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On the sigma chromatic number of the zero-divisor graphs of the ring of integers modulo n

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Abstract. The zero-divisor graph of a commutative ring R with unity is the graph $\Gamma(R)$ whose vertex set is the set of nonzero zero divisors of R , where two vertices are adjacent if and only if their product in R is zero. A vertex coloring $c : V(G) \rightarrow \mathbb{N}$ of a non-trivial connected graph G is called a *sigma coloring* if $\sigma(u) \neq \sigma(v)$ for any pair of adjacent vertices u and v . Here, $\sigma(x)$ denotes the sum of the colors assigned to vertices adjacent to x . The *sigma chromatic number* of G , denoted by $\sigma(G)$, is defined as the least number of colors needed to construct a sigma coloring of G . In this paper, we analyze the structure of the zero-divisor graph of rings \mathbb{Z}_n , where $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, where m, n_1, n_2, \dots, n_m are positive integers and p_1, p_2, \dots, p_m are distinct primes. The analysis is carried out by partitioning the vertex set of such zero-divisor graphs and analyzing the adjacencies, cardinality, and the degree of the vertices in each set of the partition. Using these properties, we determine the sigma chromatic number of these zero-divisor graphs.

1. Introduction

Graph coloring has been the subject of many researches in recent years and has been recognized for its importance as a tool for solving various problems on networks. Given a graph G , a coloring of G is a mapping $c : V(G) \rightarrow \mathbb{N}$, where $V(G)$ is the set of vertices of the graph and \mathbb{N} is the set of positive integers, also referred to as the set of colors. A *proper coloring* is one in which $c(u) \neq c(v)$ for any two adjacent vertices u and v in G . Other types of coloring have been introduced and developed by several graph theorists. Among them is a neighbor-distinguishing coloring, called *sigma coloring*, introduced by Chartrand, Okamoto and Zhang in [1]. Let G be a nontrivial connected graph and let c be a coloring (not necessarily a proper coloring) of G . For a vertex v of G , the *color sum* of v is $\sigma(v) = \sum_{x \in N(v)} c(x)$, where $N(v)$ is the set of vertices which are adjacent to v . The coloring c is called a *sigma coloring* of the graph G if $\sigma(u) \neq \sigma(v)$ for every pair of adjacent vertices u and v . The minimum number of colors that can be used in a sigma coloring of G is called the *sigma chromatic number* of G and is denoted by $\sigma(G)$.

Chartand et al [1] proved that $\sigma(G) \leq \chi(G)$, where $\chi(G)$ is the chromatic number of G . In the same paper, the authors determined the sigma chromatic numbers of paths, cycles, bipartite and complete multipartite graphs. On the other hand, Luzon, Ruiz and Tolentino [2] obtained the sigma chromatic number of some family of circulant graphs. More recently, the sigma chromatic number of the Sierpinski gasket, the Hanoi graphs, join of a finite number of paths and cycles, graph coronas involving complete graphs, power of graphs, and some families of snarks have been



studied in [3–6]. The notion of sigma coloring is closely related to the vertex colorings/labellings that are discussed in [7–9].

Adapting the notion introduced by Anderson and Livingston in [10], the zero divisor graph of a commutative ring R is defined as follows: Given a commutative ring R , the *zero-divisor graph* of R , denoted by $\Gamma(R)$, is the graph whose vertices are the nonzero zero divisors of R and in which two vertices x and y are adjacent if and only if $xy = 0$ in R . Some studies, which mainly focus on the structure and parameters of zero-divisor graphs of some ring families, may be found in [11–14]. A review on zero-divisor graphs of finite commutative rings can be found in [15].

In this paper, we analyze the structure of the zero-divisor graph of rings \mathbb{Z}_n , where $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, where m, n_1, n_2, \dots, n_m are positive integers and p_1, p_2, \dots, p_m are distinct primes. In Section 2, we carry out this analysis by partitioning the vertex set of such zero-divisor graphs and analyzing the adjacencies, cardinality, and the degree of the vertices in each element of a partition. In Section 3, we make use of these results to determine the sigma chromatic number of these zero-divisor graphs.

2. The structure of $\Gamma(\mathbb{Z}_n)$

We begin by recalling the following theorem.

