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Research Article

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On the critical behavior for inhomogeneous wave inequalities with Hardy potential in an exterior domain

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Abstract: We study the wave inequality with a Hardy potential

$$\partial_{tt}u - \Delta u + \frac{\lambda}{|x|^2}u \geq |u|^p \quad \text{in } (0,\infty)\times\Omega,$$

where Ω is the exterior of the unit ball in $\mathbb{R}^N$, $N \geq 2$, $p > 1$, and $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, under the inhomogeneous boundary condition

$$\alpha\frac{\partial u}{\partial \nu}(t,x) + \beta u(t,x) \geq w(x) \quad \text{on } (0,\infty)\times\partial\Omega,$$

where $\alpha, \beta \geq 0$ and $(\alpha,\beta) \neq (0,0)$. Namely, we show that there exists a critical exponent $p_c(N,\lambda) \in (1,\infty]$ for which, if $1 < p < p_c(N,\lambda)$, the above problem admits no global weak solution for any $w \in L^1(\partial\Omega)$ with $\int_{\partial\Omega} w(x)\,d\sigma > 0$, while if $p > p_c(N,\lambda)$, the problem admits global solutions for some $w > 0$. To the best of our knowledge, the study of the critical behavior for wave inequalities with a Hardy potential in an exterior domain was not considered in previous works. Some open questions are also mentioned in this paper.

Keywords: wave inequalities, Hardy potential, exterior domain, global weak solutions, critical exponent

MSC: 35L05, 35B33, 35B44

1 Introduction

In this paper, we are concerned with the study of existence and nonexistence of global weak solutions to the wave inequality

$$\Box u + \frac{\lambda}{|x|^2}u \geq |u|^p \quad \text{in } (0,\infty)\times\Omega. \tag{1.1}$$

Here, $\Box := \partial_{tt} - \Delta$ is the wave operator, $\Omega = \{x \in \mathbb{R}^N : |x| \geq 1\}$, $N \geq 2$, $p > 1$, and $\lambda \geq -\left(\frac{N-2}{2}\right)^2$. We will investigate (1.1) under the inhomogeneous boundary condition

$$\alpha\frac{\partial u}{\partial \nu}(t,x) + \beta u(t,x) \geq w(x) \quad \text{on } (0,\infty)\times\partial\Omega, \tag{1.2}$$

where $\alpha, \beta \geq 0$, $(\alpha,\beta) \neq (0,0)$, $w \in L^1(\partial\Omega)$, and ν denotes the outward unit normal vector on $\partial\Omega$ relative to Ω. Notice that (1.2) includes different types of inhomogeneous boundary conditions. Namely, the Dirichlet type

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boundary condition (in the case $(\alpha, \beta) = (0, 1)$)

$$u(t, x) \geq w(x) \quad \text{on } (0, \infty) \times \partial\Omega,$$

the Neumann type boundary condition (in the case $(\alpha, \beta) = (1, 0)$)

$$\frac{\partial u}{\partial \nu}(t, x) \geq w(x) \quad \text{on } (0, \infty) \times \partial\Omega,$$

and the Robin type boundary condition (in the case $\alpha = 1$ and $\beta > 0$)

$$\frac{\partial u}{\partial \nu}(t, x) + \beta u(t, x) \geq w(x) \quad \text{on } (0, \infty) \times \partial\Omega.$$

Let us consider the semilinear wave equation

$$\begin{cases} \Box u + V(x)u & = & |u|^p \quad \text{in } (0, \infty) \times \mathbb{R}^N, \\ (u(0, x), \partial_t u(0, x)) & = & (u_0(x), u_1(x)) \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $V = V(x)$ is a potential, and let $p_c(N)$ be the positive root of the quadratic equation

$$(N - 1)p^2 - (N + 1)p - 2 = 0.$$

In the special case $V \equiv 0$, (1.3) has been investigated by several authors. Namely, John [12] proved that, if the initial values are compactly supported and nonnegative, then for $N = 3$ and $1 < p < p_c(3) = 1 + \sqrt{2}$, nontrivial solutions must blow-up in finite time, while if $p > p_c(3)$, global solutions exist for small initial values. Next, a similar result has been derived by Glassey [6] in the case $N = 2$. In [19], Shaffer proved that in the case $N \in \{2, 3\}$, $p_c(N)$ belongs to the blow-up case. Georgiev et al. [5] (see also [15, 21]) proved that, if $p > p_c(N)$ and $N \geq 3$, then global solutions exist for small initial values. A blow-up result was shown by Sideris [20] (see also [9, 18]) in the case $1 < p < p_c(N)$ and $N \geq 4$. In [23], Yordanov and Zhang proved that for all $N \geq 4$, $p_c(N)$ belongs to the blow-up case.

In [22], Yordanov and Zhang studied (1.3) when $N \geq 3$ and V is a nonnegative potential satisfying the following conditions:
"There exist functions $\phi_i \in C^2(\mathbb{R}^N)$, $i = 0, 1$, such that

$$\Delta\phi_0 - V\phi_0 = 0 \quad \text{and} \quad \Delta\phi_1 - V\phi_1 = \phi_1,$$

where $C_0^{-1} \leq \phi_0(x) \leq C_0$ and $0 < \phi_1(x) \leq C_1(1 + |x|)^{\frac{-(N-1)}{2}} e^{|x|}$ with positive constants C_i, $i = 0, 1$".

It was shown that, if the initial values are nonnegative and compactly supported, then a blow-up occurs when $1 < p < p_c(N)$.

In [7], Hamidi and Laptev considered semilinear evolution inequalities of the form

$$\frac{\partial^k u}{\partial t^k} - \Delta u + \frac{\lambda}{|x|^2} u \geq |u|^p \quad \text{in } (0, \infty) \times \mathbb{R}^N, \tag{1.4}$$

where $k \geq 1$ (integer), $N \geq 3$ and $\lambda \geq -\left(\frac{N-2}{2}\right)^2$. It was shown that when the initial values are nonnegative, if

$$\lambda \geq 0 \quad \text{and} \quad 1 < p \leq 1 + \frac{2}{s^\star + \frac{2}{k}}$$

or

$$-\left(\frac{N-2}{2}\right)^2 \leq \lambda < 0 \quad \text{and} \quad 1 < p \leq 1 + \frac{2}{-s_\star + \frac{2}{k}},$$

where $s_\star < s^\star$ are the roots of the polynomial

$$s^2 + (N - 2)s - \lambda = 0,$$

then (1.4) admits no nontrivial global weak solution.

