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On the sigma value and sigma range of the join of a finite number of even cycles of the same order

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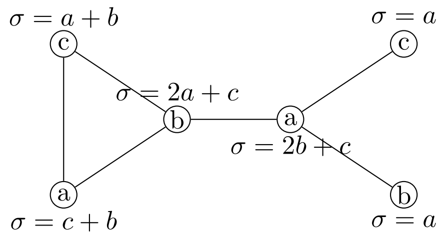
Abstract. Let $c : V(G) \to \mathbb{N}$ be a vertex coloring of a simple, connected graph G. For a vertex v of G, the *color sum* of v, denoted by $\sigma(v)$, is the sum of the colors of the neighbors of v. If $\sigma(u) \neq \sigma(v)$ for any two adjacent vertices u and v of G, then c is called a *sigma coloring* of G. The *sigma chromatic number of* G, denoted by $\sigma(G)$, is the minimum number of colors required in a sigma coloring of G. Let $\max(c)$ be the largest color assigned to a vertex of G by a coloring c. The *sigma value* of G, denoted by $\nu(G)$, is the minimum value of $\max(c)$ over all sigma $k-$colorings c of G for which $\sigma(G) = k$. On the other hand, the *sigma range* of G, denoted by $\rho(G)$, is the minimum value of $\max(c)$ over all sigma colorings c of G. In this paper, we determine the sigma value and the sigma range of the join of a finite number of even cycles of the same order. In particular, if $n \geq 4$ and n is even, then we will show that $\rho(kC_n) = \nu(kC_n) = 2$ if and only if (i) $k \leq \lfloor \frac{n}{6} \rfloor + 1$, whenever $n \equiv 0 \pmod 4$, and (ii) $k \leq \lfloor \frac{n-2}{6} \rfloor + 1$, whenever $n \equiv 2 \pmod 4$.

1. Introduction

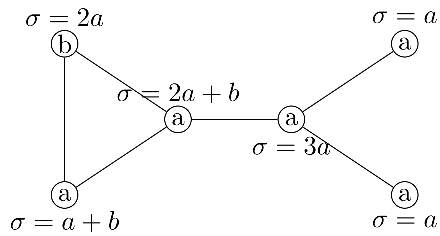
Let $G = (V(G), E(G))$ be a simple, connected graph with vertex set $V(G)$ and edge set $E(G)$. A *(vertex) coloring* of G is a function $c : V(G) \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers and is often referred to as the set of colors. A coloring c is said to be a *proper coloring* if $c(x) \neq c(y)$ whenever x and y are adjacent vertices in G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors required in a proper coloring of G.

A new type of coloring, called *sigma coloring* was introduced by Chartrand, Okamoto, and Zhang in [1]. Suppose c is a coloring of a graph G where adjacent vertices may possibly be assigned the same color. The *color sum* of a vertex v is given by $\sigma_G(v) = \sum_{x \in N(x)} c(x)$, where $N(x)$ is the neighborhood of x. For simplicity, the color sum of v will also be denoted by $\sigma(v)$ when the graph G is clear. We say that c is a *sigma coloring* of G if and only if $\sigma(u) \neq \sigma(v)$ whenever u and v are adjacent vertices in G. The *sigma chromatic number* of G is the minimum number of colors required in a sigma coloring of G, and is denoted by $\sigma(G)$. Thus, from the definition, a sigma coloring induces a proper coloring of a graph where each vertex v is assigned the color sum $\sigma(v)$.

Figure 1(a) below shows a sigma coloring of G using three colors and the color sums are indicated above or below each vertex. However, Figure 1(b) shows that it is possible to have a sigma coloring of G using only 2 colors. Since there are adjacent vertices with the same degree, it is not possible to use only one color. Thus, $\sigma(G) = 2$.



(a) A sigma 3-coloring; $a, b, c \in \mathbb{N}$



(b) A sigma 2-coloring; $a, b \in \mathbb{N}$

Figure 1. Sigma colorings of G

For a graph G, if $\chi(G) = k$, then there is always a proper coloring of G using elements of the set $\{1, 2, \cdots, k\}$. Although another set of colors may be used, we know that k is the smallest among all the largest colors in proper colorings of G. On the other hand, such need not be the case in a sigma coloring of G. Suppose $\sigma(G) = k$ and let c be a sigma coloring of G using k colors. Let $\max(c) = \max\{c(v)|v \in V(G)\}$ and $\min(c) = \min\{c(v)|v \in V(G)\}$. In [1], the following graph parameters associated to sigma colorings were defined. The *sigma value* of G is the number $\nu(G) = \min\{\max(c)|c$ is a sigma $k-$coloring of $G\}$. On the other hand, the *sigma range* of G is the number $\rho(G) = \min\{\max(c)|c$ is a sigma coloring of $G\}$. It was shown in [1] and [8] that $\sigma(G) \leq \rho(G) \leq \nu(G)$. Note that it is possible for a graph G to have $\sigma(G) \neq \rho(G)$ and $\rho(G) \neq \nu(G)$ as shown in [1] and [8].