Theorem 2.1 (Anderson and Livingston [10]). *Let R be a commutative ring with unity. Then $\Gamma(R)$ is finite if and only if either R is finite or an integral domain. In particular, if $1 \leq |\Gamma(R)| < \infty$, then R is finite and not a field.*

It follows that the zero-divisor graph of a finite field is empty. Thus, we restrict our attention to rings \mathbb{Z}_n that are not fields. In this case, the zero-divisor graph is finite and connected [10]. Moreover, if n has prime factorization $p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, where m, n_1, n_2, \dots, n_m are positive integers and p_1, p_2, \dots, p_m are distinct primes, then the ring \mathbb{Z}_n is isomorphic to $\mathbb{Z}_{p_1}^{n_1} \times \mathbb{Z}_{p_2}^{n_2} \times \cdots \times \mathbb{Z}_{p_m}^{n_m}$.

Proposition 2.2. *Let m be a positive integer, and let $R = \prod_{k=1}^m \mathbb{Z}_{p_k}^{n_k}$, where m, n_1, n_2, \dots, n_m are positive integers and p_1, p_2, \dots, p_m are primes (not necessarily distinct). Then a nonzero element $x = (x_1, x_2, \dots, x_m) \in R$ is a zero divisor of R if and only if $p_k | x_k$ for some $k = 1, 2, \dots, m$.*

Proof.

Without loss of generality, suppose $p_1 | x_1$. Then $x_1 = p_1^s r_1$ where $p_1 \nmid r_1$.

- If $s = 0$, then $x_1 = 0$. Moreover, $x \neq 0$ implies $m > 1$ and so $x_t \neq 0$ for some $t \geq 2$. Then for any $r \neq 0$, $(r, 0, \dots, 0) \cdot (x_1, x_2, \dots, x_m) = (0, 0, \dots, 0)$.
- If $1 \leq s \leq n_1 - 1$, then $p_1^{n_1-s} \neq 0$ and $(x_1, x_2, \dots, x_m) \cdot (p_1^{n_1-s}, 0, 0, \dots, 0) = (0, 0, \dots, 0)$.

Hence, in either case, x is a zero divisor of R .

Now, suppose x is a zero divisor of R and that $p_k \nmid x_k$ for all $k = 1, 2, \dots, m$. Then $x \cdot r = 0$ only if $r = 0$, contrary to the assumption that x is a zero divisor. \square

The Sets R_{i_1, i_2, \dots, i_m}

As mentioned in Section 1, we will analyze the structure of $\Gamma(\mathbb{Z}_n)$ by partitioning its vertex set. We define the following sets: For each $k \in \{1, 2, \dots, m\}$, let $i_k \in \{0, 1, \dots, n_k\}$. We denote by R_{i_1, i_2, \dots, i_m} the set of all $(x_1, x_2, \dots, x_m) \in R$ such that for each $k = 1, 2, \dots, m$, we have

- $x_k = 0$ if $i_k = n_k$, and
- $p_k^{i_k} | x_k$ and $p_k^{i_k+1} \nmid x_k$ if $i_k = 0, 1, \dots, n_k - 1$.

Note that $R_{n_1, n_2, \dots, n_m} = \{(0, 0, \dots, 0)\}$ while, by Proposition 2.2, the set $R_{0, 0, \dots, 0}$ is the set of elements that are not zero divisors of R . Hence,

$$V(\Gamma(R)) = \bigcup_{\substack{(i_1, i_2, \dots, i_m) \neq (0, 0, \dots, 0), \\ (i_1, i_2, \dots, i_m) \neq (n_1, n_2, \dots, n_m)}} R_{i_1, i_2, \dots, i_m}.$$

For convenience, throughout this paper, we will refer to the sets R_{i_1, i_2, \dots, i_m} , where $(i_1, i_2, \dots, i_m) \notin \{(0, 0, \dots, 0), (n_1, n_2, \dots, n_m)\}$, as *blocks* of $V(\Gamma(R))$. The usefulness of these blocks in analyzing the structure of $\Gamma(R)$ is captured by the following proposition.

