Dynamics of one-dimensional quantum many-body systems in time-periodic linear potentials

A. Colcelli,[1] G. Mussardo,[1] G. Sierra,[2] and A. Trombettoni[3,1,4]
[1]*SISSA and INFN, Sezione di Trieste, Via Bonomea 265, I-34136 Trieste, Italy*
[2]*Instituto de Física Teórica, UAM/CSIC, Universidad Autónoma de Madrid, E-28049 Madrid, Spain*
[3]*Department of Physics, University of Trieste, Strada Costiera 11, I-34151 Trieste, Italy*
[4]*CNR-IOM DEMOCRITOS Simulation Center, Via Bonomea 265, I-34136 Trieste, Italy*

(Received 2 July 2020; accepted 18 August 2020; published 8 September 2020)

We study a system of one-dimensional interacting quantum particles subjected to a time-periodic potential linear in space. After discussing the cases of driven one- and two-particle systems, we derive the analogous results for the many-particle case in the presence of a general interaction two-body potential and the corresponding Floquet Hamiltonian. When the undriven model is integrable, the Floquet Hamiltonian is shown to be integrable too. We determine the micromotion operator and the expression for a generic time evolved state of the system. We discuss various aspects of the dynamics of the system both at stroboscopic and intermediate times, in particular the motion of the center of mass of a generic wave packet and its spreading over time. We also discuss the case of accelerated motion of the center of mass, obtained when the integral of the coefficient strength of the linear potential on a time period is nonvanishing, and we show that the Floquet Hamiltonian gets in this case an additional static linear potential. We also discuss the application of the obtained results to the Lieb-Liniger model.

DOI: 10.1103/PhysRevA.102.033310

I. INTRODUCTION

Time-periodic driven quantum systems have become recently the subject of an intense research activity. These out of equilibrium systems give rise to interesting novel physical properties as, for instance, dynamic localization effects [1], suppression of tunneling subjected to a strongly driven optical lattice [2–8] (see [9] for more references), topological Floquet phases [10,11], time crystals [12–19], dynamics in driven systems [20–22], and Floquet prethermalization [23,24]. All these concepts and phenomena can be collected together under the heading of "Floquet engineering" [9,25], a very active field both from experimental and theoretical points of view.

The name itself came from a famous paper by Floquet [26], who was interested in the study of differential equations with coefficients given by time-periodic functions. The formalism he developed turns out to be very helpful in dealing with the Schrödinger equation of a quantum mechanical system with a time-periodic Hamiltonian [27,28]. Preparing the system in an initial state $\chi(t=0)$ and letting the periodic driving act on it, the Floquet Hamiltonian $\hat{H}_F$ is the operator that formally gives the state of the system at multiples of the period T:

$$\chi(t=nT) = e^{-i\frac{nT}{\hbar}\hat{H}_F}\,\chi(t=0). \tag{1}$$

In other words, the Floquet Hamiltonian $\hat{H}_F$ determines the stroboscopic evolution of the system. It depends on the parameters of the original undriven Hamiltonian $\hat{H}_0$ and on the time-dependent perturbation. $\hat{H}_F$ is a Hermitian operator whose eigenvalues are the so-called quasienergies $\mathcal{E}_F$. On the other hand, the evolution of the state $\chi(t=0)$ at generic times $t \in (0,T)$ is determined by the *micromotion operator* $\hat{U}_F(t,0)$, defined by the following decomposition of the time evolution operator of the system $\hat{U}(t,0)$:

$$\hat{U}(t,0) = \hat{U}_F(t,0)\,e^{-i\frac{t}{\hbar}\hat{H}_F}. \tag{2}$$

Applying the micromotion operator $\hat{U}_F(t,0)$ on the eigenstates of the Floquet Hamiltonian and multiplying by a complex exponential containing the quasienergies, one obtains the *Floquet states* $|\psi_F(t)\rangle$. They form a complete and orthonormal set of functions and therefore any solution of the original time-dependent Schrödinger equation can be written as a superposition in terms of them,

$$\chi(t) = \int dk\, A(k)\,|\psi_F(t)\rangle,$$

where k is a momentum variable, related to the energy of the system ($k \propto \sqrt{E}$), and the $A(k)$'s are time-independent coefficients. Therefore finding $\hat{H}_F$ and $\hat{U}_F(t,0)$ gives access to the full quantum dynamics of the system.

Finding the Floquet Hamiltonian and the micromotion operator for an interacting many-body system in the presence of a time-dependent driving is in general a challenging and highly interesting task, relevant for a variety of applications in the field of Floquet engineering. Tuning the form and the parameters of the undriven system and of the periodic perturbation, one aims at controlling the (desired) effective Hamiltonian of the quantum dynamics of the system itself.

In general, even if the undriven model is integrable, when we subject it to a time-periodic potential, we end up in a non-integrable Floquet Hamiltonian. In a recent paper [29] we addressed the question whether it would be possible to have

an integrable Floquet Hamiltonian by perturbing an integrable $1D$ bosonic model with a time-periodic perturbation, finding a positive answer. Namely, we considered the integrable Hamiltonian that describes a one-dimensional gas of bosons with contact interactions, i.e. the Lieb-Liniger Hamiltonian [30], in the presence of a linear in space, time-periodic one-body potential of the form

$$V(x,t) = f(t)\,x, \tag{3}$$

with a driving function $f(t)$ with period T: $f(t) = f(t+T)$. It was shown in [29] that under the condition

$$\int_0^T f(\tau)\,d\tau = 0, \tag{4}$$

the resulting Floquet Hamiltonian is integrable and has a Lieb-Liniger form, with a shift on the momenta of the particles.

Despite the fact that other exactly solvable time-dependent Hamiltonians can be constructed using different approaches [31,32], the problem of finding an integrable Floquet Hamiltonian from an undriven interacting one is in general a difficult task. The goal of the present paper is twofold: (a) first, we provide a derivation valid for general one-dimensional many-particle systems, extending the results of [29] to an arbitrary two-body interaction potential $V_{2b}(x_j - x_i)$ and giving explicit results for the micromotion operator $\hat{U}_F(t,0)$; (b) secondly, we present a detailed discussion of the case in which the condition (4) does not hold, emphasizing its role for the time dependence of the energy of the system.

We will show that if the undriven Hamiltonian is integrable and perturbed with a linear time-periodic potential, then also the Floquet Hamiltonian is integrable if the driving function has a vanishing integral over a period of oscillation, as it occurs for the Lieb-Liniger case. If, on the contrary, the condition (4) does not hold, we will see that the Floquet Hamiltonian can be still recast in a time-independent expression but with the addition of a linear potential. Expressions for the value of the energy during the stroboscopic dynamics are found and the micromotion operator explicitly written down. The method we use is based on first applying a gauge transformation on the wave function to wash out the linear term and then solving the time-dependent Schrödinger equation with Hamiltonian $\hat{H}_F$. It is worth underlining that, in general, one of the difficulties in identifying integrable Floquet Hamiltonians is that the integrability of these Hamiltonians is not at all guaranteed by the integrability of the original time-independent undriven model (see, for instance, [33] where starting from the original BCS model the corresponding BCS gap equation in the presence of a periodic driving is derived and solved numerically). For the class of one-dimensional interacting many-particle systems considered here, we show instead that it is not the case, as far as the periodic driving is a linear function on the position variables.

We remark that we are referring to the integrability of the Floquet Hamiltonian of the system, which is a time-independent Hamiltonian and therefore by *integrable* we mean that there exist an infinite number of conserved charges and hence one can find an exact solution for the stroboscopic time-independent problem. In particular the integrability of the Floquet Hamiltonian allows one to find the eigenfunctions of $\hat{H}_F$ and to write the behavior of a generic wave packet at stroboscopic times.

In the following we present a detailed analysis of all these aspects of the problem and, in particular, we show how to extract the time evolution of generic wave functions at all times by first computing the micromotion operators and then the Floquet states, with which we can expand the wave function. After discussing a general two-body interaction term, we focus on the paradigmatic and experimentally relevant case where the particles interact with contact interactions, i.e. the Lieb-Liniger model. This model constitutes an ideal playground for integrable models in one-dimensional continuous space. It is indeed exactly solvable using Bethe ansatz techniques [30,34–36], related to the nonrelativistic limit of the Sinh-Gordon model [37] and routinely used to describe (quasi-) one-dimensional bosonic gases realized in ultracold atoms experiments (see the reviews [38–40]).

The paper is organized as follows. In order to set the notations and present initially the general results in the simpler form, in Sec. II we discuss the dynamics of the one-particle case, i.e. the Schrödinger equation for a particle of mass m in a linear time-periodic potential in one dimension. In Sec. III we consider the interacting two-particle case, where both particles, in addition to their relative potential, are also subjected to a periodic driving potential proportional to their position. In Sec. IV, we address the many-body interacting case. Our conclusions are finally gathered in Sec. V.

II. ONE-BODY PROBLEM

A. Generic driving function

Let us consider the one-dimensional Schrödinger equation for a particle of mass m in a linear potential with a time varying strength:

$$i\hbar\frac{\partial\chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\chi}{\partial x^2} + x\,f(t)\,\chi(x,t). \tag{5}$$

In what follows, $f(t)$ is a generic driving function that will be taken to be periodic at the end of this section. In the literature, Eq. (5) has been studied and solved in different ways [41–44]. Here we solve it with a method that will be particularly useful to study the Floquet dynamics.

The key point of the solution of Eq. (5) is to perform a gauge transformation on the wave function,

$$\chi(x,t) = e^{i\theta(x,t)}\,\eta(y(t),t), \tag{6}$$

where $y(t) = x - \xi(t)$, while $\xi(t)$ and $\theta(x,t)$ are two functions that are determined below. Substituting Eq. (6) into (5), and imposing

$$\frac{d\xi}{dt} = \frac{\hbar}{m}\frac{\partial\theta}{\partial x}, \tag{7}$$

and

$$-\hbar\frac{\partial\theta}{\partial t} = \frac{\hbar^2}{2m}\left(\frac{\partial\theta}{\partial x}\right)^2 + x f(t), \tag{8}$$

we find that $\eta(y,t)$ satisfies the Schrödinger equation with no external potential in the spatial variable y:

$$i\hbar\frac{\partial\eta}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\eta}{\partial y^2}. \tag{9}$$

Hence, once $\theta(x,t)$ is known, $\eta(y,t)$ will be readily determined from the free dynamics. To find the gauge phase $\theta(x,t)$ we make the ansatz,

$$\theta(x,t) = \frac{m}{\hbar}\frac{d\xi}{dt}x + \Gamma(t), \tag{10}$$

that leads to the conditions,

$$m\frac{d^2\xi}{dt^2} = -f(t), \quad \hbar\frac{d\Gamma}{dt} = -\frac{m}{2}\left(\frac{d\xi}{dt}\right)^2, \tag{11}$$

which give the translational parameter $\xi(t)$ and the function $\Gamma(t)$ in terms of $f(t)$. Notice that the equation for $\xi(t)$ is the Newton's second law equation of motion, where $d^2\xi/dt^2$ represents the acceleration of the center of mass of the system, and $-f(t)$ the driving force.

Solving the equations (11), with the initial conditions $\xi(0) = d\xi(0)/dt = 0$ and $\Gamma(0) = 0$, we get

$$\theta(x,t) = -\frac{x}{\hbar}\int_0^t f(\tau)\,d\tau - \frac{1}{2m\hbar}\int_0^t d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2, \tag{12}$$

which, together with Eq. (6) and Eq. (9), completely solves Eq. (5).

Since $\theta(x,0) = 0$ and $y(0) = x$, we have from Eq. (6) that

$$\chi(x,0) = \eta(x,0), \tag{13}$$

for which the solution of the Schrödinger equation (5) reads

$$\chi(x,t) = e^{i\theta(x,t)}e^{-i\frac{t}{\hbar}\frac{\hat{p}^2}{2m}}\eta(y,0) = e^{i\theta(x,t)}e^{-i\frac{t}{\hbar}\frac{\hat{p}^2}{2m}}e^{-i\frac{\xi(t)}{\hbar}\hat{p}}\chi(x,0), \tag{14}$$

where we have used the definition of the translation operator and the free time evolution operator. Notice that no boundary conditions in the wave function have been considered in the above calculations, i.e. $x \in \mathbb{R}$.

