed problem has been overcome by a non-legitimate redefinition of the scalar field, by means of the Planck mass. Although the results reached in [98] recover the equations of quantum field theory in curved spacetime, it is worth stressing how, in the considered example, the distinction between "macroscopic" and quantum matter appears really fictitious.

Then, by using a paradigmatic model, with a single classical variable, e.g. the case of a de-Sitter WKB universe on which lives a quantum scalar field (useful to model the inflation phase of the universe), we analyse the two proposals contained in [53] and in [103] to solve the problem of the non-unitary dynamics of the matter quantum field in the presence of quantum gravity corrections.

In [53], an extended (gauge invariant) BO approximation is developed, by recovering the concept of average on the quantum variable, when calculating the classical system evolution, see also [74, 84, 102]. Apart from completing the analysis by properly rescaling the background wave function (which implies an important cancellation of the backreaction that quantum matter exerts on the background), we clarify how the evolution operator remains clearly affected by the same non-unitary features outlined in [38]. In fact, a conserved concept of probability is defined only by subtracting to the evolution operator its average on an assigned quantum state. However, the evolution operator still contains those second derivatives of the quantum system wave function with respect to the classical coordinates, that are source of non-unitarity, as soon as they are interpreted via the introduced time variable. Finally, no real Hilbert space is constructed in this approach, since the scalar product of two different states is clearly not dynamically preserved.

The approach followed in [103] faces the problem by passing from the matter Hamiltonian and the corrected Hamiltonian operators to their eigenvalues. Then, the non-unitarity is translated into the complex nature of the corrected Hamiltonian eigenvalues. The technique to remove the non-desired terms consists of eliminating the imaginary part of the corrected Hamiltonian spectrum via a phase redefinition of the quantum system wave function. The weakness of this proposal relies in the two following requirements: i) the time derivative of the corrected Hamiltonian operator has the spectrum formed with the time derivatives of the corrected Hamiltonian eigenvalues; ii) the matter Hamiltonian operator and its time derivative must commute. These two features are here shown to be not valid in general and therefore the considered procedure is just an ad hoc algorithm, appropriate to very special situations. A more subtle problem of this approach is the lack of gauge invariance under the phase transformations performed on the wave functions, a feature that is instead present in [53].

The main merit the first part of the present study consists of the fine investigation we perform on the WKB method applied to quantum gravity, outlining how the problem of dealing with non-unitary contributions is a non-trivial conceptual question which calls attention for being solved at a more fundamental level, see [88, 106, 110], or by introducing new theoretical paradigms, see for instance [108, 109] and the following papers of this series.

Eventually, we make a new proposal to recover time through the introduction of the kinematical action defined in [25]. This way, we free the time definition from the ill-defined background Laplacian that appears at quantum gravity level, ad we are able to recover a unitary evolution for the quantum gravity Schrödinger equation.

5.2 Planck mass semiclassical expansion and quantum gravity effects

We summarize here the results of [38], which presents an expansion in the Planck mass, analogue to the one of sec. 4.3, along with its extension to the quantum gravity order. We then compare the two models and extend them to arbitrary orders. The main differences of [38], respect to [36], are the expansion in the Planck mass instead of the Planck constant, the application of the model to a generic WDW equation in superspace, not restricted to a minisuperspace model, and the extension of the procedure to the quantum gravity order. However, the quantum gravity effects appear to violate unitarity.

We start from the WDW equation (4.39), including the factor $1/\sqrt{h}$ inside the supermetric G_{ijkl} defining

$$\mathcal{G}_{ab} = \frac{1}{c^2\sqrt{h}}G_{ijkl}, \qquad a, b = \{i, j\}, \{k, l\}. \tag{5.1}$$

It should not be confused with the minisupermetric, but the idea behind the indices grouping is the same. Following [38], we also define the parameter

$$M \equiv \frac{1}{4\mathcal{K}c^2} = \frac{cM_{\mathrm{P}}^2}{4\hbar}, \tag{5.2}$$

where M_{P} is the *reduced Planck mass*. Since it has the dimensions of a mass over a length, we can expect this expansion to be sensible if, for a particle, its rest mass divided by its Compton length is much smaller than M. The WDW equation can now be written as[1]

$$\left(-\frac{\hbar^2}{2M}\mathcal{G}_{ab}\frac{\delta^2}{\delta c_a\,\delta c_b} + MV(c_a) + H_q(c_a, q_\nu)\right)\Psi = 0, \tag{5.3}$$

where c_a are the degrees of freedom of h_{ij}, q_ν the ones of the matter fields ϕ_α,

$$V = -2c^2\sqrt{h}\,{}^{(3)}R \tag{5.4}$$

is the gravitational potential and

$$H_q = -\frac{\hbar^2}{2\sqrt{h}}\frac{\delta^2}{\delta\phi^2} + u\left(h_{ij}, \phi, \partial_i\phi\right) = -\hbar^2\nabla_q^2 + u \tag{5.5}$$

is the matter Hamiltonian with potential u. The universe wave functional is $\Psi = \Psi\left[h_{ij}(\boldsymbol{x}), \phi(\boldsymbol{x})\right]$.

In this expansion, the gravitational degrees of freedom will always be the classical ones, while the matter fields will always behave as quantum variables. This is due to the choice of M as expansion parameter: we will not be able to describe a classical matter component. Indeed, the expansion ic carried in the limit $M \gg 1$, which means $\mathcal{K} \to 0$: in such limit, it cannot exist a classical matter component in the Einstein equations, and the only allowed classical solutions are the vacuum ones. The advantage for doing this is that we do not need to make any "smallness" assumption for the quantum system, like the one of eq. (4.62), because it is intrinsic in the expansion itself. Moreover, we chose to use a notation similar to sec. 4.3 to allow an easy comparison between the results.

5.2.1 Classical order and Schrödinger equation

We now proceed to the expansion of the wave functional as

$$\Psi = \mathrm{e}^{\mathrm{i}S/\hbar}, \qquad S = MS_0 + S_1 + \frac{1}{M}S_2 + \dots \tag{5.6}$$

and we substitute it in the WDW equation (5.3). At the highest order $\mathcal{O}(M^2)$ we have (for simplicity, we use a single quantum variable q)

$$\left(\frac{\delta S_0}{\delta q}\right)^2 = 0. \tag{5.7}$$

[1] In [38] there is an additional term, to keep account of the factor ordering choice. Since our focus in not to solve the factor ordering issue, we refer to the demonstration done in that paper for the independence of the result on the factor ordering choice, and we write our equations in a simpler form by assuming the natural ordering, i.e. the derivatives on the right. Moreover, the functional derivatives are treated in a formal sense as ordinary ones.

In presence of more matter fields, it would be replaced by a summation of similar terms. It means that the highest order term in the action expansion should not depend on the quantum variables, that are the matter fields, i.e. $S_0 = S_0(c^a)$.

Order $\mathcal{O}(M)$ yields the Hamilton-Jacobi equation

$$\frac{1}{2}\mathcal{G}_{ab}\frac{\delta S_0}{\delta c_a}\frac{\delta S_0}{\delta c_b} + V = 0. \tag{5.8}$$

Here we can see that the matter fields have disappeared, and they cannot appear at a classical level.

The next order $\mathcal{O}(M^0)$ gives

$$\mathcal{G}_{ab}\frac{\delta S_0}{\delta c_a}\frac{\delta S_1}{\delta c_b} - \frac{\mathrm{i}\hbar}{2}\mathcal{G}_{ab}\frac{\delta^2 S_0}{\delta c_a\,\delta c_b} + \left(\frac{\delta S_1}{\delta q}\right)^2 - \mathrm{i}\hbar\frac{\delta^2 S_1}{\delta q^2} + u = 0. \tag{5.9}$$

We define the functionals

$$\psi_1(c) = \frac{1}{D}\mathrm{e}^{\mathrm{i}MS_0/\hbar} \tag{5.10a}$$

$$\chi_1(c,q) = D\mathrm{e}^{\mathrm{i}S_1/\hbar}, \tag{5.10b}$$

where $D = D(c)$ is a generic functional of the classical variables. The wave functional of the system at this order is $\Psi = \psi_1\chi_1$. We choose D to satisfy the condition

$$\mathcal{G}_{ab}\frac{\delta S_0}{\delta c_a}\frac{\delta D}{\delta c_b} - \frac{1}{2}D\mathcal{G}_{ab}\frac{\delta^2 S_0}{\delta c_a\,\delta c_b} = 0. \tag{5.11}$$

thus, D play the role of a Van Vleck determinant. For a minisuperspace model with a single degree of freedom c it holds $D = \sqrt{\mathrm{d}S_0\,/\,\mathrm{d}c}$. We notice that eq. (5.11) is equivalent to eq. (4.54), and it is nothing more than a WKB expansion for the classical degrees of freedom alone. It means that we are imposing the quantum variables not to influence the classical ones, at least at this perturbation order. Eq. (5.9) now becomes

$$\mathrm{i}\hbar\mathcal{G}_{ab}\frac{\delta S_0}{\delta c_a}\frac{\delta \chi_1}{\delta c_b} \equiv \mathrm{i}\hbar\frac{\delta\chi_1}{\delta\tau} = H_q\chi_1, \tag{5.12}$$

where time has been defined as before through

$$\frac{\delta}{\delta\tau} \equiv \mathcal{G}_{ab}\frac{\delta S_0}{\delta c_a}\frac{\delta}{\delta c_b}, \tag{5.13}$$

see eq. (4.71) (the different factor 2 lies in the constant $1/(2M)$ in the WDW equation (5.3)). The time τ contains the lapse function, i.e. it is invariant under time reparametrization. As before, it labels the trajectories in superspace which run orthogonal to the hypersurfaces $S_0 = \text{const}$. It is usually called *WKB time*.

5.2.2 Corrections to the Schrödinger equation

At order $\mathcal{O}(M^{-1})$ we decompose S_2 as

$$S_2 = \sigma(c) + \eta(c,q), \tag{5.14}$$

and we define the functionals

$$\psi_2(c) = \frac{1}{D}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(MS_0+\sigma/M)} = \psi_1 \mathrm{e}^{\mathrm{i}\sigma/(\hbar M)} \tag{5.15a}$$

$$\chi_2(c,q) = D\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_1+\eta/M)} = \chi_1 \mathrm{e}^{\mathrm{i}\eta/(\hbar M)}. \tag{5.15b}$$

At this order $\Psi = \psi_2\chi_2$. We choose σ to satisfy the second order of the classical part WKB expansion

$$\mathcal{G}_{ab}\frac{\delta S_0}{\delta c_a}\frac{\delta \sigma}{\delta c_b} - \frac{\hbar^2}{D^2}\mathcal{G}_{ab}\frac{\delta D}{\delta c_a}\frac{\delta D}{\delta c_a} + \frac{\hbar^2}{2D}\mathcal{G}_{ab}\frac{\delta^2 D}{\delta c_a\,\delta c_b} = 0. \tag{5.16}$$

Using this condition, the WDW equation expansion leads to the corrected Schrödinger equation

$$\mathrm{i}\hbar\frac{\delta\chi_2}{\delta\tau} = H_q\chi_2 + \frac{\hbar^2}{2M\chi_1}\left(\frac{2}{D}\mathcal{G}_{ab}\frac{\delta\chi_1}{\delta c_a}\frac{\delta D}{\delta c_b} - \mathcal{G}_{ab}\frac{\delta^2\chi_1}{\delta c_a\,\delta c_b}\right)\chi_2. \tag{5.17}$$

In [38], these corrections are studied through a procedure based on the projection of the gradient in the background indexes along the normal (i.e. temporal) and tangential directions to the $S_0 = \text{const}$ hypersurfaces. After defining the tangential unit vector l and by using eqs. (5.8) and (5.12), the following relation is said to hold

$$\nabla_g\chi_1 = \frac{\mathrm{i}}{2\hbar V}\nabla S_0 H_m \chi_1 + (\nabla_g\chi_1 l)l. \tag{5.18}$$

It is clear that this decomposition breaks down if $V = 0$, i.e. if ${}^{(3)}R = 0$. This equation must be then substituted into eq. (5.17). Under the assumption that the quantum Hamiltonian H_q depends adiabatically on the geometric variables, χ_1 depends on c only through τ and the tangential terms can be neglected. By making use of eqs. (5.8) and (5.11) and noting that $\nabla_q\chi_2 = (\nabla_q\chi_1)\chi_2/\chi_1 + \mathcal{O}(1/M)$, we finally find

$$\mathrm{i}\hbar\frac{\delta\chi_2}{\delta\tau} = H_q\chi_2 - \frac{1}{4MV}\left(H_q^2 + \mathrm{i}\hbar\frac{\partial H_q}{\partial\tau} - \mathrm{i}\hbar\frac{1}{V}\frac{\partial V}{\partial\tau}H_q\right)\chi_2. \tag{5.19}$$

From this equation it becomes clear that some of the quantum gravity corrections that are part of the Hamiltonian operator acting on χ_2 are non-hermitian. These terms induce a unitarity violation in the quantum sector of the theory, that impairs the standard interpretation of quantum mechanics at this order. It is worth noting that, at this order and under the hypotheses made to pass from eq. (5.17) to eq. (5.19), all three corrective terms in eq. (5.19) emerge from $\nabla_c^2\chi_1$, since the only normal contribution coming from $2\nabla_c \ln D \cdot \nabla_c\chi_1$ is suppressed by the use of eq. (5.11).

5.3 Comparison of the $\hbar$ and M expansions and extension to arbitrary orders

Both the expansions of sec. 4.3 and the one presented above make use of an adiabatic approximation to separate the semiclassical background from the quantum subsystem, in a way that is similar to a Born-Oppenheimer approximation. What is missing here with respect to a true BO approximation is the procedure of averaging over the quantum variables, that may allow for the introduction of backreaction. However, the adiabatic approximation is mathematically realized in a different way, depending on the choice of the expansion parameter.

The choice of $\hbar$ requires the assumption of smallness of the quantum subsystem, in order to obtain the adiabatic decomposition between classical and quantum subspaces. In the case of the M expansion, this decomposition is natural and is due to the choice of a parameter that contains the gravitational constant G. The price for this simplicity is a huge drawback: the only possible decomposition in the M expansion approach is between quantum matter and classical geometry. As a consequence, in the M expansion there is no way to treat classical matter fields or to include quantum geometrical degrees of freedom. In [98], the authors try to overcome one part of the problem, by introducing "macroscopic" matter fields obtained by scaling the matter variables with the Planck mass. We think that this is more likely an attempt to work around the problem, since redefining the fields through the expansion parameter is not conceptually satisfying, and sound fictitious.

It is worth noting that this nasty difference between the two expansion has a very simple origin: by looking at eqs. (5.3) and (5.5), we can see that with respect to $\hbar$ there is a perfect symmetry between geometric and matter terms, while with respect to M there is one order gap between them (eq. (5.7) is a direct consequence of that). In the $\hbar$ expansion, this gap is precisely recovered with the additional hypothesis of smallness.

With the exception of this difference, the two approaches yield similar results up to the quantum mechanics order (i.e. $\mathcal{O}(\hbar)$ in (4.54) and (4.70) and $\mathcal{O}(M^0)$ in (5.11) and (5.12)). The Hamilton-Jacobi equation is found in both expansions, although it is more general in the $\hbar$ expansion, since it corresponds to the Einstein equations in presence of matter sources. Similarly, the Schrödinger equation is found for the quantum subsystem, after imposing eq. (4.54) in the $\hbar$ expansion and eq. (5.11) in the M expansion. To see that the two equations coincide, we just have to put $A = 1/D$. To carry on the comparison at the quantum gravity order and to reach a better understanding of the structure of the theories, we will now extend both expansions to arbitrary orders. We borrow the method from [103], although we apply it with some small modifications.