Proposition 2.3. *Let m be a positive integer, and let $R = \prod_{k=1}^m \mathbb{Z}_{p_k}^{n_k}$, where m, n_1, n_2, \dots, n_m are positive integers and p_1, p_2, \dots, p_m are primes (not necessarily distinct). Then the following statements hold.*

- (i) *If $u, v \in R_{i_1, i_2, \dots, i_m}$, then $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.*
- (ii) *The vertices in R_{i_1, i_2, \dots, i_m} are adjacent to all the vertices in R_{j_1, j_2, \dots, j_m} if and only if $i_k + j_k \geq n_k$ for all $k = 1, 2, \dots, m$.*
- (iii) *The vertices in R_{i_1, \dots, i_m} form a clique in $\Gamma(R)$ if and only if $2i_k \geq n_k$ for all $k = 1, 2, \dots, m$.*

Proof. (i) It is sufficient to show that $N(u) \setminus \{v\} \subseteq N(v) \setminus \{u\}$. Suppose $w \neq v$ is adjacent to u . Let $u = (u_1, u_2, \dots, u_m), v = (v_1, v_2, \dots, v_m)$, and $w = (w_1, w_2, \dots, w_m)$. Let $k \in \{1, \dots, m\}$:

- If $i_k = n_k$, then $u_k = v_k = 0$ and $v_k \cdot w_k = 0$.
- If $0 \leq i_k \leq n_k - 1$, then since $u_k \cdot w_k = 0$, $p_k^{i_k} | u_k$, and $p_k^{i_k+1} \nmid u_k$, it follows that $p_k^{n_k - i_k} | w_k$. Now, $p_k^{i_k}$ also divides v_k and so $p_k^{n_k} | v_k \cdot w_k$, which implies that $v_k \cdot w_k = 0$.

Hence, $v \cdot w = 0$ and w is also adjacent to v .

- (ii) Let $x = (x_1, x_2, \dots, x_n) \in R_{i_1, i_2, \dots, i_m}$ and $y = (y_1, y_2, \dots, y_n) \in R_{j_1, j_2, \dots, j_m}$. Then x and y are adjacent if and only if $x_k y_k \equiv 0 \pmod{p_k^{n_k}}$ for $k = 1, 2, \dots, n$.
 - If, for some $k \in \{1, 2, \dots, n\}$, either x_k or y_k is 0, then, by definition, either i_k or j_k is equal to n_k and it follows that $i_k + j_k \geq n_k$.
 - If neither x_k nor y_k is 0, then by definition, $x_k = p_k^{i_k} \cdot s$ and $y_k = p_k^{j_k} \cdot t$, where neither s nor t is divisible by p_k . Hence, $x_k y_k \equiv 0 \pmod{p_k^{n_k}}$ if and only if $i_k + j_k \geq n_k$.
- (iii) This follows directly from (ii). □

In light of Proposition 2.3 (ii), we say that a block R_{i_1, \dots, i_m} is *adjacent* to another block R_{j_1, \dots, j_m} if all the vertices in R_{i_1, \dots, i_m} are adjacent to all the vertices in R_{j_1, \dots, j_m} . Figure 1 shows the zero-divisor graph of the ring $\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2}$ depicted using the sets R_{i_1, i_2} . Yellow vertices represent cliques.

Proposition 2.3 (iii) implies that the vertices in a block are either all adjacent to each other (i.e. the block is a clique) or no two are adjacent to each other within the block. Thus, by Proposition 2.3 (i), it follows that if a block is a clique, then $N(u) \cup \{u\} = N(v) \cup \{v\}$; while on the other hand, if a block is not a clique, then $N(u) = N(v)$. This implies that the vertices in the same block have equal degrees. The following proposition gives a formula for the degree of a vertex in any block as well as a formula for the cardinality of each block.

Proposition 2.4. *Let m be a positive integer, and let $R = \prod_{k=1}^m \mathbb{Z}_{p_k}^{n_k}$, where m, n_1, n_2, \dots, n_m are positive integers and p_1, p_2, \dots, p_m are primes (not necessarily distinct). Then the following statements hold.*

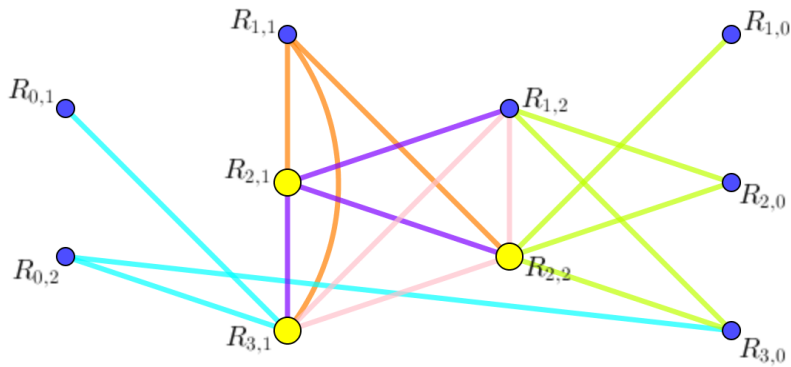


Figure 1. $\Gamma(\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2})$

$$(i) |R_{i_1, i_2, \dots, i_m}| = \prod_{k=1}^m \phi(p_k^{n_k - i_k}) = \prod_{k=1}^m [p_k^{n_k - i_k} - p_k^{n_k - i_k - 1}].$$