In terms of the solution (14), one can easily compute the expectation values of various physical quantities, such as momentum, position, as well as their variances. Assuming as initial values $\langle\hat{x}\rangle(t=0) = x_0$ and $\langle\hat{p}\rangle(t=0) = p_0$, and using the canonical commutation relations among different powers of position and momentum operators, we have

$$\langle\hat{x}\rangle(t) \equiv \langle\chi(x,t)|\hat{x}|\chi(x,t)\rangle = x_0 + \frac{t}{m}p_0 + \xi(t). \tag{15}$$

This means that the mean position of a generic wave packet, under the action of a linear time-dependent potential, is governed by the parameter $\xi(t)$ which is readily determined by Eq. (11). Moreover, concerning the expectation value of the momentum we have

$$\langle\hat{p}\rangle(t) \equiv \langle\chi(x,t)|\hat{p}|\chi(x,t)\rangle = p_0 - \int_0^t f(\tau)\,d\tau, \tag{16}$$

meaning that the value of the momentum is shifted away from its initial value by a term that depends on the driving function $f(t)$. As expected, the motion of the center of the wave packet in Eq. (15) is the same of a classical particle moving in one dimension under the action of a time-dependent gravitational force. Concerning the variance of the position, we have

$$\Delta x(t) \equiv \sqrt{\langle\hat{x}^2\rangle(t) - \langle\hat{x}\rangle^2(t)} = \Delta x_{\text{undriven}}(t), \tag{17}$$

where the subscript "undriven" stands for the undriven evolution of the variance, which is calculated using the wave function $\eta(x,t)$ instead of $\chi(x,t)$, i.e.

$$\Delta x_{\text{undriven}}(t) \equiv \sqrt{\langle\eta(x,t)|\hat{x}^2|\eta(x,t)\rangle - \langle\eta(x,t)|\hat{x}|\eta(x,t)\rangle^2}. \tag{18}$$

For the variance of the momentum we have

$$\Delta p(t) \equiv \sqrt{\langle\hat{p}^2\rangle(t) - \langle\hat{p}\rangle^2(t)} = \Delta p_{\text{undriven}}(t). \tag{19}$$

This means that it remains constant and equal to its initial value at $t = 0$.

The solution presented so far, and its consequences, are valid for any driving function. In the sequel, as a preparation for later sections, we shall focus our attention on periodic drivings.

B. Floquet approach

When $f(t)$ is periodic with period T, the Schrödinger equation (5) becomes a differential equation with periodic coefficients where we can apply the Floquet theory. This leads us to define the Floquet Hamiltonian $\hat{H}_F$, which, according to Eq. (1), controls the time evolution of the wave function at stroboscopic times $t = nT$, with $n \in \mathbb{N}$. Switching for simplicity to the bra-ket notation, Eq. (1) reads

$$|\chi(nT)\rangle = e^{-i\frac{nT}{\hbar}\hat{H}_F}|\chi(t=0)\rangle. \tag{20}$$

The eigenvalues of the Floquet Hamiltonian will be denoted by $\mathcal{E}_F$ and are known as the quasienergies. Since $\hat{H}_F$ is Hermitian, they are real numbers. The quasienergies are the timelike analogs of the quasimomenta in the study of crystalline solids. Let $\hat{U}(t,0) = e^{-i\frac{t}{\hbar}\hat{H}}$ be the time evolution operator, i.e. the quantum operator that, when applied to a wave function describes its evolution from time 0 to time t. According to the Floquet theory and the notation of [9], we can decompose $\hat{U}(t,0)$ as in Eq. (2): $\hat{U}(t,0) = \hat{U}_F(t,0)\,e^{-i\frac{t}{\hbar}\hat{H}_F}$. This relation defines the micromotion operator $\hat{U}_F(t,0)$ in terms of the Floquet Hamiltonian $\hat{H}_F$ and $\hat{U}(t,0)$. $\hat{U}_F$ is periodic in time and equals to the unity at every stroboscopic time, implying that $\hat{U}(nT,0) = e^{-i\frac{nT}{\hbar}\hat{H}_F}$. Therefore $\hat{U}(t+T,0) = \hat{U}(t,0)\hat{U}(T,0)$. This means that it is enough to know the evolution operator for times $t \in [0,T]$ in order to obtain the evolution of the system at all times $t \geqslant 0$.

The importance of these concepts becomes clear once one realizes that any solution of the time-dependent periodic Schrödinger equation (5) can be expressed in terms of the Floquet operator and their eigenfunctions. Indeed, writing the eigenvalue equation for the Floquet Hamiltonian,

$$\hat{H}_F|\tilde{u}\rangle = \mathcal{E}_F|\tilde{u}\rangle, \tag{21}$$

one can apply the micromotion operator on the wave functions $|\tilde{u}\rangle$ to write the Floquet modes (or Floquet functions according

to the notation of [21]) as

$$|u(t)\rangle = \hat{U}_F(t,0)|\tilde{u}\rangle, \tag{22}$$

which are time-periodic states, as follows from the properties of the micromotion operator stated above. It is now straightforward to construct the Floquet states, which are solutions of the time-dependent Schrödinger equation (5) with periodic $f(t)$:

$$|\psi_F(t)\rangle = |u(t)\rangle e^{-i\frac{t}{\hbar}\mathcal{E}_F}. \tag{23}$$

These states form a complete and orthonormal set of eigenfunctions of the time evolution operator over a driving period:

$$|\psi_F(t+T)\rangle = \hat{U}(t+T,t)|\psi_F(t)\rangle = e^{-i\frac{T}{\hbar}\mathcal{E}_F}|\psi_F(t)\rangle.$$

Hence, any solution of the Schrödinger equation (5) can be written as a superposition of Floquet states as

$$|\chi(t)\rangle = \int A(k)|u(t)\rangle e^{-i\frac{t}{\hbar}\mathcal{E}_F}\,dk = \int A(k)|\psi_F(t)\rangle\,dk, \tag{24}$$

weighted with time-independent coefficients A, which depend on the momenta of the particle k. Looking at the last expression, notice that the Floquet states have occupation probabilities $|A|^2$ (preserved in time) and a phase factor $e^{-i\frac{t}{\hbar}\mathcal{E}_F}$, resembling the usual factor $e^{-i\frac{t}{\hbar}E}$ present in any time evolution of energy eigenstates with eigenvalues E when their Hamiltonian does not depend on time. Therefore the quasienergies look as if they were effective energies and these are the quantities which determine the linear phase evolution of the system. Finally, notice that if the system is prepared in a Floquet state, its time evolution is periodic in time and in this case it is called a "quasistationary evolution".

Before obtaining an expression for the micromotion operator $\hat{U}_F$ from Eq. (2), it is convenient first to derive an expression for the Floquet Hamiltonian $\hat{H}_F$ of the system which will be useful in the many-body case. To get an equation for $\hat{H}_F$ we need to rewrite Eq. (14) for $t = nT$ in a single exponential operator as in Eq. (20). To do this, we can use the Baker-Campbell-Hausdorff formula between momentum and position exponential operators, arriving at

$$\begin{aligned}\hat{H}_F = {}& \frac{\hat{p}^2}{2m} + \left[\frac{\xi(nT)}{nT} + \frac{1}{2m}\int_0^{nT} f(\tau)\,d\tau\right]\hat{p} - \hbar\frac{\theta(x,nT)}{nT} + \\ & - \frac{1}{2mnT}\left[\int_0^{nT} f(\tau)\,d\tau\right]\cdot\int_0^{nT} d\tau\left[\int_0^{\tau} f(\tau')\,d\tau'\right] \\ & + \frac{1}{12m}\left[\int_0^{nT} f(\tau)\,d\tau\right]^2.\end{aligned} \tag{25}$$

From this expression one is tempted to say that the translation of the center of mass of the wave packet at different stroboscopic times, would be $\frac{\xi(nT)}{nT} + \frac{1}{2m}\int_0^{nT} f(\tau)\,d\tau$, since this is the factor that multiplies the operator $\hat{p}$. However, this is not true since to evaluate $\langle\chi(x,nT)|\hat{x}|\chi(x,nT)\rangle$, one has to split the operators in the exponential recovering the state Eq. (14), where the translation factor is simply $\frac{\xi(nT)}{nT}$. Notice that one could consider periodic boundary conditions requiring that the periodic driving function satisfies the two conditions $f(nT) = 0$, for $n \in \mathbb{N}$, and $\int_0^T f(t)\,dt = 0$, as discussed in [29].

Moreover, it is not manifest from Eq. (25) that the Floquet Hamiltonian is independent of n, as it should be the case [9]. To clarify this issue we study in more detail the translational parameter and the gauge phase. From the first equation in (11), we derive

$$\xi(t) = -\frac{1}{m}\int_0^t d\tau\left[\int_0^{\tau} f(\tau')\,d\tau'\right], \tag{26}$$

from which follows that

$$\xi(t+T) = \xi(T) + \xi(t) - \frac{t}{m}\int_0^T f(\tau)\,d\tau. \tag{27}$$

In a similar way, one gets for the gauge phase:

$$\begin{aligned}\theta(x,t+T) = {}& \theta(x,T) + \theta(x,t) - \frac{t}{2m\hbar}\left[\int_0^T f(\tau)\,d\tau\right]^2 \\ & - \frac{1}{m\hbar}\left[\int_0^T f(\tau)\,d\tau\right]\int_0^t d\tau\left[\int_0^{\tau} f(\tau')\,d\tau'\right].\end{aligned} \tag{28}$$

Setting $t = nT$, with $n \in \mathbb{N}$, in the above equations yields

$$\xi(nT) = n\xi(T) - \frac{T}{m}\frac{n(n-1)}{2}\int_0^T f(\tau)\,d\tau, \tag{29}$$

and

$$\begin{aligned}\theta(x,nT) = {}& n\theta(x,T) - \frac{T}{2m\hbar}\frac{n(n-1)}{2}\left[\int_0^T f(\tau)\,d\tau\right]^2 \\ & - \frac{1}{m\hbar}\frac{n(n-1)(2n-1)}{6}\left[\int_0^T f(\tau)\,d\tau\right] \\ & \times\int_0^T d\tau\left[\int_0^{\tau} f(\tau')\,d\tau'\right].\end{aligned} \tag{30}$$

To continue with the proof of the n independence of the Floquet Hamiltonian, we split the analysis in two cases: (1) when the integral of the driving function over one period vanishes, and (2) when it does not.

1. $\int_0^T f(t)\,dt = 0$

When the integral on a time period is vanishing, from Eq. (29) we have $\xi(nT) = n\xi(T)$ and therefore the term linear in momentum of the Floquet Hamiltonian in (25) is n independent. Moreover, since $\xi(nT)$ is linear in terms of the stroboscopic factor n, the stroboscopic motion of the wave packet has a constant velocity, as can be inferred from Eq. (15). The constant term in the Floquet Hamiltonian is also trivially n independent since $\theta(x,nT) = n\theta(x,T)$, as follows from Eq. (30). Hence, in this case the Floquet Hamiltonian can be simply written as

$$\hat{H}_F = \frac{\hat{p}^2}{2m} + \frac{\xi(T)}{T}\hat{p} - \hbar\frac{\theta(T)}{T}, \tag{31}$$

where $\theta(x,T) \equiv \theta(T)$, since the gauge phase is x independent [in the considered case of $\int_0^T f(t)\,dt = 0$], as one can see from Eq. (12). Moreover, the Floquet Hamiltonian can be rewritten as

$$\begin{aligned}\hat{H}_F = {}& \frac{\hat{p}^2}{2m} - \frac{\hat{p}}{m}\frac{1}{T}\int_0^T d\tau\int_0^{\tau} f(\tau')\,d\tau' \\ & + \frac{1}{2m}\frac{1}{T}\int_0^T d\tau\left[\int_0^{\tau} f(\tau')\,d\tau'\right]^2.\end{aligned}$$

Notice that we can also express the Hamiltonian in Eq. (31) as

$$\hat{H}_F = \frac{[\hat{p} + m\xi(T)/T]^2}{2m} + C, \tag{32}$$

where $C = -\hbar\theta(T)/T - (m/2)[\xi(T)/T]^2$. Now, applying the unitary transformation,

$$\hat{\tilde{H}}_F \equiv e^{ia\hat{x}/\hbar}\hat{H}_F e^{-ia\hat{x}/\hbar}, \tag{33}$$

with $a = m\xi(T)/T$, we get finally

$$\hat{\tilde{H}}_F = \frac{\hat{p}^2}{2m} + C. \tag{34}$$

Using these results we can derive the micromotion operator $\hat{U}_F$. First of all, from Eq. (14), the time evolution operator is

$$\hat{U}(t,0) = e^{i\theta(x,t)} e^{-i\frac{t}{\hbar}\frac{\hat{p}^2}{2m}} e^{-i\frac{\xi(t)}{\hbar}\hat{p}}. \tag{35}$$

Hence, inverting Eq. (2) and knowing the Floquet Hamiltonian from Eq. (31), we get

$$\hat{U}_F(t,0) = e^{\frac{i}{\hbar}t\{[\frac{\xi(T)}{T} - \frac{\xi(t)}{t}]\hat{p} - \hbar[\frac{\theta(T)}{T} - \frac{\theta(x,t)}{t}] + \frac{1}{2}[\int_0^t f(\tau)\,d\tau][\frac{\xi(T)}{T} - \frac{\xi(t)}{t}]\}}, \tag{36}$$

where we used the Baker-Campbell-Hausdorff formula. An alternative expression of the micromotion operator is

$$\hat{U}_F(t,0) = e^{it[\frac{\theta(x,t)}{t} - \frac{\theta(T)}{T}]}\, e^{\frac{i}{\hbar}t[\frac{\xi(T)}{T} - \frac{\xi(t)}{t}]\hat{p}}, \tag{37}$$

which has been derived using the Zassenhaus formula.