First, we need to generalise the hypotheses made in [36] in order to have a clear separation between classical and quantum subspaces, valid at each order of the expansion. In this regard, the conditions (4.63) on the minisupermetric components are assumed to be verified exactly:

$$\mathcal{G}_{ab}(c,q) = \mathcal{G}^c_{ab}(c), \qquad \mathcal{G}_{a\nu}(c,q) = 0. \tag{5.20}$$

The Laplacian operator in the background indexes ∇^2_c, being defined through $\mathcal{G}_{ab}(c)$, now depends exactly on the classical indexes only. Similarly, the background potential U_c is assumed to be independent from the quantum variables at every order.

We will write on the left the equations for the $\hbar$ expansion, and on the right the ones for the M expansion. Moreover, we will write the derivatives as ∂ for brevity of notation, but they must be intended as functional derivatives in the sense of the above M expansion. Finally, we must remember that, for the M expansion, classical variable must always coincide with gravitational ones, and the quantum ones with the matter fields. The WDW equation reads

$$\left(-\hbar^2\nabla_c^2 + U_c + H_q\right)\Psi = 0 \qquad (5.21)$$

$$\left(-\frac{\hbar^2}{2M}\nabla_c^2 + MV + H_q\right)\Psi = 0, \qquad (5.22)$$

where the Hamiltonian of the quantum subsystem has not changed

$$H_q = -\hbar^2\nabla_q^2 + u; \qquad (5.23)$$

for the $\hbar$ expansion, it satisfies the condition of smallness (4.62)

$$\frac{H_q\Psi}{H_c\Psi} = \mathcal{O}(\hbar), \qquad (5.24)$$

i.e. $u = \mathcal{O}(\hbar)$ and $\nabla_q = \mathcal{O}\left(\hbar^{-1/2}\right)$.

Let us write the wave functional as

$$\Psi(c,q) = \mathrm{e}^{\mathrm{i}S(c,q)/\hbar} \qquad (5.25)$$

and expand the complex phase S in powers of the expansion parameter

$$S = \sum_{n=0}^{\infty}\hbar^n S_n \qquad (5.26)$$

$$S = \sum_{n=0}^{\infty} M^{1-n} S_n. \qquad (5.27)$$

To obtain the factorized form of the wave functional, we assume that each order of the expansion of S after the first can be separated as

$$S_n = \sigma_n(c) + \eta_n(c,q), \qquad n \geq 1 \qquad (5.28)$$

This way, we obtain $S = S_0 + P + Q$, with

$$P(c) = \sum_{n=1}^{\infty}\hbar^n\sigma_n \qquad (5.29)$$

$$Q(c,q) = \sum_{n=1}^{\infty}\hbar^n\eta_n \qquad (5.30)$$

$$P(c) = \sum_{n=1}^{\infty} M^{-n}\sigma_n \qquad (5.31)$$

$$Q(c,q) = \sum_{n=1}^{\infty} M^{-n}\eta_n. \qquad (5.32)$$

The wave functional takes the BO-like form

$$\Psi(c,q) = \psi(c)\chi(c,q), \qquad (5.33)$$

where

$$\psi(c) = \mathrm{e}^{\mathrm{i}(S_0+P)/\hbar} \qquad (5.34)$$

$$\chi(c,q) = \mathrm{e}^{\mathrm{i}Q/\hbar} \qquad (5.35)$$

$$\psi(c) = \mathrm{e}^{\mathrm{i}M(S_0+P)/\hbar} \qquad (5.36)$$

$$Q(c,q) = \mathrm{e}^{\mathrm{i}MQ/\hbar}. \qquad (5.37)$$

The background wave functional is assumed to satisfy the WKB expansion for the classical part alone, i.e.

$$\left(-\hbar^2\nabla_c^2 + U_c\right)\psi = 0 \qquad (5.38)$$

$$\left(-\frac{\hbar^2}{2M}\nabla_c^2 + MV\right)\psi = 0. \qquad (5.39)$$

By substituting to ψ its expansion (5.34) or (5.36), this equation yields, order by order, the HJ equation for the classical action S_0 and the equations of the WKB expansion for each σ_n. We report the first orders here:

$$(\nabla_c S_0)^2 + U_c = 0 \qquad (5.40)$$

$$2\nabla_c S_0 \cdot \nabla_c \sigma_1 - \mathrm{i}\nabla_c^2 S_0 = 0 \qquad (5.41)$$

$$\begin{aligned} &2\nabla_c S_0 \cdot \nabla_c \sigma_2 \\ &\quad + (\nabla_c \sigma_1)^2 - \mathrm{i}\nabla_c^2 \sigma_1 = 0 \end{aligned} \qquad (5.42)$$

$$\begin{aligned} &2\nabla_c S_0 \cdot \nabla_c \sigma_3 \\ &\quad + 2\nabla_c \sigma_1 \cdot \nabla_c \sigma_2 - \mathrm{i}\nabla_c^2 \sigma_2 = 0 \end{aligned} \qquad (5.43)$$

$$\tfrac{1}{2}(\nabla_c S_0)^2 + V = 0 \qquad (5.44)$$

$$\nabla_c S_0 \cdot \nabla_c \sigma_1 - \tfrac{\mathrm{i}\hbar}{2}\nabla_c^2 S_0 = 0 \qquad (5.45)$$

$$\begin{aligned} &\nabla_c S_0 \cdot \nabla_c \sigma_2 \\ &\quad + \tfrac{1}{2}(\nabla_c \sigma_1)^2 - \tfrac{\mathrm{i}\hbar}{2}\nabla_c^2 \sigma_1 = 0 \end{aligned} \qquad (5.46)$$

$$\begin{aligned} &\nabla_c S_0 \cdot \nabla_c \sigma_3 \\ &\quad + \nabla_c \sigma_1 \cdot \nabla_c \sigma_2 - \tfrac{\mathrm{i}\hbar}{2}\nabla_c^2 \sigma_2 = 0 \end{aligned} \qquad (5.47)$$

It is easy to check the correspondence with the preceding expansions, through

$$A = \mathrm{e}^{\mathrm{i}\sigma_1} \qquad (5.48)$$

$$D = \mathrm{e}^{-\mathrm{i}\sigma_1/\hbar}. \qquad (5.49)$$

The equation for the quantum subsystem is obtained by plugging eq. (5.33) into the WDW equation, removing the classical part through eq. (5.38) or eq. (5.39) and dividing by ψ:

$$2\hbar^2 \nabla_c \ln\psi \cdot \nabla_c \chi = H_q \chi - \hbar^2 \nabla_c^2 \chi \qquad (5.50)$$

$$\tfrac{\hbar^2}{M}\nabla_c \ln\psi \cdot \nabla_c \chi = H_q \chi - \tfrac{\hbar^2}{2M}\nabla_c^2 \chi \qquad (5.51)$$

After substituting to ψ its expansion and using the usual definition of time

$$\partial_\tau = 2\nabla_c S_0 \cdot \nabla_c \qquad (5.52)$$

$$\partial_\tau = \nabla_c S_0 \cdot \nabla_c, \qquad (5.53)$$

where we remind the difference is due to the additional factor 2 together with M, this yields the corrected Schrödinger equation

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial \chi}{\partial \tau} &= H_q \chi \\ &\quad - 2\mathrm{i}\hbar\nabla_c P \cdot \nabla_c \chi - \hbar^2 \nabla_c^2 \chi \end{aligned} \qquad (5.54)$$

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial \chi}{\partial \tau} &= H_q \chi \\ &\quad - \mathrm{i}\hbar\nabla_c P \cdot \nabla_c \chi - \tfrac{\hbar^2}{2M}\nabla_c^2 \chi. \end{aligned} \qquad (5.55)$$

At orders $\mathcal{O}(\hbar)$ and $\mathcal{O}(M^0)$, it reduces to the exact Schrödinger equation for the quantum wave functional χ_1, given by

$$\mathrm{i}\hbar\frac{\partial \chi_1}{\partial \tau} = H_q \chi_1. \qquad (5.56)$$

At orders $\mathcal{O}(\hbar^2)$ and $\mathcal{O}(1/M)$, it is easy to find

$$
\begin{aligned}
\mathrm{i}\hbar\frac{\partial\chi_2}{\partial\tau} &= H_q\chi_2 \\
&- \hbar^2\left(2\mathrm{i}\nabla_c\sigma_1\cdot\nabla_c + \nabla_c^2\right)\chi_2
\end{aligned}
\tag{5.57}
$$

$$
\begin{aligned}
\mathrm{i}\hbar\frac{\partial\chi_2}{\partial\tau} &= H_q\chi_2 \\
&- \left(\tfrac{\mathrm{i}\hbar}{M}\nabla_c\sigma_1\cdot\nabla_c + \tfrac{\hbar^2}{2M}\nabla_c^2\right)\chi_2,
\end{aligned}
\tag{5.58}
$$

where the corrective terms are of the same kind of those in eq. (5.17). This result shows that, by restricting the classical subspace to the geometrical variables only, the $\hbar$ expansion yields precisely the same results of the M expansion, also at the quantum gravity and subsequent orders.

Following a procedure described in [103], eqs. (5.54) and (5.55) can be written in a nicer form. We assume that there is a total (non necessarily hermitian) Hamiltonian operator H such that

$$
\mathrm{i}\hbar\frac{\partial\chi}{\partial\tau} = H\chi,
\tag{5.59}
$$

and also that

$$
\nabla_c\chi = \alpha(c)\nabla_c S_0,
\tag{5.60}
$$

which is some sort of adiabatic approximation. Eqs. (5.40) and (5.44) give

$$
\alpha = -\tfrac{1}{2U}\partial_\tau\chi = \tfrac{\mathrm{i}}{2U\hbar}H\chi
\tag{5.61}
$$

$$
\alpha = -\tfrac{1}{2V}\partial_\tau\chi = \tfrac{\mathrm{i}}{2V\hbar}H\chi.
\tag{5.62}
$$

Using eqs. (5.41) and (5.45), after some calculations we can write eqs. (5.54) and (5.55) as

$$
\begin{aligned}
\mathrm{i}\hbar\tfrac{\partial\chi}{\partial\tau} &= H\chi = H_q\chi \\
&-\tfrac{1}{2U}\left(H^2 + \mathrm{i}\hbar\tfrac{\partial H}{\partial\tau} - \mathrm{i}\hbar KH\right)\chi
\end{aligned}
\tag{5.63}
$$

$$
K = \frac{1}{U}\frac{\partial U}{\partial\tau} - \frac{\mathrm{i}}{\hbar}\sum_{n=2}^{\infty}\hbar^n\frac{\partial\sigma_n}{\partial\tau}
\tag{5.64}
$$

$$
\begin{aligned}
\mathrm{i}\hbar\tfrac{\partial\chi}{\partial\tau} &= H\chi = H_q\chi \\
&-\tfrac{1}{4MV}\left(H^2 + \mathrm{i}\hbar\tfrac{\partial H}{\partial\tau} - \mathrm{i}\hbar KH\right)\chi
\end{aligned}
\tag{5.65}
$$

$$
K = \frac{1}{V}\frac{\partial V}{\partial\tau} - \frac{2\mathrm{i}M}{\hbar}\sum_{n=2}^{\infty}\frac{1}{M^n}\frac{\partial\sigma_n}{\partial\tau}.
\tag{5.66}
$$

H is an abstract Hamiltonian operator containing H_q and all the corrections at every order. These expressions show even more the equivalence of the two expansions, except for the differences already noticed.

The procedure we just performed is the generalization of that used before to derive eq. (5.19) from (5.17), being based on the adiabatic approximation $\nabla_c\chi \propto \nabla_c S_0$ (i.e. the contributions tangential to $S_0 = \text{const}$ are neglected). The use of eqs. (5.41) and (5.45) causes the sum in the expression of K to begin from $n = 2$. At the quantum gravity order $\mathcal{O}(\hbar^2)$ and $\mathcal{O}(1/M)$, eq. (5.65) yields eq. (5.19). At higher orders, the quantum gravity corrections not only arise from the ∇_c^2 term in eqs. (5.54) and (5.55), but also from the term containing P. As noted in [103], the same result can be obtained considering σ_n, V (or U_c, in the $\hbar$ expansion) and χ depending on τ only from the beginning and dropping all the components of the supermetric of the geometric subspace with the exception of the $\mathcal{G}_{\tau\tau}$ component.

As a concluding remark to this section, let us take stock of the situation. Both the $\hbar$ and the M expansions recover the already established theories through a HJ equation for GR, that fixes a background, and a Schrödinger equation in curved spacetime for quantum mechanics. The $\hbar$ expansion is more general, since it admits backgrounds generated by matter sources and quantum geometry. At the quantum gravity order, both expansions yield non-hermitian corrections, that break the unitarity of the theory. A further common feature of the two approaches is that the backreaction of the quantum subsystem on the background is not present: the inclusion of such a non-adiabatic effect would allow for quantum gravitational effects on the semiclassical sector.

5.4 Non-unitarity in the revisited M expansion

In this section, we show that the procedure used in [103] to solve the non-unitarity problem at the quantum gravity order is based on wrong assumptions. Let us briefly apply this procedure to the simple case of one geometric variable, that we identify with the time τ from the beginning. Once we use the ansatz $\Psi(\tau, q) = \psi(\tau)\chi(\tau, q)$, the WDW equation (5.3) reads

$$\frac{\hbar^2}{M}\mathcal{G}_{\tau\tau}\partial_\tau \ln\psi\partial_\tau\chi = H_q\chi - \frac{\hbar^2}{2M}\mathcal{G}_{\tau\tau}\partial_\tau^2\chi + \rho_\psi\chi, \tag{5.67}$$

where the background term

$$\rho_\psi = \frac{1}{\psi}\left[-\frac{\hbar^2}{2M}\mathcal{G}_{\tau\tau}\partial_\tau^2 + MV\right]\psi \tag{5.68}$$

corresponds to the quantity set to zero in eq. (5.39). Differently from [38], in [103], after writing $\psi_0 = \exp(\mathrm{i}MS_0/\hbar)$, the background term ρ_{ψ_0} is required to be of order $\mathcal{O}(M^0)$. In order to satisfy this request, the HJ equation

$$\frac{1}{2}M\mathcal{G}_{\tau\tau}(\partial_\tau S_0)^2 + MV = 0 \tag{5.69}$$

has to hold at order $\mathcal{O}(M)$. Hence, the expression of ρ_{ψ_0} at order $\mathcal{O}(M^0)$ is

$$\rho_{\psi_0} = -\frac{\mathrm{i}\hbar}{2}\frac{\partial_\tau V}{V}. \tag{5.70}$$

Using eq. (5.70) and assuming the existence of an abstract Hamiltonian operator H similar to that defined in (5.65), we find

$$\mathrm{i}\hbar\partial_\tau\chi_0 \equiv H\chi_0 = H_q\chi_0 - \frac{\mathrm{i}\hbar}{2}\frac{\partial_\tau V}{V}\chi_0 - \frac{1}{4MV}\left[H^2 + \mathrm{i}\hbar\partial_\tau H\right]\chi_0, \tag{5.71}$$

where χ_0 is the quantum wave functional, such that $\Psi = \psi_0\chi_0$. This equation still exhibits non-hermitian corrections. To deal with them, in [103], the authors assume the existence of two eigenvalue functions $E(\tau)$ (complex) and $\epsilon(\tau)$ (real) such that

$$H\chi_0 = E(\tau)\chi_0, \tag{5.72a}$$

$$H_q\chi_0 = \epsilon(\tau)\chi_0, \tag{5.72b}$$

and expand them in powers of $1/M$. Written in terms of these expansions, the WDW eq. (5.71) yields, at each order, an expression for the eigenvalue of the abstract Hamiltonian operator. Let us report the first two orders ($\mathcal{O}(M^0)$ and $\mathcal{O}(1/M)$):

$$E^{(0)} = \epsilon^{(0)} - \frac{\mathrm{i}\hbar}{2}\frac{\partial_\tau V}{V}, \tag{5.73a}$$

$$E^{(1)} = \epsilon^{(1)} - \frac{1}{4V}\left[\left(\epsilon^{(0)}\right)^2 - \frac{3\hbar^2}{4}\left(\frac{\partial_\tau V}{V}\right)^2 + \hbar^2\frac{\partial_\tau^2 V}{2V}\right] - \frac{\mathrm{i}\hbar}{4}\partial_\tau\left(\frac{\epsilon^{(0)}}{V}\right). \tag{5.73b}$$