(ii) Let $u \in R_{i_1, i_2, \dots, i_m}$. Then

$$\deg u = \begin{cases} -1 + \prod_{\ell=1}^m p_\ell^{i_\ell}, & \text{if } R_{i_1, i_2, \dots, i_m} \text{ is not a clique,} \\ -2 + \prod_{\ell=1}^m p_\ell^{i_\ell}, & \text{if } R_{i_1, i_2, \dots, i_m} \text{ is a clique.} \end{cases}$$

Proof. (i) First, if $0 \leq i_1 < n_1$, then

$$\begin{aligned} \#\{r \in \mathbb{Z}_{p_1^{n_1}} : p_1^{i_1} | r \text{ and } p_1^{i_1+1} \nmid r\} &= \frac{p_1^{n_1}}{p_1^{i_1}} - \frac{p_1^{n_1}}{p_1^{i_1+1}} \\ &= p_1^{n_1 - i_1} - p_1^{n_1 - i_1 - 1} \\ &= p_1^{n_1 - i_1 - 1} (p_1 - 1) \\ &= [p_1^{n_1 - i_1} - p_1^{n_1 - i_1 - 1}]. \end{aligned}$$

Moreover, if $i_1 = n_1$, note that

$$[p_1^{n_1 - i_1} - p_1^{n_1 - i_1 - 1}] = \left[1 - \frac{1}{p_1} \right] = 1.$$

The result follows immediately.

(ii) If R_{i_1, i_2, \dots, i_m} is not a clique,

$$\begin{aligned} \deg(u) &= \left[\sum_{j_m=n_m-i_m}^{n_m} \cdots \sum_{j_1=n_1-i_1}^{n_1} |R_{j_1, j_2, \dots, j_m}| \right] - |R_{n_1, n_2, \dots, n_m}| \\ &= -1 + \sum_{j_m=n_m-i_m}^{n_m} \cdots \sum_{j_1=n_1-i_1}^{n_1} |R_{j_1, j_2, \dots, j_m}| \\ &= -1 + \sum_{j_m=n_m-i_m}^{n_m} \cdots \sum_{j_1=n_1-i_1}^{n_1} \left[\prod_{\ell=1}^m [p_\ell^{n_\ell-j_\ell} - p_\ell^{n_\ell-j_\ell-1}] \right] \\ &= -1 + \prod_{\ell=1}^m \sum_{j_\ell=n_\ell-i_\ell}^{n_\ell} [p_\ell^{n_\ell-j_\ell} - p_\ell^{n_\ell-j_\ell-1}] \\ &= -1 + \prod_{\ell=1}^m (p_\ell^{i_\ell} - 1 + 1) \\ &= -1 + \prod_{\ell=1}^m p_\ell^{i_\ell}. \end{aligned}$$

If R_{i_1, i_2, \dots, i_m} is a clique, we subtract one from the expression above since u is not adjacent to itself. □

Note that Proposition 2.4 (ii) also appears in [15, 16] but in a different form. The following corollary, which holds when p_1, p_2, \dots, p_n are distinct, will be important in determining the sigma chromatic number of $\Gamma(\mathbb{Z}_n)$.

Corollary 2.5. *Let $R = \prod_{k=1}^m \mathbb{Z}_{p_k^{n_k}}$, where m, n_1, n_2, \dots, n_m are positive integers and p_1, p_2, \dots, p_m are distinct primes. If $u \in R_{i_1, i_2, \dots, i_m}$, $v \in R_{j_1, j_2, \dots, j_m}$, and $R_{i_1, i_2, \dots, i_m} \neq R_{j_1, j_2, \dots, j_m}$, then $\deg u \neq \deg v$.*

Proof.