Let's discuss a simple, yet instructive, application of these results. Imagine we are interested in describing the time evolution of a Gaussian wave packet with initial variance σ in the infinite homogeneous space, i.e. $\chi(x,0) = \frac{1}{\sqrt[4]{2\pi\sigma^2}} e^{-x^2/(2\sigma)^2}$. As we saw in the previous section, in order to determine its time evolution, we have first to find the eigenvalues and eigenfunctions of the Floquet Hamiltonian in (31). In this case the complete set of eigenfunctions is simply the plane wave set, and the associated quasienergies are then easy to determine:

$$\tilde{u}(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad \mathcal{E}_F = \frac{\hbar^2 k^2}{2m} + \frac{\xi(T)}{T}\hbar k - \hbar\frac{\theta(T)}{T}, \tag{38}$$

where k is the plane wave's momentum. The Floquet modes can be easily obtained from the action of $\hat{U}_F$ from Eq. (37) on the eigenstates $|\tilde{u}\rangle$:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} e^{ix[k - \frac{1}{\hbar}\int_0^t f(\tau)\,d\tau]} e^{-it\{\frac{1}{2m\hbar}[\int_0^t d\tau(\int_0^\tau f(\tau')\,d\tau')^2 - \frac{t}{T}\int_0^T d\tau(\int_0^\tau f(\tau')\,d\tau')^2] + k[\frac{\xi(t)}{t} - \frac{\xi(T)}{T}]\}}, \tag{39}$$

where we used Eq. (12). The Floquet modes are plane waves with a momentum that varies in time,

$$\langle u(t)|\hat{k}|u(t)\rangle = k - \frac{1}{\hbar}\int_0^t f(\tau)\,d\tau,$$

and which return to their initial value k at stroboscopic times. As required, the Floquet modes are time periodic with period T. The Floquet states are obtained from Eqs. (23) and (38),

$$\psi_F(x,t) = \frac{1}{\sqrt{2\pi}} e^{i[kx + \theta(x,t)] - it\frac{\hbar k^2}{2m} - ik\xi(t)}. \tag{40}$$

They are plane waves, periodic in time with period T, and their momentum expectation value varies in the same way as it does for the Floquet modes. One can now evaluate the time evolution of the Gaussian wave packet from Eq. (24). In order to do so, we compute the amplitude $A(k)$,

$$A(k) = \int_{-\infty}^{\infty} \chi(x,0)\psi_F^*(x,0) = \sqrt[4]{\frac{2\sigma^2}{\pi}} e^{-(k\sigma)^2},$$

and perform the Gaussian integration in Eq. (24), arriving at

$$\chi(x,t) = \frac{1}{\sqrt[4]{2\,\pi\,\sigma^2}} \frac{e^{i\,\theta(x,t)}}{\sqrt{1 + i\,\frac{\hbar t}{2m\sigma^2}}} e^{-\frac{[x - \xi(t)]^2}{4(\sigma^2 + i\frac{\hbar t}{2m})}}. \tag{41}$$

The wave packet has a Gaussian shape centered at $\xi(t)$ and spreads in time as

$$\Delta x(t) = \sqrt{\sigma^2 + \frac{\hbar^2 t^2}{4\,m^2\,\sigma^2}}, \tag{42}$$

in agreement with Eq. (17). The left side of Fig. 1 shows an example, where $f(t) = \ell\sin(\omega t)$. The center of mass of the wave packet is located at $\xi(t) = \frac{\ell}{m\omega^2}[\sin(\omega t) - \omega t]$, and it spreads according to Eq. (42). We use the parametrization $\ell = l\cdot\tilde{\ell}$ and $\omega = u\cdot\tilde{\omega}$, where $\tilde{\ell}$ and $\tilde{\omega}$ are dimensionless, and define $\tilde{t} = t/u$ and $\tilde{x} = x\sqrt[3]{\frac{ml}{\hbar^2}}$. In the left side of Fig. 1 we set $\tilde{\sigma} = \sigma\sqrt[3]{\frac{ml}{\hbar^2}} = 2^{-1/2}$, $\tilde{\ell} = 10$, and $\tilde{\omega} = 10$.

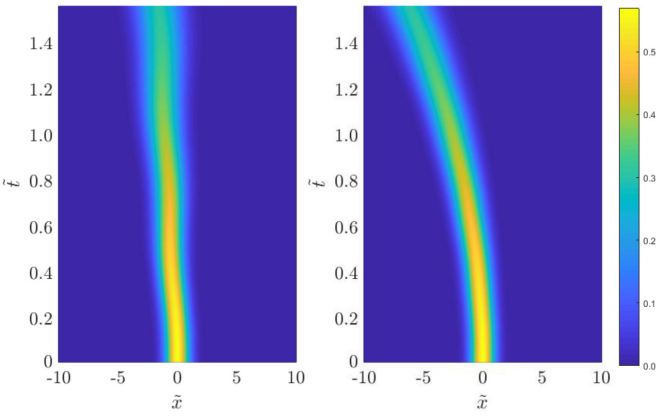

FIG. 1. Time evolution of density profiles of Gaussian wave packets $|\chi(x,t)|^2$ for a single particle in a potential: $xf(t)$. The left plot shows an evolution with a driving force $f(t) = \ell\sin(\omega t)$: The motion proceeds with a constant stroboscopic velocity towards the left. The right plot shows the evolution under a driving force $f(t) = \ell\sin^2(\omega t)$: The motion is uniformly accelerated to negative values of x. The figures are calculated via the split-step Fourier method and in both $\tilde{\sigma} = 2^{-1/2}$, $\tilde{\ell} = 10$, and $\tilde{\omega} = 10$.

2. $\int_0^T f(t)\,dt \neq 0$

In this case, the independence of the Floquet Hamiltonian (25) on n is more difficult to demonstrate. Let us define a function $F(t)$, such that $\frac{dF}{dt} = f(t)$. We have $\int_0^T f(t)\,dt = F(T) = c$, where c depends on the driving parameters and, by definition, $F(0) = 0$. It follows that $F(nT) = nF(T) = nc$. It is easy to prove that $F(t+T) = F(T) + F(t) = c + F(t)$. Therefore $F(nT + t) = nc + F(t)$ and $\xi(nT)$ can be written as

$$\xi(nT) = -\frac{1}{m}\int_0^{nT} F(t)\,dt = -\frac{n^2}{m} I, \tag{43}$$

where $I = \int_0^T F(t)\,dt$. Thus $\xi(nT)$ depends quadratically on the stroboscopic factor n, and the stroboscopic motion experiences a uniform acceleration $-\frac{1}{m}I$. Next, since $\xi(nT) \propto n^2$, one has $\xi(-T) = \xi(T)$ and, choosing $n = -1$ in Eq. (29), yields $\xi(T) = -\frac{T}{2m}\int_0^T f(t)\,dt$. This can be substituted back into Eq. (29) to obtain

$$\xi(nT) = -\frac{n^2 T}{2m}\int_0^T f(t)\,dt. \tag{44}$$

If we now take $t = nT$ in (26) and use (44), we derive the relevant equation,

$$\int_0^{nT} d\tau \int_0^{\tau} f(\tau')\,d\tau' = \frac{n^2 T}{2}\int_0^T f(t)\,dt,$$

that holds when the integral of the driving function over a driving period does not vanish. Using these results into (25), we can write

$$\hat{H}_F = \frac{\hat{p}^2}{2m} - \hbar\frac{\theta(x,T)}{T} - \frac{1}{6m}\left[\int_0^T f(\tau)\,d\tau\right]^2, \tag{45}$$

or, equivalently,

$$\hat{H}_F = \frac{\hat{p}^2}{2m} + x\frac{1}{T}\int_0^T f(\tau)\,d\tau + \frac{1}{2m}\frac{1}{T}\int_0^T d\tau\left[\int_0^{\tau} f(\tau')\,d\tau'\right]^2 - \frac{1}{6m}\left[\int_0^T f(\tau)\,d\tau\right]^2.$$

This expression is independent on n, a fact which completes the proof. Unlike the case where $\int_0^T f(t)\,dt = 0$, the Floquet Hamiltonian does not contain a term proportional to $\hat{p}$, but a static linear potential. This term forces the particle to move to the left (right) for positive (negative) values of $\int_0^T f(t)\,dt$. An example is given in Fig. 1 (right) where $\frac{1}{T}\int_0^T f(\tau)\,d\tau = \frac{\ell T}{2} > 0$, so that the wave packet moves with an acceleration of $-\frac{\ell T^2}{4m}$. However, its spread does not depend on the external driving force as predicted in Eq. (17).

The eigenfunctions of the Floquet Hamiltonian are the Airy function Ai [45] of the form,

$$\tilde{u}(x) = CAi\left\{\left(\frac{2mT^2}{\hbar^2\left[\int_0^T f(\tau)\,d\tau\right]^2}\right)^{1/3}\left(\frac{x}{T}\int_0^T f(\tau)\,d\tau - \mathcal{E}_F + \Omega\right)\right\}, \tag{46}$$

where C is a normalization constant and

$$\Omega = \frac{1}{2m}\frac{1}{T}\int_0^T d\tau\left[\int_0^{\tau} f(\tau')\,d\tau'\right]^2 - \frac{1}{6m}\left[\int_0^T f(\tau)\,d\tau\right]^2.$$

The Floquet Hamiltonian has a continuous spectrum spanning the whole range of energy values $\mathcal{E}_F$ from $-\infty$ to $+\infty$.

The micromotion operator is obtained inverting Eq. (2), and it leads to

$$\hat{U}_F(t,0) = e^{\frac{i}{\hbar}\{t\hbar[\frac{\theta(x,t)}{t} - \frac{\theta(x,T)}{T}] - \frac{t}{2m}\int_0^T f(\tau)\,d\tau\cdot[\frac{1}{3}(1+2\frac{t^2}{T^2})\int_0^T f(\tau)\,d\tau] - \xi(t)\frac{t}{T}\int_0^T f(\tau)\,d\tau\}}\, e^{-\frac{i}{\hbar}[\xi(t) + \frac{t^2}{2mT}\int_0^T f(\tau)\,d\tau]\hat{p}}. \tag{47}$$

This expression makes it complicated to determine the time evolution, even for a Gaussian wave packet, using Eq. (24). To circumvent this problem we perform the unitary transformation,

$$\chi(x,t) = \hat{U}_F(t,0)\tilde{\chi}(x,t),$$

where the transformed wave function satisfies [21]

$$i\hbar\frac{\partial\tilde{\chi}}{\partial t} = \hat{H}_F\tilde{\chi}(x,t).$$

Since $\hat{H}_F$ has a linear potential term, we can apply the same reasoning used to solve the original equation (5) for a constant driving function $\tilde{f} = \frac{1}{T}\int_0^T f(\tau)\,d\tau$, therefore we translate and gauge transform the wave function $\tilde{\chi}(x,t)$ in order to wash out the x-linear term in the Floquet Hamiltonian. By doing so, we finally get Eq. (14), which is thus the convenient way to obtain the time-evolved wave packet. In summary, we need first to calculate the free expansion of $\chi(x,0)$, then to translate the solution and finally to multiply it by the gauge phase.

