Fractional Diffusion-Telegraph Equations and their Associated Stochastic Solutions

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April 10, 2017

Abstract

We present the stochastic solution to a generalized fractional partial differential equation involving a regularized operator related to the so-called Prabhakar operator and admitting, amongst others, as specific cases the fractional diffusion equation and the fractional telegraph equation. The stochastic solution is expressed as a Lévy process time-changed with the inverse process to a linear combination of (possibly subordinated) independent stable subordinators of different indices. Furthermore a related SDE is derived and discussed.

Keywords: Time-changed processes; Lévy processes; Prabhakar operators; Regularized Prabhakar derivative; Fractional derivatives; Stochastic solution.

1 Introduction

In the last few decades considerable effort has been devoted to the study of fractional partial differential equations (fPDEs) that is of PDEs in which usual differential operators are substituted by fractional differential operators (for example two rather recent references are **?** and **?**). The simplest equation of this class is the so-called fractional diffusion equation, also known as diffusion-wave equation [see amongst others **????**]. Another well-known and well-studied fPDE is the fractional telegraph equation [**?????**]. In the more recent years, moreover, an increasing number of papers presented results connecting the study of fractional PDEs to that of some time-changed stochastic processes. The aim of this paper is to clarify this connection for a very general class of fPDEs which includes as specific cases both parabolic and hyperbolic fPDEs as well as more general integral and differential equations. In order to be more specific and for the sake of comprehension, we will start by recalling here the definitions of the classical fractional operators of Riemann–Liouville type and the Dzhrbashyan–Caputo derivative (see for the latter **???** — see **?** for a reference book).

Definition 1.1 (Riemann–Liouville integral)**.** Let $f \in L^1_{\text{loc}}(0,b)$, $0 < t < b \le \infty$, be a locally integrable real-valued function. The operator

$$J_t^\alpha f = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u)}{(t-u)^{1-\alpha}} \mathrm{d}u, \qquad \alpha > 0, \tag{1.1}$$

is called Riemann–Liouville integral of order α.

Definition 1.2 (Riemann–Liouville derivative)**.** Let $f \in L^1(0,b)$, $-\infty \le a < t < b \le \infty$, and define the power-law kernel $\mathscr{L}_\beta(t) = t^{\beta-1}/\Gamma(\beta)$, $\beta > 0$. Consider $\alpha > 0$ and write $m = \lceil \alpha \rceil$ for the smallest integer greater than or equal to α. For $f * \mathscr{L}_{m-\alpha} \in W^{m,1}(0,b)$, where $W^{m,1}(0,b)$ is the Sobolev space

$$W^{m,1}(0,b) = \left\{ h \in L^1(0,b) \colon \frac{\mathrm{d}^m}{\mathrm{d}t^m} h \in L^1(0,b) \right\}, \tag{1.2}$$

the Riemann–Liouville derivative of order α is defined as

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t (t-s)^{m-1-\alpha} f(s)\,\mathrm{d}s. \tag{1.3}$$

In order to introduce the definition of the Dzhrbashyan–Caputo derivative, let us denote by $AC^n(0,b)$, $n \in \mathbb{N}$, the space of real-valued functions $h(t)$ with continuous derivatives up to order $n-1$ on $(0,b)$ and such that $h^{(n-1)}(t)$ belongs to the space of absolutely continuous functions $AC(0,b)$, i.e.

$$AC^n(0,b) = \left\{ h:(0,b) \to \mathbb{R} \colon \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} f(x) \in AC(0,b) \right\}. \tag{1.4}$$

Definition 1.3 (Dzhrbashyan–Caputo derivative)**.** Let $\alpha > 0$, $m = \lceil \alpha \rceil$. The Dzhrbashyan–Caputo derivative of order $\alpha > 0$ is defined as

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} \frac{\mathrm{d}^m}{\mathrm{d}s^m} f(s)\,\mathrm{d}s \tag{1.5}$$

for $f \in AC^m(0,b)$ such that (1.5) exists.

Let us now recall one of the most famous applications of the Dzhrbashyan–Caputo derivative: the fractional diffusion equation in dimension one in its simplest form. Let us consider thus the Cauchy problem

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \lambda^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x,t), & t>0,\ x \in \mathbb{R}, \\ u(x,0) = \delta(x), & 0 < \alpha \le 2, \\ \frac{\mathrm{d}}{\mathrm{d}t} u(x,t)\Big|_{t=0} = 0, & 1 < \alpha \le 2. \end{cases} \tag{1.6}$$

It has been proven that the solution to (1.6) can be written as [**??**]

$$\frac{1}{2\lambda t^{\alpha/2}} W_{-\alpha/2, 1-\alpha/2}\left(-\frac{|x|}{\lambda t^{\alpha/2}} \right), \qquad t \ge 0,\ x \in \mathbb{R},\ 0 < \alpha \le 2, \tag{1.7}$$

where $W_{a,b}(z)$ is the Wright function [**?**, Chapter 1]. The solution (1.7) has the remarkable property that it reduces to the Gaussian function for $\alpha = 1$ and to the classical d'Alambert's solution to the wave equation for $\alpha \to 2$ while keeping an intermediate behaviour for $\alpha \in (1,2)$.
Aside the analytical point of view, by starting from the well-known fact that diffusion processes are strictly connected via their distributional structure to parabolic equations, the most part of the recent research on the subject has been in fact dedicated to construct stochastic processes that can be related to various classes of fractional PDEs and that can therefore furnish them with a microscopic interpretation. For example the fractional diffusion equation (1.6) can be related to a time-changed Brownian motion (see e.g. **????**). Indeed, let us call B_t, $t \ge 0$, a standard Brownian motion and V_t^α, $t \ge 0$, $\alpha \in (0,1)$ an α-stable subordinator, independent of B_t, from which the time-change will be constructed. Note that stable subordinators are a particularly well-behaved class of Lévy processes which are increasing and have a very simple Laplace exponent. Then, after having defined the right-inverse process as $K_t^\alpha = \inf\{s \ge 0 \colon V_s^\alpha > t\}$, $t \ge 0$ (see **?** for details) we have that the marginal distribution $\mathbb{P}\{B_{K_t^\alpha} \in \mathrm{d}x\}/\mathrm{d}x$ is the solution to (1.6). Similar considerations can be done for hyperbolic PDEs and some stochastic processes describing particles moving with finite velocity. With respect to this the reader can consult the papers by **????** and the references therein. For what concerns fractional evolution equations in abstract spaces we refer to **?????**.
In this paper we study fractional PDEs in which the operator acting on time generalizes the Dzhrbashyan–Caputo fractional derivative and connect them to time-changed Lévy processes. The general fractional PDE that we study in Section 3 generalizes both diffusion-like and telegraph-like differential equations.
Let us thus start by considering the operator $\mathbf{D}^\xi_{\alpha,\eta,\zeta;0+}$ that we call Prabhakar derivative. It was first introduced by **?** and it is defined as

$$\left(\mathbf{D}^\xi_{\alpha,\eta,\zeta;0+} f(\cdot) \right)(t) = \frac{\mathrm{d}^{\eta+\theta}}{\mathrm{d}t^{\eta+\theta}} \int_0^t (t-y)^{\theta-1} E^{-\xi}_{\alpha,\theta} \left[\zeta (t-y)^\alpha \right] f(y)\,\mathrm{d}y, \tag{1.8}$$

where $\theta > 0$, $\eta > 0$, $\zeta \in \mathbb{R}$, $t \ge 0$ and

$$E^\xi_{\alpha,\eta}(x) = \sum_{r=0}^\infty \frac{x^r (\xi)_r}{r!\,\Gamma(\alpha r + \eta)}, \qquad \alpha, \eta, \xi \in \mathbb{R},\ \alpha > 0, \tag{1.9}$$

is known as the generalized Mittag–Leffler function (see for example **?** or **?**). The symbol $(\xi)_r$ in (1.9) is the so-called Pochhammer symbol. Recall that the operator in (1.8) is the Riemann–Liouville derivative and notice that the operator $\mathbf{D}^{\xi}_{\alpha,\eta,\zeta;0+}$ is the left-inverse to the convolution-type operator [**?**]

$$\left(\mathbf{E}^{\xi}_{\alpha,\eta,\zeta;0+}f(\cdot)\right)(t) = \int_0^t (t-y)^{\eta-1}E^{\xi}_{\alpha,\eta}\left[\zeta(t-y)^{\alpha}\right]f(y)\,\mathrm{d}y, \tag{1.10}$$

which was originally introduced by **?**. Consider now the following operator:

Definition 1.4 (Regularized Prabhakar derivative)**.** Let $\eta > 0$, $\xi, \zeta \in \mathbb{R}$, $\alpha > 0$, $m = \lceil \eta \rceil$, $\kappa = \lceil \xi \rceil$, $f \in \mathbb{A}^{\kappa}(\mathbb{R}^+)$, where

$$\mathbb{A}^{\kappa}(\mathbb{R}^+) = \left\{u\colon \mathbb{R}^+ \mapsto \mathbb{R}^+ \text{ s.t. } \sum_{j=0}^{\kappa-1} a_j \frac{\partial^{\eta-\alpha j}}{\partial t^{\eta-\alpha j}}u \in C(\mathbb{R}^+),\ \alpha \in (0,1],\ a_j > 0\ \forall\, j,\ \left|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right| \leq t^{\beta-1},\ \beta > 0\right\}.$$

The operator

$$\left(\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+}f(\cdot)\right)(t) = \left(\mathbf{D}^{\xi}_{\alpha,\eta,\zeta;0+}f(\cdot)\right)(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)t^{k-\eta}E^{-\xi}_{\alpha,k-\eta+1}(\zeta t^{\alpha}), \tag{1.11}$$

is called regularized Prabhakar derivative.

Note that the regularized Prabhakar derivative has Laplace transform

$$\int_0^{\infty} e^{-st}\left(\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+}f(\cdot)\right)(t)\,\mathrm{d}t = s^{\eta}(1-\zeta s^{-\alpha})^{\xi}\tilde{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)s^{\eta-k-1}(1-\zeta s^{-\alpha})^{\xi}, \tag{1.12}$$

where $\tilde{f}(s)$ is the Laplace transform of function f.
In Section 2 we introduce a stochastic process M^{δ}_t, $t \geq 0$, $\delta \geq 0$, built as a suitable linear combination of stable subordinators that are made dependent by a common random time-change with a further independent stable subordinator. Indeed formula (2.31) tells us that $M^{\delta}_t = \sum_{r=0}^{n}\binom{n}{r}^{\delta/(n\omega_r)}{}_rV^{\omega_r}_t$, $t \geq 0$, where ${}_rV^{\omega_r}_t$, $r = 1, \dots n$, are the dependent ω_r-stable subordinators with $\omega_r = \gamma + \nu - r\nu\delta/n$, $n = \lceil \delta \rceil$ and $\nu\delta < \gamma + \nu < 1$. The main result of Section 2 is Theorem 2.2 which states that the process $Z^{\delta}_t := \inf\{s \geq 0\colon M^{\delta}_t \notin (0,t)\}$, $t \geq 0$, i.e. the right-inverse process to M^{δ}_t, has marginal distribution which solves the Cauchy problem

$$\begin{cases} \left(\mathbb{D}^{\delta}_{\nu,\gamma+\nu,-1;0+}h(x,\cdot)\right)(t) = -\frac{\partial}{\partial x}h(x,t), & t \geq 0,\ x \geq 0, \\ h(x,0^+) = \delta(x). \end{cases} \tag{1.13}$$

In Section 3 the most general fPDE we deal with is

$$\left(\mathbb{D}^{\delta}_{\nu,\gamma+\nu,-1;0+}g(x,\cdot)\right)(t) = \mathscr{A}g(x,t), \qquad x \in \mathbb{R}^d,\ t > 0, \tag{1.14}$$

with $\delta > 0$, $\delta\nu < \gamma + \nu \leq 1$, $\gamma, \nu \in (0,1)$, and where $\mathscr{A}$ is the infinitesimal generator of a Lévy process A^x_t, $t \geq 0$, starting from $x \in \mathbb{R}^d$. We prove in Theorem 3.1 that the time-changed process $A^x_{Z^{\delta}_t}$, $t \geq 0$, has one-dimensional distribution which solves (1.14) with a suitable choice of the initial datum.
Section 4 contains all the details for the case $\mathscr{A} = \partial^2/\partial x^2$. However, in that section $\delta\nu < \gamma + \nu \leq 2$, hence including important specific cases such as the classical telegraph equation. This clearly does not permit us to relate the solution to $A^x_{Z^{\delta}_t}$ but still several results from an analytical point of view are obtained.