Defining

$$\chi_1 = \mathrm{e}^{-\frac{1}{\hbar}\int \mathbb{Im}(E^{(0)})\,\mathrm{d}\tau}\chi_0 = \mathrm{e}^{\int \frac{\partial_\tau V}{2V}}\chi_0, \tag{5.74}$$

and substituting into eq. (5.71) we find

$$\mathrm{i}\hbar\partial_\tau\chi_1 = H_q\chi_1. \tag{5.75}$$

From the time derivative of the redefined quantum state it comes a term that exactly compensates the non-hermitian correction on the right hand side of eq. (5.71), due to eq. (5.70) at this order $\mathcal{O}(M^0)$. The background term must now be calculated for a ψ_1 defined in such a way that $\Psi = \psi_1\chi_1$, i.e.

$$\psi_1 = \mathrm{e}^{-\int \frac{\partial_\tau V}{2V}}\psi_0 = \mathrm{e}^{\mathrm{i}MS_0/\hbar+\sigma_1}, \tag{5.76}$$

where $\sigma_1 = -\ln V/2$. By doing so, we find that ρ_{ψ_1} vanishes at order $\mathcal{O}(M^0)$, yielding the continuity equation

$$\partial_\tau^2 S_0 + \partial_\tau S_0 \partial_\tau \sigma_1 = 0. \tag{5.77}$$

We can easily see that this equation vanishes naturally. Thus, ρ_{ψ_1} is of order $\mathcal{O}(1/M)$ and is given by the expression

$$\rho_{\psi_1} = \frac{\hbar^2}{4MV}\left[\frac{3}{4}\left(\frac{\partial_\tau V}{V}\right)^2 - \frac{\partial_\tau^2 V}{2V}\right]. \tag{5.78}$$

The same steps can be followed at order $\mathcal{O}(1/M)$, including the term in eq. (5.78) into eq. (5.71) and redefining the quantum state as

$$\chi_2 = \mathrm{e}^{-\frac{1}{M\hbar}\int \mathbb{Im}(E^{(1)})\,\mathrm{d}\tau}\chi_1. \tag{5.79}$$

The corrected Schrödinger equation will have only the hermitian part of the Hamiltonian operator H, exhibiting unitary evolution. The background term calculated for a ψ_2 such that $\Psi = \psi_2\chi_2$ will not vanish naturally at this order, as an effect of the backreaction of the quantum subsystem.

This procedure is based on the nice idea that the non-hermitian part of the operator H may be eliminated from the dynamical equation of the quantum subsystem by suitable redefinitions of the wave functions of the product $\Psi = \psi\chi$. However, the H operator is unknown in general and can only be constructed order by order. Moreover, in order to redefine the wave functions through phase factors one has to use the eigenvalues of H. The problem is that assuming eqs. (5.72) means that H_m and H commute (at every order) and can be diagonalized simultaneously. Unfortunately, this is clearly not true at every order, as one can see from the expression of $E^{(1)}$ in eqs. (5.73). Indeed, $E^{(1)}$ contains $\epsilon^{(0)}$ and its time derivative $\partial_\tau\epsilon^{(0)}$ and highlights that the Hamiltonian H at the order $\mathcal{O}(1/M)$ contains the matter Hamiltonian H_q and its time derivative $\dot{H}_q$, coherently with eq. (5.19). In general it is not true that H_q and $\dot{H}_q$ commute: the reason why this happens is that one may let the conjugated momenta to the classical variables appear in $\dot{H}_q$.

To convince ourselves about this, let us consider a Friedmann-Robertson-Walker (FRW) model with cosmological constant and a scalar field as matter component. The Hamiltonian constraint reads

$$H_{FRW} = -\frac{G}{32c^3\pi a}p_a^2 + \frac{c}{4\pi^2 a^3}p_\phi^2 - V, \tag{5.80a}$$

$$V(a;\Lambda) = \frac{3\pi c^3}{4G}\left(a - \frac{\Lambda}{3}a^3\right), \tag{5.80b}$$

where V is the FRW superpotential. An important remark is that the conjugated momenta to the volume of the Universe a is proportional to the time derivative of a:

$$p_a \sim \frac{a}{N}\frac{\mathrm{d}a}{\mathrm{d}t} = a\partial_\tau a. \tag{5.81}$$

The matter Hamiltonian of this simple model is just

$$H_q = \frac{c}{4\pi^2}a^{-3}p_\phi^2, \tag{5.82}$$

and its time derivative yields

$$\partial_\tau H_q = -\frac{3c}{4\pi^2}a^{-4}\partial_\tau a p_\phi^2 \sim a^{-5}p_a p_\phi \tag{5.83}$$

The appearance of p_a in $\partial_\tau H_q$ clearly leads to $[H_q, \partial_\tau H_q] \neq 0$.

A further issue of the procedure followed in [103] concerns the absence of gauge invariance in this approach: even if the total wave function Ψ is invariant under the redefinitions performed on ψ and χ, the equations of motion are not, differently from what happens in [53]. Thus, such redefinitions cannot be fully justified on theoretical grounds.

5.5 Exact expansion

Another attempt to treat a quantum subsystem on a WKB semiclassical background is provided by [53, 101]. In these works, the authors develop a decomposition in classical and quantum variables through an extended BO approach, but more accurate than the traditional BO approximation. This approach is based on an exact decomposition of the wave function, given an initial ansatz, and it is largely used in chemistry [74, 84, 91, 102], where it finds experimental verification.

We will shortly illustrate here such decomposition on the system given by eqs. (5.3) and (5.3), together with some useful considerations. In order to make simpler the comparison to [38], we will assume the matter degrees of freedom as quantum and the gravitational ones as classical, although different choices are possible with similar results.

We start making the ansatz

$$\Psi(c, q) = \psi(c)\chi(q; c) \tag{5.84a}$$

$$\langle\chi|\chi\rangle = \int \chi^*(q; c)\chi(q; c)\,\mathrm{d}q = 1. \tag{5.84b}$$

The equation for the background wave function can be written as an average of eq. (5.3) on the the quantum function χ. to do so, we need the definitions

$$\langle \mathcal{O} \rangle = \langle \chi | \mathcal{O} | \chi \rangle \tag{5.85a}$$

$$A = -\mathrm{i}\hbar \langle \nabla_c \rangle \tag{5.85b}$$

$$-\mathrm{i}\hbar\, \mathrm{D} = -\mathrm{i}\hbar \nabla_c + A, \tag{5.85c}$$

$$-\mathrm{i}\hbar\, \bar{\mathrm{D}} = -\mathrm{i}\hbar \nabla_c - A, \tag{5.85d}$$

where $\mathcal{O}$ is a generic operator and D and $\bar{\mathrm{D}}$ are covariant derivatives, in a sense that will be clear soon. The quantity A will play the role of a Berry connection. We get for the background equation

$$\left[-\frac{\hbar^2}{2M} \left(\mathrm{D}^2 + \left\langle \bar{\mathrm{D}}^2 \right\rangle \right) + MV + \langle H_q \rangle \right] \psi = 0, \tag{5.86}$$

while the equation for the quantum subsystem is found as the difference between the complete equation (5.3) and the classical one (5.86)

$$\left[-\frac{\hbar^2}{2M} \left(\bar{\mathrm{D}}^2 - \left\langle \bar{\mathrm{D}}^2 \right\rangle + 2 \frac{\mathrm{D}\,\psi}{\psi} \cdot \bar{\mathrm{D}} \right) + H_q - \langle H_q \rangle \right] \chi = 0. \tag{5.87}$$

As a bonus of the ansatz (5.84), we notice that if the total wave function is normalized to unity

$$\int \Psi^*(c, q) \Psi(c, q)\, \mathrm{d}c\, \mathrm{d}q = 1, \tag{5.88}$$

then the background wave function is naturally normalized

$$\begin{aligned} 1 &= \int \Psi^*(c, q) \Psi(c, q)\, \mathrm{d}c\, \mathrm{d}q \\ &= \int \psi^*(c) \psi(c) \int \chi^*(q; c) \chi(q; c)\, \mathrm{d}q\, \mathrm{d}c = \int \psi^*(c) \psi(c)\, \mathrm{d}c = 1. \end{aligned} \tag{5.89}$$

Eqs. (5.86) and (5.87) can be more generally derived from a variational principle [74, 84]. These considerations underline that this is a more solid and advanced model than both the traditional Born-Oppenheimer approximation and the Born-Oppenheimer like approach followed in [38, 103]. Moreover, eqs. (5.84) imply no freedom to the decomposition of the total wave function into classical and quantum components, except for a phase factor depending on the classical variables only (because ψ cannot depend on the quantum ones, even after such a transformation). However, eq. (5.86) and eq. (5.87) are invariant under such a phase change, due to the covariant derivatives of eqs. (5.85). Hence the decomposition (5.84) is unique and characterized by a gauge symmetry. This gives even more value to this formalism.

Following [53], we can absorb the covariant derivatives D and $\bar{\mathrm{D}}$ in the wave functions through the redefinitions

$$\psi = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} \int A \mathrm{d}c} \widetilde{\psi} \tag{5.90a}$$

$$\chi = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int A \mathrm{d}c} \widetilde{\chi}. \tag{5.90b}$$

This way, we find the equations for the semiclassical background

$$\left[-\frac{\hbar^2}{2M}\left(\nabla_c^2+\widetilde{\langle\nabla_c^2\rangle}\right)+MV+\widetilde{\langle H_q\rangle}\right]\widetilde{\psi}=0 \tag{5.91}$$

and for the quantum subsystem

$$\left[-\frac{\hbar^2}{2M}\left(\nabla_c^2-\widetilde{\langle\nabla_c^2\rangle}+2\frac{\nabla_c\widetilde{\psi}}{\widetilde{\psi}}\cdot\nabla_c\right)+H_m-\widetilde{\langle H_q\rangle}\right]\widetilde{\chi}=0, \tag{5.92}$$

where $\widetilde{\langle\mathcal{O}\rangle}$ is the average of the operator $\mathcal{O}$ over the new wave function $\widetilde{\chi}$:

$$\widetilde{\langle\mathcal{O}\rangle}=\langle\widetilde{\chi}|\mathcal{O}|\widetilde{\chi}\rangle\,. \tag{5.93}$$

If an operator $\mathcal{O}_q$ acts only on the quantum variables, the two averages match

$$\widetilde{\langle\mathcal{O}_q\rangle}=\langle\mathcal{O}_q\rangle\,. \tag{5.94}$$

We stress that (5.90) is not a simple phase transformation, because $A\propto\langle\nabla_c\rangle$ depends on the state χ.

We still need another step to find the Schrödinger equation for the quantum subsystem. First, we define time through a WKB expansion of $\widetilde{\psi}$ in powers of $\hbar$

$$\widetilde{\psi}=\frac{1}{N}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\text{eff}}},\qquad N,S_{\text{eff}}\in\mathbb{R} \tag{5.95}$$

$$\mathrm{i}\hbar\partial_\tau=\frac{\mathrm{i}\hbar}{M}\nabla_cS_{\text{eff}}\cdot\nabla_c, \tag{5.96}$$

where the inverse of the first order quantum amplitude has been denoted with N, not to create confusion with the covariant derivative. We plug this into eq. (5.91) and find, at order $\mathcal{O}(\hbar^0)$, the Hamilton-Jacobi equation

$$\frac{1}{2M}(\nabla_cS_{\text{eff}})^2+MV+\langle H_q\rangle=0 \tag{5.97}$$

and, at order $\mathcal{O}(\hbar)$, the equation for the Van Vleck determinant N

$$\frac{1}{2N}\nabla_c^2S_{\text{eff}}+\nabla_c\left(\frac{1}{N}\right)\cdot\nabla_cS_{\text{eff}}=0. \tag{5.98}$$

We can see that, differently from [38], we now have a backreaction from the quantum subsystem on the the semiclassical background.

We now redefine the quantum wave function to obtain the correct Schrödinger equation

$$\widetilde{\chi}=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\langle H_q\rangle\mathrm{d}\tau}\chi_s \tag{5.99}$$

where, as before, the phase depends on the state χ. After some calculations, by using the time definition of eq. (5.96), we find

$$(H_q-\mathrm{i}\hbar\partial_\tau)\chi_s=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int\langle H_q\rangle\mathrm{d}\tau-\frac{\mathrm{i}}{\hbar}\int A\mathrm{d}c}\frac{\hbar^2}{2M}\left[\bar{\mathrm{D}}^2-\left\langle\bar{\mathrm{D}}^2\right\rangle+2(\mathrm{D}\ln N)\cdot\bar{\mathrm{D}}\right]\chi, \tag{5.100}$$

where the traditional Born-Oppenheimer approximation consists in neglecting the right hand side [53].

In conclusion, in [53] the authors show the unitarity of the theory through

$$
\begin{aligned}
&\mathrm{i}\hbar\partial_\tau \langle\chi_s|\chi_s\rangle = \int (\chi_s^* \mathrm{i}\hbar\partial_\tau\chi_s - \mathrm{CC})\,\mathrm{d}q = \\
&\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int(A-A^\dagger)\mathrm{d}c}\left[\left(\langle H_q\rangle - \frac{\hbar^2}{M}(\mathrm{D}\ln N)\left\langle\bar{\mathrm{D}}\right\rangle - \frac{\hbar^2}{2M}\left\langle\bar{\mathrm{D}}^2 - \left\langle\bar{\mathrm{D}}^2\right\rangle\right\rangle\right) - \mathrm{CC}\right] = 0,
\end{aligned}
\tag{5.101}
$$

where CC is the complex conjugate of the previous term.

5.6 Non-unitarity of the exact expansion

There are some open issues in the procedure developed in [53]. First, the quantum wave function is χ_s and the semiclassical wave function is $\widetilde{\psi}$, while their product should yield exactly the total wave function: this implies a breaking of the gauge symmetry of the theory.

Second, the Schrödinger equation (5.100) contains derivatives with respect to the background variables, which in turn contain also the time: these derivatives must be clearly expressed and analysed. Indeed, as shown in [38] and in sec. 5.3, the Laplacian operator on the right hand side of eq. (5.100) in particular is responsible for the unitarity breaking terms at the quantum gravity order.

Furthermore, an additional problem is that the right hand side of eq. (5.101) vanishes only if one takes the norm of the states, but it does not for different quantum states. This means that a proper dynamical Hilbert space can not be built in this approach, since a conserved scalar product can not be defined for all the states.

We now improve the method proposed in [53] and in last section, in order to deal with the open issues just discussed. Despite the generality of the $\hbar$ expansion, we will here expand the semiclassical wave function in powers of M; this way the comparison with [38] will be simpler. The expansion in $\hbar$ follows similar calculations, beside the differences noted in sec. 5.3.