The study of blow-up phenomena for semilinear wave equations in exterior domains was considered by many authors (see e.g. [8, 10, 11, 13, 14, 24, 25] and the references therein). In particular, Zhang [24] studied the semilinear wave equation

$$\Box u = |u|^p \quad \text{in } (0,\infty)\times\Omega \tag{1.5}$$

under the inhomogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(t,x) = w(x) \quad \text{on } (0,\infty)\times\partial\Omega, \tag{1.6}$$

where $N \geq 3$, $w \in L^1(\partial\Omega)$, $w \geq 0$, and $w \not\equiv 0$. Namely, it was shown that (1.5)–(1.6) admits as critical exponent the real number $p^\star = 1 + \frac{2}{N-2}$, i.e. if $1 < p < p^\star$, then (1.5)–(1.6) admits no global weak solution, while if $p > p^\star$, global solutions exist for some $w > 0$. Later, the same critical exponent was obtained for (1.5) under the inhomogeneous Dirichlet boundary condition [10]

$$u(t,x) = w(x) \quad \text{on } (0,\infty)\times\partial\Omega, \tag{1.7}$$

and the Robin boundary condition [8]

$$\frac{\partial u}{\partial \nu}(t,x) + u = w(x) \quad \text{on } (0,\infty)\times\partial\Omega. \tag{1.8}$$

To enlarge the literature review on the main topic of this article, we recall the study of blow-up of solutions carried out by Mohammed et al. [17], for fully nonlinear uniformly elliptic equations. Also, we mention the recent work of Bahrouni et al. [1], where the authors dealt with a class of double phase variational functionals related to the study of transonic flow, and established useful integral inequalities. In a series of remarkable papers, Cîrstea and Rădulescu [2–4] focused on special classes of semilinear elliptic equations (namely, logistic equations) and linked the nonregular variation of the nonlinearity at infinity with the blow-up rate of the solutions. They also established existence and uniqueness results for related problems, in the cases of homogeneous Dirichlet, Neumann or Robin boundary condition.

To the best of our knowledge, the study of critical behavior for wave inequalities with Hardy potential in an exterior domain was not considered in previous works. In this paper, we investigate the critical behavior for (1.1) under the inhomogeneous boundary condition (1.2). Namely, we will show that there exists a critical exponent $p_c(N,\lambda) \in (1,\infty]$ for which, when $1 < p < p_c(N,\lambda)$ and $\int_{\partial\Omega} w(x)\,d\sigma > 0$, (1.1)–(1.2) has no global weak solution; when $p > p_c(N,\lambda)$, the problem admits global solutions for some $w > 0$.

Before presenting our results, let us mention in which sense the solutions to (1.1)–(1.2) are considered. Let

$$\mathcal{O} = (0,\infty)\times\Omega \quad \text{and} \quad \partial\mathcal{O} = (0,\infty)\times\partial\Omega.$$

We introduce the test function space

$$\Phi_{\alpha,\beta} = \left\{\varphi \in C_c^2(\mathcal{O}) : \varphi \geq 0,\ \frac{\partial\varphi}{\partial\nu}\Big|_{\partial\mathcal{O}} \leq 0 \text{ if } \alpha = 0,\ \alpha\frac{\partial\varphi}{\partial\nu} + \beta\varphi\Big|_{\partial\mathcal{O}} = 0\right\},$$

where $C_c^2(\mathcal{O})$ denotes the space of C^2 functions compactly supported in $\mathcal{O}$. Notice that Ω is closed and $\partial\mathcal{O} \subset \mathcal{O}$.

Definition 1.1. *A function $u \in L^p_{loc}(\mathcal{O})$ is a global weak solution to* (1.1)–(1.2), *if*

$$\int_{\mathcal{O}} |u|^p\varphi\,dx\,dt + L_\varphi(w) \leq \int_{\mathcal{O}} u\left(\Box\varphi + \frac{\lambda}{|x|^2}\varphi\right)dx\,dt, \tag{1.9}$$

for all $\varphi \in \Phi_{\alpha,\beta}$, where

$$L_\varphi(w) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_{\partial\mathcal{O}} w(x)\varphi\,d\sigma\,dt & \textit{if} \quad \alpha > 0, \\[2ex] \dfrac{-1}{\beta}\displaystyle\int_{\partial\mathcal{O}} w(x)\frac{\partial\varphi}{\partial\nu}\,d\sigma\,dt & \textit{if} \quad \alpha = 0. \end{cases}$$

Now, we are ready to state our main results. We discuss separately the cases $\lambda = -\left(\frac{N-2}{2}\right)^2$ and $\lambda > -\left(\frac{N-2}{2}\right)^2$. For $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, let

$$\lambda_N = \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda}.$$

Theorem 1.1. *Let $N \geq 2$, $\alpha, \beta \geq 0$, $(\alpha, \beta) \neq (0, 0)$ and $\lambda = -\left(\frac{N-2}{2}\right)^2$.*

(*i*) *If $N = 2$, $w \in L^1(\partial\Omega)$ and $\int_{\partial\Omega} w(x)\,d\sigma > 0$, then for all $p > 1$,* (1.1)–(1.2) *admits no global weak solution.*

(*ii*) *If $N \geq 3$, $w \in L^1(\partial\Omega)$ and $\int_{\partial\Omega} w(x)\,d\sigma > 0$, then for all*

$$1 < p < 1 + \frac{4}{N-2},$$

(1.1)–(1.2) *admits no global weak solution.*

(*iii*) *If $N \geq 3$ and*

$$p > 1 + \frac{4}{N-2},$$

then (1.1)–(1.2) *admits global solutions (stationary solutions) for some $w > 0$.*

Theorem 1.2. *Let $N \geq 2$, $\alpha, \beta \geq 0$, $(\alpha, \beta) \neq (0, 0)$ and $\lambda > -\left(\frac{N-2}{2}\right)^2$.*

(*i*) *If $w \in L^1(\partial\Omega)$ and $\int_{\partial\Omega} w(x)\,d\sigma > 0$, then for all*

$$1 < p < 1 + \frac{4}{N-2+2\lambda_N},$$

(1.1)–(1.2) *admits no global weak solution.*

(*ii*) *If*

$$p > 1 + \frac{4}{N-2+2\lambda_N},$$

then (1.1)–(1.2) *admits global solutions (stationary solutions) for some $w > 0$.*

Remark 1.1. *Let*

$$p_c(N,\lambda) = \begin{cases} \infty & \text{if} \quad N-2+2\lambda_N = 0, \\ 1 + \frac{4}{N-2+2\lambda_N} & \text{if} \quad N-2+2\lambda_N > 0. \end{cases}$$

From Theorems 1.1 and 1.2, one deduces that,

(i) if $1 < p < p_c(N,\lambda)$ and $\int_{\partial\Omega} w(x)\,d\sigma > 0$, then (1.1)–(1.2) *has no global weak solution;*

(ii) if $p > p_c(N,\lambda)$, then (1.1)–(1.2) *admits global solutions for some $w > 0$.*

The above statements show that the exponent $p_c(N,\lambda)$ is critical for (1.1)–(1.2).

Notice that in the case $\lambda = 0$, one has

$$p_c(N,0) = \begin{cases} \infty & \text{if} \quad N = 2, \\ 1 + \frac{2}{N-2} & \text{if} \quad N \geq 3, \end{cases}$$

which is the same critical exponent obtained for the semilinear wave equation (1.5) *under the inhomogeneous Neumann boundary condition* (1.6) (see [24]), *the inhomogeneous Dirichlet boundary condition* (1.7) (see [10]), *and the inhomogeneous Robin boundary condition* (1.8) (see [8]).

Remark 1.2. *From Theorems 1.1 and 1.2, we deduce that $p_c(N,\lambda)$ is also critical for the exterior problem*

$$\begin{cases} -\Delta u + \frac{\lambda}{|x|^2}u \geq |u|^p & \text{in } \Omega, \\ \alpha \frac{\partial u}{\partial \nu} + \beta u \geq w & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

Namely, if $1 < p < p_c(N,\lambda)$ *and* $\int_{\partial\Omega} w(x)\,d\sigma > 0$, *then* (1.10) *admits no weak solution, while if* $p > p_c(N,\lambda)$, *then* (1.10) *admits solutions for some* $w > 0$.