2 The operator $\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+}$ and its relation to the hitting time of linear combinations of stable subordinators

In the following we study the connection of the operator $\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+}$ to some stochastic processes constructed as inverse processes to some linear combinations of dependent stable subordinators.

2.1 Connections of $\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+}$ to the hitting time of linear combinations of stable subordinators

Here we analyze the relations of some inverse processes of linear combinations of dependent stable subordinators to differential equations involving the operator $\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+}$ with a restriction on the range of the parameter η and considered as an operator acting on functions $t \mapsto f(\cdot,t)$. When $\eta \in (0,1)$ from formula (1.11) we have

$$\left(\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+} f(x,\cdot)\right)(t) = \left(\mathbf{D}^{\xi}_{\alpha,\eta,\zeta;0+} f(x,\cdot)\right)(t) - f(x,0^+)\, t^{-\eta} E^{-\xi}_{\alpha,1-\eta}(\zeta t^{\alpha}), \tag{2.1}$$

with Laplace transform

$$\int_0^{\infty} e^{-st} \left(\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+} f(x,\cdot)\right)(t)\,\mathrm{d}t = s^{\eta}(1-\zeta s^{-\alpha})^{\xi} \tilde{f}(x,s) - f(x,0^+) s^{\eta-1}(1-\zeta s^{-\alpha})^{\xi}. \tag{2.2}$$

Let us now assume $(\gamma+\nu) \in (0,1)$, $\delta \in \mathbb{R}$, $k = \lceil \delta \rceil$, and call $\Omega^{d+1} := \Omega \times (0,\infty)$ with $\Omega \subseteq \mathbb{R}^d$. We define the function space

$$\begin{aligned}\mathbb{A}^k(\Omega^{d+1}) = \Bigg\{ u\colon \Omega^{d+1} \mapsto \mathbb{R}^+ \text{ s.t. } &\sum_{j=0}^{k-1} a_j \frac{\mathfrak{d}^{\eta-\alpha j}}{\mathfrak{d} t^{\eta-\alpha j}} u \in C(\Omega^{d+1}),\ \alpha \in (0,1],\ a_j > 0\ \forall\, j, \\ &\Big|\frac{\partial u}{\partial t}(x,t)\Big| \leq \mathfrak{g}(x) t^{\beta-1},\ \beta > 0,\ \mathfrak{g} \in L^{\infty}(\Omega) \Bigg\}.\end{aligned} \tag{2.3}$$

Our aim is to find the solution to the following Cauchy problem, $u \in \mathbb{A}^k(\mathbb{R}^{1+1}_+) \cap C_0(\mathbb{R}_+)$,

$$\begin{cases} \left(\mathbb{D}^{\delta}_{\nu,\gamma+\nu,-1;0+} h(x,\cdot)\right)(t) = -\frac{\partial}{\partial x} h(x,t), & t > 0,\ x > 0, \\ h(x,0^+) = \delta(x). \end{cases} \tag{2.4}$$

It is interesting to note that when $\delta = 0$, the above equation reduces to that satisfied by the density law of the inverse $(\gamma+\nu)$-stable subordinator (see e.g. **?**, Section 4). Indeed, by using Remark A.1 and after some calculations, we arrive at

$$\frac{\mathfrak{d}^{\gamma+\nu}}{\mathfrak{d} t^{\gamma+\nu}} h(x,t) = -\frac{\partial}{\partial x} h(x,t), \qquad t > 0,\ x > 0, \tag{2.5}$$

where, we recall, $\mathfrak{d}^{\varpi}/\mathfrak{d}t^{\varpi}$ is the Dzhrbashyan–Caputo fractional deivative of order ϖ. For $\delta = 1$, we obtain

$$\frac{\mathfrak{d}^{\gamma+\nu}}{\mathfrak{d} t^{\gamma+\nu}} h(x,t) + \frac{\mathfrak{d}^{\gamma}}{\mathfrak{d} t^{\gamma}} h(x,t) = -\frac{\partial}{\partial x} h(x,t), \qquad t > 0,\ x > 0. \tag{2.6}$$

In order to solve (2.4) we apply a Laplace–Laplace transform to (2.4) with respect to both variables.

Proposition 2.1. *The Laplace–Laplace transform* $\tilde{\tilde{h}}(z,s) = \int_0^{\infty}\int_0^{\infty} e^{-zx-st} h(x,t)\,\mathrm{d}x\,\mathrm{d}t$ *of the solution to* (2.4) *is*

$$\tilde{\tilde{h}}(z,s) = \frac{s^{\gamma+\nu-1}\left(1+s^{-\nu}\right)^{\delta}}{s^{\gamma+\nu}\left(1+s^{-\nu}\right)^{\delta} + z}, \qquad z > 0,\ s > 0. \tag{2.7}$$

Proof. By means of direct calculation and considering (1.12) and the initial condition we have that

$$s^{\gamma+\nu}\tilde{\tilde{h}}(z,s)\left(1+s^{-\nu}\right)^{\delta} = -z\,\tilde{\tilde{h}}(z,s) + s^{\gamma+\nu-1}\left(1+s^{-\nu}\right)^{\delta}. \tag{2.8}$$

Rewriting (2.8) with respect to $\tilde{\tilde{h}}(z,s)$ we obtain the thesis (2.7). □

Proposition 2.2. *The t-Laplace transform and x-Laplace transform of $h(x,t)$ read respectively*

$$\tilde{h}(x,s) = \int_0^{\infty} e^{-st} h(x,t)\,\mathrm{d}t = s^{\gamma+\nu-1}\left(1+s^{-\nu}\right)^{\delta} e^{-xs^{\gamma+\nu}\left(1+s^{-\nu}\right)^{\delta}}, \qquad x \geq 0,\ s > 0, \tag{2.9}$$

$$\tilde{h}'(z,t) = \int_0^{\infty} e^{-zx} h(x,t)\,\mathrm{d}x = \sum_{r=0}^{\infty} (-z)^r\, t^{r(\gamma+\nu)} E^{r\delta}_{\nu,r(\gamma+\nu)+1}(-t^{\nu}), \qquad t \geq 0,\ z > 0, \tag{2.10}$$

where $|z/(s^{\gamma+\nu}(1+s^{-\nu})^{\delta})| < 1$.

Proof. Formula (2.9) is straightforward as it follows from the expression of the Laplace transform of an exponential. For what concerns formula (2.10), from (2.7), we first write

$$\tilde{\tilde{h}}(z,s) = s^{-1}\left(1+zs^{-(\gamma+\nu)}\left(1+s^{-\nu}\right)^{-\delta}\right)^{-1} = \sum_{r=0}^{\infty}(-z)^r s^{-(\gamma+\nu)r-1}\left(1+s^{-\nu}\right)^{-r\delta}, \tag{2.11}$$

which holds if $|z/(s^{\gamma+\nu}(1+s^{-\nu})^{\delta})| < 1$. We then recall the following formula for the Laplace transform of a generalized Mittag–Leffler function [**?**, formula (2.3.24)]

$$\int_0^\infty t^{\eta-1}e^{-pt}E^{\xi}_{\alpha,\eta}(\zeta t^\alpha)\,\mathrm{d}t = p^{-\eta}\left(1-\zeta p^{-\eta}\right)^{-\xi}, \qquad \alpha>0,\ \eta>0,\ p>|\zeta|^{1/\alpha}, \tag{2.12}$$

by means of which result (2.10) is easily found. □

Remark 2.1. Note how the above t-Laplace transform (2.9), for $\delta=0$, reduces to the well-known t-Laplace transform of the inverse $(\gamma+\nu)$-stable subordinator, namely $s^{\gamma+\nu-1}\exp(-xs^{\gamma+\nu})$ [**?**, formula (2.14)]. Furthermore, the x-Laplace transform, again in the case $\delta=0$, can be written as

$$\tilde{h}'(z,t) = \sum_{r=0}^{\infty}\frac{(-zt^{\gamma+\nu})^r}{\Gamma(r(\gamma+\nu)+1)} = E_{\gamma+\nu,1}(-zt^{\gamma+\nu}), \tag{2.13}$$

which, as expected, concides with the classical result [**?**, formula (2.13)].

The remaining pages of this section are devoted to explain in which sense the integral operator we are analyzing is connected to some stochastic processes. For the sake of clarity we explain our results first in the specific case $\delta\in(0,1]$ leaving the presentation of the more general case $\delta>0$ at the end of this section.

Consider a filtered probability space $(\Omega,\mathscr{F},\mathfrak{G},\mathbb{P})$, where $\mathfrak{G}=(\mathscr{G}_t)_{t\geq 0}$ is the associated filtration, and the process

$$U_t^{(\alpha_1,\alpha_2)} = {}_1V_t^{\alpha_1} + {}_2V_t^{\alpha_2}, \qquad t\geq 0,\ \alpha_1,\alpha_2\in(0,1), \tag{2.14}$$

adapted to $\mathfrak{G}$, where ${}_jV_t^{\alpha_j}$, $t\geq 0$, $j=1,2$, are independent stable subordinators of order α_j. Let us further consider the stable subordinator V_t^δ, $t\geq 0$, $\delta\in(0,1]$ also adapted to $\mathfrak{G}$ and independent of $U_t^{(\alpha_1,\alpha_2)}$, $t\geq 0$. We focus now on the subordinated process

$$U^{(\alpha_1,\alpha_2)}_{V_t^\delta} = {}_1V^{\alpha_1}_{V_t^\delta} + {}_2V^{\alpha_2}_{V_t^\delta}, \qquad t\geq 0, \tag{2.15}$$

clearly adapted to the time-changed filtration $\left(\mathscr{G}_{V_t^\delta}\right)_{t\geq 0}$ and, in particular we have that its Laplace transform is

$$\mathbb{E}\exp\left(-zU^{(\alpha_1,\alpha_2)}_{V_t^\delta}\right) = \mathbb{E}\exp\left(-z^{\alpha_1}V_t^\delta - z^{\alpha_2}V_t^\delta\right) = \exp\left(-t(z^{\alpha_1}+z^{\alpha_2})^\delta\right). \tag{2.16}$$