Given the previous considerations, let us define the background wave function ψ_s associated with χ_s through

$$
\widetilde{\psi} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int\langle H_q\rangle\mathrm{d}\tau}\psi_s, \tag{5.102}
$$

in such a way that the total wave function reads

$$
\Psi = \psi\chi = \widetilde{\psi}\widetilde{\chi} = \psi_s\chi_s. \tag{5.103}
$$

We now perform a semiclassical expansion on the background wave function, similar to those made in sec. 5.3

$$
\psi_s = \mathrm{e}^{\mathrm{i}M(S_0+P)/\hbar}, \qquad P(c) = \sum_{n=1}^{\infty} M^{-n}\sigma_n. \tag{5.104}
$$

This time we decompose P in its real and imaginary parts as $P = \zeta - \mathrm{i}\rho$, such that

$$
\psi_s = \mathrm{e}^{M\rho/\hbar}\mathrm{e}^{\mathrm{i}M(S_0+\zeta)/\hbar}, \tag{5.105}
$$

where

$$\mathbb{R}\mathrm{e}(P) \equiv \zeta = \frac{1}{M}\zeta_1 + \frac{1}{M^2}\zeta_2 + \dots \tag{5.106a}$$

$$-\mathbb{I}\mathrm{m}(P) \equiv \rho = \frac{1}{M}\rho_1 + \frac{1}{M^2}\rho_2 + \dots \tag{5.106b}$$

Defining time as usual by now, that is through eq. (5.96) with the substitution $S_{\text{eff}} \to MS_0$, we have

$$\mathrm{i}\hbar\partial_\tau = \mathrm{i}\hbar\nabla_c S_0 \cdot \nabla_c \tag{5.107}$$

and eq. (5.100) at order $\mathcal{O}(M^0)$ yields the Schrödinger equation

$$(-\mathrm{i}\hbar\partial_\tau + H_q)\chi_s = 0, \tag{5.108}$$

as expected.

The first interesting differences from [53] appears in the equation for the background expansion. Eq. (5.86) yields at order $\mathcal{O}(M)$ the usual Hamilton-Jacobi equation

$$\frac{1}{2}(\nabla_c S_0)^2 + V = 0, \tag{5.109}$$

while at order $\mathcal{O}(M^0)$

$$-\frac{\mathrm{i}\hbar}{2}\nabla_c^2 S_0 + \nabla_c S_0 \cdot \nabla_c \zeta_1 - \mathrm{i}\nabla_c S_0 \cdot \nabla_c \rho_1 - \nabla_c S_0 \cdot \nabla_c \int \langle H_q \rangle \, \mathrm{d}\tau + \langle H_q \rangle = 0. \tag{5.110}$$

By using the definition of time (5.107) we see that the backreaction disappears. The fact that the backreaction shifted by one order in the expansion is just because in [53] the authors performed an $\hbar$ expansion without the hypothesis of smallness of the quantum subsystem (i.e. $H_q \sim \hbar$), that would have caused the backreaction to appear in eq. (5.98). After separating the real and imaginary parts, we find

$$\frac{\hbar}{2}\nabla_c^2 S_0 + \nabla_c S_0 \cdot \nabla_c \rho_1 = 0 \tag{5.111a}$$

$$\nabla_c S_0 \cdot \nabla_c \zeta_1 = 0. \tag{5.111b}$$

The first equation corresponds exactly to eq. (5.45), while the second points out that ζ_1 has no dynamical relevance. Indeed, by using eq. (5.107), eq. (5.111b) reads

$$\partial_\tau \zeta_1 = 0. \tag{5.112}$$

Until now, we recovered exactly the results of [38] (and equivalently [36]), but with the adoption of the more advanced formalism of [53].

Now we can look at the quantum gravity order $\mathcal{O}(1/M)$. We will show that the quantum gravity corrections calculated with this approach differ from those calculated in [38], but still yield a unitarity violation, in contrast with what is declared in [53]. For simplicity, we restrict our study to the case of a single gravitational degree of freedom, which will be denoted as α; this is consistent with [53] and it does not alter the results. This way we will not need to deal with the projection of the gradients in the geometrical indexes with respect to the $S_0 = \text{const}$ hypersurfaces. Following this assumption, we have [103]

$$\partial_\tau = \mathcal{G}_{\alpha\alpha}(\partial_\alpha S_0)\partial_\alpha \tag{5.113a}$$

$$\partial_\tau S_0 = -2V \tag{5.113b}$$

$$\mathcal{G}_{\tau\tau} = -\frac{1}{2V} \tag{5.113c}$$

$$\partial_\alpha = \frac{\mathrm{d}}{\mathrm{d}\alpha} = \frac{\mathrm{d}S_0}{\mathrm{d}\alpha}\frac{1}{\mathrm{d}S_0/\,\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\tau} = -\frac{1}{2V}(\partial_\alpha S_0)\partial_\tau, \tag{5.113d}$$

where eq. (5.113c) comes from writing eq. (5.109) directly in τ (that means choosing time as the classical variable instead of α from the beginning). We stress that, even with a single geometrical degree of freedom, this procedure is valid only if $V \neq 0$, otherwise we would have $\partial_\tau S_0 = V = 0$.

By using the previous relations and eq. (5.108), it is easy to find

$$\mathrm{i}\hbar\partial_\alpha\widetilde{\chi} = -\frac{1}{2V}(\partial_\alpha S_0)(H_m - \langle H_m\rangle)\widetilde{\chi} \tag{5.114a}$$

$$\begin{aligned}\mathcal{G}_{\alpha\alpha}(\mathrm{i}\hbar\partial_\alpha)^2\widetilde{\chi} = \Bigg[&\frac{\mathrm{i}\hbar}{2V}\frac{\dot{V}}{V}(H_m - \langle H_m\rangle) - \frac{\mathrm{i}\hbar}{2V}\mathcal{G}_{\alpha\alpha}(\partial_\alpha^2 S_0)(H_m - \langle H_m\rangle)\\ &-\frac{\mathrm{i}\hbar}{2V}\left(\dot{H}_m - \left\langle \dot{H}_m\right\rangle\right) - \frac{1}{2V}(H_m - \langle H_m\rangle)^2\Bigg]\widetilde{\chi}\end{aligned} \tag{5.114b}$$

$$\left\langle \mathcal{G}_{\alpha\alpha}(\mathrm{i}\hbar\partial_\alpha)^2\right\rangle = -\frac{1}{2V}\left(\left\langle H_m^2\right\rangle - \langle H_m\rangle^2\right) \tag{5.114c}$$

where we indicated time derivatives with a dot and we used the identity

$$\partial_\tau\langle H_m\rangle = \langle\partial_\tau H_m\rangle\,, \tag{5.115}$$

due to eq. (5.108). After some cumbersome calculations, making use of eqs. (5.114), eqs. (5.111) and eq. (5.99), and starting from eq. (5.92) we can rewrite eq. (5.100) up to order $\mathcal{O}(1/M)$ as

$$\begin{aligned}&\mathrm{i}\hbar\partial_\tau\chi_s = H_q\chi_s\\ &\quad-\frac{1}{4MV}\left[\left(H_q^2 - \left\langle H_q^2\right\rangle\right) + \mathrm{i}\hbar\left(\dot{H}_q - \left\langle\dot{H}_q\right\rangle\right) - \mathrm{i}\hbar\frac{\dot{V}}{V}(H_q - \langle H_q\rangle)\right]\chi_s.\end{aligned} \tag{5.116}$$

The last equation is the equivalent of eq. (5.19), that is eq. (42) of [38], but in the framework of [53]. We see that the non-hermiticity of the quantum gravity Hamiltonian is still a problem, unless one takes the norm of a state, hence eq. (5.101). In this case, differently from [38], all quantum gravity corrections vanish and this may be interpreted as a prediction of this approach, that is that quantum gravity effects are smaller than we thought and may appear only at subsequent orders, if not at all. This last interpretation, however, seems somewhat awkward.

We now turn our attention to the background wave function at order $\mathcal{O}(1/M)$. Rewriting ∂_α through eq. (5.113d), with the help of eqs. (5.111) and of (5.109) we find

$$\begin{aligned}-\frac{\hbar^2}{2}\mathcal{G}_{\alpha\alpha}\frac{\partial_\alpha^2\widetilde{\psi}}{\widetilde{\psi}} &= \partial_\tau\zeta_2 - \mathrm{i}\partial_\tau\rho_2 - \frac{1}{4V}\langle H_q\rangle^2 + \frac{\mathrm{i}\hbar\dot{V}}{4V^2}\langle H_q\rangle - \frac{\mathrm{i}\hbar}{4V}\left\langle\dot{H}_q\right\rangle\\ &\quad-\frac{1}{4V}(\partial_\tau\rho_1)^2 - \frac{\hbar\dot{V}}{4V^2}\partial_\tau\rho_1 + \frac{\hbar}{4V}\partial_\tau^2\rho_1.\end{aligned} \tag{5.117}$$

Through the last equation and eq. (5.114c), we can rewrite eq. (5.91) at the desired order. After separating the real and imaginary parts, we find

$$\partial_\tau\zeta_2 - \frac{1}{4V}\left[(\partial_\tau\rho_1)^2 + \hbar\frac{\dot{V}}{V}\partial_\tau\rho_1 - \hbar\partial_\tau^2\rho_1\right] - \frac{1}{4V}\left\langle H_q^2\right\rangle = 0, \tag{5.118a}$$

$$\partial_\tau\rho_2 - \frac{\hbar\dot{V}}{4V^2}\langle H_q\rangle + \frac{\hbar}{4V}\left\langle \dot{H}_q\right\rangle = 0. \tag{5.118b}$$

This perturbative order clearly shows the backreaction of the quantum subsystem, at the same order expected in [103], although the solutions are very different, as well as for the corrected Schrödinger equation (5.116), because of the formalism. By writing the equations in the time component from the beginning, eq. (5.111a) becomes

$$\frac{\hbar}{2}\mathcal{G}_{\tau\tau}\partial_\tau^2 S_0 + \partial_\tau\rho_1 = 0, \tag{5.119}$$

and making use of eqs. (5.113) we can write

$$\partial_\tau\rho_1 = -\frac{\hbar}{2}\frac{\dot{V}}{V}. \tag{5.120}$$

With this result, we can simplify eq. (5.118a) and obtain

$$\partial_\tau\zeta_2 - \frac{\hbar^2}{4V}\left[\frac{\ddot{V}}{2V} - \frac{3}{4}\frac{(\dot{V})^2}{V^2}\right] - \frac{1}{4V}\left\langle H_q^2\right\rangle = 0. \tag{5.121}$$

5.7 Some notes about time

Given the strong importance of time in the non unitarity problem, a few additional remarks on some aspect related to it are required. The first one is that the WKB time is an intrinsic time of the system and is related to the general time t through the lapse function N (not to be confused with the N from the WKB expansion of ψ)

$$\partial_\tau = \frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}t}. \tag{5.122}$$

This does not alter the structure of our solutions, because the lapse function is always associated with time. About that, see for example the Schrödinger eq. (34) of [36], i.e. eq. (4.70).

The second consideration is less evident, and it is related to the exact decomposition of [53] and the transformations performed on the wave functions. Going from the initial functions ψ, χ to the final functions ψ_s, χ_s requires two transformations, one that involves $A \propto \langle\nabla_c\rangle$ and one that involves $\langle H_q\rangle \propto \langle\partial_\tau\rangle$. Hence, the total transformation is given by eqs. (5.90), (5.99) and (5.102)

$$\psi = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int A\mathrm{d}c}\widetilde{\psi} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int A\mathrm{d}c}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int\langle H_q\rangle\mathrm{d}\tau}\psi_s \tag{5.123}$$

$$\chi = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int A\mathrm{d}c}\widetilde{\chi} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int A\mathrm{d}c}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\langle H_q\rangle\mathrm{d}\tau}\chi_s, \tag{5.124}$$

where, given the definition of A, the first phase resembles a Berry phase. The interesting fact is that ∂_τ and ∇_c are related through eq. (5.96), or equivalently through eq. (5.107). It is argued that such transformations cannot be taken individually, but form a unique transformation on the system [59]. Moreover, if we write the exponent through the derivatives we get

$$\frac{\mathrm{i}}{\hbar}\int (A\,\mathrm{d}c + \langle H_q\rangle\,\mathrm{d}\tau) = \int (\langle\nabla_c\rangle\,\mathrm{d}c - \langle\partial_\tau\rangle\,\mathrm{d}\tau), \tag{5.125}$$

where we used eq. (5.85b) and (5.108) (or equivalently eq. (5.100), neglecting the fluctuations). If we choose the classical variables set to be $\{\tau, h_i\}$ from the beginning, where the h_i are the degrees of freedom orthogonal to time, the last equation reads (a summation on index i is implied)

$$\int (\langle\partial_\tau\rangle\,\mathrm{d}\tau + \langle\partial_{h_i}\rangle\,\mathrm{d}h_i) - \int \langle\partial_\tau\rangle\,\mathrm{d}\tau = \int \langle\partial_{h_i}\rangle\,\mathrm{d}h_i\,. \tag{5.126}$$

If we now define

$$A_i = -\mathrm{i}\hbar\,\langle\partial_{h_i}\rangle\,, \tag{5.127}$$

the full transformation reads

$$\psi = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int A_i \mathrm{d}h_i}\psi_s \tag{5.128}$$

$$\chi = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int A_i \mathrm{d}h_i}\chi_s\,: \tag{5.129}$$

we can easily notice that such transformation is performed on the hyperplane orthogonal to the time coordinate. This is an interesting feature of the model and one of its core definitions, and probably it deserves more investigation.

Moreover, from all the analyses states until now it appears that the most important term responsible for the non-unitarity of the models is the classical Laplacian ∇_c^2. Be it through some adiabatic assumption on the quantum wave function, some projection parallelly and orthogonally to the hypersurfaces $S_0 = \mathrm{const}$ or simply by having time as the classical variable from the beginning, at some point that Laplacian generates $\nabla_\tau^2\chi$. this is the crucial point that always generates non-unitarity, because it holds

$$-\hbar^2\nabla_\tau^2\chi = \mathrm{i}\hbar\partial_\tau(H\chi) = \mathrm{i}\hbar\dot{H}\chi + H^2\chi, \tag{5.130}$$

and we just found the incriminated term. Thus, until time is defined through ∇_c, the model is probably doom to find a non.hermitian Hamiltonian at the quantum gravity level.

5.8 Unitarity through the introduction of the kinematical action

In order to overcome the problems discussed above, we propose a way to obtain the time parameter through the addition of *the kinematical* action term.

5.8.1 The kinematical action

The kinematical action was first introduced in [25] as a tool to maintain the constraint equations of a system by adding variables in the Lagrangian and Hamiltonian formalisms. For curved spacetimes in ADM formalism, the kinematical action reads

$$S_{\text{kin}} = \int \mathrm{d}^4x \, (p_\mu \partial_t y^\mu - N^\mu p_\mu), \tag{5.131}$$

where the coordinates u^μ are those defining the parametric equations of the hypersurfaces in the ADM splitting, see eq. (4.7), and p_μ are the associated momenta. We will now apply this procedure to the case of a matter field in a curved background.

To understand the meaning of this addition to the action, we firstly analyse a model consisting of only the kinematical action and a scalar field. The trivial equations of motion show that

$$p_\mu = 0, \qquad \dot{p}_\mu = 0, \qquad \dot{u}^\mu = \partial_t u^\mu = N^\mu = N\eta^\mu + N^i b_i^\mu, \tag{5.132}$$

where the other variables are the basis of the ADM foliation, as in eq. (4.9). The last equation closely relates u^μ and N. Moreover, we have the additional contributions to the super-Hamiltonian and supermomentum constraints given by

$$H^{\text{kin}} = \eta^\mu p_\mu \tag{5.133a}$$

$$H_i^{\text{kin}} = b_i^\mu p_\mu, \tag{5.133b}$$

which are the key components to define a meaningful time variable for the matter field dynamics, in a different way than those proposed in the previous sections.