The detailed analysis performed so far is valid for a single particle subjected to a linear potential which varies periodically in time. We shall show below that it can be extended straightforwardly to two or many particles interacting with a generic interacting potential $V_{2b}(x_j - x_i)$.

III. INTRODUCING INTERACTIONS: THE TWO-BODY PROBLEM

Let us now consider a one-dimensional system of two interacting particles subjected to a linear time-periodic potential.

The Schrödinger equation reads

$$i\hbar\frac{\partial\chi}{\partial t}=\sum_{j=1}^{2}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_j^2}+x_j f(t)\right]\chi+V_{2b}(x_2-x_1)\chi, \tag{48}$$

where $V_{2b}(x_2-x_1)$ is a generic potential between the two particles. To solve the Schrödinger equation (48), we can employ the same method discussed in the previous section: First we perform the gauge transformation,

$$\chi(x_1,x_2,t)=e^{i[\theta(x_1,t)+\theta(x_2,t)]}\eta(y_1(t),y_2(t),t), \tag{49}$$

where $y_j(t)=x_j-\xi(t)$, for $j=1,2$. The wave function $\eta(y_1,y_2,t)$ satisfies the Schrödinger equation for two interacting particles with no external potential:

$$i\hbar\frac{\partial\eta}{\partial t}=-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\right]\eta+V_{2b}(y_2-y_1)\,\eta, \tag{50}$$

while $\xi(t)$ and $\theta(x_j,t)$ obey Eqs. (26) and (12), once we use the same initial conditions of the previous case.

Notice that $V_{2b}(y_1-y_2)=V_{2b}(x_1-x_2)$, because $y_j(t)=x_j-\xi(t)$. Moreover, since $\xi(0)=0$, the two wave functions coincide at initial time: $\chi(x_1,x_2,0)=\eta(x_1,x_2,0)$, hence the solution of (48) can be written as

$$\chi(x_1,x_2,t)=e^{i\theta(x_1,t)+i\theta(x_2,t)e^{-i\frac{\xi(t)}{\hbar}(\hat{p}_1+\hat{p}_2)}e^{-i\frac{t}{\hbar}\left[\frac{\hat{p}_1^2+\hat{p}_2^2}{2m}+V_{2b}(x_2-x_1)\right]}}\times\chi(x_1,x_2,0). \tag{51}$$

With this expression, using the procedure discussed in the previous section, we can compute the expectation values of physical observables and their variances. More precisely, the expectation value of a single particle operator $\hat{O}_j$ is defined as

$$\langle\hat{O}_j\rangle(t)\equiv\langle\chi(x_1,x_2,t)|\hat{O}_j|\chi(x_1,x_2,t)\rangle=\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\,\chi^*(x_1,x_2,t)\,\hat{O}_j\,\chi(x_1,x_2,t), \tag{52}$$

and expectation values of position and momentum can be computed using the Baker-Campbell-Hausdorff formula.

We will show below that there is a decoupling between the linear potential term and the interacting one. This decoupling arises from the separation of the center of mass motion (which is determined by the external potential), and the relative motion (determined by the interacting potential). The diffusion of the wave packet evolves as it would be free from the linear time-dependent potential, but of course depends on the interaction.

The undriven Hamiltonian is given by

$$\hat{H}_0=\frac{\hat{p}_1^2+\hat{p}_2^2}{2m}+V_{2b}(x_2-x_1).$$

This implies that the total momentum of the system $\hat{P}=\hat{p}_1+\hat{p}_2$ is conserved, i.e. $[\hat{H}_0,\hat{P}]=0$. An example is the contact interaction $V_{2b}(x_2-x_1)=\lambda\,\delta(x_2-x_1)$, with λ the coupling strength. This property allows us to calculate the total energy of the state:

$$E(t)=\langle\hat{H}\rangle(t)=\langle\chi(x_1,x_2,t)|\left[\frac{\hat{p}_1^2+\hat{p}_2^2}{2m}+f(t)(x_1+x_2)+V_{2b}(x_2-x_1)\right]|\chi(x_1,x_2,t)\rangle. \tag{53}$$

After a lengthy calculation, using the canonical commutation relations and Eq. (51), we obtain for a generic driving function $f(t)$, including as well the nonperiodic cases:

$$E(t)=E(0)+\frac{1}{m}\left[\int_0^t f(\tau)d\tau\right]^2+\sum_{j=1}^{2}p_{0,j}\left[\frac{t}{m}f(t)-\frac{1}{m}\int_0^t f(\tau)d\tau\right]+-\frac{2f(t)}{m}\int_0^t d\tau\int_0^\tau f(\tau')d\tau'+\sum_{j=1}^{2}x_{0,j}[f(t)-f(0)], \tag{54}$$

where $E(0)$ is the initial energy of the state, containing all the interaction effects. The remaining terms arise from the linear driving potential and depend on the position $x_{0,j}$ and momenta $p_{0,j}$, of the jth particle at time $t=0$. If $f(t)$ is constant, as for a constant (gravitational or electric) force, then the energy is conserved. On the other hand, if $f(t)$ is periodic, its integral over a time period vanishes, and $f(t=0)=0$, then the energy is conserved at stroboscopic times.

Next we shall study the models where $f(t)$ is periodic. As done in the previous section, we shall consider two cases: $\int_0^T f(t)\,dt=0$ and $\int_0^T f(t)\,dt\neq0$. The evolution operator can be read from (51)

$$\hat{U}(t,0)=e^{i[\theta(x_1,t)+\theta(x_2,t)]}e^{-i\frac{\xi(t)}{\hbar}(\hat{p}_1+\hat{p}_2)}e^{-i\frac{t}{\hbar}\left[\frac{\hat{p}_1^2+\hat{p}_2^2}{2m}+V_{2b}(x_2-x_1)\right]}. \tag{55}$$

It is convenient to use the center of mass and relative coordinates: $x=x_2-x_1$ and $X=\frac{x_1+x_2}{2}$. In these variables the effects of the linear time-dependent potential and the interactions are completely decoupled. The time evolution in these coordinates reads

$$\hat{U}(t,0)=\hat{U}^{\rm com}(t,0)\hat{U}^{\rm rel}(t,0)=e^{-\frac{i}{\hbar}\{2X\int_0^t f(\tau)\,d\tau+\frac{1}{m}\int_0^t d\tau[\int_0^\tau f(\tau')\,d\tau']^2\}}e^{-i\frac{\xi(t)}{\hbar}\hat{P}}e^{-i\frac{t}{\hbar}\frac{\hat{p}^2}{4m}}\times e^{-i\frac{t}{\hbar}\left[\frac{\hat{p}^2}{m}+V_{2b}(x)\right]}, \tag{56}$$

where $\hat{P}$ is the total momentum, that commutes with the undriven Hamiltonian, and $\hat{p}=\hat{p}_2-\hat{p}_1$ is the relative momentum of the particles.

1. $\int_0^T f(t)\,dt = 0$

In this case one finds

$$\hat{H}_F = \sum_{j=1}^{2}\left[\frac{\hat{p}_j^2}{2\,m} + \frac{\xi(T)}{T}\hat{p}_j - \hbar\frac{\theta(T)}{T}\right] + V_{2b}(x_2 - x_1), \quad (57)$$

where $\theta(x_j, T) = \theta(T)$, as follows from Eq. (12).

From the analysis performed so far, and for the similarities with the one-body case, we know that the stroboscopic motion described by the Floquet Hamiltonian occurs with a constant velocity, since the translational parameter is $\xi(nT) \propto n$. Notice that if the Schrödinger equation with the original undriven Hamiltonian is solvable, then also the Floquet Hamiltonian associated with the motion under the action of a linear time-dependent potential is solvable, since it is described by the same two-body potential of the original problem with no driving, apart from a momentum shift. We observe that it is not convenient to solve the dynamics via Eq. (24) with respect to the eigenfunctions of the Floquet Hamiltonian in Eq. (57), while it is instead more advantageous to pass to relative and center of mass coordinates. Using the center of mass and relative coordinates the Floquet Floquet Hamiltonian decouples in two parts:

$$\hat{H}_F^{\rm com} = \frac{\hat{P}^2}{4m} + \frac{\xi(T)}{T}\hat{P} - 2\hbar\frac{\theta(T)}{T}, \quad (58)$$

and

$$\hat{H}_F^{\rm rel} = \frac{\hat{p}^2}{m} + V_{2b}(x). \quad (59)$$

The same factorization occurs for the micromotion operators, by defining

$$\hat{U}(t,0) = \hat{U}_F^{\rm com}(t,0)e^{-i\frac{t}{\hbar}\hat{H}_F^{\rm com}}\,\hat{U}_F^{\rm rel}(t,0)e^{-i\frac{t}{\hbar}\hat{H}_F^{\rm rel}}. \quad (60)$$

Using Eq. (56), the micromotion operator for the center of mass evolution has a form,

$$\hat{U}_F^{\rm com}(t,0) = e^{-it\left\{\frac{2X}{\hbar t}\int_0^t f(\tau)\,d\tau + \frac{1}{m\hbar t}\int_0^t d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2 + 2\frac{\theta(T)}{T}\right\}}, \times e^{i\frac{t}{\hbar}\left[\frac{\xi(T)}{T} - \frac{\xi(t)}{t}\right]\hat{P}}, \quad (61)$$

while the micromotion operator for the relative coordinate is instead trivial,

$$\hat{U}_F^{\rm rel}(t,0) = \hat{\mathbb{1}}. \quad (62)$$

The time evolution for the relative motion depends of course on the interacting potential $V_{2b}(x)$. Concerning the center of mass motion, we notice the similarity of Eq. (58) with the Floquet Hamiltonian (31) for a single particle, that allow us to use the results of the previous section. The eigenfunctions of the Floquet Hamiltonian (58) are plane waves with a continuous spectrum of quasienergies:

$$\tilde{u}^{\rm com}(X) = \frac{1}{\sqrt{2\pi}}e^{iKX}, \qquad \mathcal{E}_F^{\rm com} = \frac{\hbar^2K^2}{4m} + \frac{\xi(T)}{T}\hbar K - 2\hbar\frac{\theta(T)}{T}, \quad (63)$$

where K is the center-of-mass momentum. Next, we can get the Floquet modes by applying $\hat{U}_F^{\rm com}(t,0)$ onto $\tilde{u}^{\rm com}(X)$, obtaining

$$u^{\rm com}(X,t) = \frac{1}{\sqrt{2\pi}}e^{iX[K - \frac{2}{\hbar}\int_0^t f(\tau)\,d\tau]}e^{-it\{\frac{1}{m\hbar t}[\int_0^t d\tau(\int_0^\tau f(\tau')\,d\tau')^2 - \frac{2t}{T}\int_0^T d\tau(\int_0^\tau f(\tau')\,d\tau')^2] + K[\frac{\xi(t)}{t} - \frac{\xi(T)}{T}]\}}, \quad (64)$$

where we used Eq. (12). As in the one-body problem, the Floquet modes are plane waves with a momentum varying in time as

$$\langle u(t)|\hat{K}|u(t)\rangle = K - \frac{2}{\hbar}\int_0^t f(\tau)\,d\tau,$$

which implies that $\langle K\rangle(nT) = K$. We finally get the Floquet states from Eqs. (23) and (63),

$$\psi_F^{\rm com}(X,t) = \frac{1}{\sqrt{2\pi}}e^{i\{KX - \frac{2X}{\hbar t}\int_0^t f(\tau)\,d\tau - \frac{1}{m\hbar t}\int_0^t d\tau[\int_0^\tau f(\tau')\,d\tau']^2\} - it\frac{\hbar K^2}{4m} - iK\xi(t)}, \quad (65)$$

that are plane waves, periodic in time with period T, and whose average center-of-mass momentum behaves like that of the Floquet modes. Therefore the center-of-mass component of the wave function, the solution of (48), reads as

$$\phi(X,t) = \int A(K)\psi_F^{\rm com}(X,t)\,dK, \quad (66)$$

where we have written $\chi(x_1, x_2, t) = \phi(X,t)\varphi(x,t)$.