Notice that for (2.15) we obtain

$$U^{(\alpha_1,\alpha_2)}_{V_t^\delta} \overset{\mathrm{d}}{=} {}_1V_t^{\delta\alpha_1} + {}_2V_t^{\delta\alpha_2}, \qquad t\geq 0, \tag{2.17}$$

where ${}_jV_t^{\delta\alpha_j}$, $t\geq 0$, $j=1,2$ are now $\mathfrak{G}$-adapted dependent stable subordinators. The dependence is due to the time-change and vanishes in the degenerate case $\delta=1$. For $\delta\in(0,1)$ the processes in (2.17) possess dependent increments but non-decreasing paths, that is the increments are non-negative. Thus we argue that

$$C_t^{(\delta\alpha_1,\delta\alpha_2)} = \inf\left\{s\geq 0\colon {}_1V_s^{\delta\alpha_1} + {}_2V_s^{\delta\alpha_2} \notin (0,t)\right\} \overset{\mathrm{d}}{=} \inf\left\{s\geq 0\colon U^{(\alpha_1,\alpha_2)}_{V_s^\delta}\notin(0,t)\right\}, \qquad t\geq 0, \tag{2.18}$$

is the first exit time of the process (2.17) from the interval $(0,t)$ whose distribution coincides with that of the first exit time of (2.15) from the same interval $(0,t)$. We refer to (2.18) also as the inverse to (2.15) as

$$\mathbb{P}\left\{C_t^{(\delta\alpha_1,\delta\alpha_2)} > x\right\} = \mathbb{P}\left\{U^{(\alpha_1,\alpha_2)}_{V_x^\delta} < t\right\}. \tag{2.19}$$

Note that if we let the inverse process $C_t^{(\delta\alpha_1,\delta\alpha_2)}$, $t\geq 0$, be adapted to filtration $\mathfrak{F}=(\mathscr{F}_t)$, we have that $\mathfrak{G}=(\mathscr{G}_t)_{t\geq 0} = \left(\mathscr{F}_{C_t^{(\delta\alpha_1,\delta\alpha_2)}}\right)_{t\geq 0}$, that is $C_t^{(\delta\alpha_1,\delta\alpha_2)}$ is a time change on the filtered probability space $(\Omega,\mathscr{F},\mathfrak{F},\mathbb{P})$. Relation (2.18) allows us to derive the t-Laplace transform of the density law of (2.18):

$$-\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty e^{-st}\mathbb{P}\left\{C_t^{(\delta\alpha_1,\delta\alpha_2)}>x\right\}\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty e^{-st}\mathbb{P}\left\{U^{(\alpha_1,\alpha_2)}_{V_x^\delta}<t\right\}\mathrm{d}t \tag{2.20}$$

$$=-\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty e^{-st}\mathbb{P}\left\{U^{(\alpha_1,\alpha_2)}_{V_x^\delta}\in \mathrm{d}t\right\}=-\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\exp\left(-sU^{(\alpha_1,\alpha_2)}_{V_x^\delta}\right).$$

Now, by using the Laplace transform (2.16) we arrive at

$$\int_0^\infty e^{-st}\left(\mathbb{P}\left\{C_t^{(\delta\alpha_1,\delta\alpha_2)}\in \mathrm{d}x\right\}/\mathrm{d}x\right)\mathrm{d}t=-\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}x}\exp\left(-x(s^{\alpha_1}+s^{\alpha_2})^\delta\right) \tag{2.21}$$
$$=\frac{1}{s}(s^{\alpha_1}+s^{\alpha_2})^\delta\exp\left(-x(s^{\alpha_1}+s^{\alpha_2})^\delta\right).$$

First we present the following result.

Proposition 2.3. *We have that*

$$C_t^{(\delta\alpha_1,\delta\alpha_2)}\overset{\mathrm{d}}{=}L^\delta_{C_t^{(\alpha_1,\alpha_2)}},\qquad t\ge 0,\ \delta\in(0,1], \tag{2.22}$$

where $L_t^\delta=\inf\{x\ge 0:\ V_x^\delta\notin(0,t)\}$, $t\ge 0$ *is the inverse to the stable subordinator* V_t^δ, $t\ge 0$ *in the sense that*

$$\mathbb{P}\{L_t^\delta<x\}=\mathbb{P}\{V_x^\delta>t\} \tag{2.23}$$

and $C_t^{(\alpha_1,\alpha_2)}=\inf\left\{s\ge 0:\ {}_1V_s^{\alpha_1}+{}_2V_s^{\alpha_2}\notin(0,t)\right\}$, $t\ge 0$, *is the inverse to* $U_t^{(\alpha_1,\alpha_2)}$, $t\ge 0$ *in the sense that*

$$\mathbb{P}\left\{C_t^{(\alpha_1,\alpha_2)}>x\right\}=\mathbb{P}\left\{{}_1V_x^{\alpha_1}+{}_2V_x^{\alpha_2}<t\right\}. \tag{2.24}$$

Proof. It suffices to consider formula (2.21) for $\delta=1$ and the integral

$$\int_0^\infty e^{-st}\left(\mathbb{E}\exp\left(-zL^\delta_{C_t^{(\alpha_1,\alpha_2)}}\right)\right)\mathrm{d}t=\int_0^\infty \mathbb{E}\left[e^{-zL_x^\delta}\right]\int_0^\infty e^{-st}\mathbb{P}\left\{C_t^{(\alpha_1,\alpha_2)}\in\mathrm{d}x\right\}\mathrm{d}t \tag{2.25}$$
$$=\int_0^\infty\mathbb{E}\left[e^{-zL_x^\delta}\right]\frac{1}{s}(s^{\alpha_1}+s^{\alpha_2})\exp\left(-x(s^{\alpha_1}+s^{\alpha_2})\right)\mathrm{d}x$$
$$=\frac{1}{s}(s^{\alpha_1}+s^{\alpha_2})\int_0^\infty E_\delta(-zx^\delta)\exp\left(-x(s^{\alpha_1}+s^{\alpha_2})\right)\mathrm{d}x=\frac{1}{s}\frac{(s^{\alpha_1}+s^{\alpha_2})^\delta}{z+(s^{\alpha_1}+s^{\alpha_2})^\delta}$$

which coincides with the x-Laplace transform of (2.21). □

In the following, when we refer to stochastic solution of a pde, we mean the stochastic process whose density function is the fundamental solution to such pde.

Remark 2.2. We observe that the process $C_t^{(\alpha_1,\alpha_2)}$, $t\ge 0$ has been investigated in **?** and is, for $\alpha_1=\gamma+\nu\in(0,1)$ and $\alpha_2=\gamma\in(0,1)$ the stochastic solution to the fractional telegraph equation

$$\frac{\partial^{\gamma+\nu}}{\partial t^{\gamma+\nu}}u(x,t)+\frac{\partial^\gamma}{\partial t^\nu}u(x,t)=-\frac{\partial}{\partial x}u(x,t),\qquad x\ge 0,\ t\ge 0, \tag{2.26}$$

subject to the initial and the boundary conditions

$$u(x,0)=\delta(x),\qquad u(0,t)=\frac{t^{-\gamma-\nu}}{\Gamma(1-\gamma-\nu)}+\frac{t^{-\nu}}{\Gamma(1-\nu)}. \tag{2.27}$$

Now we are ready to prove the following result which shows the relation to the Cauchy problem (2.4).

Theorem 2.1. *The stochastic solution to* (2.4), $\delta\in(0,1]$ *is given by the hitting time* (2.18) *of the subordinated process* (2.15) *with* $\alpha_1=(\gamma+\nu)/\delta\in(0,1]$, $\alpha_2=(\gamma+\nu)/\delta-\nu\in(0,1]$. *Furthermore, the process* (2.18) *becomes*

$$Z_t^\delta:=C_t^{(\gamma+\nu,\gamma+\nu-\delta\nu)}=\inf\left\{s\ge 0:\ {}_1V_s^{\gamma+\nu}+{}_2V_s^{\gamma+\nu-\delta\nu}\notin(0,t)\right\},\qquad t\ge 0, \tag{2.28}$$

where ${}_1V_t^{\gamma+\nu}$ *and* ${}_2V_t^{\gamma+\nu-\delta\nu}$ *are dependent stable subordinators.*

Proof. From (2.21) we obtain the Laplace–Laplace transform of the density law of the process (2.18) as follows

$$\int_0^\infty e^{-st}\,\mathbb{E}\exp\left(-zC_t^{(\delta\alpha_1,\delta\alpha_2)}\right)\mathrm{d}t = \int_0^\infty\int_0^\infty e^{-st-zx}\,\mathbb{P}\left\{C_t^{(\delta\alpha_1,\delta\alpha_2)}\in \mathrm{d}x\right\}\mathrm{d}t \tag{2.29}$$
$$= \int_0^\infty e^{-zx}\frac{1}{s}(s^{\alpha_1}+s^{\alpha_2})^\delta \exp\left(-x(s^{\alpha_1}+s^{\alpha_2})^\delta\right)\mathrm{d}x = \frac{1}{s}\frac{(s^{\alpha_1}+s^{\alpha_2})^\delta}{z+(s^{\alpha_1}+s^{\alpha_2})^\delta}.$$

For $\alpha_1=(\gamma+\nu)/\delta$, $\alpha_2=(\gamma+\nu)/\delta-\nu$, we have that

$$\int_0^\infty e^{-st}\,\mathbb{E}\exp\left(-zZ_t^\delta\right)\mathrm{d}t = \frac{s^{\gamma+\nu-1}(1+s^{-\nu})^\delta}{z+s^{\gamma+\nu}(1+s^{-\nu})^\delta}, \tag{2.30}$$

which coincides with (2.7). □

We now move to analyzing the more general case $\delta>0$. In light of Theorem 2.1, the results stated in the following theorem will appear natural. What changes is basically that now we are dealing with a linear combination of subordinated stable subordinators whose hitting time will be the stochastic solution to (2.4).

Theorem 2.2. *Given the filtered probability space $(\Omega,\mathscr{F},\mathfrak{F},\mathbb{P})$, the stochastic solution to (2.4), $\delta>0$, is given by the $\mathfrak{F}$-hitting time Z_t^δ, $t\ge 0$, of the $\mathfrak{G}$-adapted process*

$$M_t^\delta = \sum_{r=0}^n \binom{n}{r}^{1/[(\gamma+\nu)n/\delta-r\nu]} {}_rV_t^{\gamma+\nu-r\nu\delta/n} = \sum_{r=0}^n {}_rV_{\binom{n}{r}t}^{\gamma+\nu-r\nu\delta/n}, \qquad t\ge 0, \tag{2.31}$$

where $\mathfrak{F}=(\mathscr{F}_t)_{t\ge0}=\left(\mathscr{G}_{Z_t^\delta}\right)_{t\ge0}$, is the associated filtration (with $\mathfrak{G}=(\mathscr{G}_t)_{t\ge0}$), ${}_rV_t^{\gamma+\nu-r\nu\delta/n}$, $r=1,\dots n$, are dependent stable subordinators, $n=\lceil\delta\rceil$ is the ceiling of δ and $\nu\delta<\gamma+\nu<1$.