To show this, let us consider a matter field ϕ immersed in a fixed gravitational background. Its action in ADM variables reads

$$S_\phi = \int \mathrm{d}x^0 \, \mathrm{d}^3x \left(\pi\dot{\phi} - NH^\phi - N^i H_i^\phi \right), \tag{5.134}$$

where π is the momentum conjugated to the scalar field. The super-Hamiltonian of the scalar field reads

$$H^\phi = \frac{1}{2\sqrt{h}}\pi^2 + \frac{1}{2}\sqrt{h}h^{ij}(\partial_i\phi)(\partial_j\phi) + U(\phi), \tag{5.135}$$

where U is the field potential, and its supermomentum

$$H_i^\phi = (\partial_i\phi)\pi. \tag{5.136}$$

Having fixed the background, we not make the variations with respect to N and N^i, thus they loose the physical definition of the ADM foliation.

The addition of the kinematical action allows to recover the definition of the deformation vector and so the structure of the space-time foliation, which would otherwise be lost in this case. Adding S_{kin}, independent from both metric and matter field, the dynamics of the scalar field and assigned gravitational background is left unchanged

$$\begin{aligned} S &= S^\phi + S^{\text{kin}} \\ &= \int \mathrm{d}x^0 \, \mathrm{d}^3x \left(p_\mu \dot{u}^\mu + \pi\dot{\phi} - N(H^\phi + H^{\text{kin}}) - N^i(H_i^\phi + H_i^{\text{kin}}) \right). \end{aligned} \tag{5.137}$$

The meaning of the deformation vector is recovered, and the super-Hamiltonian and supermomentum constraints become

$$H^{\phi} = -H^{\text{kin}} = -p_{\mu}n^{\mu} \tag{5.138a}$$

$$H_i^{\phi} = -H_i^{\text{kin}} = -p_{\mu}b_i^{\mu}. \tag{5.138b}$$

In the canonical quantization procedure, the momenta p^{μ} are transformed into derivative operators, and they will be crucial in the construction of the time variable.

We will now show how this procedure can be applied to obtain a matter field dynamics without non-unitary terms arising from the previous proposals, which would prevent the predictability of the theory.

5.8.2 Scalar matter fields immersed on a WKB gravitational background with kinematical action

We propose here a theory consisting of a single matter scalar field ϕ with potential U, immersed in an assigned quantum gravity background, with the addition of the kinematical action. The generalization to the case of n matter fields is straightforward by replacing ϕ with $\sum_{\alpha} \phi_{\alpha}$ and inserting the cross-interaction terms into U.

The total action of the system reads

$$\begin{aligned} S = S^{\text{grav}} + S^{\phi} + S^{\text{kin}} = \int \mathrm{d}x^0\, \mathrm{d}^3x \Big(\Pi_{ij}\dot{h}^{ij} + p_{\mu}\dot{y}^{\mu} + \pi\dot{\phi} \\ - N(H^{\text{grav}} + H^{\phi} + H^{\text{kin}}) - N^i(H_i^{\text{grav}} + H_i^{\phi} + H_i^{\text{kin}}) \Big) \end{aligned} \tag{5.139}$$

and writing the momentum p_{μ} as a derivative operator, the super-Hamiltonian and supermomentum constraints of the system are

$$H^{\text{grav}} + H^{\phi} = -H^{\text{kin}} = i\hbar\eta^{\mu}\frac{\delta}{\delta u^{\mu}} \tag{5.140a}$$

$$H_i^{\text{grav}} + H_i^{\phi} = -H_i^{\text{kin}} = i\hbar b_i^{\mu}\frac{\delta}{\delta u^{\mu}}. \tag{5.140b}$$

We assume the scalar field to be small, so we can safely use the M expansion of sec. 5.3. The wave function is

$$\Psi(h_{ij}, \phi, u^{\mu}) = \psi(h_{ij})\chi(\phi, u^{\mu}; h_{ij}), \tag{5.141}$$

where the slow semiclassical part depends only on the 3-metric, while the fast quantum part depends on the matter field and the kinematical action variable and parametrically on the 3-metric. This separation is justified by considering the different energy scales of the two components, in a case where the scalar fields act as test fields giving negligible contribution to the background and with a fast dynamics that can be computed at nearly fixed values of the 3-metric tensor. For simplicity of notation, we will adopt the variables h_a and q, as before, where we use h instead of c because it will appear also in the supermomentum constraint.

As before, we make the expansion

$$\Psi = \psi\chi = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(MS_0 + \sigma_1 + \frac{1}{M}\sigma_2)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\eta_1 + \frac{1}{M}\eta_2)}. \tag{5.142}$$

We assume ψ to satisfy the gravitational (classical) part of the constraints at any order, i.e.

$$H^{\text{grav}}\psi = \left(-\frac{\hbar^2}{2M}\nabla_h^2 + MV\right)\psi = 0 \tag{5.143}$$

$$H_i^{\text{grav}}\psi = 2\mathrm{i}\hbar h_i\,{}^{(3)}\nabla_h \partial_{h_k}\psi = 0, \tag{5.144}$$

while the total constraints read

$$\begin{aligned}(H^{\text{grav}} + H^{\phi})\Psi &= \left(-\frac{\hbar^2}{2M}\nabla_h^2 + MV - \hbar^2\nabla_q^2 + U\right)\Psi \\ &= -H^{\text{kin}}\Psi = \mathrm{i}\hbar\eta^{\mu}\partial_{u^\mu}\Psi\end{aligned} \tag{5.145}$$

$$(H_i^{\text{grav}} + H_i^{\phi})\Psi = \left(2\mathrm{i}\hbar h_i\,{}^{(3)}\nabla_h\partial_{h_k} - \mathrm{i}\hbar(\partial_i q)\nabla_q\right)\Psi = -H_i^{\text{kin}}\Psi = \mathrm{i}\hbar b_i^{\mu}\partial_{u^\mu}\Psi. \tag{5.146}$$

We will also need the additional assumption

$$\nabla_h\chi = \mathcal{O}(1/M). \tag{5.147}$$

The first perturbative order is $\mathcal{O}(M)$, at which we find

$$\frac{1}{2}(\nabla_h S_0)^2 + V = 0 \tag{5.148}$$

$$-2h_i\,{}^{(3)}\nabla_h(\partial_{h_k}S_0) = 0. \tag{5.149}$$

The last one is the invariance of the theory under diffeomorphism at this perturbative order. We will find it again order by order, i.e. the theory is completely invariant under diffeomorphisms, as it should.

At the next order, that is $\mathcal{O}(M^0)$, things become more interesting. We have

$$\nabla_h S_0 \cdot \nabla_h\sigma_1 - \frac{\mathrm{i}\hbar}{2}\nabla_h^2 S_0 = 0 \tag{5.150}$$

$$-2h_i\,{}^{(3)}\nabla_h(\partial_{h_k}\sigma_1) = 0 \tag{5.151}$$

$$(-\hbar^2\nabla_q^2 + U)\chi = \mathrm{i}\hbar\eta^{\mu}\partial_{u^\mu}\chi \tag{5.152}$$

$$-\mathrm{i}\hbar(\partial_i q)\nabla_q\chi = \mathrm{i}\hbar b_i^{\mu}\partial_{u^\mu}\chi. \tag{5.153}$$

Using the last two equations, keeping in mind the relation between $\dot{u}^\mu$ and N in (5.132) and integrating to remove the spatial dependence, we get

$$\begin{aligned}\mathrm{i}\hbar\partial_t\chi &= \mathrm{i}\hbar\int_\Sigma \mathrm{d}^3x\,\dot{u}^\mu\partial_{u^\mu}\chi = \mathrm{i}\hbar\int_\Sigma \mathrm{d}^3x\,N^\mu\partial_{u^\mu}\chi \\ &= \mathrm{i}\hbar\int_\Sigma \mathrm{d}^3x\,(N\eta^\nu + N^i b_i^\mu)\partial_{u^\mu}\chi = \int_\Sigma \mathrm{d}^3x\,(NH^\phi + N^i H_i^\phi)\chi = \mathcal{H}\chi.\end{aligned} \tag{5.154}$$

Yet another time, we have the Schrödinger equation.

We can summarize our procedure until now with an exact set of equations:

$$H^{\text{grav}}\psi = 0 \tag{5.155}$$

$$H_i^{\text{grav}}\psi = 0 \tag{5.156}$$

$$\widetilde{H}^{\text{grav}}\Psi \equiv H^{\text{grav}}\psi\chi - \chi H^{\text{grav}}\psi \tag{5.157}$$

$$\widetilde{H}_i^{\text{grav}}\Psi \equiv H_i^{\text{grav}}\psi\chi - \chi H_i^{\text{grav}}\psi \tag{5.158}$$

$$\mathrm{i}\hbar\partial_t\Psi = \int_\Sigma \mathrm{d}^3x\left[N(\widetilde{H}^{\text{grav}} + H^\phi) + N^i(\widetilde{H}_i^{\text{grav}} + H_i^\phi)\right]\Psi. \tag{5.159}$$

Because ψ does not depend on u^μ, it does not depend on t neither. Moreover, its diffeomorphism invariance means that it can be brought outside of the integral, because it does not depend on the spatial coordinate, but only on the equivalence class of 3-geometries. Doing so the last equation yields the Schrödinger equation. The presence of ψ, however, leaves behind some pieces, depending on the perturbative order. Moreover, we still need to face the big issue: will the quantum gravity order preserve unitarity?

For simplicity of notation, let us analyse together the orders $\mathcal{O}(M^0)$ and $\mathcal{O}(1/M)$. Subtracting eqs. (5.143) and (5.144) respectively from eqs. (5.145) and (5.146), we have

$$(-\hbar^2\nabla_q^2 - 2\mathrm{i}\hbar M\nabla_h S_0 \cdot \nabla_h + U)\chi = \mathrm{i}\hbar\eta^\mu\partial_{u^\mu}\chi \tag{5.160}$$

$$(2\mathrm{i}\hbar h_i{}^{(3)}\nabla_h\partial_{h_k} - \mathrm{i}\hbar(\partial_i q)\nabla_q)\chi = \mathrm{i}\hbar b_i^\mu\partial_{u^\mu}\chi. \tag{5.161}$$

The additional term is the one that used to be the temporal derivative. It appears only now due to the "smallness" assumption (5.147). Thus, the corrected Schrödinger equation at this order is

$$\mathrm{i}\hbar\partial_t\chi = \mathcal{H}\chi + \int_\Sigma \mathrm{d}^3x \left(-2N\mathrm{i}\hbar M\nabla_h S_0 \cdot \nabla_h + 2\mathrm{i}\hbar h_i{}^{(3)}\nabla_h\partial_{h_k}\right)\chi. \tag{5.162}$$

Both the corrections are unitary.

We stress here the difference in the choice of the temporal coordinate from the proposals of [36] and [38], since time is not recovered from the dependence from the "classical" variables h_{ij}, but from the kinematical action variable u^μ. Nonetheless, until the quantum mechanics order, the results are formally the same as in [36] and [38], since the Schrödinger equation is recovered in all cases. The main difference and consequence of this approach is visible at the quantum gravity order, where it differs deeply from theirs.

5.9 Final remarks and possible extensions

Let us go through the steps of our analysis. In sec. 5.3 the two WKB expansions in $\hbar$ and in the Planck mass proposed in [36, 38] have been carefully analysed and compared. We have offered a derivation of both expansions in a formalism that is similar to that adopted in [103]. By doing so, we have extended the $\hbar$ expansion to arbitrary orders and found quantum gravity corrections to the quantum sector of the theory, starting from the second order in the expansion parameter. This can be seen in the corrected Schrödinger equation in curved spacetime (5.54) and in eq. (5.57), i.e. its version at order $\mathcal{O}(\hbar^2)$. The comparison with the Planck mass expansion has revealed that the corrections are of the same kind, see eq. (5.55). The non-hermitian nature of the corrections is better highlighted once the derivatives of the wave function in the classical indexes are expressed in terms of time, as reported for the Planck mass expansion in eq. (5.65). We have also pointed out that at order $\hbar^2$ the only source of unitarity breaking terms is the Laplacian operator in the background variables in the right hand side of eqs. (5.54) and (5.55).

As for the background sector, the $\hbar$ expansion has yielded a HJ equation, corresponding to the Einstein equations in presence of a matter source, and the usual equations of a WKB expansion, see eqs. (5.40) and following ones and [26]. We have discussed the fact that this feature is not completely shared by the Planck mass expansion, since, even if the background equations have the same form at each order, in this case the classical limit of matter is excluded. Though the Planck constant expansion needs for the additional hypothesis of smallness of the quantum subsystem to derive the Schrödinger equation, it gains in generality: one may think of the more elegant Planck mass expansion as just the sub-case of the $\hbar$ expansion with a purely geometrical background. This sentence can be seen as the synthesis of the result of the comparison between the two expansions. Moreover, we have stressed that the origin of this difference can be traced in the way the adiabatic separation between slow and fast degrees of freedom is mathematically realized in the two expansions. The Planck mass is the natural adiabatic parameter to split quantum matter from classical geometry and in this sense it does not admit the matter component in the HJ equation. This is acceptable if one is only interested in the recovery of quantum field theory on curved spacetime and gives no importance to the nature of the fixed background. However the Planck mass expansion can not be applied to cosmology without manually rescaling the matter fields with the Planck mass itself, when the theory is applied to inflation, as discussed in the Introduction.

Eventually, in sec. 5.4 we have shown that the solution proposed in [103] to solve the non-unitarity problem within the framework of the Planck mass expansion is based on very strong hypotheses, and, thus, it solves the problem only for very particular models. The procedure developed in [103] is based on the eigenvalue equations (5.72). Passing from the Hamiltonian operators to their eigenvalues allows for the absorption of the non-hermitian corrections in the background wave function: this is done through redefinitions of the background wave function at each order in correspondence of the redefinitions of the quantum wave function made in eq. (5.74) and (5.79). We have argued that the relations (5.72) can not hold at the same time, since it is not true, in general, that H and H_q commute. The reason of this statement is that H contains the time derivative of the matter Hamiltonian and, in general, H_q and $\dot{H}_q$ do not commute. As a counter-example to the procedure of [103], we have shown the non commutation of the matter Hamiltonian with its time derivative for the toy model of inflation described by eqs. (5.80). However our procedure can be applied to all the models with H_q that depends on the background variables: indeed, in this case $\dot{H}_q$ contains the time derivatives of the background variables that can be used to make their conjugate momenta appear in its expression.

Next we dealt with the problem of unitarity breaking at the quantum gravity order. In sec. 5.6, after having reviewed the expansion based on the exact decomposition of the wave function of the Universe proposed in [53], we have completed the analysis by addressing the two major issues of this study. On one side, we have restored the gauge invariance of the theory, that was clearly broken by the authors. This has been done in eq. (5.102), by defining the background wave function ψ_s correspondent to the purely quantum wave function χ_s defined in [53]. By doing this, we have shown that the backreaction experiences a two order shift in the expansion parameter from the order of the HJ equation, where it appeared in [53]. The first shift is due to the fact that the authors made an $\hbar$ expansion on the background wave function without the hypothesis of smallness of the quantum subsystem, i.e. $H_q \sim \hbar$. This hypothesis would have made the backreaction appear in the continuity equation, at order $\hbar$.