2. $\int_0^T f(t)\,dt \neq 0$

Using the methods presented in previous sections, we find

$$\hat{H}_F = \sum_{j=1}^{2}\left[\frac{\hat{p}_j^2}{2m} - \hbar\frac{\theta(x_j,T)}{T}\right] - \frac{1}{3m}\left[\int_0^T f(\tau)\,d\tau\right]^2 + V_{2b}(x_2 - x_1). \quad (67)$$

This expression contains a linear potential, hidden in the gauge phases $\theta(x_j, T)$. Analogously to the one-body example, the stroboscopic motion of the particles is uniformly accelerated:

$$\frac{d^2\langle x_j\rangle}{dt^2}(nT) = -\frac{1}{m}\int_0^T d\tau \int_0^\tau f(\tau')\,d\tau'.$$

Using the center of mass and relative coordinates, the Floquet Hamiltonian (67) splits in two parts:

$$\hat{H}_F^{\rm com} = \frac{\hat{P}^2}{4m} + \hat{X}\frac{1}{T}\int_0^T f(\tau)\,d\tau + \frac{1}{m}\frac{1}{T}\int_0^T d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2 - \frac{1}{3m}\left[\int_0^T f(\tau)\,d\tau\right]^2, \tag{68}$$

while the Floquet Hamiltonian of the relative motion is given by Eq. (59). The difference between the cases (1) and (2) stems only from the center-of-mass motion which has an additional linear dependence on $\hat{P}$ in the first case, and $\hat{X}$ in the second. The micromotion operator can be split as well, obtaining Eq. (62) for the relative part, and

$$\hat{U}_F^{\rm com}(t,0) = e^{\frac{i}{\hbar}\left\{t\left[X\left(\frac{1}{T}\int_0^T f(\tau)\,d\tau - \frac{1}{t}\int_0^t f(\tau)\,d\tau\right) + \frac{1}{mT}\int_0^T d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2 + \frac{1}{mt}\int_0^t d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2\right] + \right.}$$
$$\left. - \frac{t}{m}\int_0^T f(\tau)\,d\tau\cdot\left[\frac{1}{3}\left(1+2\frac{t^2}{T^2}\right)\int_0^T f(\tau)\,d\tau\right] + -2\xi(t)\frac{t}{T}\int_0^T f(\tau)\,d\tau\right\} e^{-\frac{i}{\hbar}\hat{P}\left[\xi(t)+\frac{t^2}{2mT}\int_0^T f(\tau)\,d\tau\right]}, \tag{69}$$

for the center of mass.

The dynamics of the relative part can be analyzed once the two-body potential is given, while the analysis performed on the center-of-mass part follows the same line of the one-body case. By this we mean that one has to perform a unitary transformation on the center-of-mass wave function: $\Phi(X,t) = \hat{U}_F\tilde{\Phi}(X,t)$, and therefore the new wave function $\tilde{\Phi}(X,t)$ satisfies a time-dependent Schrödinger equation with the Floquet Hamiltonian (68). Washing away the X-linear dependence of the Floquet Hamiltonian by means of a translation and a gauge transformation, for the center-of-mass part of Eq. (51) we have

$$\Phi(X,t) = e^{-\frac{i}{\hbar}\left\{2X\int_0^t f(\tau)\,d\tau + \frac{1}{m}\int_0^t d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2\right\}}$$
$$\times\, e^{-i\frac{\xi(t)}{\hbar}\hat{P}} e^{-i\frac{t}{\hbar}\frac{\hat{P}^2}{4m}}\,\Phi(X,0), \tag{70}$$

where Eq. (56) has been used.

As an example, we use the above results to study the time evolution of two particles with contact interactions initially prepared in a Gaussian wave packet.

A. Contact interactions

Let consider a contact potential: $V_{2b}(x_2 - x_1) = \lambda\delta(x_2 - x_1)$, where $\lambda > 0$ is the repulsive interaction parameter. At the initial time we prepare a Gaussian wave packet with variance σ,

$$\chi(x_1, x_2, 0) = \frac{1}{\sqrt{\pi\sigma^2}} e^{-\left(x_1^2+x_2^2\right)/2\sigma^2}, \tag{71}$$

that factorizes into the center of mass and relative parts,

$$\Phi(X,0) = \sqrt[4]{\frac{2}{\pi\sigma^2}}\, e^{-X^2/\sigma^2}, \quad \varphi(x,0) = \frac{1}{\sqrt[4]{2\pi\sigma^2}}\, e^{-x^2/4\sigma^2}. \tag{72}$$

Let us start with the case: $\int_0^T f(\tau)\,d\tau = 0$. Finding the time-independent coefficient $A(K)$ appearing in Eq. (66) at $t = 0$, and using (72), yields

$$\Phi(X,t) = \sqrt[4]{\frac{2}{\pi\sigma^2}}\,\frac{e^{i\theta(X,t)}}{\sqrt{1 + i\frac{\hbar t}{m\sigma^2}}}\, e^{-\frac{[X-\xi(t)]^2}{\sigma^2\left(1+i\frac{\hbar t}{m\sigma^2}\right)}}. \tag{73}$$

Concerning the relative motion, we use the propagator $G(x, x'; t, 0)$ in the presence of a Dirac δ potential [46,47],

$$\varphi(x,t) = \int_{-\infty}^{\infty} G(x, x'; t, 0)\,\varphi(x', 0)\,dx', \tag{74}$$

with

$$G(x, x'; t, 0) = \frac{1}{\sqrt{4\pi\, i\hbar t/m}}\, e^{i\frac{m(x-x')^2}{4\hbar t}} - \frac{m\lambda}{4\hbar^2}\, e^{\frac{m\lambda}{2\hbar^2}(|x|+|x'|)+i\frac{m\lambda^2 t}{4\hbar}}$$
$$\times\,\mathrm{erfc}\left(\frac{|x| + |x'| + i\frac{\lambda t}{\hbar}}{\sqrt{4\, i\hbar t/m}}\right), \tag{75}$$

with erfc being the complementary error function:

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt.$$

The numerical integration of (74), provides the wave function $\chi(x_1, x_2, t)$ for any value of $\lambda > 0$. In the limit of hard-core interactions, $\lambda \to \infty$, the integral (74) can be computed analytically and gives

$$\varphi(x,t) = \frac{1}{(2\pi)^{1/4}}\sqrt{\frac{im\sigma/\hbar t}{-1 + im\sigma^2/\hbar t}}$$
$$\times\,\mathrm{erf}\left(\frac{m\,\sigma\,x}{2\hbar t\sqrt{-1 + im\sigma^2/\hbar t}}\right) e^{-\frac{m}{4\hbar t}\frac{x^2}{i+m\sigma^2/\hbar t}}, \tag{76}$$

where $\mathrm{erf}(z) = 1 - \mathrm{erfc}(z)$. We have studied the time evolution of the density matrix,

$$\rho(x_1, t) = 2\int_{-\infty}^{\infty} |\chi(x_1, x_2, t)|^2\,dx_2, \tag{77}$$

in order to visualize the evolution of the wave packet. The density matrix (77) reads in the center of mass and relative wave functions, as

$$\rho(x,t) = 2\int_{-\infty}^{\infty}\left|\Phi\left(\frac{x_1}{2} + x, t\right)\right|^2 |\varphi(x_1, t)|^2\,dx_1. \tag{78}$$

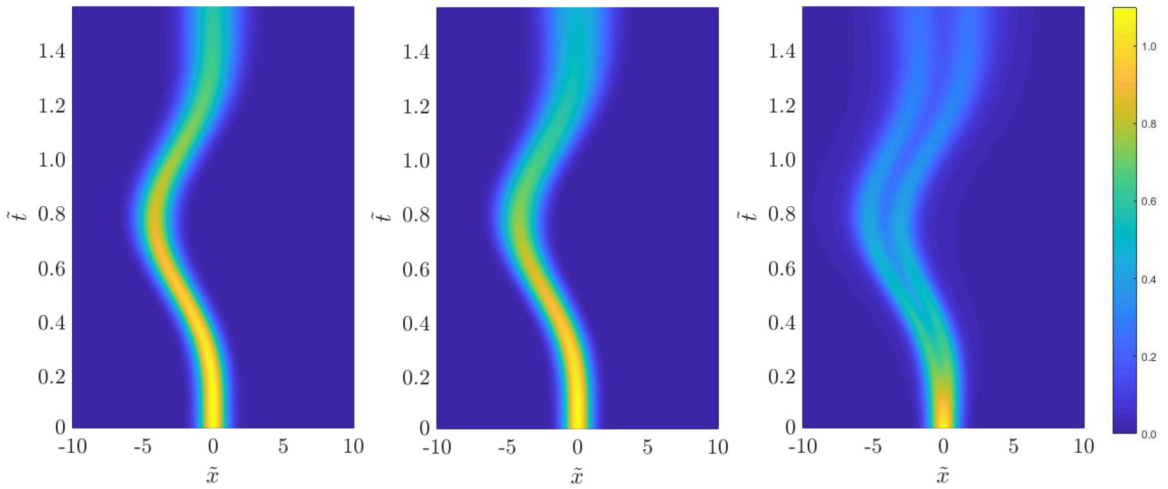

FIG. 2. Time evolution of density matrix profiles (77) for a Gaussian wave packet (71), under the action of a linear external potential: $xf(t)$, with driving function $f(t) = \ell[\cos^2(\omega t) - 1 + \frac{4}{3}\sin^4(\omega t)]$. The left side plot is the free case, $\tilde{\lambda} = 0$, the central plot has $\tilde{\lambda} = 1$, and the right side plot has $\tilde{\lambda} = \infty$. The center of mass moves with constant stroboscopic velocity, as predicted analytically, and the wave packet spreads over time as it would do for the undriven case $\ell = 0$. As one can see from the right side plot, for very large interactions, the wave packet rapidly tends to split in two specular parts. In all the figures the values $\tilde{\ell} = 200$, $\tilde{\omega} = 2$, and $\tilde{\sigma} = 1$ have been chosen.

The results are reported in Fig. 2 for different times and coupling strengths λ, using the driving function

$$f(t) = \ell\left[\cos^2(\omega t) - 1 + \frac{4}{3}\sin^4(\omega t)\right].$$

We choose the same dimensionless variables as in the one-body case: dimensionless coupling strength $\tilde{\lambda} = l\frac{m\lambda}{\hbar^2}$, $\tilde{\ell} = 200$, $\tilde{\omega} = 2$, and $\tilde{\sigma} = 1$. The values $\tilde{\lambda} = 0$, 1, and ∞, correspond to the left, center, and right sides of Fig. 2. Here

$$\xi(t) = \frac{\ell}{12m\omega^2}\sin^4(\omega t)$$

vanishes at stroboscopic times, as checked in the numerical simulations. We have also verified that the wave packet expands as it was not subjected to the linear oscillating potential, in agreement with the theoretical prediction.

Figure 2 shows that increasing the parameter λ, the variance of the wave packet increases in time more rapidly. We have been able to fit this behavior with the approximation:

$$\Delta x_j(t) \approx \frac{\sigma}{\sqrt{2}}\sqrt{1 + \left(\frac{\hbar t}{m\sigma^2}\right)^2\left(1 + \mathcal{B}\frac{m\lambda\sigma}{2\hbar^2}\right)}, \qquad (79)$$

where $\mathcal{B} \approx 1.23$. For $\lambda = 0$ one retrieves an expression similar to Eq. (42), while in the limit $\lambda \to \infty$, Eq. (79) diverges for all t because the tail of the density matrix decays as $\propto x^2$, even starting from a Gaussian.