Proof. We start, similarly to proof of Theorem 2.1, by considering the process

$$U_t = \sum_{r=0}^n \binom{n}{r}^{1/[(\gamma+\nu)n/\delta-r\nu]} {}_rV_t^{(\gamma+\nu)n/\delta-r\nu}, \tag{2.32}$$

with ${}_rV_t^{(\gamma+\nu)n/\delta-r\nu}$, $r=1,\dots n$, independent stable subordinators and $\nu\delta<\gamma+\nu<\delta/n$. This process must be subordinated to a further stable subordinator $V_t^{\delta/n}$ independent of ${}_rV_t^{(\gamma+\nu)n/\delta-r\nu}$, $r=1,\dots n$. We thus obtain

$$U_{V_t^{\delta/n}} = \sum_{r=0}^n \binom{n}{r}^{1/[(\gamma+\nu)n/\delta-r\nu]} {}_rV_{V_t^{\delta/n}}^{(\gamma+\nu)n/\delta-r\nu}. \tag{2.33}$$

Note that, as U_t is a linear combination of independent stable subordinators, the process $U_{V_t^{\delta/n}} \overset{\mathrm{d}}{=} M_t^\delta$. Due to the nature of the time-change considered it is clear that M_t^δ is a linear combination of dependent subordinators. This is easily explained by noticing that the shared time-change turns the independency $({}_0V_t^{(\gamma+\nu)n/\delta}, {}_1V_t^{(\gamma+\nu)n/\delta-1\nu}, {}_2V_t^{(\gamma+\nu)2/\delta-\nu}\dots, {}_nV_t^{(\gamma+\nu)n/\delta-n\nu})$ into the dependent collection of subordinators $({}_0V_t^{\gamma+\nu}, {}_1V_t^{\gamma+\nu-\nu\delta/n}, {}_2V_t^{\gamma+\nu-2\nu\delta/n},\dots, {}_nV_t^{\gamma+\nu-n\nu\delta/n})$. The Laplace transform of the density law of $U_{V_t^{\delta/n}}$ reads

$$\mathbb{E}\exp\left\{-zU_{V_t^{\delta/n}}\right\} = \mathbb{E}\exp\left\{-V_t^{\delta/n}\sum_{r=0}^n\binom{n}{r}z^{(\gamma+\nu)n/\delta-r\nu}\right\} = \exp\left\{-t\left[\sum_{r=0}^n\binom{n}{r}z^{(\gamma+\nu)n/\delta-r\nu}\right]^{\delta/n}\right\} \tag{2.34}$$
$$= \exp\left\{-t\left[z^{(\gamma+\nu)n/\delta}(1+z^{-\nu})^n\right]^{\delta/n}\right\} = \exp\left\{-t\,z^{\gamma+\nu}(1+z^{-\nu})^\delta\right\}.$$

Let us define now the right-inverse process to M_t^δ as

$$Z_t^\delta := \inf\left\{s\ge0\colon M_t^\delta\notin(0,t)\right\} \overset{\mathrm{d}}{=} \inf\left\{s\ge0\colon U_{V_t^{\delta/n}}\notin(0,t)\right\}, \qquad t\ge0,\ \delta>0. \tag{2.35}$$

In particular note that $\mathbb{P}\{Z_t^\delta>x\}=\mathbb{P}\{U_{V_x^{\delta/n}}<t\}$.
The time-Laplace transform related to the inverse process Z_t^δ, $t\ge0$, can be determined as in the following.

$$-\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty e^{-st}\mathbb{P}\{Z_t^\delta>x\}\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty e^{-st}\mathbb{P}\{U_{V_x^{\delta/n}}<t\}\mathrm{d}t \tag{2.36}$$

$$= -s^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty e^{-st}\mathbb{P}\{U_{V_x^{\delta/n}}\in \mathrm{d}t\} = -s^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}e^{-sU_{V_x^{\delta/n}}}.$$

Therefore

$$\int_0^\infty e^{-st}\left[\mathbb{P}\{Z_t^\delta\in \mathrm{d}x\}/\mathrm{d}x\right]\mathrm{d}t = -s^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\exp\left\{-x\,s^{\gamma+\nu}(1+s^{-\nu})^\delta\right\} \tag{2.37}$$
$$= s^{\gamma+\nu-1}(1+s^{-\nu})^\delta\exp\left\{-x\,s^{\gamma+\nu}(1+s^{-\nu})^\delta\right\}.$$

Finally we calculate the complete Laplace–Laplace transform.

$$\int_0^\infty e^{-st}\mathbb{E}e^{-zZ_t^\delta}\mathrm{d}t = \int_0^\infty\int_0^\infty e^{-st-zx}\mathbb{P}\{Z_t^\delta\in\mathrm{d}x\}\mathrm{d}t \tag{2.38}$$
$$= \int_0^\infty e^{-zx}s^{\gamma+\nu-1}(1+s^{-\nu})^\delta\exp\left\{-x\,s^{\gamma+\nu}(1+s^{-\nu})^\delta\right\}\mathrm{d}x = \frac{s^{\gamma+\nu-1}(1+s^{-\nu})^\delta}{z+s^{\gamma+\nu}(1+s^{-\nu})^\delta}.$$

As the latter expression coincides with (2.7) the proof of the theorem is complete. □

Remark 2.3. If we consider $n=\lceil\delta\rceil$ and $m>n$ we have that the corresponding process becomes

$$M^\delta_{V_t^{n/m}} = \sum_{r=0}^n\binom{n}{r}^{1/[(\gamma+\nu)n/\delta-r\nu]}{}_rV_t^{(\gamma+\nu)n/m-r\nu\delta/m},\qquad t\ge 0,\ \forall m>n, \tag{2.39}$$

where $\nu\delta<\gamma+\nu<1$ and $V_t^{n/m}$ is an independent n/m-stable subordinator. It is worthy noticing that in this case the hitting time of the above process is not a stochastic solution to equation (2.4).

Aside the results contained in Theorem 2.2 we are able also to prove a subordination relation for the stochastic solution Z_t^δ, $t\ge 0$, in the following proposition.

Proposition 2.4. *We have that*

$$Z_t^\delta \overset{\mathrm{d}}{=} L^{\delta/n}_{F_t},\qquad t\ge 0, \tag{2.40}$$

where $n=\lceil\delta\rceil$ is the ceiling of $\delta>0$ and where $L_t^{\delta/n}=\inf\{x\ge 0:\ V_x^{\delta/n}\notin(0,t)\}$, $t\ge 0$, is the right-inverse process to the stable subordinator $V_t^{\delta/n}$, $t\ge 0$, in the sense that $\mathbb{P}\{L_t^{\delta/n}<x\}=\mathbb{P}\{V_x^{\delta/n}>t\}$ and the hitting time $F_t=\inf\{s\ge 0\colon U_s\notin(0,t)\}$, $t\ge 0$ is the inverse to U_t, $t\ge 0$ in the sense that $\mathbb{P}\{F_t>x\}=\mathbb{P}\{U_x<t\}$.

Proof. In order to prove the subordination relation it is sufficient to consider formula (2.37) for $\delta=n$ and the following calculations.

$$\int_0^\infty e^{-st}\Big(\mathbb{E}\exp\big(-zL^{\delta/n}_{F_t}\big)\Big)\mathrm{d}t = \int_0^\infty\mathbb{E}\left[e^{-zL_x^{\delta/n}}\right]\int_0^\infty e^{-st}\mathbb{P}\{F_t\in\mathrm{d}x\}\,\mathrm{d}t \tag{2.41}$$
$$= \int_0^\infty\mathbb{E}\left[e^{-zL_x^{\delta/n}}\right]s^{(\gamma+\nu)n/\delta-1}(1+s^{-\nu})^n\exp\left(-x\,s^{(\gamma+\nu)n/\delta}(1+s^{-\nu})^n\right)\mathrm{d}x$$
$$= s^{(\gamma+\nu)n/\delta-1}(1+s^{-\nu})^n\int_0^\infty E_{\delta/n,1}(-zx^{\delta/n})\exp\left(-x\,s^{(\gamma+\nu)n/\delta}(1+s^{-\nu})^n\right)\mathrm{d}x$$
$$= s^{(\gamma+\nu)n/\delta-1}(1+s^{-\nu})^n\frac{\left[s^{(\gamma+\nu)n/\delta}(1+s^{-\nu})^n\right]^{\delta/n-1}}{z+\left[s^{(\gamma+\nu)n/\delta}(1+s^{-\nu})^n\right]^{\delta/n}} = \frac{s^{\gamma+\nu-1}(1+s^{-\nu})^\delta}{z+s^{\gamma+\nu}(1+s^{-\nu})^\delta}.$$

The last expression exactly coincides with the x-Laplace transform of (2.37). Notice that $E_{\delta/n,1}(x)=E^1_{\delta/n,1}(x)$ is the classical two-parameter Mittag–Leffler function. □

Remark 2.4. In the specific case of $\delta=n\in\mathbb{N}\cup\{0\}$, equation (2.4) takes a peculiar form. We have

$$\left(\mathbb{D}^n_{\nu,\gamma+\nu,-1;0+}h(x,\cdot)\right)(t) = -\frac{\partial}{\partial x}h(x,t) \Leftrightarrow \frac{\partial^{\gamma+\nu+\theta}}{\partial t^{\gamma+\nu+\theta}}\int_0^t(t-y)^{\theta-1}E^{-n}_{\nu,\theta}\left[-(t-y)^\nu\right]h(x,y)\,\mathrm{d}y \tag{2.42}$$

$$= -\frac{\partial}{\partial x}h(x,t) + \delta(x)\, t^{-(\gamma+\nu)} E^{-n}_{\nu,1-(\gamma+\nu)}(-t^{\nu}).$$

By recalling again that $(-n)_r = (-1)^r(n-r+1)_r = (-1)^r n!/(n-r)!$, we obtain

$$\frac{\partial^{\gamma+\nu+\theta}}{\partial t^{\gamma+\nu+\theta}} \int_0^t (t-y)^{\theta-1} \sum_{r=0}^{n} \binom{n}{r} \frac{(t-y)^{\nu r}}{\Gamma(\nu r+\theta)} h(x,y)\,\mathrm{d}y = -\frac{\partial}{\partial x}h(x,t) + \delta(x) \sum_{r=0}^{n} \binom{n}{r} \frac{t^{-(\gamma-\nu(r-1))}}{\Gamma(1-(\gamma-\nu(r-1)))} \tag{2.43}$$

$$\Leftrightarrow \quad \sum_{r=0}^{n} \binom{n}{r} \frac{\partial^{\gamma-\nu(r-1)}}{\partial t^{\gamma-\nu(r-1)}} h(x,t) = -\frac{\partial}{\partial x}h(x,t) + \delta(x) \sum_{r=0}^{n} \binom{n}{r} \frac{t^{-(\gamma-\nu(r-1))}}{\Gamma(1-(\gamma-\nu(r-1)))}$$

$$\Leftrightarrow \quad \sum_{r=0}^{n} \binom{n}{r} \frac{\mathfrak{d}^{\gamma-\nu(r-1)}}{\mathfrak{d} t^{\gamma-\nu(r-1)}} h(x,t) = -\frac{\partial}{\partial x}h(x,t)$$

with $h(x,0) = \delta(x)$, $0 < \gamma - \nu(r-1) < 1$ and thus $n\nu < \gamma + \nu < 1$. Note furthermore that equations (2.43) and (4.20) are consistent with Theorem 3.1 of **?**.

Before moving to the next section where the introduced process is related to the the stochastic solution of different abstract Cauchy problems, we first underline in the following remark that the inverse process Z^{δ}_t, $t \geq 0$, $\delta > 0$, is well-behaved and can be used as a time-change.

Remark 2.5. The inverse process Z^{δ}_t, $t \geq 0$, $\delta > 0$, is a continuous time-change on the probability space $(\Omega, \mathscr{F}, \mathfrak{F}, \mathbb{P})$. This simply ensues from the construction of the process (2.31) as a linear combination with non-negative coefficients of dependent stable subordinators which are clearly increasing processes (right-continuous and taking values in $[0,\infty])$ and M^{δ}_t, $t \geq 0$, is adapted to $\mathfrak{G}$.

We conclude this section by studying the case $\delta < 0$. When δ is strictly negative calculations become more complicated. We present the following result for $-1/2 < \delta < 0$, $\nu = -\beta$.