Remark 1.3. *At this time, if* $N - 2 + 2\lambda_N > 0$, *we do not know whether* $p = p_c(N,\lambda)$ *belongs to the nonexistence case or not. This question is open.*

Remark 1.4. *(i) In this paper, the inhomogeneous term* w *depends only on the variable space. It would be interesting to study the critical behavior for* (1.1)–(1.2) *when* $w = w(t,x)$.
(ii) It would be also interesting to study the critical behavior for (1.1)–(1.2) *when* $w \equiv 0$.

The rest of the paper is organized as follows. In Section 2, we establish some lemmas and provide some estimates that will be used in the proofs of our main results. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2. Namely, we first prove the nonexistence results (parts (i) and (ii) of Theorem 1.1, and part (i) of Theorem 1.2), next we prove the existence results (part (iii) of Theorem 1.1 and part (ii) of Theorem 1.2).

2 Preliminaries

For $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, let Δ_λ be the differential operator defined by

$$\Delta_\lambda := \Delta - \frac{\lambda}{|x|^2}.$$

For $\alpha, \beta \geq 0$ and $(\alpha,\beta) \neq (0,0)$, we introduce the function $H_{\alpha,\beta}$ defined in Ω by

$$H_{\alpha,\beta}(x) = \begin{cases} H^{(1)}_{\alpha,\beta}(x) & \text{if } \lambda = -\left(\frac{N-2}{2}\right)^2, \\ H^{(2)}_{\alpha,\beta}(x) & \text{if } \lambda > -\left(\frac{N-2}{2}\right)^2, \end{cases}$$

where

$$H^{(1)}_{\alpha,\beta}(x) = |x|^{\frac{2-N}{2}}\left[\alpha + \left(\beta + \frac{(N-2)\alpha}{2}\right)\ln|x|\right]$$

and

$$H^{(2)}_{\alpha,\beta}(x) = |x|^{\frac{2-N}{2}+\lambda_N}\left[\beta + \left(\frac{N-2}{2} + \lambda_N\right)\alpha + \left(\left(\frac{2-N}{2} + \lambda_N\right)\alpha - \beta\right)|x|^{-2\lambda_N}\right].$$

One can check easily that $H_{\alpha,\beta}$ is a nonnegative solution to the exterior problem

$$\begin{cases} -\Delta_\lambda H_{\alpha,\beta} = 0 & \text{in } \Omega, \\ \alpha \frac{\partial H_{\alpha,\beta}}{\partial \nu} + \beta H_{\alpha,\beta} = 0 & \text{on } \partial\Omega. \end{cases}$$

We need also to introduce two cut-off functions. Let $\eta, \xi \in C^\infty(\mathbb{R})$ be such that

$$\eta \geq 0, \quad \eta \not\equiv 0, \quad \operatorname{supp}(\eta) \subset (0,1)$$

and

$$0 \le \xi \le 1, \quad \xi(s) = 1 \text{ if } |s| \le 1, \quad \xi(s) = 0 \text{ if } |s| \ge 2.$$

For $0 < T < \infty$, let

$$H_T(x) = H_{\alpha,\beta}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\ell}, \quad x \in \Omega$$

and

$$\eta_T(t) = \eta\left(\frac{t}{T}\right)^{\ell}, \quad t > 0,$$

where $\ell \ge 2$ and $\theta > 0$ are constants to be chosen later.

Lemma 2.1. *For all $\ell \ge 2$, $\theta > 0$, and sufficiently large T, the function*

$$\varphi_T(t,x) := \eta_T(t)H_T(x), \quad (t,x) \in \mathcal{O}$$

belongs to the test function space $\Phi_{\alpha,\beta}$.

Proof. It can be easily seen that $\varphi_T \ge 0$, and for sufficiently large T, $\varphi_T \in C_c^2(\mathcal{O})$. On the other hand, for $1 < |x| < 1 + \epsilon$ ($\epsilon > 0$ is sufficiently small), one has

$$\nabla H_T(x) = \xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\ell} \nabla H_{\alpha,\beta}(x) + 2\ell T^{-2\theta}|x|H_{\alpha,\beta}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\ell-1} \nabla\xi\left(\frac{|x|^2}{T^{2\theta}}\right).$$

By the definition of the cut-off function ξ, since T is supposed to be large enough, one obtains

$$\nabla H_T(x) = \nabla H_{\alpha,\beta}(x), \quad 1 < |x| < 1 + \epsilon.$$

Similarly, one has

$$H_T(x) = H_{\alpha,\beta}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\ell} = H_{\alpha,\beta}(x), \quad 1 < |x| < 1 + \epsilon.$$

Then, since $H_{\alpha,\beta}$ satisfies the boundary condition

$$\alpha\frac{\partial H_{\alpha,\beta}}{\partial \nu} + \beta H_{\alpha,\beta} = 0 \quad \text{on } \partial\Omega,$$

one deduces that

$$\alpha\frac{\partial \varphi_T}{\partial \nu} + \beta\varphi_T\Big|_{\partial\mathcal{O}} = \eta(t)\left(\alpha\frac{\partial H_{\alpha,\beta}}{\partial \nu} + \beta H_{\alpha,\beta}\Big|_{\partial\Omega}\right) = 0.$$

Next, we take $\alpha = 0$. If $\lambda = -\left(\frac{N-2}{2}\right)^2$, for $r = |x|$, one has

$$\frac{\partial H_T}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial H_{\alpha,\beta}}{\partial \nu}\Big|_{\partial\Omega} = -\frac{\partial H_{\alpha,\beta}^{(1)}}{\partial r}\Big|_{r=1} = -\beta < 0. \tag{2.1}$$

If $\lambda > -\left(\frac{N-2}{2}\right)^2$, one has

$$\frac{\partial H_T}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial H_{\alpha,\beta}}{\partial \nu}\Big|_{\partial\Omega} = -\frac{\partial H_{\alpha,\beta}^{(2)}}{\partial r}\Big|_{r=1} = -2\lambda_N\beta < 0. \tag{2.2}$$

Hence, if $\alpha = 0$, in both cases, we have

$$\frac{\partial H_T}{\partial \nu}\Big|_{\partial\Omega} < 0,$$

which yields (since $\eta \ge 0$)

$$\frac{\partial \varphi_T}{\partial \nu}\Big|_{\partial\mathcal{O}} \le 0,$$

and the lemma is proved. □

Throughout this paper, C denotes a positive constant (independent of T) whose value may change from line to line.