As an additional check, we have calculated numerically the total energy of a two-particle system driven with $f(t) = \ell\sin^3(\omega t)$, separating its center of mass and relative components. The analytical value can be obtained from Eq. (54), and is represented by the solid, dashed, and dotted lines in Fig. 3. The circular dots represent the values calculated numerically. We have used $\tilde{\ell} = 200$, $\tilde{\omega} = 60$, $\tilde{\sigma} = 2^{-1/2}$, and $p_{0,j} = x_{0,j} = 0$ for $j = 1, 2$. The interaction strengths, $\tilde{\lambda} =$ 0.1, 1, and 10, only displace the curves since their effects are encoded in the initial energy factor $E(0)$ of Eq. (54), as can be seen from the inset of the plot. For this driving function we have $f(nT) = \int_0^T f(\tau)\,d\tau = 0$, therefore from Eq. (54) the energies at the stroboscopic times are equal to the initial energy, i.e. $E(nT) = E(0)$ for every n, and there is no heating of the system, in agreement with theoretical results [7,8] and experimental findings [48].

In the case where $\int_0^T f(\tau)\,d\tau \neq 0$, we used Eq. (70) for the center-of-mass initial wave function of Eq. (72), obtaining the same result as when $\int_0^T f(\tau)\,d\tau = 0$, i.e. we retrieved Eq. (73). For the relative motion we have applied the same reasoning as before, by which we know that the relative

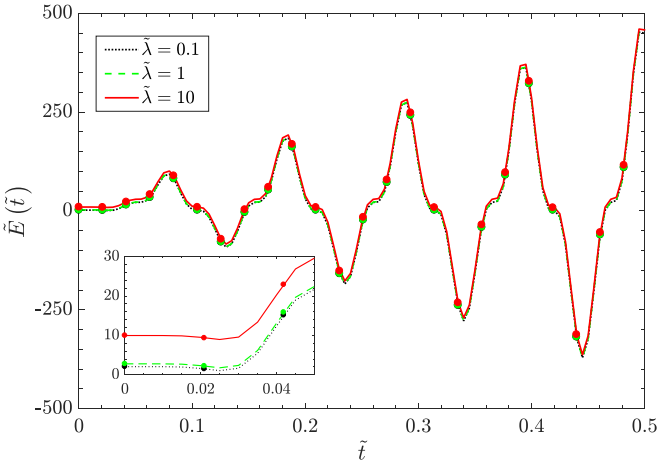

FIG. 3. Time evolution of the energy $\tilde{E} = \sqrt[3]{\frac{m}{\hbar^2 l}}E$ for two interacting particles subjected to a linear external potential: $xf(t)$, with driving function $f(t) = \ell\sin^3(\omega t)$. The system is prepared in the Gaussian wave packet state (71). The curves represent different values of the parameter $\tilde{\lambda}$, which only shifts the total energy, as shown in the inset for short times $\tilde{t}$ and different coupling strengths. The circular dots represent the energy values calculated from the numerical computation.

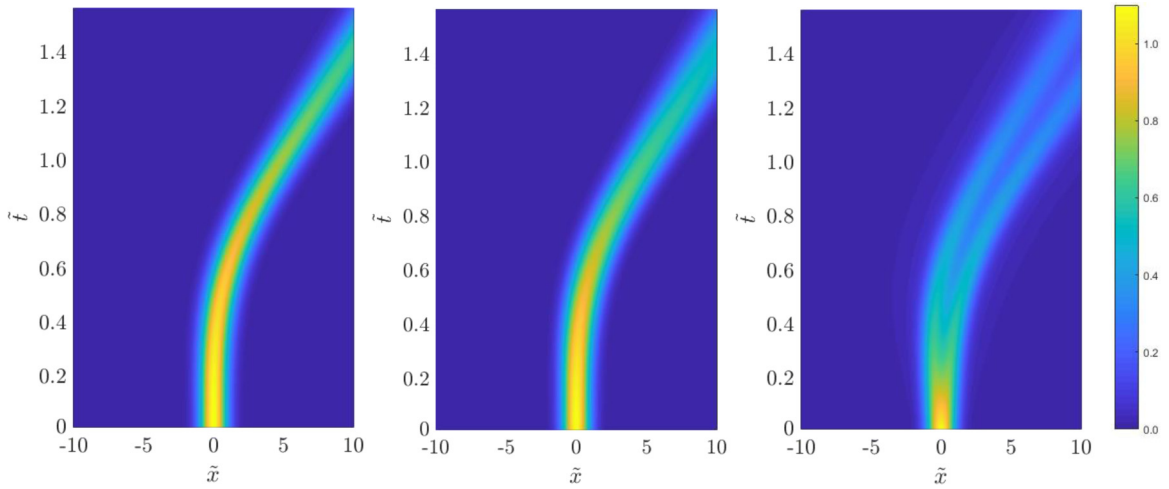

FIG. 4. Evolution of density matrix (77) for a Gaussian wave packet (71) under the action of a linear external potential: $xf(t)$, where the driving function is $f(t) = \ell[\cos(\omega t) - 1]$. Notice that the center of mass motion is uniformly accelerated to the right, as predicted analytically, and the wave packet spreads over time as it would do for the undriven case. From left to right panels one has $\tilde{\lambda} = 0, 1, \infty$; moreover, $\tilde{\ell} = 10$, $\tilde{\omega} = 5$, and $\tilde{\sigma} = 1$.

part of the wave packet evolves according to Eq. (74). We have performed a numerical simulation of a system made of two δ-interacting particles under the action of a linear potential with driving function: $f(t) = \ell[\cos(\omega t) - 1]$. The results for different interaction strengths λ are reported in Fig. 4, where the density matrix calculation (78) is plotted, in correspondence with $\tilde{\ell} = 10$, $\tilde{\omega} = 5$, and $\tilde{\sigma} = 1$. In this case the motion is uniformly accelerated to the right side of the x axis; indeed the translational parameter reads $\xi(t) = \frac{\ell}{2m\omega^2}[\omega^2 t^2 - 2 + 2\cos(\omega t)]$. This has to be compared with the case $\int_0^T f(t)\,dt = 0$, where the center of mass does not accelerate.

Concerning the spreading of the wave packet, it is the same as in the case without a driving potential and it also satisfies Eq. (79) with $\mathcal{B} \approx 1.23$. In conclusion, there is no difference for the wave packet spreading between the results of a driving function whose integral over a period vanishes or not.

IV. MANY-BODY PROBLEM

The analysis done so far can be generalized to many-body systems with N interacting particles, a generic interacting potential $V_{2b}(x_j - x_i)$, and under the action of an external linear time-dependent potential. The Schrödinger equation reads

$$i\hbar\frac{\partial\chi}{\partial t} = \sum_{j=1}^{N}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_j^2} + x_j f(t)\right]\chi + \sum_{j>i} V_{2b}(x_j - x_i)\chi. \tag{80}$$

Performing the translation and a gauge transformation,

$$\chi(x_1,\ldots,x_N,t) \equiv \prod_{j=1}^{N} e^{i\theta(x_j,t)}\eta(y_1,\ldots,y_N,t), \tag{81}$$

the wave function $\eta(y_1,\ldots,y_N,t)$ satisfies the Schrödinger equation without the external driving, i.e.

$$i\hbar\frac{\partial\eta}{\partial t} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2\eta}{\partial y_j^2} + \sum_{j>i} V_{2b}(y_j - y_i)\eta, \tag{82}$$

where $y_j(t) = x_j - \xi(t)$, $\forall j$, therefore the interacting potential is invariant under these transformations: $V_{2b}(y_j - y_i) = V_{2b}(x_j - x_i)$.

Using the initial conditions $\xi(0) = 0$ and $\theta(x_j, 0) = 0$, $\forall j$, the parameter $\xi(t)$ and the gauge phase $\theta(x_j, t)$ satisfy Eqs. (26) and (12). Hence, the two wave functions coincide at initial time $t = 0$.

The complete solution of the Schrödinger equation (80) can be formally written as

$$\chi(x_1,\ldots,x_N,t) = \prod_{j=1}^{N}\left[e^{i\theta(x_j,t)}e^{-i\frac{\xi(t)}{\hbar}\hat{p}_j}\right]e^{-i\frac{t}{\hbar}\hat{H}_0}\eta(x_1,\ldots,x_N,0), \tag{83}$$

where the undriven Hamiltonian of one-dimensional many-particle systems has the general form,

$$\hat{H}_0 = \sum_{j=1}^{N}\frac{\hat{p}_j^2}{2m} + \sum_{j>i} V_{2b}(x_j - x_i). \tag{84}$$

In (83) the momentum operator $\hat{p}_j$ is the generator of the translation for the jth particle, and η is the solution of the Schrödinger equation with no linear driving.

The generalization of the two-body results for the expectation values of physical observables is straightforward. First, we can compute the total energy of the system evaluating the expectation value of the driven Hamiltonian. In the calculation we use the conservation of the total momentum $\hat{P} = \sum_{j=1}^{N}\hat{p}_j$ for the undriven Hamiltonian $\hat{H}_0$, i.e. $[\hat{H}_0, \hat{P}] = 0$, valid in the considered case in which the interaction V_{2b} depends on the relative distance between the particles (see more comments in Sec. IV A). Using the commutation relations we find for a

general (also nonperiodic) driving function $f(t)$:

$$E(t) = E(0) + \frac{N}{2m}\left[\int_0^t f(\tau)d\tau\right]^2 + \sum_{j=1}^{N} p_{0,j}\left[\frac{t}{m}f(t) - \frac{1}{m}\int_0^t f(\tau)d\tau\right] + - \frac{Nf(t)}{m}\int_0^t d\tau \int_0^\tau f(\tau')d\tau' + \sum_{j=1}^{N} x_{0,j}[f(t) - f(0)], \tag{85}$$

which generalizes Eq. (54). As for the two-body case, if $f(t)$ is periodic in time and its integral over a time period vanishes, then the energy is conserved at stroboscopic times if $f(t=0) = 0$. Once again, there is a decoupling between the interactions and the external linear driving potential, since the effect of the interactions among particles is encoded in the initial value of the energy $E(0)$, while the remaining terms collect the effect of the external potential.

Thanks to the simple rewriting of the many-body wave function in Eq. (83), we are also able to write the one-body density matrix of the driven system in terms of the undriven one. The one-body density matrix is defined as [49]

$$\rho(x, x', t) = N \int dx_2 \dots dx_N \, \chi^*(x, x_2, \dots, x_N, t) \times \chi(x', x_2, \dots, x_N, t). \tag{86}$$

Therefore using Eq. (83) we can rewrite the density matrix as

$$\rho(x, x', t) = N\, e^{i[\theta(x',t)-\theta(x,t)]} \int dy_2 \dots dy_N \, \eta^*(y, y_2, \dots, y_N, t) \times \eta(y', y_2, \dots, y_N, t), \tag{87}$$

since $dx_j = dy_j$ for every j, while $y(t) = x - \xi(t)$, $y'(t) = x' - \xi(t)$. So, finally,

$$\rho(x, x', t) = e^{i[\theta(x',t)-\theta(x,t)]}\, \rho_{\text{undriven}}(y, y', t), \tag{88}$$

where $\rho_{\text{undriven}}(y, y', t)$ is defined in terms of the wave function η solution of the Schrödinger equation without the driving term.

For a translational invariant system, the above equation may be further simplified by writing everything in terms of the relative coordinate $r \equiv x - x'$. In this case, since it is also true that $r = y - y'$, then Eq. (88) may be rewritten as

$$\rho(r, t) = e^{i\frac{r}{\hbar}\int_0^t f(\tau)\,d\tau}\, \rho_{\text{undriven}}(r, t). \tag{89}$$

We may further analyze the eigenvalues of the one-body density matrix for a translational invariant system. In the equilibrium, the one-body density matrix satisfies the eigenvalue equation [49]

$$\int \rho(x, x')\, \phi_i(x)\, dx = \lambda_i\, \phi_i(x'), \tag{90}$$

where λ_i is the occupation number of the ith natural orbital eigenvector $\phi_i(x)$. The λ_i are such that $\sum_i \lambda_i = N$. For the dynamics, when the Galilean invariance is not broken (correspondingly requiring appropriate initial conditions), the quantum number labeling the occupation of the natural orbitals is the wave number k, and the natural orbitals at time t are simply plane waves. Therefore we may write Eq. (90) for a driven translational invariant many-body system as

$$\lambda_k(t) = \int \rho(r, t)\, e^{ikr}\, dr. \tag{91}$$

Now, thanks to Eq. (89), we can write the following relation between the natural orbitals occupation numbers of the driven system with those of the undriven one:

$$\lambda_k(t) = \lambda_{\tilde{k}}^{\text{undriven}}(t), \tag{92}$$

where $\tilde{k}(t) = k + \frac{1}{\hbar}\int_0^t f(\tau)\, d\tau$, and we have defined the occupation numbers of the system without driving as

$$\lambda_k^{\text{undriven}}(t) = \int \rho_{\text{undriven}}(r, t)\, e^{ikr}\, dr. \tag{93}$$

From the above relations, one may observe that there is only a time-dependent translation over the momentum wave number which identifies the occupation numbers of the driven system with respect to the undriven case.