Theorem 2.3. *Let $0 < \gamma - \beta < 1$, (with $\gamma, \beta \in (0,1)$) and $\epsilon = -\delta$ with $-1/2 < \delta < 0$. In this case the solution to* (2.4) *is*

$$h(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{\mathrm{d}z}{\sqrt{z}} \phi(x,z) \int_0^{\infty} \mathrm{d}u\, \varphi(u,z,t), \quad x,t > 0, \tag{2.44}$$

where

$$\varphi(u,z,t) = \int_0^t \int_0^{\infty} e^{-y}\, v_{2\epsilon}(y, u^2/4z)\, v_{\beta}(t-k, y^{\beta})\, l_{\gamma-\beta}(u,k)\,\mathrm{d}y\,\mathrm{d}k. \tag{2.45}$$

$v_a(x,t) = \mathbb{P}\{V^a_t \in \mathrm{d}x\}/\mathrm{d}x$ is the density law of the a-stable subordinator, $l_a(x,t) = \mathbb{P}\{L^a_t \in \mathrm{d}x\}/\mathrm{d}x$ is the density law of the inverse process to an a-stable subordinator, and ϕ is a function such that $\tilde{\phi}(\mu,z) = \mu e^{-z\mu^2}$.

Proof. First we show that

$$\tilde{\varphi}(u,z,s) = e^{-\frac{u^2}{4z}(1+s^{\beta})^{2\epsilon}} s^{\gamma-\beta-1} e^{-us^{\gamma-\beta}}. \tag{2.46}$$

Indeed, the first exponential term in the right-hand side of (2.46) can be written as

$$\begin{aligned} e^{-\frac{u^2}{4z}(1+s^{\beta})^{2\epsilon}} &= \mathbb{E}\, e^{-(1+s^{\beta})V^{2\epsilon}_{u^2/4z}} = \mathbb{E}\,\mathbb{E}\, e^{-(1+sV^{\beta}_1)V^{2\epsilon}_{u^2/4z}} \\ &= \int_0^{\infty} e^{-y}\, \mathbb{E}\, e^{-ysV^{\beta}_1} v_{2\epsilon}(y,u^2/4z)\,\mathrm{d}y = \int_0^{\infty} e^{-y-(ys)^{\beta}} v_{2\epsilon}(y,u^2/4z)\,\mathrm{d}y, \end{aligned} \tag{2.47}$$

where $e^{-(ys)^{\beta}} = \int_0^{\infty} e^{-sx} v_{\beta}(x,y^{\beta})\,\mathrm{d}x$. Also, the remaining terms of equation (2.46) are in fact the Laplace transform of the density law of an inverse $(\gamma-\beta)$-stable subordinator, that is

$$s^{\gamma-\beta-1} e^{-us^{\gamma-\beta}} = \int_0^{\infty} e^{-sx} l_{\gamma-\beta}(u,x)\,\mathrm{d}x. \tag{2.48}$$

Now, note that the following product can be represented as the Laplace transform of a convolution, as it is shown in the following formula.

$$e^{-(ys)^{\beta}} s^{\gamma-\beta-1} e^{-us^{\gamma-\beta}} = \int_0^{\infty} e^{-st} \int_0^t v_{\beta}(t-w, y^{\beta}) l_{\gamma-\beta}(u,w)\,\mathrm{d}w\,\mathrm{d}t. \tag{2.49}$$

Therefore we have

$$\int_0^\infty e^{-st}\varphi(u,z,t)\,\mathrm{d}t = \int_0^\infty e^{-y} v_{2\epsilon}(y,u^2/4z) e^{-(ys)^\beta} s^{\gamma-\beta-1} e^{-us^{\gamma-\beta}}\mathrm{d}y. \tag{2.50}$$

Last step has been obtained by using (2.49). Considering (2.47) we immediately obtain result (2.46).
The t-Laplace transform of h becomes therefore

$$\tilde{h}(x,s) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\mathrm{d}z}{\sqrt{z}}\phi(x,z)\int_0^\infty e^{-\frac{u^2(1+s^\beta)^{2\epsilon}}{4z}} s^{\gamma-\beta-1} e^{-us^{\gamma-\beta}}\,\mathrm{d}u \tag{2.51}$$

and the double Laplace transform is given by

$$\tilde{\tilde{h}}(\mu,s) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\mathrm{d}z}{\sqrt{z}}\left(\mu e^{-z\mu^2}\right)\int_0^\infty e^{-\frac{u^2(1+s^\beta)^{2\epsilon}}{4z}} s^{\gamma-\beta-1} e^{-us^{\gamma-\beta}}\,\mathrm{d}u. \tag{2.52}$$

By considering that $K_\alpha(2\sqrt{ab}) = \frac{1}{2}\left(\frac{b}{a}\right)^{\frac{\alpha}{2}}\int_0^\infty y^{\alpha-1}e^{-yb-\frac{a}{y}}\,\mathrm{d}y$ is the modified Bessel function of the second kind, we have that

$$\int_0^\infty \frac{\mathrm{d}z}{\sqrt{z}}\left(\mu e^{-z\mu^2}\right)e^{-\frac{u^2(1+s^\beta)^{2\epsilon}}{4z}} = 2\mu\left(\frac{u^2(1+s^\beta)^{2\epsilon}}{4\mu^2}\right)^{\frac{1}{4}} K_{\frac{1}{2}}\left(\sqrt{\mu^2u^2(1+s^\beta)^{2\epsilon}}\right). \tag{2.53}$$

Since $K_{1/2}(z) = \sqrt{\frac{\pi}{z}}e^{-z}$, we get

$$\int_0^\infty \frac{\mathrm{d}z}{\sqrt{z}}\left(\mu e^{-z\mu^2}\right)e^{-\frac{u^2(1+s^\beta)^{2\epsilon}}{4z}} = 2\mu\left(\frac{u^2(1+s^\beta)^{2\epsilon}}{4\mu^2}\right)^{\frac{1}{4}}\sqrt{\frac{\pi}{2}}\left(\mu^2u^2(1+s^\beta)^{2\epsilon}\right)^{-\frac{1}{4}} e^{-\mu u(1+s^\beta)^\epsilon} = \sqrt{\pi}e^{-\mu u(1+s^\beta)^\epsilon}. \tag{2.54}$$

Hence,

$$\tilde{\tilde{h}}(\mu,s) = s^{\gamma-\beta-1}\int_0^\infty \mathrm{d}u\, e^{-us^{\gamma-\beta}-\mu u(1+s^\beta)^\epsilon} = \frac{s^{\gamma-\beta-1}}{s^{\gamma-\beta}+\mu(1+s^\beta)^\epsilon} = \frac{s^{\gamma-\beta-1}(1+s^\beta)^{-\epsilon}}{s^{\gamma-\beta}(1+s^\beta)^{-\epsilon}+\mu} \tag{2.55}$$

which coincides with (2.7) for $\beta = -\nu$ and $\epsilon = -\delta$. □

3 Time changed Lévy processes

Let A_t^x, $t \geq 0$, be an $\mathbb{R}^d$-valued $\mathfrak{F}$-adapted Lévy process starting from $x \in \mathbb{R}^d$, with characteristics (a, Q, Π). We introduce the convolution semigroup

$$T_t f(x) = \mathbb{E}f(A_t^x) = \int_{\mathbb{R}^d} f(y)\mathbb{P}(A_t^x \in \mathrm{d}y) \tag{3.1}$$

with infinitesimal generator

$$\lim_{t\to 0}\frac{1}{t}(T_t f - f) = \mathscr{A}f, \tag{3.2}$$

where the strong limit exists in the domain

$$D(\mathscr{A}) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2\left(1+|\Psi(\xi)|^2\right)\mathrm{d}\xi < \infty \right\}. \tag{3.3}$$

In (3.3), $\hat{f}$ represents the Fourier transform of f. The cumulant generating function (also known in some literature as Fourier symbol) of the process $A_t := A_t^0$, $t \geq 0$, clearly is

$$\Psi(\xi) = i\langle a, \xi\rangle + \frac{1}{2}\langle \xi, Q\xi\rangle + \int_{\mathbb{R}^d\backslash\{0\}}\left(1 - e^{i\langle z,\xi\rangle} + i\langle z,\xi\rangle \mathbb{I}_{|z|<1}\right)\Pi(\mathrm{d}z). \tag{3.4}$$

The Borel measure $\Pi(\cdot)$ is the so-called Lévy measure satisfying $\int_{\mathbb{R}^d\setminus\{0\}}(1\wedge|z|^2)\,\Pi(dz)<\infty$, where, as usual, $|z|^2=\langle z,z\rangle$. We have that

$$\mathscr{A}f(x)=\lim_{t\to 0}\frac{T_t f(x)-f(x)}{t}=\int_{\mathbb{R}^d}e^{i\langle x,\xi\rangle}\lim_{t\to 0}\frac{e^{-t\Psi(\xi)}-1}{t}\hat{f}(\xi)\,\mathrm{d}\xi=\int_{\mathbb{R}^d}e^{i\langle x,\xi\rangle}(-\Psi(\xi))\hat{f}(\xi)\,\mathrm{d}\xi \tag{3.5}$$

and therefore, $-\Psi$ is the Fourier multiplier of $\mathscr{A}$. An explicit representation in terms of Lévy measure can be also given, we consider some special cases below. The semigroup $\mu_t(y,x)=\mathbb{P}(A^x_t\in \mathrm{d}y)/\mathrm{d}y$ denotes the density of the Lévy process A^x_t on $\mathbb{R}^d$ starting from $x\in\mathbb{R}^d$. This means that

$$\mathbb{E}\exp(i\xi A_t)=\int_{\mathbb{R}^d}e^{i\xi y}\mu_t(\mathrm{d}y)=\exp(-t\Psi(\xi)) \tag{3.6}$$

and that the function $\Psi(\cdot)$ completely determines the density of A_t, $t\geq 0$. In (3.6), $\mu_t(\mathrm{d}y)=\mu_t(\mathrm{d}y,0)$.
We introduce the time-change operator

$$H^{\gamma,\nu,\delta}_t=\int_0^\infty h(\mathrm{d}y,t)T_y \tag{3.7}$$

where $h(y,t)$ is the solution to (2.4) and hence the density law of Z^δ_t, $t\geq 0$, with $\delta\in(0,\infty)$ and T_y is the convolution semigroup in (3.1). For $\delta\in(0,1)$, the function $h(y,t)$ coincides with the density law of (2.18). For the operator (3.7), we obtain that

$$\|H^{\gamma,\nu,\delta}_t f\|\leq\int_0^\infty|h(\mathrm{d}y,t)|\,\|T_y f\|\leq\|f\|\int_0^\infty|h(\mathrm{d}y,t)|=\|f\|. \tag{3.8}$$

Indeed, T_t is a strongly continuous contraction semigroup on $\mathrm{C}^\infty(\mathbb{R}^d)$ and $h(\cdot,t)$ is a probability measure on $(0,\infty)$. Consider now the space $L^p((0,\infty),e^{-t}\mathrm{d}t)$ of all measurable functions $t\mapsto f(t)$ equipped with the norm $\|f\|_p^p=\int_0^\infty|f(t)|^pe^{-t}\mathrm{d}t$. Recalling that $n=\lceil\delta\rceil$ and therefore that $\delta/n\in(0,1]$, we have

$$\|H^{\gamma,\nu,\delta}_t f\|_1\leq\int_0^\infty\|h(\mathrm{d}y,t)\|_1\,|T_y f|=\|T_{t/2^{\delta/n}}f\|_1. \tag{3.9}$$

Indeed, from (2.37) we also have that

$$\|h(\mathrm{d}y,\cdot)\|_1=\int_0^\infty h(\mathrm{d}y,t)e^{-t}\mathrm{d}t=2^{\delta/n}e^{-y2^{\delta/n}}\mathrm{d}y. \tag{3.10}$$