Since we performed the expansion in the Planck mass to simplify the comparison with [38, 103], we expected the backreaction in the continuity equation. However the redefinition of the background wave function has led to a term that exactly compensates the backreaction in the continuity equation, see eq. (5.110). Then, we have shown that the first contribution of the backreaction in the background equations appears at the quantum gravity order, accordingly to [103]. For simplicity, we have restricted our analysis to the case of a minisuperspace model with a single geometrical variable. The main result of our calculation is contained in eqs. (5.118a), that exhibit the backreaction of the quantum subsystem.

On the other side, we have made explicit the Laplacian operators in the corrected Schrödinger equation (5.100) in terms of time derivatives, for the single geometrical variable model. This is what has to be done to check properly the unitarity of the time evolution at the quantum gravity order. The result of this analysis is contained in eq. (5.116), where the analogue of the corrected Schrödinger equation obtained in [38] (see eq. (5.19)) have been derived in the formalism of [53]. This equation shows that, once the complete form of the time evolution operator is made explicit, the problem of unitarity breaking at the quantum gravity order affects the approach proposed in [53], as well as the others discussed in this paper.

We concluded this chapter with sec. 5.8, where we proposed an alternate way of defining the time variable through the kinematical action and we showed this definition to lead to unitary corrections at the quantum gravity order.

Summarizing, the analysis above has demonstrated the following major points.

On one hand, we have clarified that the proper parameter to construct a WKB approach to the slow-varying part of the quantum system necessarily is the Planck constant, according to standard quantum mechanical criteria. This statement relies on the possibility to get also the matter contribution on the classical limit (i.e. in the HJ equation), according to the idea that quantum boson fields can be characterized by so high occupation numbers to be described by a classical energy-momentum tensor, as in the case of the electromagnetic field and of the scalar field in cosmology. Also fermion fields can admit a classical limit, when the fermion density is sufficiently high. However for these fields such associated classical limit is more commonly regarded as a phenomenological source and its presence in the HJ equation could be inferred independently of the classical limit.

On the other hand, we have clarified how the problem of a non-unitary evolution, emerging at the second order in the expansion parameter, is independent of the specific nature of such a parameter, if the Planck constant or the Planck mass. This shortcoming of the WKB formulation seems to be an intrinsic feature of the assumed decomposition of the quantum state into a slow-varying and a fast-varying component. We also argued that neither of the proposed solutions for the non-unitarity problem is actually viable, because while in [53] the real meaning of the Laplacian operator in the slow variables is not properly addressed (the time evolution operator is not unitary), in the proposal of [103] the removal of the undesired terms is operated by assumptions which are not valid in general, holding only for special ad hoc cases. Meanwhile, we managed to enhance the model of [53], correcting the background behaviour and calculating the true explicit form of the corrected Schrödinger equation.

However, the idea proposed in [103] contains some physical insight in suggesting that the a priori assumption of a WKB expansion for the slow-varying system component is too restrictive. In fact, limiting our attention to a quantum field on a semiclassical gravitational background, it appears a reasonable conjecture that, at least in general, the quantum field backreaction be not completely negligible. By other words, the nature of the semiclassical system can not be pre-determined, but it should be consistent with the quantum field dynamics, order by order in the parameter expansion. In [53], the problem of a quantum matter backreaction is considered, but, by completing the redefinition of the semiclassical wave function, we have demonstrated that the backreaction cancels out from the semiclassical equations, up to the quantum gravity order.

We then made a new proposal for a definition of time that shows to be unitary also at quantum gravity levels. This was constructed through the use af the kinematical action. This way, time is related to the other variables through the equation on motion of the system, and not directly to the S_0 functional of the background model. This allow us to prevent the problems related to the usual definition of time, and summarised in sec. 5.7.

We also admit that a more radical point of view could state that the emergence of a non-unitary contribution in the dynamics — when quantum gravity corrections are considered on quantum field theory — is the evidence that the standard BO decomposition be not appropriate to the gravitational sector. The reason could be in the intrinsic coupling that matter and gravity maintain, in principle at any order of a common WKB expansion, so that postulating the existence of a fast quantum system would break the natural feature of the gravity-matter coupling also on a quantum level. The BO approximation holds up to first order in $\hbar$ as demonstrated in [36] only because the quantum matter backreaction is expected to be of order $\hbar^2$, neglected on that footing. For a recent reformulation of this problem in terms of a Weyl quantization procedure, allowing the inspection of the quantum phase space in place of the configurational variables only, see [108, 109].

Chapter 6

Application of the WKB scheme to the perturbations of a FRW universe

The unification of quantum theory and gravity is one of the central problems in physics. Several approaches have been developed, see for example [86, 94] and references in chapter 5, but in order to decide which gives the best description of nature we need to be able to test them. However, finding predictions is problematic because of the scales in play.

Promising scenarios are highly energetic ones, or physical phenomena characterized by quantities in play able to enhance the otherwise small effects of quantum gravity. One of them, is the highly energetic early inflationary phase of the universe. Thus, the study of quantum primordial perturbations is important both for insights into quantum gravity and for the primordial perturbation themselves.

Among the studies regarding such phenomena, [98] adopts the formalism developed in section 5.2. However, the calculation contains some weak points that should be better addressed.

Following [115], we analyse the WKB scheme applied in [98], in which the authors study the scalar perturbations to a flat FRW background in the inflationary epoch. Because of the very high energies at play during this epoch, quantum-gravitational effects are expected to be relevant in this model, at least at a perturbative level. Therefore, the Hamiltonian constraints for background and perturbative degrees of freedom are derived treating background and perturbation variables as distinct physical degrees of freedom, and then the WKB semiclassical scheme of [38] is applied to the Wheeler-DeWitt equation.

6.1 Review of the original procedure

We start reviewing the original procedure followed by [98]. We describe the physical system through the same formalism presented in sec. 5.2. The ADM Hamiltonian density of the system is a combination of the super-Hamiltonian and supermomentum constraints

$$\mathcal{H} = \frac{2\mathcal{K}}{\sqrt{h}}\left(\Pi^{ij}\Pi_{ij} - \frac{1}{2}\Pi^2\right) - \frac{\sqrt{h}}{2\mathcal{K}}\,{}^{(3)}R\frac{\sqrt{h}}{2}\left(\frac{\pi_\phi^2}{h} + h^{ij}\partial_i\phi\partial_j\phi + 2V(\phi)\right) = 0 \quad (6.1a)$$

$$\mathcal{H}_i = -2\,{}^{(3)}\nabla_j\Pi_i^{\ j} + \pi_\phi\partial_i\phi = 0, \quad (6.1b)$$

where Π^{ij} and π_ϕ are respectively the conjugate momenta to h_{ij} and ϕ and $V(\phi)$ is the scalar field potential.

Assuming a flat isotropic FRW background metric in conformal time τ

$$\mathrm{d}s^2 = a^2(\tau)(-\,\mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{x}^2), \quad (6.2)$$

along with a homogeneous scalar field $\phi(\tau)$, the background ADM Hamiltonian constraint becomes

$$\mathcal{H}_0 = -\frac{1}{2a^2 m_{\mathrm{P}}^2}\pi_a^2 + \frac{1}{2a^2}\pi_\phi^2 + a^4 V(\phi), \quad (6.3)$$

where $m_{\mathrm{P}}^2 = \frac{3}{4\pi G}$ is the "rescaled Planck mass" defined by [98], and we will assume for the rest of this chapter $\hbar = c = 1$. This expression may be written in a more symmetric fashion by means of a change of variable involving the scale factor

$$\alpha \equiv \ln a \quad (6.4)$$

and by a rescaling of the scalar field with the Planck mass[1]

$$\phi \to m_{\mathrm{P}}\phi. \quad (6.5)$$

The latter is also necessary in order to recover the correct background equations involving the scalar field at leading order in the WKB semiclassic expansion, as noted in sec. 5.3. For the same reason it is necessary to rescale the potential as well, so we define

$$\mathcal{V}(\alpha, \phi) \equiv \frac{2}{m_{\mathrm{P}}^2}\mathrm{e}^{4\alpha}V(\phi). \quad (6.6)$$

Using these relations we can rewrite the background Hamiltonian (6.3) as

$$\mathcal{H}_0 = \frac{\mathrm{e}^{-2\alpha}}{2}\left(-\frac{\pi_\alpha^2}{m_{\mathrm{P}}^2} + \frac{\pi_\phi^2}{m_{\mathrm{P}}^2} + m_{\mathrm{P}}^2\mathrm{e}^{2\alpha}\mathcal{V}\right). \quad (6.7)$$

6.1.1 Perturbative Hamiltonian

For simplicity we focus on the scalar metric perturbations, that can be parametrized by means of four scalar functions of time and coordinates, which appear in the line element of the perturbed metric

$$\begin{aligned}\mathrm{d}s^2 = a^2(\tau)\Big\{&(1 + 2A(x,\tau))\,\mathrm{d}\tau^2 - 2(\partial_i B(x,\tau))\,\mathrm{d}x^i\,\mathrm{d}\tau \\ &- [(1 - 2\psi(x,\tau))\delta_{ij} + 2\partial_i\partial_j E(x,\tau)]\,\mathrm{d}x^i\,\mathrm{d}x^j\Big\}.\end{aligned} \quad (6.8)$$

[1]Even if we strongly criticised such rescaling before, we prefer to stick to the notation and assumption of [98], apart from the aspect we want to clarify more, in order to have a better comparison.

In addition to the metric part of the perturbations, we must consider a perturbed scalar field $\delta\phi(x,\tau)$.

Scalar cosmological perturbations are gauge-dependent, so it is useful to resort to gauge-invariant quantities constructed as combinations of scalar perturbations. These combinations, known as Bardeen potentials, take the form [29, 44]

$$\Phi_{\mathrm{B}}(\tau,\boldsymbol{x}) \equiv A + \frac{1}{a}\partial_\tau\left[a(B-\dot{E})\right] \tag{6.9}$$

$$\varphi^{\mathrm{GI}}(\tau,\boldsymbol{x}) \equiv \delta\phi + \dot{\phi}(B-\dot{E}), \tag{6.10}$$

where the dot represents as before a derivative with respect to conformal time τ. These two quantities can be combined to form a single gauge-invariant perturbation for both matter and geometry, called the Mukhanov-Sasaki variable

$$v(\tau,\boldsymbol{x}) \equiv a\left[\varphi^{\mathrm{GI}} + \dot{\phi}\frac{\Phi_{\mathrm{B}}}{\mathscr{H}}\right], \tag{6.11}$$

where $\mathscr{H} \equiv \dot{a}/a$ is the Hubble parameter in conformal time.

It is then possible, following [34, 44], to write the second order perturbation of the Einstein-Hilbert action around a flat FRW background in the form

$$\frac{1}{2}\delta^2 S = \frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^3x\left((\dot{v})^2 - \delta^{ij}\partial_i v\partial_j v + \frac{\ddot{z}}{z}v^2\right), \tag{6.12}$$

where the parameter z is defined as

$$z \equiv \sqrt{1-\frac{\dot{\mathscr{H}}}{\mathscr{H}^2}}. \tag{6.13}$$

The last expression leads to the Hamiltonian in Fourier space for the k-mode [98]

$$\mathcal{H}_{\boldsymbol{k}} = \frac{1}{2}\left[\pi_{\boldsymbol{k}}^2 + v_{\boldsymbol{k}}^2\left(k^2-\frac{\ddot{z}}{z}\right)\right]. \tag{6.14}$$

This corresponds to the Hamiltonian of an harmonic oscillator with a time-dependent mass. The frequency of the oscillator thus takes the form:

$$\omega_{\boldsymbol{k}}^2(\tau) \equiv k^2 - \frac{\ddot{z}}{z}. \tag{6.15}$$

6.1.2 Quantization and WKB approach

The total Hamiltonian constraint for each k-mode of the classical system is obtained by adding the background part (6.7) and the perturbative part (6.14)

$$\mathcal{H} = \frac{\mathrm{e}^{-2\alpha}}{2}\left[-\frac{\pi_\alpha^2}{m_{\mathrm{P}}^2} + \frac{\pi_\phi^2}{m_{\mathrm{P}}^2} + m_{\mathrm{P}}^2\mathrm{e}^{2\alpha}\mathcal{V}\right] + \frac{1}{2}\left[\pi_{\boldsymbol{k}}^2 + \omega_{\boldsymbol{k}}^2 v_{\boldsymbol{k}}^2\right] = 0. \tag{6.16}$$

The canonical quantization of this model can be achieved by formally substituting each momentum with a derivative taken on the respective conjugate variable

$$\pi_\alpha \to -\mathrm{i}\frac{\partial}{\partial\alpha}, \qquad \pi_\phi \to -\mathrm{i}\frac{\partial}{\partial\phi}, \qquad \pi_{\boldsymbol{k}} \to -\mathrm{i}\frac{\partial}{\partial v_{\boldsymbol{k}}}. \tag{6.17}$$

This way, we obtain a Wheeler-DeWitt equation for a single k-mode of the background and perturbative degrees of freedom, which reads

$$\frac{1}{2}\left[\frac{\mathrm{e}^{-2\alpha}}{m_{\mathrm{P}}^2}\left(\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\phi^2}+m_{\mathrm{P}}^2\mathrm{e}^{2\alpha}\mathcal{V}\right)+\left(-\frac{\partial^2}{\partial v_{\boldsymbol{k}}^2}+\omega_{\boldsymbol{k}}(\tau)^2 v_{\boldsymbol{k}}^2\right)\right]\Psi_{\boldsymbol{k}}=0. \tag{6.18}$$

To make the notation more compact it is useful to relabel the background variables as q^A, where the index A takes the values 0 and 1

$$q^0\equiv\alpha,\qquad q^1\equiv\phi \tag{6.19}$$

In addition, a background supermetric in configuration space is introduced in the form

$$\mathcal{G}_{ab}\equiv\mathrm{diag}\left(-\mathrm{e}^{-2\alpha},\mathrm{e}^{-2\alpha}\right). \tag{6.20}$$

Equation (6.18) then can be written as:

$$\frac{1}{2}\left[-\frac{1}{m_{\mathrm{P}}^2}\mathcal{G}_{ab}\frac{\partial^2}{\partial q_a\partial q_b}+m_{\mathrm{P}}^2\mathcal{V}(q^A)-\frac{\partial^2}{\partial v_{\boldsymbol{k}}^2}+\omega_{\boldsymbol{k}}^2(\tau)v_{\boldsymbol{k}}^2\right]\Psi_{\boldsymbol{k}}(q^A,v_{\boldsymbol{k}})=0. \tag{6.21}$$

the last eq. can be solved perturbatively by applying the semiclassical WKB method of [38] and sec. 5.2. indeed, it is easy to identify the various terms: doing this, we can safely write the final quantum gravity Hamiltonian as

$$\mathrm{i}\frac{\partial}{\partial\tau}\chi_{\boldsymbol{k}}^{(1)}=\mathcal{H}_{\boldsymbol{k}}\chi_{\boldsymbol{k}}^{(1)}-\frac{1}{2m_{\mathrm{P}}^2}\left[\frac{(\mathcal{H}_{\boldsymbol{k}})^2}{\mathcal{V}}+\mathrm{i}\frac{\partial}{\partial\tau}\left(\frac{\mathcal{H}_{\boldsymbol{k}}}{\mathcal{V}}\right)\right]\chi_{\boldsymbol{k}}^{(1)}, \tag{6.22}$$

where

$$\mathcal{H}_{\boldsymbol{k}}=\frac{1}{2}\left[-\frac{\partial^2}{\partial_{\boldsymbol{k}}^2}+v_{\boldsymbol{k}}^2\omega_{\boldsymbol{k}}^2(\tau)\right]. \tag{6.23}$$

Obviously, this suffers from all the issues we described earlier in sec. 5.2. Here, however, we want to analyse a different kind of problem.