Lemma 2.2. *For all* $0 < T < \infty$ *and* $\ell \geq 2$, *we have*

$$\int_0^\infty \eta_T(t)\,dt = CT.$$

Proof. By the definition of the function η_T, and using the properties of the cut-off function η, one obtains

$$\begin{aligned}\int_0^\infty \eta_T(t)\,dt &= \int_0^\infty \eta\left(\frac{t}{T}\right)^\ell dt\\ &= \int_0^T \eta\left(\frac{t}{T}\right)^\ell dt\\ &= T\int_0^1 \eta(s)^\ell\,ds,\end{aligned}$$

and the lemma is proved. □

Lemma 2.3. *Let* $m > 1$. *For all* $0 < T < \infty$ *and* $\ell \geq \frac{2m}{m-1}$, *we have*

$$\int_0^\infty \eta_T(t)^{\frac{-1}{m-1}}|\eta_T''(t)|^{\frac{m}{m-1}}\,dt \leq CT^{1-\frac{2m}{m-1}}.$$

Proof. It can be easily seen that

$$|\eta_T''(t)| \leq CT^{-2}\eta\left(\frac{t}{T}\right)^{\ell-2}, \quad 0 < t < T.$$

Hence, one obtains

$$\begin{aligned}\int_0^\infty \eta_T(t)^{\frac{-1}{m-1}}|\eta_T''(t)|^{\frac{m}{m-1}}\,dt &= \int_0^T \eta_T(t)^{\frac{-1}{m-1}}|\eta_T''(t)|^{\frac{m}{m-1}}\,dt\\ &\leq CT^{\frac{-2m}{m-1}}\int_0^T \eta_T(t)^{\ell-\frac{2m}{m-1}}\,dt\\ &= CT^{1-\frac{2m}{m-1}}\int_0^1 \eta(s)^{\ell-\frac{2m}{m-1}}\,ds,\end{aligned}$$

which yields the desired estimate. □

Lemma 2.4. *Let* $\lambda = -\left(\frac{N-2}{2}\right)^2$. *For all* $\theta > 0$, $\ell \geq 2$, *and sufficiently large* T, *we have*

$$\int_\Omega H_T(x)\,dx \leq CT^{\frac{\theta(N+2)}{2}}\ln T.$$

Proof. By the definition of the function H_T (as well as the function $H_{\alpha,\beta}$), and the properties of the cut-off function ξ, for sufficiently large T, one has

$$\begin{aligned}\int_\Omega H_T(x)\,dx &= \int_\Omega H_{\alpha,\beta}^{(1)}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell dx\\ &= \int_{|x|>1} |x|^{\frac{2-N}{2}}\left[\alpha + \left(\beta + \frac{(N-2)\alpha}{2}\right)\ln|x|\right]\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell dx\end{aligned}$$

$$\leq T^{\frac{\theta(N+2)}{2}} \int_{T^{-\theta}<|y|<\sqrt{2}} |y|^{\frac{2-N}{2}} \left[\alpha + \left(\beta + \frac{(N-2)\alpha}{2}\right) \ln\left(T^{\theta}|y|\right)\right] dy.$$

Observe that

$$\beta + \frac{(N-2)\alpha}{2} = 0 \iff \beta = 0 \text{ and } N = 2.$$

So, if $\beta = 0$ and $N = 2$, one obtains

$$\begin{aligned} \int_{\Omega} H_T(x)\,dx &\leq \alpha T^{\frac{\theta(N+2)}{2}} \int_{T^{-\theta}<|y|<\sqrt{2}} |y|^{\frac{2-N}{2}}\,dy \\ &\leq \alpha T^{\frac{\theta(N+2)}{2}} \int_{0<|y|<\sqrt{2}} |y|^{\frac{2-N}{2}}\,dy \\ &\leq C T^{\frac{\theta(N+2)}{2}} \int_{\rho=0}^{\sqrt{2}} \rho^{\frac{N}{2}}\,d\rho, \end{aligned}$$

that is,

$$\int_{\Omega} H_T(x)\,dx \leq C T^{\frac{\theta(N+2)}{2}}. \tag{2.3}$$

If $\beta > 0$ or $N \geq 3$, one obtains $\beta + \frac{(N-2)\alpha}{2} > 0$ and

$$\begin{aligned} \int_{\Omega} H_T(x)\,dx &\leq C T^{\frac{\theta(N+2)}{2}} \ln T \int_{T^{-\theta}<|y|<\sqrt{2}} |y|^{\frac{2-N}{2}}\,dy \\ &\leq C T^{\frac{\theta(N+2)}{2}} \ln T \int_{\rho=0}^{\sqrt{2}} \rho^{\frac{N}{2}}\,d\rho, \end{aligned}$$

that is,

$$\int_{\Omega} H_T(x)\,dx \leq C T^{\frac{\theta(N+2)}{2}} \ln T. \tag{2.4}$$

Hence, (2.3) and (2.4) yield the desired estimate. □

Lemma 2.5. *Let $\lambda > -\left(\frac{N-2}{2}\right)^2$. For all $\theta > 0$, $\ell \geq 2$, and sufficiently large T, we have*

$$\int_{\Omega} H_T(x)\,dx \leq C T^{\theta\left(\frac{N+2}{2}+\lambda_N\right)}.$$

Proof. In this case, one has

$$H_{\alpha,\beta}(x) = H^{(2)}_{\alpha,\beta}(x) = O\left(|x|^{\frac{2-N}{2}+\lambda_N}\right), \text{ as } |x| \to \infty.$$

Hence, for sufficiently large T, we obtain

$$\begin{aligned} \int_{\Omega} H_T(x)\,dx &= \int_{|x|>1} H^{(2)}_{\alpha,\beta}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\ell} dx \\ &= T^{N\theta} \int_{T^{-\theta}<|y|<\sqrt{2}} H^{(2)}_{\alpha,\beta}(T^{\theta}y)\xi(|y|^2)^{\ell}\,dy \\ &\leq C T^{\theta\left(\frac{N+2}{2}+\lambda_N\right)} \int_{0<|y|<\sqrt{2}} |y|^{\frac{2-N}{2}+\lambda_N}\,dy \end{aligned}$$

$$= \quad CT^{\theta\left(\frac{N+2}{2}+\lambda_N\right)} \int_{\rho=0}^{\sqrt{2}} \rho^{\frac{N}{2}+\lambda_N}\, d\rho,$$

which yields the desired estimate. □

Lemma 2.6. *Let $\lambda = -\left(\frac{N-2}{2}\right)^2$ and $m > 1$. For all $\theta > 0$, $\ell \geq \frac{2m}{m-1}$, and sufficiently large T, we have*

$$\int_\Omega H_T(x)^{\frac{-1}{m-1}} |\Delta_\lambda H_T|^{\frac{m}{m-1}}\, dx \leq CT^{\theta\left(\frac{N+2}{2}-\frac{2m}{m-1}\right)} \ln T.$$

Proof. For all $x \in \Omega$, one has

$$\begin{aligned}
-\Delta_\lambda H_T(x) &= \left(-\Delta + \frac{\lambda}{|x|^2}\right)\left[H^{(1)}_{\alpha,\beta}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell\right] \\
&= -\Delta\left[H^{(1)}_{\alpha,\beta}\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell\right] + \frac{\lambda}{|x|^2}H^{(1)}_{\alpha,\beta}\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell \\
&= -\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell \Delta H^{(1)}_{\alpha,\beta}(x) - H^{(1)}_{\alpha,\beta}\Delta\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell - 2\nabla H^{(1)}_{\alpha,\beta}\cdot\nabla\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell \\
&\quad + \frac{\lambda}{|x|^2}H^{(1)}_{\alpha,\beta}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell \\
&= -\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell \Delta_\lambda H^{(1)}_{\alpha,\beta}(x) - H^{(1)}_{\alpha,\beta}(x)\Delta\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell - 2\nabla H^{(1)}_{\alpha,\beta}(x)\cdot\nabla\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell,
\end{aligned}$$

where " $\cdot$ " denotes the inner product in $\mathbb{R}^N$. Since $\Delta_\lambda H^{(1)}_{\alpha,\beta} = 0$, it holds that