From Proposition 2.4 we have that $h(x,t)=\int_0^\infty l_{\delta/n}(x,r)k(\mathrm{d}r,t)$ and the operator (3.7) takes the form

$$H^{\gamma,\nu,\delta}_t=\int_0^\infty k(\mathrm{d}r,t)\int_0^\infty l_{\delta/n}(\mathrm{d}y,r)\,T_y \tag{3.11}$$

where, $l_{\delta/n}(\mathrm{d}y,r)=\mathbb{P}\{L^{\delta/n}_r\in\mathrm{d}y\}$ and $k(\mathrm{d}r,t)=\mathbb{P}\{F_t\in\mathrm{d}r\}$.
For $n=1$ the density law of the process $F_t\stackrel{\mathrm{d}}{=}C^{(\gamma+\nu,\nu)}_t$, $t\geq 0$ has the explicit representation (see **?**)

$$k(x,t)=\int_0^t l_{\gamma+\nu}(x,y)v_\nu(t-y,x)\mathrm{d}y+\int_0^t l_\nu(x,y)v_{\gamma+\nu}(t-y,x)\mathrm{d}y. \tag{3.12}$$

For the sake of completeness we show that the Laplace transform of the density (3.12) is written as

$$\tilde{k}(x,s)=\frac{1}{s}\left(s^{\gamma+\nu}+s^\nu\right)e^{-x(s^{\gamma+\nu}+s^\nu)}. \tag{3.13}$$

Indeed, from the Laplace transforms $\tilde{l}_\alpha(x,s)=s^{\alpha-1}e^{-xs^\alpha}$ and $\tilde{v}_\alpha(s,x)=e^{-xs^\alpha}$, we arrive at

$$\int_0^\infty e^{-st}k(x,t)dt=\tilde{l}_{\gamma+\nu}(x,s)\tilde{v}_\nu(s,x)+\tilde{l}_\nu(x,s)\tilde{v}_{\gamma+\nu}(s,x)=\left(s^{\gamma+\nu-1}+s^{\nu-1}\right)e^{-x(s^{\gamma+\nu}+s^\nu)}. \tag{3.14}$$

Note also that when $\delta=n=0$, the process $F_t\stackrel{\mathrm{d}}{=}L^{\gamma+\nu}_t$, which is an inverse $(\gamma+\nu)$-stable subordinator.
Recall that $\mathbb{R}^{d+1}:=\mathbb{R}^d\times(0,\infty)$ and that the function space $\mathbb{A}^k$ is defined in (2.3).

Theorem 3.1. *Let $\delta > 0$, $n = \lceil \delta \rceil$, and $\delta\nu < \gamma + \nu \leq 1$, $\gamma, \nu \in (0,1)$. The unique solution to*

$$\begin{cases} g \in \mathbb{A}^n(\mathbb{R}^{d+1}), \\ \left(\mathbb{D}^{\delta}_{\nu,\gamma+\nu,-1;0+} g(x,\cdot)\right)(t) = \mathscr{A} g(x,t), \quad x \in \mathbb{R}^d, \; t > 0, \\ g(x,0) = f(x), \end{cases} \tag{3.15}$$

with $f \in D(\mathscr{A})$, is written as $g(x,t) = H_t^{\gamma,\nu,\delta} f(x) = \mathbb{E} f\left(A^x_{Z_t^\delta}\right)$, where $H_t^{\gamma,\nu,\delta}$ is the time-change operator (3.11) *and A_t^x, $t \geq 0$, is the Lévy process started at $x \in \mathbb{R}^d$ with infinitesimal generator* (3.2).

Proof. From (2.9) and (3.7) we obtain the Laplace transform

$$\tilde{g}(x,s) = \int_0^\infty \tilde{h}(\mathrm{d}y,s) T_y f(x) = \int_0^\infty \tilde{h}(\mathrm{d}y,s) \mathbb{E} f(A^x_y). \tag{3.16}$$

We need now the Fourier transform

$$\int_{\mathbb{R}^d} e^{i\xi\cdot x}\, \tilde{g}(x,s)\, \mathrm{d}x = \hat{f}(\xi) \int_0^\infty \tilde{h}(\mathrm{d}y,s)\, \hat{\mu}_y(\xi) \tag{3.17}$$

where recall that $\hat{\mu}_y(\xi) = e^{-y\Psi(\xi)}$. From (2.9) and (3.6), (3.17) takes the form

$$\hat{\tilde{g}}(\xi,s) = \hat{f}(\xi) \frac{s^{\gamma+\nu-1}(1+s^{-\nu})^\delta}{\Psi(\xi) + s^{\gamma+\nu}(1+s^{-\nu})^\delta}. \tag{3.18}$$

The Fourier transform of (3.15) leads to the equation

$$\left(\mathbb{D}^{\delta}_{\nu,\gamma+\nu,-1;0+} \hat{g}(\xi,\cdot)\right)(t) = -\Psi(\xi)\, \hat{g}(\xi,t). \tag{3.19}$$

By taking into account formula (2.2) we get the Fourier-Laplace transform

$$s^{\gamma+\nu}(1+s^{-\nu})^\delta \hat{\tilde{g}}(\xi,s) - \hat{g}(\xi,0^+) s^{\gamma+\nu-1}(1+s^{-\nu})^\delta + \Psi(\xi)\, \hat{\tilde{g}}(\xi,s) = 0 \tag{3.20}$$

and therefore, we get that

$$\hat{\tilde{g}}(\xi,s) = \hat{g}(\xi,0^+) \frac{s^{\gamma+\nu-1}(1+s^{-\nu})^\delta}{\Psi(\xi) + s^{\gamma+\nu}(1+s^{-\nu})^\delta}. \tag{3.21}$$

with $g(x,0) = f(x)$. □

Remark 3.1. Consider the integral (3.11). We observe that

$$\int_0^\infty l_{\delta/n}(\mathrm{d}y,r)\, T_y f(x) = \mathbb{E} f(A^x_{L_r^{\delta/n}}) \tag{3.22}$$

is the solution to the fractional problem

$$\begin{cases} \frac{\partial^{\delta/n}}{\partial r^{\delta/n}} u(x,r) = \mathscr{A} u(x,r), \quad x \in \mathbb{R}^d, \; r > 0, \; \delta/n \in (0,1), \\ u(x,0) = f(x). \end{cases} \tag{3.23}$$

Furthermore, for $\delta = 1$, $\int_0^\infty k(\mathrm{d}r,t) T_r f(x) = \mathbb{E} f(A_t^x)$ is the solution to

$$\begin{cases} \frac{\partial^{\gamma+\nu}}{\partial t^{\gamma+\nu}} u(x,r) + \frac{\partial^\nu}{\partial t^\nu} u(x,r) = \mathscr{A} u(x,t), \quad x \in \mathbb{R}^d, \; t > 0, \; \nu < \gamma+\nu \leq 1, \\ u(x,0) = f(x). \end{cases} \tag{3.24}$$

Remark 3.2. Note that even though Theorem 3.1 requires that $n\nu < \gamma + \nu \leq 1$, the Fourier-Laplace transform (3.18) is still valid for $\delta\nu < \gamma + \nu \leq 2$. In this case however it cannot be related to the process $A^x_{Z_t^\delta}$, $t \geq 0$.

Remark 3.3. Formula (3.4) defines the Fourier multiplier of $\mathscr{A}$ which is the infinitesimal generator of a Lévy process A_t, $t \geq 0$. We mention below some specific cases:

- if $\Psi(\xi) = |\xi|^{2\alpha}$ with $\alpha \in (0,1]$, then $\mathscr{A} = -(-\triangle)^\alpha$ is the fractional Laplacian. The process A_t is an isotropic stable process and becomes a Brownian motion for $\alpha = 1$. Thus, for $\alpha \in (0,1)$, we have that (for a well-defined function f)

$$-\mathscr{A}f(x) = (-\triangle)^\alpha f(x) = \mathscr{C}(\alpha,d)\int_{\mathbb{R}^d} \frac{f(x+y)+f(x-y)-2f(x)}{|y|^{2\alpha+d}}\mathrm{d}y, \tag{3.25}$$

  where $\mathscr{C}(\alpha,d)$ is a constant depending on α, d. It is well known that, in this case, the process $A_t = B_{V_t^\alpha}$ where B is a multidimensional Brownian motion and V^α is a stable subordinator (the Bochner subordination rule). In our case, therefore we get that $g(x,t) = \mathbb{E}f(x+B_{\tau_t})$, where $\tau_t = V^\alpha_{Z_t^\delta}$ is a time-changed subordinator, is the solution to the problem (3.15) with generator (3.25).

- if $d = 1$ and $\Psi(\xi) = \lambda(1-e^{i\xi})$, then A_t is a Poisson process on $\mathbb{Z}_+ := \{0,1,2,\dots\}$ with rate $\lambda > 0$ and

$$\mathscr{A}f(x) = \lambda\{f(x)-f(x-1)\}1_{\mathbb{Z}_+}(x) \tag{3.26}$$

  is the governing operator written as λ times the discrete gradient on $\mathbb{Z}_+$. Also, we usually write $\mathscr{A}f = \lambda(I-B)f$.

- if $d = 1$ and $\Psi(\xi) = \lambda(1+i\xi-e^{i\xi})$, the corresponding process is the compensated Poisson on $\mathbb{R}$ with rate $\lambda > 0$. The generator takes the form

$$\mathscr{A}f = \lambda(I-B)f - \lambda f'. \tag{3.27}$$

4 One-dimensional case with $0 < \gamma + \nu \le 2$

Here the results obtained in the previous section are specialized for $\mathscr{A} = c\partial^2/\partial x^2$, $0 \ne c \in \mathbb{R}$. Note however that in this section the order of the operator $\mathbb{D}^\delta_{\nu,\gamma+\nu,-\lambda;0+}$ is allowed to reach 2. Specifically we need $0 < \gamma+\nu \le 2$. Consider thus the Cauchy problem

$$\begin{cases} \left(\mathbb{D}^\delta_{\nu,\gamma+\nu,-\lambda;0+}g(x,\cdot)\right)(t) = c\frac{\partial^2}{\partial x^2}g(x,t), & x\in\mathbb{R},\ t>0,\\ g(x,0^+) = \delta(x), \\ \frac{\partial}{\partial t}g(x,t)\Big|_{t\downarrow 0} = 0. \end{cases} \tag{4.1}$$

In the above equation $\delta \in \mathbb{R}$, $\gamma \in (0,\infty)$, $\nu \in (0,\infty)$, $0 < \gamma+\nu \le 2$, $c \ne 0$ is a real constant. Note therefore that here $\gamma+\nu \in (0,2]$ so that (4.1) is not in fact a direct specialization of (3.15). This explains also the presence in (4.1) of the addictional initial condition $\frac{\partial}{\partial t}g(x,t)\Big|_{t\downarrow 0} = 0$.