A critical point in the previous procedure is the derivation of the $\ddot{z}/z$ term contained in the potential $\omega_{\boldsymbol{k}}^2 v_{\boldsymbol{k}}^2$ of the Hamiltonian (6.14). It is clear from its definition (6.13) that it contains higher-order time derivatives of the background variable a or, equivalently, of α. In the original papers [34, 44], the authors obtain this term performing integration by parts over the time variable in the action integral, and do so legitimately since their aim was to quantize only the perturbations around a classical background. Thus they are allowed to keep $\ddot{z}/z$ after quantization as a function of classical variables.

On the other hand, in [98] the authors' goal is to quantize the background as well as the perturbations, therefore the higher-order time derivatives of canonical coordinates become, in the Hamiltonian formalism, higher-order derivatives of the momenta, whose treatment in the quantization process is unclear. This concern is expressed by the authors themselves, but then passed over [98]. Therefore, in order to verify the validity of their result, it is necessary to adopt an approach that avoids the mathematical manipulations that lead to the Hamiltonian (6.14).

6.2 Gauge invariant perturbations Hamiltonian

We now describe a procedure that allows to formulate the Hamiltonian of scalar perturbations in a gauge-invariant fashion. This method bears many similarities to the Hamilton-Jacobi method for finding the equations of motion of a system, except in our case the goal is to determine the true physical degrees of freedom of such system, which must be gauge-invariant. To do this, we first have to clearly separate the scalar degrees of freedom in the perturbed ADM 3-metric from the vector and tensor ones and compute the scalar part of the Hamiltonian and momentum constraints.

Let us consider a flat FRW 3-metric $h_{ij} = \mathrm{e}^{2\alpha}\delta_{ij}$ and its perturbations $\gamma_{ij} \equiv \delta h_{ij}$. The latter comprises six degrees of freedom: two scalar ones, two vector ones and two tensor ones. Therefore, we can consider a decomposition of the following kind [50]

$$\gamma_{ij} = \sum_{m=1}^{6} A^m_{ij}\gamma_m, \tag{6.24}$$

where $m = 1, 2$ are the scalar, $m = 3, 4$ the vector, and $m = 5, 6$ the tensor perturbations, A^m_{ij} being six linear operators. The matrices $A^m_{ij}\gamma_m$ form an orthogonal basis in the space of 3×3 symmetric matrices, with the definition of scalar product between two elements M and N of the space being

$$(M, N) \equiv \mathrm{Tr}(MN) = h^{ik}h^{jl}M_{ij}N_{kl}. \tag{6.25}$$

For the elements of the basis we have the conditions [86]

$$h^{ij}A^m_{ij} = 0, \qquad m = 2, \ldots, 6 \tag{6.26a}$$

$$\partial^i\partial^j A^m_{ij} = 0, \qquad m = 3, 4 \tag{6.26b}$$

$$\partial^i A^m_{ij} = 0, \qquad m = 5, 6. \tag{6.26c}$$

Since we are interested only in the scalar part of (6.24), we just need the expression of the first two operators of the basis. These are [86]

$$A^1_{ij} = h_{ij} \tag{6.27a}$$

$$A^2_{ij} == \mathrm{e}^{2\alpha}\left(\partial_i\partial_j - \frac{h_{ij}}{3}\partial_k\partial^k\right), \tag{6.27b}$$

thus the scalar part of the metric perturbation can be written as the sum of two terms

$$\gamma_{ij} = \frac{\gamma}{3}h_{ij} + \mathrm{e}^{2\alpha}\left(\partial_i\partial_j - \frac{h_{ij}}{3}\partial_k\partial^k\right)\mu. \tag{6.28}$$

The first term is the trace part of the perturbation (γ being the trace of γ_{ij}), whereas the second term is called the longitudinal part of the perturbation, with μ the longitudinal scalar degree of freedom.

It is more convenient to work in Fourier space because A^2_{ij} is a differential operator, so that we have the formal substitution $\partial_i \to ik_i$. Then the expression for A^2_{ij} becomes

$$\tilde{A}^2_{ij} = -\mathrm{e}^{2\alpha}\left(k_ik_j - \frac{h_{ij}}{3}k^2\right) \tag{6.29}$$

where $k^2 \equiv h^{ij}k_ik_j$. Therefore, in Fourier space the expression of the metric perturbation reads

$$\tilde{\gamma}_{ij} = \frac{\tilde{\gamma}}{3}h_{ij} - \mathrm{e}^{2\alpha}\left(k_ik_j - \frac{h_{ij}}{3}k^2\right)\tilde{\mu} \tag{6.30}$$

From now on we will always refer to the Fourier transforms of scalar perturbations and operators: we will therefore drop the tilde and imply an integration over $\mathrm{d}^3k/(2\pi)^3$.

From equation (6.28) we can see that the coefficients of the scalar part of the decomposition are

$$\gamma_1 = \frac{\gamma}{3} \tag{6.31a}$$

$$\gamma_2 = \mu \tag{6.31b}$$

and, considering a perturbation $\delta\alpha$ to the scale factor, we have

$$\gamma_1 = 2\delta\alpha. \tag{6.32}$$

Our task now is to write the Hamiltonian and momentum constraints in terms of these scalar perturbations and their conjugate momenta. What we need is a canonical transformation that allows the change of variables to take place. Following [50], we define a generating function of the third kind to find such a transformation

$$S = -P^{ij}\gamma_m A^m_{ij}. \tag{6.33}$$

The decomposition (6.24) can also be used for the perturbed momentum

$$P^{ij} = \delta\Pi^{ij} = \frac{h^{ij}}{3}P - \mathrm{e}^{-2\alpha}\left(k^ik^j - \frac{h^{ij}}{3}k^2\right)P_\mu, \tag{6.34}$$

where with P_μ we indicate the longitudinal degree of freedom of P^{ij}. With these definitions, we can explicitly write the generating function as

$$S = -2\delta\alpha P - \frac{2}{3}k^4\mu P_\mu. \tag{6.35}$$

The momenta conjugate to the variables $\delta\alpha$ and μ are

$$\pi_{\delta\alpha} = -\frac{\partial S}{\partial\delta\alpha} = 2P \tag{6.36a}$$

$$\pi_\mu = -\frac{\partial S}{\partial\mu} = \frac{2}{3}k^4P_\mu. \tag{6.36b}$$

We can now write the scalar part of the first perturbations to the constraints (6.1a) and (6.1b), which will be called $\delta\mathcal{H}$ and $\delta\mathcal{H}_i$ respectively. These constitute four constraints for the perturbations. It can be shown [50] that $\delta\mathcal{H}$ already contains only scalar degrees of freedom, whereas the scalar part of $\delta\mathcal{H}_i$ may be extracted by taking its divergence: this is granted by equations (6.26b) and (6.26c), which ensures that the divergence makes vector and tensor degrees of freedom vanish. This way, we only have two constraints for the scalar perturbations, which we will relabel E and M. Using decompositions (6.30) and (6.34), these are [50]

$$\begin{aligned}\delta\mathcal{H} \equiv E = \mathrm{e}^{-3\alpha}\Bigg[&\left(-\frac{2\pi G}{3}\pi_\alpha^2 - \frac{3}{2}\pi_\phi^2 - 3\mathrm{e}^{6\alpha}V\right)\delta\alpha \\ &- \frac{4\pi G}{3}\pi_\alpha\pi_{\delta\alpha} + \pi_\phi\delta\pi_\phi + \mathrm{e}^{6\alpha}\delta\phi\partial_\phi V - 2\frac{\mathrm{e}^{6\alpha}}{8\pi G}k^2\delta\alpha - \frac{\mathrm{e}^{6\alpha}}{24\pi G}k^4\mu\Bigg] = 0\end{aligned} \tag{6.37a}$$

$$\partial^i\delta\mathcal{H}_i \equiv M = -\frac{2}{3}\delta\alpha\pi_\alpha - \frac{1}{3}\pi_{\delta\alpha} + \frac{2}{9}k^2\pi_\alpha\mu + \frac{2}{k^2}\pi_\mu + \pi_\phi\delta\phi = 0. \tag{6.37b}$$

6.2.1 Hamilton-Jacobi method for constraints

Now that we have the scalar part of both constraints expressed in terms of the canonical coordinates $\delta\alpha$, μ, $\delta\phi$ and their respective conjugate momenta, we can illustrate the method by means of which we will achieve a gauge-invariant expression for the second-order Hamiltonian.

We are dealing with a 6-dimensional scalar phase space with two constraints $E = 0$ and $M = 0$. These are first class constraints: their Poisson bracket vanishes weakly, i.e. it is either zero or a linear combination of the constraints. The two constraints restrict the freedom of the system to a 4-dimensional constraint subspace: such a system is now subject to the Hamiltonian flow (i.e. the Hamilton equations), and a constraint flow, that is the gauge transformations generated by the constraints. The constraints being first class means that a system whose initial conditions lie in the constraint subspace, will always remain confined to that subspace under the Hamiltonian and constraint flow (for a discussion of the theory of constrained systems, see [66]).

Since the two scalar constraints generate two gauge functions, the gauge orbits are 2-dimensional: therefore the physical phase space is further reduced to a 2-dimensional subspace. Thus we expect to be able to reduce the dynamics of the system to a canonical gauge-invariant coordinate and to its conjugate momentum.

The following procedure, first presented in [37], is inspired by the standard Hamilton-Jacobi theory. In order to change canonical variables to (Q_i, P^i), it is necessary to use a canonical transformation to preserve the form of the Hamiltonian, which can be derived from a generating function $S(q_i, P^i, t)$. Old and new coordinates are related by

$$p^i = \frac{\partial S}{\partial q_i} \tag{6.38a}$$

$$Q_i = \frac{\partial S}{\partial P^i}, \tag{6.38b}$$

while the new Hamiltonian then is given by

$$H_{\mathrm{new}} = H + \frac{\partial S}{\partial t}. \tag{6.39}$$

The explicit time dependence in the Hamiltonian and generating function can be due to the presence of external parameters λ, π_λ that can be canonical conjugate of a system regulated by a Hamiltonian H_λ. If this is the case, the previous equation can be reformulated as

$$H_{\text{new}} = H + \{S, H_\lambda\}, \tag{6.40}$$

where the Poisson bracket acts only on the external variables λ, π_λ. The Hamilton-Jacobi method requires to find that transformation that makes the new Hamiltonian vanish

$$H\left(q_i, p^i = \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{6.41}$$

This technique can be used to solve the equations of motion of a system, provided that one is able to find a complete solution for the Hamilton-Jacobi equation, that is with as many constants of integration as the n degrees of freedom.

The method we are about to follow is in many respects similar to Hamilton-Jacobi theory, but it is applied to the first class constraints of the system. For a system with a 2n-dimensional phase space and m first class constraints $C_a\left(q_i, p^i\right) = 0$, $a = 1, ..., m$, the physical phase space has dimension $2(n-m)$. It is possible to obtain the canonical coordinates that span the physical space by solving m Hamilton-Jacobi-like equations

$$C_a\left(q_i, p^i = \frac{\partial S}{\partial q_i}\right) = 0, \quad a = 1, \ldots, m. \tag{6.42}$$

The generating function $S\left(q_i, P^k\right)$, $k = m+1, ..., n$, depends on $n-m$ constants of integration P^k. Then the relation between the old coordinates (q_i, p^i) and the actual degrees of freedom (Q_k, P^k) is given by the canonical transformation

$$p^i = \frac{\partial S}{\partial q_i} \tag{6.43a}$$

$$Q_k = \frac{\partial S}{\partial P^k}. \tag{6.43b}$$

These equations can be inverted adding m arbitrary parameters β_a (the gauge functions) to obtain the old canonical coordinates

$$q_i = q_i\left(Q_k, P^k, \beta_a\right), \qquad p_i = p_i\left(Q_k, P^k, \beta_a\right). \tag{6.44}$$

If the Poisson bracket of the Hamiltonian with any of the constraints vanishes weakly, then the gauge-invariant Hamiltonian for the physical degrees of freedom is given by an expression similar to (6.39) or (6.40) [37]

$$H_{GI} = H\left(q_i\left(Q_k, P^k\right), p_i\left(Q_k, P^k\right)\right) + \{S, H_\lambda\}, \tag{6.45}$$

where it suffices to express q_i and p_i as functions of Q_k and P^k, since the gauge functions β_a automatically cancel.

Now we should apply this technique to our specific case, defining the canonical coordinates q_α, p^β

$$q_0 \equiv \delta\phi, \quad q_1 \equiv \delta\alpha, \quad q_2 \equiv \mu \tag{6.46a}$$

$$p^0 \equiv \delta\pi_\phi, \quad p^1 \equiv \pi_{\delta\alpha}, \quad p^2 \equiv \pi_\mu, \tag{6.46b}$$

and solving eq. (6.42) for each constraint, i.e. for

$$E\left(q_\alpha, p^\beta = \frac{\partial S}{\partial q_\beta}\right) = 0 \tag{6.47a}$$

$$M\left(q_\alpha, p^\beta = \frac{\partial S}{\partial q_\beta}\right) = 0, \tag{6.47b}$$

where $\alpha, \beta = 0, 1, 2$. The details of this calculation are described in [50], we now focus on the interesting parts.

We have only one degree of freedom left, so we find only a new gauge invariant variable

$$v = \delta\phi + \frac{3}{4\pi G}\frac{\pi_\phi}{\pi_\alpha}\left(\delta\alpha + \frac{k^2}{6}\mu\right) = \delta\phi - \frac{\dot{\phi}}{\mathscr{H}}\left(\delta\alpha + \frac{k^2}{6}\mu\right), \tag{6.48}$$

which is exactly the gauge-invariant Mukhanov-Sasaki, except that it is written in Fourier space and here we do not have the rescaling of a scale factor a, which is an ad hoc choice. The gauge invariant Hamiltonian, after performing the whole procedure, is

$$\begin{aligned}\mathcal{H}_{\boldsymbol{k}}^{\mathrm{GI}} = \frac{1}{2}\mathrm{e}^{-3\alpha}\Bigg\{P_v^2 + \Bigg[24\pi G\pi_\phi^2 - 18\frac{\pi_\phi^4}{\pi_\alpha^2} \\ - 12\mathrm{e}^{6\alpha}\frac{\pi_\phi}{\pi_\alpha}\partial_\phi V + \mathrm{e}^{6\alpha}\left(\partial_\phi\partial_\phi V + k^2\right)\Bigg] v^2\Bigg\}.\end{aligned} \tag{6.49}$$

We stress that, in this Hamiltonian, the scalar field has not been rescaled yet.

Not making the assumption about the classical behaviour of z in eq. (6.13) left us with a huge problem: the background momentum π_α now appears in the denominator and we have a nonlocal theory. This prevents us from applying canonical quantization to the Hamiltonian and thus to use the semiclassical approximation scheme. We stress that the background momenta terms in (6.49) are analogous to the term $\ddot{z}/z$ in (6.14), which contains background momenta and their derivatives. Therefore, we have to consider other ways of removing this issue, before quantization is carried out.