Let us now focus on periodic driving functions. As before, we discuss separately the cases when $\int_0^T f(\tau)\, d\tau = 0$ and $\neq 0$. In the first case, the gauge phase at stroboscopic times is independent on the position variables, while the parameter ξ is linear in the stroboscopic factor n, indicating a stroboscopic motion with constant velocity. Using the fact that $[\hat{H}_0, \hat{P}] = 0$ and the Baker-Campbell-Hausdorff formula on Eq. (83) evaluated at $t = nT$, we find the Floquet Hamiltonian

$$\hat{H}_F = \sum_{j=1}^{N}\left[\frac{\hat{p}_j^2}{2m} + \frac{\xi(T)}{T}\hat{p}_j - \hbar\frac{\theta(T)}{T}\right] + \sum_{j<i} V_{2b}(x_j - x_i). \tag{94}$$

Hence, if the undriven Hamiltonian describes an integrable model, also the Floquet Hamiltonian is exactly solvable since it has the same two-body interaction potential among particles and presents only a shift in the momenta. For the micromotion operator one finds

$$\hat{U}_F(t, 0) = e^{it\sum_{j=1}^{N}\left[\frac{\theta(x_j,t)}{t} - \frac{\theta(T)}{T}\right]} e^{i\frac{t}{\hbar}\left[\frac{\xi(T)}{T} - \frac{\xi(t)}{t}\right]\sum_{j=1}^{N}\hat{p}_j}. \tag{95}$$

If $f(t)$ has a nonvanishing integral over a driving period, then the Floquet Hamiltonian reads

$$\hat{H}_F = \sum_{j=1}^{N}\left[\frac{\hat{p}_j^2}{2m} - \hbar\frac{\theta(x_j, T)}{T}\right] - \frac{N}{6m}\left[\int_0^T f(\tau)\, d\tau\right]^2 + \sum_{j<i} V_{2b}(x_j - x_i), \tag{96}$$

which presents a time-independent x-linear potential term acting on all the particles. In this case, as we saw for the one-body problem, the system is governed by a stroboscopic dynamics with a uniform acceleration, since the translational parameter depends quadratically on the stroboscopic factor: $\xi(nT) \propto n^2$.

The micromotion operator reads

$$\hat{U}_F(t,0) = e^{\frac{i}{\hbar}\{t\hbar\sum_{j=1}^{N}[\frac{\theta(x_j,t)}{t}-\frac{\theta(x_j,T)}{T}]-N\frac{t}{2m}\int_0^T f(\tau)\,d\tau[\frac{1}{3}(1+2\frac{t^2}{T^2})\int_0^T f(\tau)\,d\tau]-N\xi(t)\frac{t}{T}\int_0^T f(\tau)\,d\tau\}} e^{-\frac{i}{\hbar}[\xi(t)+\frac{t^2}{2mT}\int_0^T f(\tau)\,d\tau]\sum_{j=1}^{N}\hat{p}_j}. \quad (97)$$

A. Comments

We pause here to comment on the generality of our findings. The main results in the case $\int_0^T f(\tau)\,d\tau = 0$ are Eqs. (94) and (95). They are valid for any form of the two-body potential V_{2b} and therefore for any interacting Hamiltonian (84), integrable or not. The crucial assumption we have made is that the two-body potential V_{2b} depends only on the relative distance $x_i - x_j$, otherwise $V_{2b}(x_i, x_j)$ would be in general different from $V_{2b}(y_i, y_j)$ when the transformation $y_j = x_j - \xi(t)$ is done. Since $V_{2b}(x_j - x_i) = V_{2b}(y_j - y_i)$ then the equations of motions for the wave function $\eta(y_1, \ldots, y_N, t)$ are exactly the same of those for the wave function $\chi(x_1, \ldots, x_N, t)$, except for the fact that the time-periodic linear potential has been removed. Notice that in presence of one-body potentials $V_{1b}(x_i)$, breaking translational invariance, this fact would be no longer valid. When the interacting many-body Hamiltonian has only the kinetic term plus a time-independent two-body potential V_{2b} depending only on the relative distance between the particles, then the conservation of the total momentum of the undriven Hamiltonian $\hat{H}_0$ is guaranteed:

$$[\hat{H}_0, \hat{P}] = 0,$$

a relation we subsequently used to determine the Floquet Hamiltonian, the micromotion operator, and the expression of the energy at time t.

We conclude that if, in addition, $\hat{H}_0$ turns out to be integrable, then the associated Floquet Hamiltonian is integrable too. We have presented the analysis for many-body systems made of bosons, but it could equally be applied to many-body systems made of fermions or Bose-Fermi mixtures. In few words, our results are valid for any one-dimensional integrable Hamiltonian in the *continuum*. This also includes the Gaudin-Yang model for one-dimensional Fermi gases, integrable Bose-Fermi mixtures, integrable multicomponent Lieb–Liniger Bose gases, and Calogero-Sutherland models (in the absence of external one-body harmonic potential) [35,36,50,51].

Hence, having in mind the broad generality of our results, we shall present below a study of the paradigmatic Lieb-Liniger model driven by an external linear time-dependent potential whose driving function has a vanishing integral over a driving period.

It is therefore clear that since the Gaudin-Yang model for a one-dimensional Fermi gas is described by the same Hamiltonian of the Lieb-Liniger model, but with the only difference of having attractive interactions (i.e., the sign of the interaction parameter will change), the method presented will apply also in that case. Finally when one considers Bose-Fermi mixtures or multicomponent Lieb-Liniger Bose gases, the Hamiltonian has more terms, for intra- and interspecies interactions, but each of them separately satisfies the required conditions of having two-body potentials which depend only on the relative distances among particles and our method can be applied as well. Let's take as an example the case of a mixture of two distinguishable bosonic species labeled with A and B. The Hamiltonian of the system reads [52]

$$H = \sum_{\sigma=A,B} H_\sigma + H_{AB},$$

where the single-species Hamiltonian is

$$H_\sigma = \sum_{j=1}^{N_\sigma}\left[-\frac{\hbar^2}{2\,m_\sigma}\frac{\partial^2}{\partial x_{\sigma,j}^2} + x_{\sigma,j}\,f(t)\right] + \lambda_\sigma \sum_{j>i}\delta(x_{\sigma,j} - x_{\sigma,i}),$$

while the interspecies Hamiltonian reads

$$H_{AB} = \lambda_{AB}\sum_{a=1}^{N_A}\sum_{b=1}^{N_B}\delta(x_{A,a} - x_{B,b}),$$

where N_A and N_B are the number of particles of species A and B, respectively. Therefore, when the driving function $f(t)$ is the same for both the species, the method which consists of making a gauge transformation of the wave function and a translation will allow one to eliminate the external linear potential separately in H_A and H_B, while the interspecies Hamiltonian will not be affected at all. In the more specific case when the undriven mixture is integrable, which happens when the masses of bosons and fermions are the same and when they have equal repulsive interactions between Bose-Fermi and Bose-Bose particles [53], then also the Floquet Hamiltonian will be integrable for the same reasons that we have seen for a generic interacting potential $V_{2b}(x_i, x_j)$.

B. Driven Lieb-Liniger gas

The Lieb-Liniger model describes a gas of N bosons with δ-contact repulsive interactions in one dimension [30], that is, $V_{2b}(x_j - x_i) = \lambda\delta(x_j - x_i)$, with $\lambda > 0$ the interaction parameter. The dynamics of the Lieb-Liniger model in a linear potential was studied in [54], while we refer to [55–57] for a study of the classical counterpart of the Lieb-Liniger model, the nonlinear Schrödinger equation, in the presence of a time-dependent linear potential. The Floquet analysis of the Lieb-Liniger model with a periodic tilting was studied in [29], where it was discussed the stroboscopic evolution written in terms of the eigenfunctions of the Floquet Hamiltonian in Eq. (94). Here we make a further step forward, giving a procedure for getting an expression for the time evolution of a generic wave packet.

The undriven Hamiltonian of this system, i.e.

$$\hat{H}_0 = \sum_{j=1}^{N}\frac{\hat{p}_j^2}{2m} + \lambda\sum_{j<i}\delta(x_j - x_i), \quad (98)$$

is an integrable Hamiltonian and an exact expression of its eigenfunction can be obtained using the Bethe ansatz technique [34,35]. Therefore we can write the eigenfunctions for

the Floquet Hamiltonian (94) as Bethe ansatz states

$$\tilde{u}(x_1,\ldots,x_N)=\sum_P A_P(Q)\,e^{\frac{i}{\hbar}\sum_{j=1}^N k_{P_j}x_j}, \tag{99}$$

where Q is the permutation index which specifies the order of the particles, while P is the permutation index of the pseudorapidities k_j, which are undetermined until boundary conditions are chosen [35,36] (we refer to [29] for a discussion on the relation between the boundary conditions and the external linear potential). The amplitudes $A_P(Q)$ can be written as

$$A_P=\mathcal{N}(-1)^P\prod_{j<l}\left(k_{P_j}-k_{P_l}+i\frac{m\lambda}{\hbar^2}\right),$$

where $\mathcal{N}$ represents the normalization factor. The respective quasienergies are given by

$$\mathcal{E}_F=\frac{\hbar^2}{2m}\sum_{j=1}^N k_j^2+\hbar\frac{\xi(T)}{T}\sum_{j=1}^N k_j-N\hbar\frac{\theta(T)}{T}. \tag{100}$$

For convenience, we will indicate the state $\tilde{u}$ as $\mathrm{BAS}(k_1,\ldots,k_N)$, where BAS stands for *Bethe Ansatz State*. In order to understand what happens for the N-body case, it is convenient to start from the two-body problem. In this case we can write [58]

$$\mathrm{BAS}(k_1,k_2)=g(x_1,x_2)\theta_H(x_2-x_1)+g(x_2,x_1)\theta_H(x_1-x_2), \tag{101}$$

where $\theta_H(x)$ is the Heaviside step function, while

$$g(x_1,x_2)=\left[i(k_1-k_2)-\frac{m\lambda}{\hbar^2}\right]e^{i(k_1x_1+k_2x_2)}+\left[i(k_1-k_2)+\frac{m\lambda}{\hbar^2}\right]e^{i(k_2x_1+k_1x_2)}.$$

Hence, $g(x_1+a,x_2+a)=g(x_1,x_2)\,e^{ia(k_1+k_2)}$ for generic a, and the action of the micromotion operator (95) on the BAS will give the following Floquet modes:

$$u(t)=\mathrm{BAS}\left(k_1-\frac{1}{\hbar}\int_0^t f(\tau)\,d\tau,k_2-\frac{1}{\hbar}\int_0^t f(\tau)\,d\tau\right)\times e^{i(k_1+k_2)t\left[\frac{\xi(T)}{T}-\frac{\xi(t)}{t}\right]}e^{-i\left\{\frac{1}{m\hbar}\int_0^t d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2+2\frac{t}{T}\theta(T)\right\}}. \tag{102}$$

Apart from a phase, the Floquet modes are then Bethe ansatz states with shifted pseudomomenta. The Floquet states from Eqs. (23) and (100) read

$$\psi_F(t)=\mathrm{BAS}\left(k_1-\frac{1}{\hbar}\int_0^t f(\tau)\,d\tau,k_2-\frac{1}{\hbar}\int_0^t f(\tau)\,d\tau\right)\times e^{-i(k_1+k_2)\xi(t)}\,e^{-\frac{i}{m\hbar}\int_0^t d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2}, \tag{103}$$

and the total momentum expectation value of the Floquet states is therefore $\langle\hat{P}\rangle_F(t)=\hbar(k_1+k_2)-\frac{2}{\hbar}\int_0^t f(\tau)\,d\tau$. These results may be easily extended to the many-body case. The Floquet modes can be written as

$$u(t)=\mathrm{BAS}\left(k_1-\frac{1}{\hbar}\int_0^t f(\tau)\,d\tau,\ldots,k_N-\frac{1}{\hbar}\int_0^t f(\tau)\,d\tau\right)\times e^{it\left[\frac{\xi(T)}{T}-\frac{\xi(t)}{t}\right]\sum_{j=1}^N k_j}\times e^{-i\left\{\frac{N}{2m\hbar}\int_0^t d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2+N\frac{t}{T}\theta(T)\right\}}, \tag{104}$$

while the Floquet states read

$$\psi_F(t)=\mathrm{BAS}\left(k_1-\frac{1}{\hbar}\int_0^t f(\tau)\,d\tau,\ldots,k_N-\frac{1}{\hbar}\int_0^t f(\tau)\,d\tau\right)\times e^{-i\xi(t)\sum_{j=1}^N k_j}e^{-i\frac{N}{m\hbar}\int_0^t d\tau\left[\int_0^\tau f(\tau')\,d\tau'\right]^2}. \tag{105}$$

The total momentum of the Floquet states is then

$$\langle\hat{P}\rangle_F(t)=\hbar\sum_{j=1}^N k_j-\frac{N}{\hbar}\int_0^t f(\tau)\,d\tau. \tag{106}$$

In particular one can calculate the time evolution of a generic wave packet for this system as

$$\chi(x_1,\ldots,x_N,t)=\int A(k_1,\ldots,k_N)\,\psi_F(t)\,d^Nk, \tag{107}$$

which is an extension of the one-body equation (24).