$$-\Delta_\lambda H_T(x) = -H^{(1)}_{\alpha,\beta}(x)\Delta\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell - 2\nabla H^{(1)}_{\alpha,\beta}(x)\cdot\nabla\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell,$$

which yields

$$\begin{aligned}
&|\Delta_\lambda H_T(x)|^{\frac{m}{m-1}} \\
&\leq \left(H^{(1)}_{\alpha,\beta}(x)\left|\Delta\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell\right| + 2|\nabla H^{(1)}_{\alpha,\beta}(x)|\left|\nabla\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell\right|\right)^{\frac{m}{m-1}} \\
&\leq C\left(H^{(1)}_{\alpha,\beta}(x)^{\frac{m}{m-1}}\left|\Delta\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell\right|^{\frac{m}{m-1}} + |\nabla H^{(1)}_{\alpha,\beta}(x)|^{\frac{m}{m-1}}\left|\nabla\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell\right|^{\frac{m}{m-1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&H_T(x)^{\frac{-1}{m-1}}|\Delta_\lambda H_T(x)|^{\frac{m}{m-1}} \\
&\leq CH^{(1)}_{\alpha,\beta}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\frac{-\ell}{m-1}}\left|\Delta\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell\right|^{\frac{m}{m-1}} \\
&\quad + CH^{(1)}_{\alpha,\beta}(x)^{\frac{-1}{m-1}}|\nabla H^{(1)}_{\alpha,\beta}(x)|^{\frac{m}{m-1}}\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\frac{-\ell}{m-1}}\left|\nabla\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell\right|^{\frac{m}{m-1}}.
\end{aligned}$$

Hence, it holds that

$$\int_\Omega H_T(x)^{\frac{-1}{m-1}}|\Delta_\lambda H_T(x)|^{\frac{m}{m-1}}\, dx \leq C\left(I_1(T) + I_2(T)\right), \tag{2.5}$$

where

$$I_1(T) = \int_\Omega H^{(1)}_{\alpha,\beta}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\frac{-\ell}{m-1}}\left|\Delta\,\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^\ell\right|^{\frac{m}{m-1}}\, dx$$

and

$$I_2(T)=\int_\Omega H^{(1)}_{\alpha,\beta}(x)^{\frac{-1}{m-1}}|\nabla H^{(1)}_{\alpha,\beta}(x)|^{\frac{m}{m-1}}\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\frac{-\ell}{m-1}}\left|\nabla\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\ell}\right|^{\frac{m}{m-1}}dx.$$

Now, let us estimate $I_i(T)$, $i=1,2$. Using the properties of the cut-off function ξ, for sufficiently large T, one has

$$\begin{aligned}I_1(T) &= \int_{T^\theta<|x|<\sqrt{2}T^\theta} H^{(1)}_{\alpha,\beta}(x)\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\frac{-\ell}{m-1}}\left|\Delta\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\ell}\right|^{\frac{m}{m-1}}dx\\ &= T^{\theta\left(N-\frac{2m}{m-1}\right)}\int_{1<|y|<\sqrt{2}} H^{(1)}_{\alpha,\beta}(T^\theta y)\xi(|y|^2)^{\frac{-\ell}{m-1}}|\Delta\xi(|y|^2)^{\ell}|^{\frac{m}{m-1}}dy.\end{aligned}$$

On the other hand, it can be easily seen that for $1<|y|<\sqrt{2}$, one has

$$|\Delta\xi(|y|^2)^\ell|\le C\xi(|y|^2)^{\ell-2}.$$

Hence, it holds that

$$\begin{aligned}I_1(T) &\le CT^{\theta\left(N-\frac{2m}{m-1}\right)}\int_{1<|y|<\sqrt{2}} H^{(1)}_{\alpha,\beta}(T^\theta y)\xi(|y|^2)^{\ell-\frac{2m}{m-1}}dy\\ &\le CT^{\theta\left(N-\frac{2m}{m-1}\right)}\int_{1<|y|<\sqrt{2}} H^{(1)}_{\alpha,\beta}(T^\theta y)\,dy.\end{aligned}\tag{2.6}$$

By the definition of the function $H^{(1)}_{\alpha,\beta}$, one has

$$H^{(1)}_{\alpha,\beta}(T^\theta y)=T^{\frac{\theta(2-N)}{2}}|y|^{\frac{2-N}{2}}\left[\alpha+\left(\beta+\frac{(N-2)\alpha}{2}\right)\ln\left(T^\theta|y|\right)\right],\quad 1<|y|<\sqrt{2}.$$

Observe that

$$\beta+\frac{(N-2)\alpha}{2}=0\iff\beta=0\text{ and }N=2.$$

Hence, if $\beta=0$ and $N=2$, one obtains

$$\int_{1<|y|<\sqrt{2}} H^{(1)}_{\alpha,\beta}(T^\theta y)\,dy=\alpha T^{\frac{\theta(2-N)}{2}}\int_{1<|y|<\sqrt{2}}|y|^{\frac{2-N}{2}}dy=CT^{\frac{\theta(2-N)}{2}}.$$

If $\beta>0$ or $N\ge 3$, for sufficiently large T, one obtains

$$\int_{1<|y|<\sqrt{2}} H^{(1)}_{\alpha,\beta}(T^\theta y)\,dy=CT^{\frac{\theta(2-N)}{2}}\ln T\int_{1<|y|<\sqrt{2}}|y|^{\frac{2-N}{2}}dy=CT^{\frac{\theta(2-N)}{2}}\ln T.$$

Hence, in both cases, by (2.6), for sufficiently large T, one deduces that

$$I_1(T)\le CT^{\theta\left(\frac{N+2}{2}-\frac{2m}{m-1}\right)}\ln T.\tag{2.7}$$

Next, one has

$$\begin{aligned}I_2(T) &= \int_{T^\theta<|x|<\sqrt{2}T^\theta} H^{(1)}_{\alpha,\beta}(x)^{\frac{-1}{m-1}}|\nabla H^{(1)}_{\alpha,\beta}(x)|^{\frac{m}{m-1}}\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\frac{-\ell}{m-1}}\left|\nabla\xi\left(\frac{|x|^2}{T^{2\theta}}\right)^{\ell}\right|^{\frac{m}{m-1}}dx\\ &= T^{\theta\left(N-\frac{m}{m-1}\right)}\int_{1<|y|<\sqrt{2}} H^{(1)}_{\alpha,\beta}(T^\theta y)^{\frac{-1}{m-1}}|\nabla H^{(1)}_{\alpha,\beta}(T^\theta y)|^{\frac{m}{m-1}}\xi(|y|^2)^{\frac{-\ell}{m-1}}|\nabla\xi(|y|^2)^{\ell}|^{\frac{m}{m-1}}dy.\end{aligned}$$