- (i) Suppose R_{i_1, i_2, \dots, i_m} is a clique while R_{j_1, j_2, \dots, j_m} is not.
 - (a) Suppose there is an ℓ for which $i_\ell > 0$ and $j_\ell > 0$. Then $\deg u \equiv -1 \pmod{p_\ell}$ while $\deg v \equiv -2 \pmod{p_\ell}$.
 - (b) Suppose $i_\ell j_\ell = 0$ for all ℓ . Since R_{i_1, i_2, \dots, i_m} is a clique, by Proposition 2.3 (iii), we must have $i_\ell > 0$ for all ℓ . Then $j_\ell = 0$ for all ℓ , which cannot happen.
- (ii) Suppose R_{i_1, i_2, \dots, i_m} and R_{j_1, j_2, \dots, j_m} are both cliques or are both not cliques. Without loss of generality, assume $i_1 > j_1$. Since p_1, p_2, \dots, p_m are distinct primes, then $\prod_{\ell=1}^m p_\ell^{i_\ell}$ is divisible by $p_1^{i_1}$ while $\prod_{\ell=1}^m p_\ell^{j_\ell}$ is not. □

3. The sigma chromatic number of $\Gamma(\mathbb{Z}_n)$

The *closed neighborhood* of a vertex u , denoted by $N[u]$, in a graph G is given by $N[u] = N(u) \cup \{u\}$. The following is an important observation.

Observation 3.1 (Chartand et al. [1]). *If H is a complete subgraph of order k in a graph G such that $N[u] = N[v]$ for every two vertices u and v of H , then $\sigma(G) \geq k$.*

Identifying the largest possible order of a subgraph H as in Observation 3.1 is important in establishing a lower bound for the sigma chromatic number. Hence, we denote by $\bar{\omega}(G)$ the order of a subgraph H of G of maximal cardinality for which

- (i) H is a clique; and
- (ii) for all $u, v \in H$, we have $N[u] = N[v]$.

Finally, through the following theorem, we obtain the sigma chromatic number of $\Gamma(\mathbb{Z}_n)$. Recall that $\Gamma(\mathbb{Z}_n)$ is empty when \mathbb{Z}_n is a field and that \mathbb{Z}_n is isomorphic to $\mathbb{Z}_{p_1}^{n_1} \times \mathbb{Z}_{p_2}^{n_2} \times \dots \times \mathbb{Z}_{p_m}^{n_m}$ if n has prime factorization $p_1^{n_1} p_2^{n_2} \dots p_m^{n_m}$, where m, n_1, n_2, \dots, n_m are positive integers and p_1, p_2, \dots, p_m are distinct primes.

Theorem 3.2. *Let $m \geq 1$ and let $R = \prod_{k=1}^m \mathbb{Z}_{p_k}^{n_k}$, where p_1, p_2, \dots, p_m are distinct primes and n_1, n_2, \dots, n_m are positive integers. If R is not a field, then*

$$\sigma(\Gamma(R)) = \bar{\omega}(\Gamma(R)) = \prod_{k=1}^m \left[p_k^{n_k - \lceil \frac{n_k}{2} \rceil - 1} (p_k - 1) \right].$$

Proof. First, we prove that $\bar{\omega}(\Gamma(R)) = \prod_{k=1}^m \left[p_k^{n_k - \lceil \frac{n_k}{2} \rceil - 1} (p_k - 1) \right]$. Let H be a subgraph of $\Gamma(R)$ of maximal cardinality satisfying (i) and (ii) above.

By Corollary 2.5, it follows that $N[u] \neq N[v]$ whenever u and v belong to different blocks. Moreover, by Proposition 2.3(i), $N[u] = N[v]$ whenever u and v belong to the same block that is a clique. Hence, $H = R_{i_1, i_2, \dots, i_m}$ for some i_1, i_2, \dots, i_m ; moreover, by Proposition 2.3(iii), we must have $2i_k \geq n_k$ for all k . Since the order of H must be the largest possible, by Proposition 2.4 (i), we must have $i_k = \lceil n_k/2 \rceil$ for all k . Therefore, $H = R_{\lceil n_1/2 \rceil, \lceil n_2/2 \rceil, \dots, \lceil n_m/2 \rceil}$ and $\bar{\omega}(\Gamma(R)) = \prod_{k=1}^m \left[p_k^{n_k - \lceil \frac{n_k}{2} \rceil - 1} (p_k - 1) \right]$.