6.3 WKB approximation for the new Hamiltonian

The Hamiltonian (6.49) of last section is nonlocal and has the conjugate momentum of α at the denominator: we cannot simply quantize this theory. To solve this issue, we first multiply the total Hamiltonian by π_α^2 and then proceed to the quantization. At this point, we expand in $1/m_{\mathrm{P}}^2$ as in [98]. For simplicity we will write the new Hamiltonian as

$$\mathcal{H} = \frac{e^{-2\alpha}}{2}\left[-\frac{\pi_\alpha^2}{m_{\mathrm{P}}^2} + \frac{\pi_\phi^2}{m_{\mathrm{P}}^2} + m_{\mathrm{P}}^2 e^{2\alpha}\mathcal{V}\right] + \mathcal{H}_v, \tag{6.50}$$

where $\mathcal{H}_v$ is given by (6.49). Therefore, we proceed to quantize and solve

$$\left[\left(\frac{1}{2}\frac{\mathcal{G}_{ab}}{m_{\mathrm{P}}^2}\pi_a\pi_b + \frac{1}{2}m_{\mathrm{P}}^2\mathcal{V}\right)\pi_\alpha^2 + \mathcal{H}_v\pi_\alpha^2\right]\psi_{\boldsymbol{k}}\chi_{\boldsymbol{k}} = 0. \tag{6.51}$$

The first thing we need to do is write the perturbed Hamiltonian, after applying the rescaling of the scalar field of [98]. We find

$$\begin{aligned}\mathcal{H}_{\boldsymbol{k}}^{\mathrm{GI}} = \frac{1}{2}\mathrm{e}^{-3\alpha}\Bigg\{P_v^2 + \Bigg[\frac{18}{m_{\mathrm{P}}^4}\left(\pi_\phi^2 - \frac{\pi_\phi^4}{\pi_\alpha^2}\right) \\ - 6\mathrm{e}^{2\alpha}\frac{\pi_\phi}{\pi_\alpha}\partial_\phi\mathcal{V} + \left(\frac{1}{2}\mathrm{e}^{2\alpha}\partial_\phi\partial_\phi\mathcal{V} + \mathrm{e}^{6\alpha}k^2\right)\Bigg]v^2\Bigg\}.\end{aligned} \tag{6.52}$$

In order to simplify our calculations, while still matching the expansion of [98], we expand through

$$\psi_{\boldsymbol{k}} = \mathrm{e}^{\mathrm{i}m_{\mathrm{P}}^2 S_0 + \zeta + \mathrm{i}m_{\mathrm{P}}^{-2}\sigma + \dots}, \qquad \chi_{\boldsymbol{k}} = \mathrm{e}^{\mathrm{i}S_1 - \zeta + \mathrm{i}m_{\mathrm{P}}^{-2}\eta + \dots}, \qquad D(q^A) = \mathrm{e}^{-\zeta(q^A)}, \tag{6.53}$$

where D is the same as in sec. 5.2 and for brevity we define $\mathrm{i}\mu = \mathrm{i}S_1 - \zeta$. This theory appears much more complex from the original one of [98]. To simplify our calculation, we leave the perturbations Hamiltonian simply written as $\mathcal{H}_v$ and proceed to the expansion.

At order $\mathcal{O}(m_{\mathrm{P}}^8)$ we find again

$$(\partial_{v_{\boldsymbol{k}}} S_0)^2 = 0. \tag{6.54}$$

At the order $\mathcal{O}(m_{\mathrm{P}}^6)$ we are left with

$$\left[G_{AB}(\partial_A S_0)(\partial_B S_0) + \mathrm{e}^{2\alpha}\mathcal{V}\right](\partial_\alpha S_0)^2 = 0, \tag{6.55}$$

which is the usual Hamilton-Jacobi equation, plus a spurious term. Before proceeding, we can use the Hamilton-Jacobi equation to simplify the perturbations Hamiltonian

$$\begin{aligned}\mathcal{H}_{\boldsymbol{k}}^{\mathrm{GI}}\partial_\alpha^2(\psi_{\boldsymbol{k}}\chi_{\boldsymbol{k}}) = \frac{\mathrm{e}^{-3\alpha}m_{\mathrm{P}}^4}{2}\Bigg\{ - (\partial_\alpha S_0)^2\psi_{\boldsymbol{k}}\partial_v^2\chi_{\boldsymbol{k}} + v^2\psi_{\boldsymbol{k}}\chi_{\boldsymbol{k}}\Bigg[18(\partial_\alpha S_0)^2\mathrm{e}^{2\alpha}\mathcal{V} \\ - 6\mathrm{e}^{2\alpha}\partial_\phi\mathcal{V}\partial_\phi S_0\partial_\alpha S_0 + \left(\frac{1}{2}\mathrm{e}^{2\alpha}\partial_\phi\partial_\phi\mathcal{V} + \mathrm{e}^{6\alpha}k^2\right)(\partial_\alpha S_0)^2\Bigg]\Bigg\} + \mathcal{O}\Big(m_{\mathrm{P}}^2\Big) \equiv \\ - \frac{m_{\mathrm{P}}^4}{2}(\partial_\alpha S_0)^2\psi_{\boldsymbol{k}}\left\{\partial_v^2\chi_{\boldsymbol{k}} - v^2\chi_{\boldsymbol{k}}\tilde{\omega}_{\boldsymbol{k}}^2\right\} + \mathcal{O}\Big(m_{\mathrm{P}}^2\Big),\end{aligned} \tag{6.56}$$

where we reabsorbed the $\mathrm{e}^{3\alpha}$ term in the other variables and the last passage is a definition for $\tilde{\omega}_{\boldsymbol{k}}$. We are able, at this order, to reproduce the correct form of the perturbations Hamiltonian. However, the match between $\tilde{\omega}_{\boldsymbol{k}}$ and $\omega_{\boldsymbol{k}}$ depends on the functional form of $\mathcal{V}$ and on the solution of the Hamilton-Jacobi equation. Fortunately, with the potential of [98], we recover the desired result [115]. This result was expected, because this is the higher perturbation order for $\mathcal{H}_{\boldsymbol{k}}^{\mathrm{GI}}$, and its classical value coincide with the one one of [34, 44, 98], see [50].

At the order $\mathcal{O}(m_{\mathrm{P}}^4)$ we have the modified WKB equation

$$\begin{aligned}2G_{AB}(\partial_\alpha S_0)^2(\partial_A S_0)\partial_B\zeta + (\partial_\alpha S_0)^2 G_{AB}\partial_A\partial_B(S_0) \\ + 4(\partial_\alpha S_0)G_{AB}(\partial_A S_0)(\partial_B\partial_\alpha S_0) = 0,\end{aligned} \tag{6.57}$$

where the last term comes from the addition of the α-momentum. Beside that, the quantum part gives

$$\left[\mathrm{i}(\partial_\alpha S_0)^2 G_{AB}(\partial_A S_0)(\partial_B \mu) + \mathcal{H}_v \partial_\alpha^2\right] \psi_{\boldsymbol{k}} \chi_{\boldsymbol{k}} = 0. \tag{6.58}$$

If the perturbations Hamiltonian was a "good" operator, we would find the Schrödinger equation. Indeed, it can be shown that this happens also in this case, although the calculations are much more cumbersome [115].

The real problems arise at order $\mathcal{O}(m_\mathrm{P}^2)$. Here, a quantity of new, spurious term populates the equations. The problem is that we don't have a perfect cancellation of the new terms, as happened until now. The modified WKB equation at this order is easily obtained as an expansion of the background only part, i.e. expanding at the desired order

$$\left[\left(-\frac{1}{2}\frac{G_{AB}}{m_\mathrm{P}^2}\partial_A\partial_B + \frac{1}{2}m_\mathrm{P}^2\mathcal{V}\right)\partial_\alpha^2\right]\psi_{\boldsymbol{k}} = 0 \text{ at order } \mathcal{O}\left(m_\mathrm{P}^2\right). \tag{6.59}$$

It is, however, the quantum gravity Schrödinger equation that deserves our attention. Making use of eqs. (6.55), (6.57), (6.58) and (6.59), after some cumbersome but trivial calculations we finally find

$$\begin{aligned}
&\frac{1}{2}G_{AB}(\partial_\alpha S_0)^2\{-2\partial_A S_0 \partial_B \eta + 2\mathrm{i}\partial_A \zeta \partial_B \mu + [-\partial_A \mu \partial_B \mu + \mathrm{i}\partial_A \partial_B \mu]\} \\
&- G_{AB}\{-[2\partial_A S_0 \partial_B \zeta \partial_\alpha S_0 + \partial_A \partial_B S_0 \partial_\alpha S_0 + 2\partial_A S_0 \partial_A \partial_\alpha S_0]\mathrm{i}\partial_\alpha \mu \\
&\qquad - 2\mathrm{i}\partial_\alpha S_0 \partial_\alpha \partial_A S_0 \partial_B \mu - 2\partial_\alpha S_0 \partial_A S_0[-\partial_\alpha \mu \partial_B \mu + \mathrm{i}\partial_\alpha \partial_B \mu]\} \\
&+ \frac{(\mathcal{H}_{\boldsymbol{k}}\partial_\alpha^2)\,\psi_{\boldsymbol{k}}\chi_{\boldsymbol{k}}}{\psi_{\boldsymbol{k}}\chi_{\boldsymbol{k}}} = 0,
\end{aligned} \tag{6.60}$$

where the last element should be expanded at the correct order. Here, if $\mathcal{H}_{\boldsymbol{k}}$ was a "good" Hamiltonian and we could simplify it with the ∂_α^2 term, we would find the corrected Schrödinger from the first line of last equation, with the leftover of $-2\chi_{\boldsymbol{k}}^{-1}\mathrm{i}\partial_\alpha S_0 \mathcal{H}_{\boldsymbol{k}}(\chi_{\boldsymbol{k}}\partial_\alpha\mu)$ from the last term. This remnant, together with the second and third line of eq. (6.60), should be identically zero; but this does not happen. Indeed, there are some spurious terms deriving from the combination of derivatives in α that are mixed with the usual ones, while others cannot be simplified because, in general, $\mathcal{H}_{\boldsymbol{k}}$ do not commute with ∂_α. The same happened in the WKB equation (6.57), altering its form, but now this prevents recovering the quantum gravitational Schrödinger equation. Moreover, other problems are related to $\mathcal{H}_{\boldsymbol{k}}$ not being "good", so that it should be expanded, finally disrupting any remaining hope to recover the expected result.

We are unable to explicitly solve this equation, or to revert to some form similar to the expected one, because the derivatives in α do not combine naturally in the definition of time, neither in the previous WKB expansion. However, we recovered the Schrödinger equation of quantum mechanics.

All of this has been caused by not approximating the z in (6.13) as classical. The effects of assuming z as classical were to be expected exactly at the subsequent expansion order, i.e. the quantum gravitational one. We can only conclude that this theory is not equivalent to the one of [98] and that assumptions on the classical behaviour of some pieces of the equations should be taken with more care.

6.4 Results and possible extensions

We analysed the procedure of quantization of [98]. In order to avoid the unjustified substitution of a momentum with a numeric function (see eq. (6.13)), we tried to change variables in the Hamiltonian before the quantization procedure. Nevertheless, we failed and we ended up with the same original problem.

We tried to circumvent this issue multiplying for the momentum at denominator before applying the quantization procedure. Even if we recovered the Schrödinger equation at the quantum mechanical order, we failed to recover the same quantum gravitational effects.

Our result suggests that, the requirements due to the momentum at denominator play some tricky role in the model. Moreover, even the authors of [98] express some concerns regarding its substitution with a numerical function. Therefore, the problem is still open and unsolved. What would have happened by considering, as it should, that term as an operator? Even if we did not solve the issue, our equations shed doubt on the validity of the original procedure.

An obvious extension of this work would be to find some simplifying scheme or assumption, or some alternative framework, to solve this problem, and compare the results to the original ones of [98]. Moreover, there have been other criticisms regarding other aspects of this work [101], but they share with it the problem analysed here. The tricky thing seems to be the necessary presence of a momentum at the denominator to construct a gauge invariant variable. If this is really so, than the only way to address this issue would be to deal directly with gauge related problems.

Chapter 7

Conclusions

In this thesis we studied the cosmological perturbations both from a classical and a quantum gravitational point of view.

In the classical part we studied the effects caused by the presence of a primordial magnetic field on cosmological perturbations. The magnetic field is believed to be the seed of the present magnetic fields, observed in galaxies and galaxy clusters [49, 57, 67, 73], as well as the cause for tye formation of large scale filament-like structures in the universe.

Given the short Debye length of the order of 10 cm, the universe can be described through a fluid theory. Moreover, the tight coupling between neutral and ionized matter [72, 89, 104] allows the universe to be described as a plasma, while the large photon to baryon ratio is responsible for keeping active a strong Thomson scattering, even after recombination and up to $z \simeq 100$ [18, 72, 89].

Thus, we relied on a general relativistic MDH formulation to account for the effects of magnetic fields on the seeds of cosmological perturbations.

Many authors already studied these phenomena, nearly always relying on isotropic models and neglecting anisotropic effects, see [77] and references therein. However, the present of the magnetic field is surely to introduce them [89], even on the general relativistic background, thus requiring an anisotropic modelization. In chapter 2, we developed a self-consistent scheme for the analysis of cosmological perturbations in presence of both a cosmological magnetic field and its related spatial anisotropy. We considered a Bianchi I model, whose anisotropy is completely due to the magnetic field presence, and we analysed the cosmological perturbations in synchronous gauge. We showed that we are able to control the gauge related problems, and that out main variables are gauge invariant for large enough times. Thus, we solved the equations for both super-horizon wavelengths in radiation dominated universe and for sub-horizon wavelength in the matter dominated era. We enhanced the present literature solutions [65, 77, 97, 105] adding the effects of anisotropy in both regimes, and we also confirmed the validity of the Newtonian solutions in matter dominated era [89]. We accompanied the analytical analysis with a numerical integration, showing the features we describe. These results are presented in [107, 114].

Then, in chapter 3, we faced the issue of viscosity in the Newtonian limit. The magnetic field presence causes shear in the system [89, 90]. If the viscous effects were relevant, then they could affect and destroy the anisotropic effects due to the magnetic field presence. This is a fundamental point to confirm the applicability of the results of chapter 2.

After a careful analysis, we showed both the effects of viscosity when it is relevant to the system and that, fortunately, it is limited to below a scale of about 5000 solar masses, so it would not interfere with the Jeans mechanism, nor with the magnetic field related anisotropy. We accompanied the analysis with a fine numeric integration, showing that the eccentricity of an initial spherical perturbation, due to the presence of a magnetic field throughout its evolution, is preserved in presence of viscosity. These analyses lead to [111].

In the second part of this thesis, we dealt with the cosmological perturbations on a quantum gravity level. We adopted the traditional approach of canonical quantization, leading to the Wheeler-DeWitt equation.

The first thing we had to face was the problem of time. This is one of the most relevant open issues in canonical quantum gravity, and although there is a huge literature about this problem, a commonly accepted solution has not been found yet.

In chapter 5 we carefully analysed and critically reviewed the current models presented by [36, 38], showing they they are deemed to be subject to non-unitarity at a quantum gravity level. We showed that the solution proposed by [103] is not acceptable, meanwhile extending the $\hbar$ expansion of [36] to arbitrary orders.

Then, we reviewed the promising exact decomposition of [53]. While it applies a way more robust formalism, we noted some discrepancies in the results. We fixed them, thus constructing a finer version of that approach. However, we also showed that that model, too, is subject to non-unitarity.

All the previous models are based on a semiclassical decomposition, similar to a Born-Oppenheimer approach, leading to the definition of a derivative with respect to time through the gradient with respect to the classical variables. We analysed these models to find the major cause of issues exactly in such definition. Learning from this, we proposed a novel solution, free of the issues of the usual definition, that we showed to be unitary even at the quantum gravity order. These results, together, are presented in [112].

Eventually, we presented in chapter 6 the application of the model of [38] to cosmological perturbations, as done in [98]. The calculations, however, rely on some assumptions that deserve a lot much care. We tried to “clean” the procedure by using a more robust formalism instead and not relying on additional assumptions on the quantum variables. However, we found ourselves stuck at the quantum gravity level. This could mean that those assumptions were to be taken with greater care, as we suggested, and that this issue still deserves attention, being our quantum gravity order not compatible with the original one. These results will appear in [115].

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