It can be easily seen that for $1 < |y| < \sqrt{2}$, one has

$$|\nabla \xi(|y|^2)^\ell| \le C\xi(|y|^2)^{\ell-1}.$$

Hence, it holds that

$$\begin{aligned} I_2(T) &\le CT^{\theta(N-\frac{m}{m-1})} \int\limits_{1<|y|<\sqrt{2}} H^{(1)}_{\alpha,\beta}(T^\theta y)^{\frac{-1}{m-1}} |\nabla H^{(1)}_{\alpha,\beta}(T^\theta y)|^{\frac{m}{m-1}} \xi(|y|^2)^{\ell-\frac{m}{m-1}}\, dy \\ &\le CT^{\theta(N-\frac{m}{m-1})} \int\limits_{1<|y|<\sqrt{2}} H^{(1)}_{\alpha,\beta}(T^\theta y)^{\frac{-1}{m-1}} |\nabla H^{(1)}_{\alpha,\beta}(T^\theta y)|^{\frac{m}{m-1}}\, dy. \end{aligned} \tag{2.8}$$

Elementary calculations show that for sufficiently large T and $1 < |y| < \sqrt{2}$, we get

$$H^{(1)}_{\alpha,\beta}(T^\theta y)^{\frac{-1}{m-1}} |\nabla H^{(1)}_{\alpha,\beta}(T^\theta y)|^{\frac{m}{m-1}} \le CT^{-\theta(\frac{N}{2}+\frac{1}{m-1})} \quad \text{if } N = 2 \text{ and } \beta = 0$$

and

$$H^{(1)}_{\alpha,\beta}(T^\theta y)^{\frac{-1}{m-1}} |\nabla H^{(1)}_{\alpha,\beta}(T^\theta y)|^{\frac{m}{m-1}} \le CT^{-\theta(\frac{N}{2}+\frac{1}{m-1})} \ln T \quad \text{if } N \ge 3 \text{ or } \beta > 0.$$

Hence, in both cases, for sufficiently large T, one has

$$H^{(1)}_{\alpha,\beta}(T^\theta y)^{\frac{-1}{m-1}} |\nabla H^{(1)}_{\alpha,\beta}(T^\theta y)|^{\frac{m}{m-1}} \le CT^{-\theta(\frac{N}{2}+\frac{1}{m-1})} \ln T, \quad 1 < |y| < \sqrt{2}.$$

Then, by (2.8), one obtains

$$I_2(T) \le CT^{\theta(\frac{N+2}{2}-\frac{2m}{m-1})} \ln T. \tag{2.9}$$

Finally, (2.5), (2.7) and (2.9) yield the desired estimate. □

Lemma 2.7. *Let $\lambda > -\left(\frac{N-2}{2}\right)^2$ and $m > 1$. For all $\theta > 0$, $\ell \ge \frac{2m}{m-1}$, and sufficiently large T, we have*

$$\int\limits_\Omega H_T(x)^{\frac{-1}{m-1}} |\Delta_\lambda H_T|^{\frac{m}{m-1}}\, dx \le CT^{\theta(\lambda_N+\frac{N+2}{2}-\frac{2m}{m-1})}.$$

Proof. Following the proof of Lemma 2.6, for sufficiently large T, one has

$$\int\limits_\Omega H_T(x)^{\frac{-1}{m-1}} |\Delta_\lambda H_T|^{\frac{m}{m-1}}\, dx \le C\left(J_1(T) + J_2(T)\right), \tag{2.10}$$

where

$$J_1(T) \le CT^{\theta(N-\frac{2m}{m-1})} \int\limits_{1<|y|<\sqrt{2}} H^{(2)}_{\alpha,\beta}(T^\theta y)\, dy \tag{2.11}$$

and

$$J_2(T) \le CT^{\theta(N-\frac{m}{m-1})} \int\limits_{1<|y|<\sqrt{2}} H^{(2)}_{\alpha,\beta}(T^\theta y)^{\frac{-1}{m-1}} |\nabla H^{(2)}_{\alpha,\beta}(T^\theta y)|^{\frac{m}{m-1}}\, dy. \tag{2.12}$$

Elementary calculations show that for sufficiently large T and $1 < |y| < \sqrt{2}$, one has

$$H^{(2)}_{\alpha,\beta}(T^\theta y) \le CT^{\theta(\frac{2-N}{2}+\lambda_N)} \tag{2.13}$$

and

$$H^{(2)}_{\alpha,\beta}(T^\theta y)^{\frac{-1}{m-1}} |\nabla H^{(2)}_{\alpha,\beta}(T^\theta y)|^{\frac{m}{m-1}} \le CT^{\theta(\lambda_N-\frac{N}{2}-\frac{1}{m-1})}. \tag{2.14}$$

Hence, (2.10), (2.11), (2.12), (2.13) and (2.14) yield the desired estimate. □

3 Proofs of the main results

In this section, we prove Theorems 1.1 and 1.2. We first establish the nonexistence results.

3.1 Nonexistence results

We prove below parts (i) and (ii) of Theorem 1.1, as well as part (i) of Theorem 1.2. The proof is based on a rescaled test-function argument (see [16] for a general account of these methods) and a judicious choice of the test function.

Proof. Let us suppose that $u \in L^p_{loc}(\mathcal{O})$ is a global weak solution to (1.1)–(1.2). By (1.9), we obtain

$$\int_{\mathcal{O}} |u|^p \varphi \, dx\, dt + L_\varphi(w) \le \int_{\mathcal{O}} |u||\partial_{tt}\varphi| \, dx\, dt + \int_{\mathcal{O}} |u||\Delta_\lambda \varphi| \, dx\, dt, \tag{3.1}$$

for every $\varphi \in \Phi_{\alpha,\beta}$. Using ε-Young inequality with $\varepsilon = \frac{1}{2}$, we get

$$\int_{\mathcal{O}} |u||\partial_{tt}\varphi| \, dx\, dt \le \frac{1}{2} \int_{\mathcal{O}} |u|^p \varphi \, dx\, dt + C \int_{\mathcal{O}} \varphi^{\frac{-1}{p-1}} |\partial_{tt}\varphi|^{\frac{p}{p-1}} \, dx\, dt \tag{3.2}$$

and

$$\int_{\mathcal{O}} |u||\Delta_\lambda \varphi| \, dx\, dt \le \frac{1}{2} \int_{\mathcal{O}} |u|^p \varphi \, dx\, dt + C \int_{\mathcal{O}} \varphi^{\frac{-1}{p-1}} |\Delta_\lambda \varphi|^{\frac{p}{p-1}} \, dx\, dt. \tag{3.3}$$

Hence, it follows from (3.1), (3.2) and (3.3) that

$$L_\varphi(w) \le C \left(\int_{\mathcal{O}} \varphi^{\frac{-1}{p-1}} |\partial_{tt}\varphi|^{\frac{p}{p-1}} \, dx\, dt + \int_{\mathcal{O}} \varphi^{\frac{-1}{p-1}} |\Delta_\lambda \varphi|^{\frac{p}{p-1}} \, dx\, dt \right), \tag{3.4}$$

for every $\varphi \in \Phi_{\alpha,\beta}$. By Lemma 2.1 and (3.4), for all $\ell \ge \frac{2p}{p-1}$, $\theta > 0$, and sufficiently large T, one has

$$L_{\varphi_T}(w) \le C \left(\int_{\mathcal{O}} \varphi_T^{\frac{-1}{p-1}} |\partial_{tt}\varphi_T|^{\frac{p}{p-1}} \, dx\, dt + \int_{\mathcal{O}} \varphi_T^{\frac{-1}{p-1}} |\Delta_\lambda \varphi_T|^{\frac{p}{p-1}} \, dx\, dt \right). \tag{3.5}$$