It is worth stressing that this is a nontrivial expansion to evaluate: Indeed, once the initial wave packet has been chosen at $t=0$, one needs to evaluate the time-independent amplitudes $A(k_1,\ldots,k_N)$ inverting the integral by multiplying by $\psi_F^*(t)$, and then evaluate the N-dimensional integral on the right-hand side.

V. CONCLUSIONS

In this paper we have studied the effect of a time-dependent linear external potential on one-dimensional quantum systems made of one, two, and many particles. The potential could physically represent a time varying gravitational linear force, or a time varying electric field acting on the system, therefore its analysis is interesting in many different contexts. The key point of our approach has been to solve the problem for a generic driving function by applying a gauge transformation on the wave function and a translation over the position variables. Doing so, we have been able to compute expectation values for different observables such as the center-of-mass position of a wave packet and its variance, and the way these observables depend on time. We have observed that the external driving does not affect the spread of a wave packet, which depends instead only on the interaction effects. This is the result of the decoupling of the external potential which takes place already from the two-particle case, due to the linearity of the potential. This decoupling acts at the level of the center of mass and relative coordinates and can be observed also in the behavior of the total energy of the system, which oscillates in time depending on the form of the driving function $f(t)$. We derived expressions for the energy of the state at any time also for nonperiodic driving function. The system in general does not conserve the energy, apart from some specific cases, e.g. if $f(t)$ is constant in time. However, when f is periodic in time and its integral on a time period vanishes, plus $f(t=0)=0$, then the energy at stroboscopic times is conserved (notice

that, at stroboscopic times, the expectation value of the full Hamiltonian does not need to be equal to the expectation value of the Floquet Hamitonian). When f is periodic, but its integral on a time period is nonvanishing, then the energy at stroboscopic times is in general not conserved.

For a periodic driving, we have analyzed in detail the dynamics of the systems. In this case we have employed the Floquet approach and written down the Floquet Hamiltonian and the micromotion operator, describing the time evolution of the system at stroboscopic times and generic intermediate times, respectively. Our results, as discussed in Sec. IV A, are valid when the two-body interaction terms depend only on the relative distance between the particles so that the total momentum commutes with the undriven Hamiltonian. If the undriven Hamiltonian is integrable, and obeys such conditions, then, when $\int_0^T f(\tau)\,d\tau = 0$, the Floquet Hamiltonian is integrable, too. Therefore, our results are valid for any one-dimensional integrable Hamiltonian on the continuum including the Gaudin-Yang model for one-dimensional Fermi gases, integrable Bose-Fermi mixtures, integrable multicomponent Lieb-Liniger Bose gases, and Calogero-Sutherland models (in the absence of external one-body harmonic potential). It would be of interest to study the integrablity of the Floquet Hamiltonian and the micromotion operator for undriven integrable lattice Hamiltonians subjected to time-periodic linear potentials (or magnetic fields) suitably extending the method presented here.

If the integral of the driving function on a period of oscillation is, on the contrary, nonvanishing, then the Floquet Hamiltonian can be shown to be time independent and it contains a linear, constant in time, external potential. In this case, such a term can be eliminated using the same recipe of a gauge transformation and a translation over the position variables. The study whether such Floquet Hamiltonians are in general formally integrable is a very interesting topic of future research.

We finally obtained expressions for the Floquet states for one-, two-, and many-body cases with contact interactions, where it has been observed that they essentially retain the form of the eigenfunctions of the original undriven Hamiltonian with a time-dependent translation over the momenta (or pseudomomenta). Our approaches can be applied to any many-body system where the particles interact with a two-body potential which depends on the difference between particle positions and are translationally invariant. It would be very interesting to consider the effects of different boundary conditions on the problem in finite-size systems, and employing a Floquet engineering approach to study ac-Stark shifts and multiphoton resonances [21] for single- and many-particle systems.

ACKNOWLEDGMENTS

Discussions with G. Santoro, J. Schmiedmayer, and M. Aidelsburger are gratefully acknowledged. The authors thank the Erwin Schrödinger Institute (ESI) in Wien for support during the Programme "Quantum Paths". G.S. acknowledges financial support from Spanish Grants No. PGC2018-095862-B-C21, No. QUITEMAD + S2013/ICE-2801, and No. SEV-2016-0597 of the "Centro de Excelencia Severo Ochoa" Programme.

[1] D. H. Dunlap and V. M. Kenkre, Phys. Rev. B **34**, 3625 (1986).
[2] C. E. Creffield, Phys. Rev. B **67**, 165301 (2003).
[3] A. Eckardt, C. Weiss, and M. Holthaus, Phys. Rev. Lett. **95**, 260404 (2005).
[4] H. Lignier, C. Sias, D. Ciampini, Y. Singh, A. Zenesini, O. Morsch, and E. Arimondo, Phys. Rev. Lett. **99**, 220403 (2007).
[5] E. Kierig, U. Schnorrberger, A. Schietinger, J. Tomkovic, and M. K. Oberthaler, Phys. Rev. Lett. **100**, 190405 (2008).
[6] A. Eckardt, M. Holthaus, H. Lignier, A. Zenesini, D. Ciampini, O. Morsch, and E. Arimondo, Phys. Rev. A **79**, 013611 (2009).
[7] C. E. Creffield and G. Sierra, Phys. Rev. A **91**, 063608 (2015).
[8] R. He, M.-Z. Ai, J.-M. Cui, Y.-F. Huang, Y.-J. Han, C.-F. Li, T. Tu, C. E. Creffield, G. Sierra, and G.-C. Guo, Phys. Rev. A **101**, 043402 (2020).
[9] A. Eckardt, Rev. Mod. Phys. **89**, 011004 (2017).
[10] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B **82**, 235114 (2010).
[11] N. Lindner, G. Refael, and V. Galitski, Nat. Phys. **7**, 490 (2011).
[12] F. Wilczek, Phys. Rev. Lett. **111**, 250402 (2013).
[13] H. Watanabe and M. Oshikawa, Phys. Rev. Lett. **114**, 251603 (2015).
[14] S. Choi *et al.*, Nature (London) **543**, 221 (2017).
[15] J. Zhang *et al.*, Nature (London) **543**, 217 (2017).
[16] A. Russomanno, F. Iemini, M. Dalmonte, and R. Fazio, Phys. Rev. B **95**, 214307 (2017).
[17] N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, Phys. Rev. Lett. **118**, 030401 (2017).
[18] K. Sacha and J. Zakrzewski, Rep. Prog. Phys. **81**, 016401 (2018).
[19] K. Giergiel, A. Kosior, P. Hannaford, and K. Sacha, Phys. Rev. A **98**, 013613 (2018).
[20] N. Goldman and J. Dalibard, Phys. Rev. X **4**, 031027 (2014).
[21] M. Holthaus, J. Phys. B **49**, 013001 (2016).
[22] A. Russomanno, A. Silva, and G. E. Santoro, Phys. Rev. Lett. **109**, 257201 (2012).
[23] S. A. Weidinger and M. Knap, Sci. Rep. **7**, 45382 (2017).
[24] A. Herrmann, Y. Murakami, M. Eckstein, and P. Werner, Europhys. Lett. **120**, 57001 (2018).
[25] T. Oka and S. Kitamura, Annu. Rev. Condens. Matter Phys. **10**, 387 (2019).
[26] G. Floquet, Ann. de l'Ecole Norm. Suppl. **12**, 47 (1883).
[27] J. H. Shirley, Phys. Rev. **138**, B979 (1965).
[28] M. Grifoni and P. Hänggi, Phys. Rep. **304**, 229 (1998).
[29] A. Colcelli, G. Mussardo, G. Sierra, and A. Trombettoni, Phys. Rev. Lett. **123**, 130401 (2019).
[30] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
[31] E. A. Yuzbashyan, Ann. Phys. **392**, 323 (2018).

[32] N. A. Sinitsyn, E. A. Yuzbashyan, V. Y. Chernyak, A. Patra, and C. Sun, Phys. Rev. Lett. **120**, 190402 (2018).
[33] A. Komnik and M. Thorwart, Eur. Phys. J. B **89**, 244 (2016).
[34] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).
[35] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993).
[36] G. Mussardo, *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics*, 2nd ed. (Oxford University Press, Oxford, 2020).
[37] M. Kormos, G. Mussardo, and A. Trombettoni, Phys. Rev. Lett. **103**, 210404 (2009); Phys. Rev. A **81**, 043606 (2010).
[38] V. A. Yurovsky, M. Olshanii, and D. S. Weiss, Adv. At. Mol. Opt. Phys. **55**, 61 (2008).
[39] I. Bouchoule, N. J. van Druten, and C. I. Westbrook, in *Atom Chips*, edited by J. Reichel and V. Vuletic (Wiley, New York, 2010), pp. 331–363.
[40] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, Rev. Mod. Phys. **83**, 1405 (2011).
[41] M. V. Berry and N. L. Balazs, Am. J. Phys. **47**, 264 (1979).
[42] A. R. P. Rau and K. Unnikrishnan, Phys. Lett. A **222**, 304 (1996).
[43] I. Guedes, Phys. Rev. A **63**, 034102 (2001).
[44] M. Feng, Phys. Rev. A **64**, 034101 (2001).
[45] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics Non-Relativistic Theory* (Pergamon Press, Oxford, 1965).
[46] D. Bauch, Nuovo Cimento B **85**, 118 (1985).
[47] M. A. Andreata and V. V. Dodonov, J. Phys. A **37**, 2423 (2004); J. Russ. Laser Res. **35**, 39 (2014).
[48] S. Pandey, H. Mas, G. Drougakis, P. Thekkeppatt, V. Bolpasi, G. Vasilakis, K. Poulios, and W. von Klitzing, Nature (London) **570**, 205 (2019).
[49] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity* (Oxford University Press, Oxford, 2016).
[50] B. Sutherland, *Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems* (World Scientific, Singapore, 2004).
[51] M. Gaudin and J.-S. Caux, *The Bethe Wavefunction* (Cambridge University Press, Cambridge, 2014).
[52] S. Zöllner, H.-D. Meyer, and P. Schmelcher, Phys. Rev. A **78**, 013629 (2008).
[53] A. Imambekov and E. Demler, Phys. Rev. A **73**, 021602(R) (2006).
[54] D. Jukić, S. Galić, R. Pezer, and H. Buljan, Phys. Rev. A **82**, 023606 (2010).
[55] H. H. Chen and C. S. Liu, Phys. Rev. Lett. **37**, 693 (1976).
[56] T. Tsurumi and M. Wadati, J. Phys. Soc. Jpn. **71**, 1044 (2002).
[57] See Appendix D in M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and Continuous Nonlinear Schrödinger Systems* (Cambridge University Press, Cambridge, 2004).
[58] F. Franchini, *An Introduction to Integrable Techniques for One-Dimensional Quantum Systems* (Springer, Cham, 2017).