Remark 4.1. For $\delta = 0$, equation (4.1) reduces to the time-fractional diffusion-wave equation [**?**]. Indeed we have

$$\begin{aligned} &\left(\mathbb{D}^0_{\nu,\gamma+\nu,-\lambda;0+}g(x,\cdot)\right)(t) = c\frac{\partial^2}{\partial x^2}g(x,t) && (4.2)\\ \Leftrightarrow\quad & \frac{\partial^{\gamma+\nu+\theta}}{\partial t^{\gamma+\nu+\theta}}\int_0^t (t-y)^{\theta-1}E^0_{\nu,\theta}[-\lambda(t-y)^\nu]g(x,y)\,\mathrm{d}y = c\frac{\partial^2}{\partial x^2}g(x,t) + \delta(x)\,t^{-(\gamma+\nu)}E^0_{\nu,1-(\gamma+\nu)}(-\lambda t^\nu) \\ \Leftrightarrow\quad & \frac{\partial^{\gamma+\nu}}{\partial t^{\gamma+\nu}}\frac{\partial^\theta}{\partial t^\theta}\frac{1}{\Gamma(\theta)}\int_0^t (t-y)^{\theta-1}g(x,y)\,\mathrm{d}y = c\frac{\partial^2}{\partial x^2}g(x,t) + \delta(x)\frac{t^{-(\gamma+\nu)}}{\Gamma(1-(\gamma+\nu))} \\ \overset{(\kappa=\gamma+\nu)}{\Leftrightarrow}\quad & \frac{\partial^\kappa}{\partial t^\kappa}g(x,t) = c\frac{\partial^2}{\partial x^2}g(x,t) + \delta(x)\frac{t^{-\kappa}}{\Gamma(1-\kappa)}, \qquad 0<\kappa\le 2 \\ \Leftrightarrow\quad & \frac{\mathfrak{d}^\kappa}{\mathfrak{d}t^\kappa}g(x,t) = c\frac{\partial^2}{\partial x^2}g(x,t). \end{aligned}$$

In the second step of the above formula we have used the fact that

$$E^0_{\nu,\theta}[-\lambda(t-y)^\nu] = 1/\Gamma(\theta), \qquad E^0_{\nu,1-(\gamma+\nu)}(-\lambda t^\nu) = 1/\Gamma(1-(\gamma+\nu)). \tag{4.3}$$

Also, as mentioned before, we considered here that the semigroup property for the Riemann–Liouville fractional derivative and hence some regularity conditions on the solution g are fulfilled (see Section A).

Remark 4.2. For $\delta = 1$, equation (4.1) reduces to the fractional telegraph equation [**?**]. In this case we have

$$\begin{aligned}
&\left(\mathbb{D}^{1}_{\nu,\gamma+\nu,-\lambda;0+} g(x,\cdot)\right)(t) = c\frac{\partial^2}{\partial x^2} g(x,t) \qquad (4.4)\\
\Leftrightarrow \quad &\frac{\partial^{\gamma+\nu+\theta}}{\partial t^{\gamma+\nu+\theta}} \int_0^t (t-y)^{\theta-1} E^{-1}_{\nu,\theta}\left[-\lambda(t-y)^\nu\right] g(x,y)\,\mathrm{d}y = c\frac{\partial^2}{\partial x^2} g(x,t) + \delta(x)\, t^{-(\gamma+\nu)} E^{-1}_{\nu,1-(\gamma+\nu)}(-\lambda t^\nu).
\end{aligned}$$

Now, by considering that

$$E^{-1}_{\nu,\theta}\left[-\lambda(t-y)^\nu\right] = \frac{1}{\Gamma(\theta)} + \frac{\lambda(t-y)^\nu}{\Gamma(\nu+\theta)}, \tag{4.5}$$

$$E^{-1}_{\nu,1-(\gamma+\nu)}(-\lambda t^\nu) = \frac{1}{\Gamma(1-(\gamma+\nu))} + \frac{\lambda t^\nu}{\Gamma(1-\gamma)}, \tag{4.6}$$

we can write

$$\begin{aligned}
&\frac{\partial^{\gamma+\nu}}{\partial t^{\gamma+\nu}} \frac{\partial^\theta}{\partial t^\theta} \frac{1}{\Gamma(\theta)} \int_0^t (t-y)^{\theta-1} g(x,y)\,\mathrm{d}y + \lambda \frac{\partial^\gamma}{\partial t^\gamma} \frac{\partial^{\nu+\theta}}{\partial^{\nu+\theta}} \frac{1}{\Gamma(\nu+\theta)} \int_0^t (t-y)^{\nu+\theta-1} g(x,y)\,\mathrm{d}y \qquad (4.7)\\
&\quad = c\frac{\partial^2}{\partial x^2} g(x,t) + \delta(x)\frac{t^{-(\gamma+\nu)}}{\Gamma(1-(\gamma+\nu))} + \delta(x)\frac{\lambda t^{-\gamma}}{\Gamma(1-\gamma)}\\
\Leftrightarrow \quad &\frac{\partial^{\gamma+\nu}}{\partial t^{\gamma+\nu}} g(x,t) + \lambda\frac{\partial^\gamma}{\partial t^\gamma} g(x,t) = c\frac{\partial^2}{\partial x^2} g(x,t) + \delta(x)\frac{t^{-(\gamma+\nu)}}{\Gamma(1-(\gamma+\nu))} + \delta(x)\frac{\lambda t^{-\gamma}}{\Gamma(1-\gamma)}\\
\Leftrightarrow \quad &\frac{\mathfrak{d}^{\gamma+\nu}}{\mathfrak{d} t^{\gamma+\nu}} g(x,t) + \lambda\frac{\mathfrak{d}^\gamma}{\mathfrak{d} t^\gamma} g(x,t) = c\frac{\partial^2}{\partial x^2} g(x,t),
\end{aligned}$$

where $0 < \gamma + \nu \le 2$.

The Laplace–Fourier transform of the solution $g(x,t)$ to equation (4.1) can be easily determined as follows. We start with the application of the Fourier transform $\hat{g}(\beta,t) = \int_{-\infty}^{\infty} e^{i\beta x} g(x,t)\,\mathrm{d}x$, immediately yielding

$$\begin{cases}
\left(\mathbb{D}^{\delta}_{\nu,\gamma+\nu,-\lambda;0+} \hat{g}(\beta,\cdot)\right)(t) = -c\beta^2 \hat{g}(\beta,t),\\
\hat{g}(\beta,0^+) = 1,\\
\frac{\partial}{\partial t}\hat{g}(x,t)\big|_{t\downarrow 0} = 0.
\end{cases} \tag{4.8}$$

Then, by using formula (2.12) and applying the Laplace transform (with parameter s) to both members of (4.8), the complete Laplace–Fourier transform of (4.1) can be written as

$$s^{\gamma+\nu+\theta} \hat{\tilde{g}}(\beta,s) s^{-\theta} \left(1+\lambda s^{-\nu}\right)^\delta = -c\beta^2 \hat{\tilde{g}}(\beta,s) + s^{\gamma+\nu-1}\left(1+\lambda s^{-\nu}\right)^\delta. \tag{4.9}$$

From this we immediately obtain the Laplace–Fourier transform of the solution to equation (4.1) as

$$\hat{\tilde{g}}(\beta,s) = \frac{s^{\gamma+\nu-1}\left(1+\lambda s^{-\nu}\right)^\delta}{s^{\gamma+\nu}\left(1+\lambda s^{-\nu}\right)^\delta + c\beta^2}. \tag{4.10}$$

Remark 4.3. Clearly, for $\delta = 0$, formula (4.10) reduces to the Laplace–Fourier transform of the solution to the fractional diffusion-wave equation [**?**, formula (2.17)], while for $\delta = 1$ it leads to that of the fractional telegraph equation [**?**, formula (2.6) for $\gamma = \nu = \alpha$].

The Fourier transform of the solution to (4.1) can be derived by inverting the Laplace transform in (4.10) as follows.

$$\hat{\tilde{g}}(\beta,s) = s^{-1}\left(1 + \frac{c\beta^2}{s^{\gamma+\nu}\left(1+\lambda s^{-\nu}\right)^\delta}\right)^{-1} = s^{-1}\sum_{r=0}^{\infty}\left[-\frac{c\beta^2}{s^{\gamma+\nu}\left(1+\lambda s^{-\nu}\right)^\delta}\right]^r. \tag{4.11}$$

The last step is valid whenever $|c\beta^2/(s^{\gamma+\nu}(1+\lambda s^{-\nu})^\delta)| < 1$. We then have

$$\hat{\tilde{g}}(\beta,s) = \sum_{r=0}^{\infty}(-c\beta^2)^r s^{-(r(\gamma+\nu)+1)}\left(1+\lambda s^{-\nu}\right)^{-r\delta}. \tag{4.12}$$

Consequently, by recalling again formula (2.12) and considering **?**, Theorem 30.1 which ensures the inversion term by term, the Fourier transform of $g(x,t)$ reads

$$\begin{aligned}\hat{g}(\beta,t) &= \sum_{r=0}^{\infty}(-c\beta^2 t^{\gamma+\nu})^r E^{r\delta}_{\nu,r(\gamma+\nu)+1}(-\lambda t^{\nu}) \\ &= \sum_{m=0}^{\infty}(-\lambda t^{\nu})^m {}_2\psi_2\left[c\beta^2 t^{\gamma+\nu}\left|\begin{array}{l}(1,1),(m,\delta)\\(0,\delta),(\gamma+\nu,\,\nu m+1)\end{array}\right.\right],\end{aligned} \tag{4.13}$$

where ${}_p\psi_q$ is the generalized Wright function [**?**, Section 1.11], defined as

$${}_p\psi_q(x) = {}_p\psi_q\left[x\left|\begin{array}{l}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{array}\right.\right] = \sum_{k=0}^{\infty}\frac{x^k}{k!}\frac{\prod_{m=1}^{p}\Gamma(a_m+\alpha_m k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}, \tag{4.14}$$

where $x, a_m, b_j \in \mathbb{C}$, $\alpha_m, \beta_j \in \mathbb{R}$, $m = 1,\dots,p$, $j = 1,\dots,q$.
The Laplace–Fourier transform (4.10) immediately yields

$$\begin{aligned}\tilde{g}(x,s) &= \int_0^{\infty} e^{-st} g(x,t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x}\frac{s^{\gamma+\nu-1}(1+\lambda s^{-\nu})^{\delta}}{s^{\gamma-\nu}(1+\lambda s^{-\nu})^{\delta}+c\beta^2}\,\mathrm{d}\beta \\ &= \frac{1}{2sc^{1/2}} s^{(\gamma+\nu)/2}\left(1+\lambda s^{-\nu}\right)^{\delta/2} e^{-\frac{|x|}{c^{1/2}}s^{(\gamma+\nu)/2}\left(1+\lambda s^{-\nu}\right)^{\delta/2}}.\end{aligned} \tag{4.15}$$

Remark 4.4. Note how the Fourier transform (4.13), for $\delta = 0$ reduces to the Fourier transform of the solution of the pure fractional diffusion equation as we recall that $(0)_m = 0$, $m = 1,2,\dots$, but $(0)_0 = 1$. Therefore,

$$\hat{g}(\beta,t) = E_{\gamma+\nu,1}\left(-c\beta^2 t^{\gamma+\nu}\right), \tag{4.16}$$

which coincides with the corresponding formula of **?**, page 212.
When $\delta = 1$ instead we arrive at

$$\begin{aligned}\hat{g}(\beta,t) &= \sum_{r=0}^{\infty}(-c\beta^2 t^{\gamma+\nu})^r E^{r}_{\nu,r(\gamma+\nu)+1}(-\lambda t^{\nu}) \\ &= \sum_{m=0}^{\infty}(-\lambda t^{\nu})^m {}_2\psi_2\left[c\beta^2 t^{\gamma+\nu}\left|\begin{array}{l}(1,1),(m,1)\\(0,1),(\gamma+\nu,\,\nu m+1)\end{array}\right.\right],\end{aligned} \tag{4.17}$$

which should be compared with formula (2.7) of **?** when $\nu = \gamma = \alpha$.