Now, we shall estimate the terms from the right-hand side of the above inequality. By the definition of the function φ_T, one has

$$\int_{\mathcal{O}} \varphi_T^{\frac{-1}{p-1}} |\partial_{tt}\varphi_T|^{\frac{p}{p-1}} \, dx\, dt = \left(\int_0^\infty \eta_T(t)^{\frac{-1}{p-1}} |\eta_T''(t)|^{\frac{p}{p-1}} \, dt \right) \left(\int_\Omega H_T(x) \, dx \right). \tag{3.6}$$

On the other hand, using Lemma 2.3 with $m = p$, we obtain

$$\int_0^\infty \eta_T(t)^{\frac{-1}{p-1}} |\eta_T''(t)|^{\frac{p}{p-1}} \, dt \le C T^{1-\frac{2p}{p-1}}. \tag{3.7}$$

Moreover, combining Lemma 2.4 with Lemma 2.5, one deduces that for all $\lambda \ge -\left(\frac{N-2}{2}\right)^2$,

$$\int_\Omega H_T(x) \, dx \le C T^{\theta\left(\frac{N+2}{2} + \lambda_N\right)} \ln T. \tag{3.8}$$

Hence, by (3.6), (3.7) and (3.8), it holds that

$$\int_{\mathcal{O}} \varphi_T^{\frac{-1}{p-1}} |\partial_{tt}\varphi_T|^{\frac{p}{p-1}} \, dx\, dt \le C T^{\theta\left(\frac{N+2}{2} + \lambda_N\right) + 1 - \frac{2p}{p-1}} \ln T. \tag{3.9}$$

Again, by the definition of the function φ_T, one has

$$\int_{\mathcal{O}} \varphi_T^{\frac{-1}{p-1}} |\Delta_\lambda \varphi_T|^{\frac{p}{p-1}} \, dx\, dt = \left(\int_0^\infty \eta_T(t) \, dt \right) \left(\int_\Omega H_T^{\frac{-1}{p-1}} |\Delta_\lambda H_T|^{\frac{p}{p-1}} \, dx\, dt \right). \tag{3.10}$$

Combining Lemma 2.6 with Lemma 2.7, and taking $m = p$, one deduces that for all $\lambda \geq -\left(\frac{N-2}{2}\right)^2$,

$$\int_\Omega H_T^{\frac{-1}{p-1}} |\Delta_\lambda H_T|^{\frac{p}{p-1}}\,dx\,dt \leq CT^{\theta\left(\lambda_N + \frac{N+2}{2} - \frac{2p}{p-1}\right)} \ln T. \tag{3.11}$$

Hence, by Lemma 2.2, (3.10) and (3.11), we obtain

$$\int_\mathcal{O} \varphi_T^{\frac{-1}{p-1}} |\Delta_\lambda \varphi_T|^{\frac{p}{p-1}}\,dx\,dt \leq CT^{\theta\left(\lambda_N + \frac{N+2}{2} - \frac{2p}{p-1}\right)+1} \ln T. \tag{3.12}$$

Consider now the term from the left-hand side of (3.5). By the definition of L_{φ_T}, if $\alpha > 0$, one has

$$L_{\varphi_T}(w) = \frac{1}{\alpha}\int_{\partial\mathcal{O}} w(x)\varphi_T(t,x)\,d\sigma\,dt = \frac{1}{\alpha}\left(\int_0^\infty \eta_T(t)\,dt\right)\left(\int_{\partial\Omega} w(x)H_T(x)\,d\sigma\right).$$

By the definition of the function H_T, and using Lemma 2.2, it holds that

$$L_{\varphi_T}(w) = CT\int_{\partial\Omega} w(x)H_{\alpha,\beta}(x)\xi\left(\frac{1}{T^{2\theta}}\right)^\ell\,d\sigma.$$

Since T is supposed to be large enough, by the definition of the cut-off function ξ, we get

$$L_{\varphi_T}(w) = CT\int_{\partial\Omega} w(x)H_{\alpha,\beta}(x)\,d\sigma.$$

On the other hand, by the definition of the function $H_{\alpha,\beta}$, for all $x \in \partial\Omega$ ($|x| = 1$), one has

$$H_{\alpha,\beta}(x) = \begin{cases} \alpha > 0 & \text{if} \quad \lambda = -\left(\frac{N-2}{2}\right)^2, \\ 2\lambda_N\alpha > 0 & \text{if} \quad \lambda > -\left(\frac{N-2}{2}\right)^2. \end{cases}$$

Then, for all $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, one obtains

$$L_{\varphi_T}(w) = CT\int_{\partial\Omega} w(x)\,d\sigma, \quad \alpha > 0. \tag{3.13}$$

If $\alpha = 0$, by the definition of L_{φ_T}, and using Lemma 2.2, one has

$$L_{\varphi_T}(w) = \frac{-1}{\beta}\int_{\partial\mathcal{O}} w(x)\frac{\partial\varphi}{\partial\nu}\,d\sigma\,dt = -CT\int_{\partial\Omega} w(x)\frac{\partial H_T}{\partial\nu}\,d\sigma.$$

Notice that by (2.1) and (2.2), one has

$$\frac{\partial H_T}{\partial\nu}\bigg|_{\partial\Omega} = \begin{cases} -\beta < 0 & \text{if} \quad \lambda = -\left(\frac{N-2}{2}\right)^2, \\ -2\lambda_N\beta < 0 & \text{if} \quad \lambda > -\left(\frac{N-2}{2}\right)^2. \end{cases}$$

Hence, for all $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, one obtains

$$L_{\varphi_T}(w) = CT\int_{\partial\Omega} w(x)\,d\sigma, \quad \alpha = 0. \tag{3.14}$$

Combining (3.13) with (3.14), one obtains

$$L_{\varphi_T}(w) = CT\int_{\partial\Omega} w(x)\,d\sigma, \quad \alpha,\beta \geq 0,\ (\alpha,\beta) \neq (0,0). \tag{3.15}$$

Now, using (3.5), (3.9), (3.12) and (3.15), we obtain

$$\int_{\partial\Omega} w(x)\,d\sigma \le C\left(T^{\theta\left(\frac{N+2}{2}+\lambda_N\right)-\frac{2p}{p-1}} + T^{\theta\left(\lambda_N+\frac{N+2}{2}-\frac{2p}{p-1}\right)}\right)\ln T. \tag{3.16}$$

Observe that for $\theta = 1$, one has

$$\theta\left(\frac{N+2}{2}+\lambda_N\right)-\frac{2p}{p-1} = \theta\left(\lambda_N+\frac{N+2}{2}-\frac{2p}{p-1}\right) = \lambda_N+\frac{N+2}{2}-\frac{2p}{p-1}.$$

Hence, taking $\theta = 1$ in (3.16), we get

$$\int_{\partial\Omega} w(x)\,d\sigma \le CT^{\lambda_N+\frac{N+2}{2}-\frac{2p}{p-1}}\ln T. \tag{3.17}$$

We discuss two cases.