Now, we prove that $\sigma(\Gamma(R)) = \bar{\omega}(\Gamma(R))$. By Observation 3.1, we have $\sigma(\Gamma(R)) \geq \bar{\omega}(\Gamma(R))$. We now construct a vertex coloring c of $\Gamma(R)$ that uses $\bar{\omega}(\Gamma(R))$ colors. Let $d = \Delta(\Gamma(R)) + 1$ and define c as follows: Let $S = R_{i_1, i_2, \dots, i_m}$ for some i_1, i_2, \dots, i_m .

- (i) If S is a clique, set $c(S) = \{1, d, d^2, \dots, d^{|S|-1}\}$.
- (ii) If S is not a clique, set $c(S) = \{1\}$.

Hence, c uses $|H| = \bar{\omega}(\Gamma(R))$ colors.

We show that c is a sigma coloring. By the choice of colors, it is enough to consider only adjacent vertices with equal degrees. By Corollary 2.5, two vertices have equal degrees only if they belong to the same block. For two vertices in the same block to be adjacent, the block has to be a clique. So suppose $u, v \in S = R_{i_1, i_2, \dots, i_m}$ for some i_1, i_2, \dots, i_m , where S is a clique. Then $N(u) \setminus \{v\} = N(v) \setminus \{u\}$ while $c(u) \neq c(v)$. This implies that

$$\sigma(u) = c(v) + \sum_{x \in N(u) \setminus \{v\}} c(x) \neq c(u) + \sum_{y \in N(v) \setminus \{u\}} c(y) = \sigma(v).$$

Therefore, c is a sigma coloring and the conclusion follows. □

By taking advantage of the properties of the zero-divisor graph of a ring of the form $R = \prod_{k=1}^m \mathbb{Z}_{p_k}^{n_k}$, where p_1, p_2, \dots, p_m are distinct primes and n_1, n_2, \dots, n_m are positive integers, we were able to determine in Theorem 3.2 the exact sigma chromatic number of such a zero-divisor graph. Consequently, we have also determined the sigma chromatic number of any ring \mathbb{Z}_n , which is isomorphic to some ring of the form $\mathbb{Z}_{p_1}^{n_1} \times \mathbb{Z}_{p_2}^{n_2} \times \dots \times \mathbb{Z}_{p_m}^{n_m}$.

4. Conclusion

In this paper, we determined the sigma chromatic number of the zero-divisor graph of the ring \mathbb{Z}_n of integers modulo n , where n is restricted so that \mathbb{Z}_n is not a field. First, we analyzed the structure of $\Gamma(R)$, where $R = \mathbb{Z}_{p_1}^{n_1} \times \mathbb{Z}_{p_2}^{n_2} \times \cdots \times \mathbb{Z}_{p_m}^{n_m}$, m, n_1, n_2, \dots, n_m are positive integers, and p_1, p_2, \dots, p_m are primes. The analysis is carried out by partitioning the vertex set of $\Gamma(R)$ and determining the adjacencies, cardinality, and the degree of vertices of each element of the partition. The sigma chromatic number of $\Gamma(\mathbb{Z}_n)$ is determined using these properties and by noting that \mathbb{Z}_n is isomorphic $\mathbb{Z}_{p_1}^{n_1} \times \mathbb{Z}_{p_2}^{n_2} \times \cdots \times \mathbb{Z}_{p_m}^{n_m}$ if n has prime factorization $p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$.

Our result is that $\sigma(\Gamma(\mathbb{Z}_n)) = \prod_{k=1}^m \left[p_k^{n_k - \lceil \frac{n_k}{2} \rceil - 1} (p_k - 1) \right]$.

Our paper focuses on the determination of the sigma chromatic number using the properties pertaining to the structure of the zero-divisor graphs of the rings \mathbb{Z}_n . It might also be worthwhile to consider how these properties can be used to determine other graph parameters, including chromatic numbers associated with other colorings, for these zero-divisor graphs. Moreover, as the aim of the paper only involves zero-divisor graphs of rings of the form \mathbb{Z}_n , we have restricted our attention to rings of the form $\mathbb{Z}_{p_1}^{n_1} \times \mathbb{Z}_{p_2}^{n_2} \times \cdots \times \mathbb{Z}_{p_m}^{n_m}$, m, n_1, n_2, \dots, n_m are positive integers, and p_1, p_2, \dots, p_m are distinct primes. However, it is also interesting to consider the zero-divisor graphs of such rings without the assumption that the primes p_1, p_2, \dots, p_m are distinct.

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