Remark 4.5. For $\delta = 2$, $\gamma > \nu$, we obtain an interesting specific case. In this case equation (4.1) reduces to

$$\begin{cases}\frac{\partial^{\gamma+\nu}}{\partial t^{\gamma+\nu}}g(x,t) + 2\lambda\frac{\partial^{\gamma}}{\partial t^{\gamma}}g(x,t) + \lambda^2\frac{\partial^{\gamma-\nu}}{\partial t^{\gamma-\nu}}g(x,t) = c\frac{\partial^2}{\partial x^2}g(x,t),\\ g(x,0^+) = \delta(x).\end{cases} \tag{4.18}$$

In the general case of $\delta = n \in \mathbb{N}\cup\{0\}$, we can work out equation (4.1) as follows.

$$\begin{aligned}\left(\mathbb{D}^{n}_{\nu,\gamma+\nu,-\lambda;0+}g(x,\cdot)\right)(t) = c\frac{\partial^2}{\partial x^2}g(x,t) \Leftrightarrow\quad & \frac{\partial^{\gamma+\nu+\theta}}{\partial t^{\gamma+\nu+\theta}}\int_0^t (t-y)^{\theta-1}E^{-n}_{\nu,\theta}\left[-\lambda(t-y)^{\nu}\right]g(x,y)\,\mathrm{d}y \\ &= c\frac{\partial^2}{\partial x^2}g(x,t) + \delta(x)\,t^{-(\gamma+\nu)}E^{-n}_{\nu,1-(\gamma+\nu)}(-\lambda t^{\nu}).\end{aligned} \tag{4.19}$$

Now, considering that $(-n)_r = (-1)^r(n-r+1)_r = (-1)^r n!/(n-r)!$, we can write

$$\frac{\partial^{\gamma+\nu+\theta}}{\partial t^{\gamma+\nu+\theta}}\int_0^t (t-y)^{\theta-1}\sum_{r=0}^{n}\binom{n}{r}\frac{\lambda^r(t-y)^{\nu r}}{\Gamma(\nu r+\theta)}g(x,y)\,\mathrm{d}y = c\frac{\partial^2}{\partial x^2}g(x,t) + \delta(x)\sum_{r=0}^{n}\binom{n}{r}\frac{\lambda^r t^{-(\gamma-\nu(r-1))}}{\Gamma(1-(\gamma-\nu(r-1)))} \tag{4.20}$$

$$\begin{aligned}\Leftrightarrow\quad & \sum_{r=0}^{n}\binom{n}{r}\lambda^r\frac{\partial^{\gamma-\nu(r-1)}}{\partial t^{\gamma-\nu(r-1)}}g(x,t) = c\frac{\partial^2}{\partial x^2}g(x,t) + \delta(x)\sum_{r=0}^{n}\binom{n}{r}\frac{\lambda^r t^{-(\gamma-\nu(r-1))}}{\Gamma(1-(\gamma-\nu(r-1)))} \\ \Leftrightarrow\quad & \sum_{r=0}^{n}\binom{n}{r}\lambda^r\frac{\partial^{\gamma-\nu(r-1)}}{\partial t^{\gamma-\nu(r-1)}}g(x,t) = c\frac{\partial^2}{\partial x^2}g(x,t).\end{aligned}$$

Here we have $0 < \gamma - \nu(r-1) \le 2$ so that $n\nu < \gamma + \nu \le 2$.

Remark 4.6 (Wave-telegraph equation). When $\gamma = \nu = 1$, the equation considered is a wave-telegraph equation. In this case, the allowed range for δ is $\delta \le 1$. The interpolating equation reads

$$\left(\mathbb{D}^{\delta}_{1,2,-\lambda;0+} g(x,\cdot)\right)(t) = c\frac{\partial^2}{\partial x^2} g(x,t), \qquad x \in \mathbb{R},\ t \ge 0. \tag{4.21}$$

Appendix A Properties of the operator $\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+}$

We analyze here some properties of the operator $\mathbb{D}^{\xi}_{\alpha,\eta,\zeta;0+}$. In the following proposition we show that the integral operator (1.10) is in fact a generalization of the left sided Riemann–Liouville fractional integral, to which it reduces for $\xi = 0$. The convolution kernel here is no longer a power-law but it is instead a generalized Mittag–Leffler kernel. It follows furthermore that the operator (1.8) represents a generalization to the left sided Riemann–Liouville fractional derivative.

Proposition A.1. *For $\xi = 0$ we have that*

$$\left(\mathbf{E}^{0}_{\alpha,\eta,\zeta;0+} f(\cdot)\right)(t) = \frac{1}{\Gamma(\eta)}\int_0^t (t-y)^{\eta-1} f(y)\mathrm{d}y, \tag{A.1}$$

$$\left(\mathbf{D}^{0}_{\alpha,\eta,\zeta;0+} f(\cdot)\right)(t) = \frac{\mathrm{d}^\eta}{\mathrm{d}t^\eta} f(t), \tag{A.2}$$

with $\eta > 0$, $\alpha > 0$, $\zeta \in \mathbb{R}$, $t \ge 0$.

Proof. Formula (A.1) can be derived by simply considering that $E^{0}_{\alpha,\eta}\left[\zeta(t-y)^\alpha\right] = 1/\Gamma(\eta)$. In order to briefly prove formula (A.2) we can write that

$$\begin{aligned}\left(\mathbf{D}^{0}_{\alpha,\eta,\zeta;0+} f(\cdot)\right)(t) &= \frac{\mathrm{d}^{\eta+\theta}}{\mathrm{d}t^{\eta+\theta}}\int_0^t (t-y)^{\theta-1} E^{0}_{\alpha,\theta}\left[\zeta(t-y)^\alpha\right] f(y)\,\mathrm{d}y \\ &= \frac{\mathrm{d}^\eta}{\mathrm{d}t^\eta}\frac{\mathrm{d}^\theta}{\mathrm{d}t^\theta}\frac{1}{\Gamma(\theta)}\int_0^t (t-y)^{\theta-1} f(y)\,\mathrm{d}y = \frac{\mathrm{d}^\eta}{\mathrm{d}t^\eta} f(t),\end{aligned} \tag{A.3}$$

as the Riemann–Liouville fractional derivative is the left-inverse operator to the Riemann–Liouville fractional integral. As before, in the second step of formula (A.3) we considered that $E^{0}_{\alpha,\theta}\left[\zeta(t-y)^\alpha\right] = 1/\Gamma(\theta)$, and that the semigroup property for the Riemann–Liouville fractional derivative is fulfilled as

$$\left.\frac{\mathrm{d}^r}{\mathrm{d}t^r}\int_0^t (t-y)^{\theta-1} f(y)\,\mathrm{d}y\right|_{t=0} = 0, \qquad \forall r \in \mathbb{N}\cup\{0\}. \tag{A.4}$$

□

Remark A.1. From Proposition (A.1) it immediately follows that the operator (1.11) represents a generalization of the left sided Dzhrbashyan–Caputo fractional derivative $\partial^\eta/\partial t^\eta$. Indeed, for $\xi = 0$, we can write

$$\begin{aligned}\left(\mathbb{D}^{0}_{\alpha,\eta,\zeta;0+} f(\cdot)\right)(t) &= \left(\mathbf{D}^{0}_{\alpha,\eta,\zeta;0+} f(\cdot)\right)(t) - f(0^+)\, t^{-\eta} E^{0}_{\alpha,1-\eta}(\zeta t^\alpha), \\ &= \frac{\mathrm{d}^\eta}{\mathrm{d}t^\eta} f(t) - f(0^+)\frac{t^{-\eta}}{\Gamma(1-\eta)} = \frac{\partial^\eta}{\partial t^\eta} f(t).\end{aligned} \tag{A.5}$$

For more information on Dzhrbashyan–Caputo derivatives, the reader can refer for example to **?**.

Proposition A.2. *The operator $\mathbf{D}^{\xi}_{\alpha,\eta,\zeta;0+}$, $\eta > 0$, $\alpha > 0$, $\zeta \in \mathbb{R}$, $t \ge 0$, can be written also as*

$$\left(\mathbf{D}^{\xi}_{\alpha,\eta,\zeta;0+} f(\cdot)\right)(t) = W_{-1,\xi+1}\left(-\zeta J^\alpha_t\right) f(t), \tag{A.6}$$

where

$$W_{a,b}(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!\Gamma(ar+b)}, \qquad a \ge 1, b \in \mathbb{R}, \tag{A.7}$$

is the classical Wright function which is convergent in $|x| < 1$ if $a = -1$, $b > 0$, and J^α_t is the Riemann–Liouville fractional integral.

Proof. We start by expanding in series the generalized Mittag–Leffler function in the kernel of the operator.

$$\begin{aligned}\left(\mathbf{D}^{\xi}_{\alpha,\eta,\zeta;0+}f(\cdot)\right)(t) &= \frac{\mathrm{d}^{\eta+\theta}}{\mathrm{d}t^{\eta+\theta}}\int_0^t (t-y)^{\theta-1}E^{-\xi}_{\alpha,\theta}\left[\zeta(t-y)^{\alpha}\right]f(y)\,\mathrm{d}y \\ &= \frac{\mathrm{d}^{\eta+\theta}}{\mathrm{d}t^{\eta+\theta}}\int_0^t (t-y)^{\theta-1}\sum_{r=0}^{\infty}\frac{\zeta^r(-\xi)_r(t-y)^{\alpha r}}{r!\Gamma(\alpha r+\theta)}f(y)\,\mathrm{d}y.\end{aligned} \tag{A.8}$$

By recalling now that $(-\xi)_r = (-1)^r(\xi-r+1)_r = (-1)^r\Gamma(\xi+1)/\Gamma(\xi-r+1)$ and considering again that the semigroup property is satisfied, we have that

$$\begin{aligned}\left(\mathbf{D}^{\xi}_{\alpha,\eta,\zeta;0+}f(\cdot)\right)(t) &= \Gamma(\xi+1)\frac{\mathrm{d}^{\eta+\theta}}{\mathrm{d}t^{\eta+\theta}}\sum_{r=0}^{\infty}\frac{(-\zeta)^r}{r!\Gamma(\xi-r+1)}\frac{1}{\Gamma(\alpha r+\theta)}\int_0^t (t-y)^{\alpha r+\theta-1}f(y)\,\mathrm{d}y \\ &= \Gamma(\xi+1)\frac{\mathrm{d}^{\eta+\theta}}{\mathrm{d}t^{\eta+\theta}}\sum_{r=0}^{\infty}\frac{(-\zeta)^r}{r!\Gamma(\xi-r+1)}J_t^{\alpha r+\theta}f(t) \\ &= \Gamma(\xi+1)\frac{\mathrm{d}^{\eta}}{\mathrm{d}t^{\eta}}\sum_{r=0}^{\infty}\frac{(-\zeta)^r}{r!\Gamma(\xi-r+1)}J_t^{\alpha r}f(t),\end{aligned} \tag{A.9}$$

provided that the series converges and where $J_t^{\alpha r}$ represents the Riemann–Liouville fractional integral of order αr. Thus we obtain that

$$\begin{aligned}\left(\mathbf{D}^{\xi}_{\alpha,\eta,\zeta;0+}f(\cdot)\right)(t) &= \Gamma(\xi+1)\frac{\mathrm{d}^{\eta}}{\mathrm{d}t^{\eta}}\sum_{r=0}^{\infty}\frac{(-\zeta J_t^{\alpha})^r}{r!\Gamma(\xi-r+1)}f(t) \\ &= \Gamma(\xi+1)\frac{\mathrm{d}^{\eta}}{\mathrm{d}t^{\eta}}W_{-1,\xi+1}\left(-\zeta J_t^{\alpha}\right)f(t).\end{aligned} \tag{A.10}$$

The obtained representation (A.10) is formal and it becomes an actual representation whenever all the requested convergence conditions are fulfilled. □