Case 1: $\lambda = -\left(\frac{N-2}{2}\right)^2$.
In this case, one has $\lambda_N = 0$. So (3.17) reduces to

$$\int_{\partial\Omega} w(x)\,d\sigma \le CT^{\frac{N+2}{2}-\frac{2p}{p-1}}\ln T. \tag{3.18}$$

Moreover, if $N = 2$, (3.18) reduces to

$$\int_{\partial\Omega} w(x)\,d\sigma \le CT^{2\left(1-\frac{p}{p-1}\right)}\ln T. \tag{3.19}$$

Hence, passing to the limit as $T \to \infty$ in (3.19), one obtains a contradiction with the assumption $\int_{\partial\Omega} w(x)\,d\sigma > 0$. This proves part (i) of Theorem 1.1. If $N \ge 3$ and

$$1 < p < 1 + \frac{4}{N-2},$$

one can check easily that

$$\frac{N+2}{2} - \frac{2p}{p-1} < 0.$$

Hence, passing to the limit as $T \to \infty$ in (3.18), we obtain a contradiction. This proves part (ii) of Theorem 1.1.

Case 2: $\lambda > -\left(\frac{N-2}{2}\right)^2$.
In this case, one has $\lambda_N > 0$. Moreover, it can be easily seen that, if

$$1 < p < 1 + \frac{4}{N-2+2\lambda_N},$$

then

$$\lambda_N + \frac{N+2}{2} - \frac{2p}{p-1} < 0.$$

Hence, passing to the limit as $T \to \infty$ in (3.17), we lead to contradiction. This proves part (i) of Theorem 1.2. □

3.2 Existence results

Now, we prove the existence results given by part (iii) of Theorem 1.1 and part (ii) of Theorem 1.2.

Proof of part (iii) *of Theorem 1.1.* Let $N \geq 3$, $\alpha, \beta \geq 0$, $(\alpha, \beta) \neq (0, 0)$, $\lambda = -\left(\frac{N-2}{2}\right)^2$, and

$$p > 1 + \frac{4}{N-2}. \tag{3.20}$$

For

$$0 < \delta < 1, \quad \mu = \frac{2-N}{2}, \quad \tau > e^{\frac{\alpha\delta}{\beta-\alpha\mu}} \geq 1, \quad \varepsilon > 0,$$

let

$$u(x) = \varepsilon |x|^{\mu} \left(\ln(\tau|x|)\right)^{\delta}, \quad x \in \Omega. \tag{3.21}$$

Elementary calculations show that

$$\begin{aligned} -\Delta_\lambda u(x) &= -\varepsilon |x|^{\mu-2} \left(\ln(\tau|x|)\right)^{\delta-2} \left[\left(\mu(N+\mu-2)-\lambda\right) \left(\ln(\tau|x|)\right)^2 + \delta(N+2\mu-2)\ln(\tau|x|) + \delta(\delta-1)\right] \\ &= \varepsilon\delta(1-\delta)|x|^{\mu-2} \left(\ln(\tau|x|)\right)^{\delta-2} \end{aligned}$$

and

$$\begin{aligned} -\Delta_\lambda u(x) - |u(x)|^p &= \varepsilon\delta(1-\delta)|x|^{\mu-2} \left(\ln(\tau|x|)\right)^{\delta-2} - \varepsilon^p |x|^{\mu p} \left(\ln(\tau|x|)\right)^{\delta p} \\ &= \varepsilon|x|^{\mu-2} \left(\ln(\tau|x|)\right)^{\delta-2} \left[\delta(1-\delta) - \varepsilon^{p-1}|x|^{\mu p-\mu+2} \left(\ln(\tau|x|)\right)^{\delta(p-1)+2}\right]. \end{aligned} \tag{3.22}$$

On the other hand, by (3.20), it holds that

$$\mu p - \mu + 2 = \frac{(2-N)(p-1)}{2} + 2 < 0.$$

Hence, there exists a constant $A > 0$ such that

$$|x|^{\mu p-\mu+2} \left(\ln(\tau|x|)\right)^{\delta(p-1)+2} \leq A, \quad x \in \Omega,$$

which yields (by (3.22))

$$-\Delta_\lambda u(x) - |u(x)|^p \geq \varepsilon|x|^{\mu-2} \left(\ln(\tau|x|)\right)^{\delta-2} \left[\delta(1-\delta) - \varepsilon^{p-1}A\right].$$

Since $0 < \delta < 1$, taking $0 < \varepsilon < \left[\frac{\delta(1-\delta)}{A}\right]^{\frac{1}{p-1}}$, one obtains

$$-\Delta_\lambda u(x) - |u(x)|^p \geq 0, \quad x \in \Omega.$$

On the other hand, for $r = |x|$, we have

$$\begin{aligned} \left(\alpha\frac{\partial u}{\partial \nu} + \beta u\right)\Big|_{\partial\Omega} &= \left(-\alpha\frac{\partial u}{\partial r} + \beta u\right)\Big|_{r=1} \\ &= \varepsilon(\ln\tau)^{\delta-1}\left[(\beta-\alpha\mu)\ln\tau - \alpha\delta\right] \\ &:= w. \end{aligned} \tag{3.23}$$

Since $\tau > e^{\frac{\alpha\delta}{\beta-\alpha\mu}}$, we deduce that $w > 0$. Hence, the function u defined by (3.21) is a stationary solution to (1.1)–(1.2), where $w > 0$ is given by (3.23). This proves part (iii) of Theorem 1.1. □

Proof of part (ii) *of Theorem 1.2.* Let $N \geq 2$, $\alpha, \beta \geq 0$, $(\alpha, \beta) \neq (0, 0)$, $\lambda > -\left(\frac{N-2}{2}\right)^2$, and

$$p > 1 + \frac{4}{N-2+2\lambda_N}. \tag{3.24}$$

For

$$\frac{2-N}{2} - \lambda_N < \delta < \min\left\{\frac{-2}{p-1}, \frac{2-N}{2} + \lambda_N\right\} \tag{3.25}$$

and

$$0 < \varepsilon \leq \left[-\delta^2 + (2-N)\delta + \lambda\right]^{\frac{1}{p-1}}, \tag{3.26}$$

let

$$u(x) = \varepsilon |x|^{\delta}, \quad x \in \Omega. \tag{3.27}$$

Notice that by (3.24), the set of δ satisfying (3.25) is nonempty. Moreover, by (3.25), since $\lambda > -\left(\frac{N-2}{2}\right)^2$, one has

$$-\delta^2 + (2-N)\delta + \lambda > 0.$$

Elementary calculations show that

$$-\Delta_\lambda u - |u|^p = \varepsilon |x|^{\delta-2}\left[\left(-\delta^2 + (2-N)\delta + \lambda\right) - \varepsilon^{p-1}|x|^{\delta p - \delta + 2}\right].$$

Hence, using (3.25) and (3.26), we obtain

$$\begin{aligned} -\Delta_\lambda u - |u|^p &\geq \varepsilon |x|^{\delta-2}\left[\left(-\delta^2 + (2-N)\delta + \lambda\right) - \varepsilon^{p-1}\right] \\ &\geq 0. \end{aligned}$$

On the other hand, for $r = |x|$, we have

$$\begin{aligned} \left(\alpha \frac{\partial u}{\partial \nu} + \beta u\right)\Big|_{\partial\Omega} &= \left(-\alpha \frac{\partial u}{\partial r} + \beta u\right)\Big|_{r=1} \\ &= \varepsilon(\beta - \alpha\delta) > 0. \end{aligned}$$

Hence, we deduce that the function u defined by (3.27) is a stationary solution to (1.1)–(1.2), where $w \equiv \varepsilon(\beta - \alpha\delta) > 0$. This proves part (ii) of Theorem 1.2. □

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