This paper discusses the sigma value and sigma range of joins of even cycles of the same order. If G_1 and G_2 are disjoint graphs, the *join* of G_1 and G_2, denoted by $G_1 + G_2$, is a graph whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv|u \in V(G_1), v \in V(G_2)\}$. Studies on the sigma chromatic number of some families of graphs can be found in [2-6]. In [7], a coloring similar to a sigma coloring was also introduced. However, little is known about the two other graph parameters, $\nu(G)$ and $\rho(G)$. While it can be easily shown that for a cycle C_n, with $n \geq 3$, $\rho(C_n) = \nu(C_n) = 2$ if n is even and $\rho(C_n) = \nu(C_n) = 3$ if n is odd, the values of $\rho(C_n)$ and $\nu(C_n)$, where kC_n is the join of k cycles C_n are yet to be determined. In this paper, we address this problem when C_n is an even cycle. The methodology involves using the least and largest possible sums of colors in a sigma 2-coloring of C_n using only the colors 1 and 2.

2. Known Results

The following observations and theorems will be used in this paper.

Observation 2.1 ([1]). *Let G be a nontrivial connected graph. Then $\sigma(G) = 1$ if and only if any two adjacent vertices of G have different degrees.*

Theorem 2.1 ([1]). *If C_n is a cycle of order n, where $n \geq 3$, then*

$$\sigma(C_n) = \begin{cases} 2, & \textit{if } n \textit{ is even} \\ 3, & \textit{if } n \textit{ is odd.} \end{cases}$$

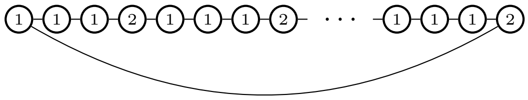
z

Theorem 2.2 ([2]). *Suppose c is a sigma $k-$coloring of $G + H$, where G and H are disjoint graphs. Then, the restricted colorings $c_1 = c|_{V(G)}$ and $c_2 = c|_{V(H)}$ are sigma colorings of G and H, respectively, that use at most k colors. Thus, $\sigma(G) \leq \sigma(G + H)$ and $\sigma(H) \leq \sigma(G + H)$.*

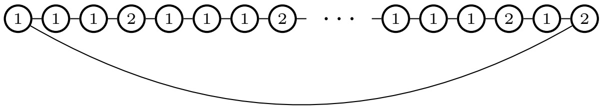
Theorem 2.3 ([1], [8]). *For a nontrivial connected graph G, $\sigma(G) \leq \rho(G) \leq \nu(G)$.*

3. Minimal and Maximal Sigma Colorings of C_n

Let $n \in \mathbb{N}$ and n be even. Also, let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$. From Theorem 2.1, we have $\sigma(C_n) = 2$. We give a sigma 2-coloring c_1 of C_n using the colors 1 and 2. Let $c_1(v_i) = 2$ if $4 \mid i$ or $i = n$, and $c_1(v_i) = 1$, otherwise. The coloring c_1 of C_n is illustrated in the diagram below.



(a) $n \equiv 0 \pmod 4$



(b) $n \equiv 2 \pmod 4$

Figure 2. The coloring c_1 of C_n

Note that when $n \equiv 0 \pmod 4$, the color sums of the vertices in C_n are as follows:

$$\sigma_{c_1}(v_i) = \begin{cases} 2, & \text{if } i \text{ is even} \\ 3, & \text{if } i \text{ is odd.} \end{cases}$$

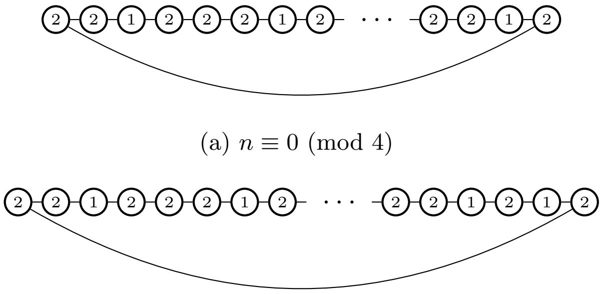
When $n \equiv 2 \pmod 4$, the color sums of the vertices in C_n are as follows:

$$\sigma_{c_1}(v_i) = \begin{cases} 2, & \text{if } i \text{ is even} \\ 3, & \text{if } i \text{ is odd and } i \neq n-1 \\ 4, & \text{if } i = n-1. \end{cases}$$

Since adjacent vertices in C_n have unequal color sums, c_1 is a sigma coloring of C_n. Since c_1 is a coloring that uses the colors 1 and 2, we then have the following result.

Proposition 3.1. *Let $n \geq 4$ be even. Then, $\rho(C_n) = \nu(C_n) = 2$.*

From the definition, the coloring c_1 maximizes the number of times that the smallest color, 1, is assigned to the vertices of C_n and minimizes the number of times that the largest color, 2, is assigned. If $S(n) = \sum_{x \in V(C_n)} c(x)$, where c is a sigma coloring of C_n, then c_1 is a sigma coloring giving the smallest possible sum $S(n)$. We say that a sigma coloring of C_n is a *minimal sigma coloring* of C_n if the corresponding value of $S(n)$ is minimal. On the other hand, a *maximal sigma coloring* of C_n is one that yields the largest value of $S(n)$ over all sigma colorings using the colors 1 and 2. A maximal sigma coloring of C_n can be constructed by interchanging the colors 1 and 2 in c_1. A maximal coloring c_2 may also be given as follows: $c_2(v_i) = 1$ if $i \equiv 3 \pmod 4$ or $i = n-1$, and $c_2(v_i) = 2$, otherwise.



(a) $n \equiv 0 \pmod 4$

(b) $n \equiv 2 \pmod 4$

Figure 3. The coloring c_2 of C_n

When $n \equiv 0 \pmod 4$, the color sums of the vertices in C_n are as follows:

$$\sigma_{c_2}(v_i) = \begin{cases} 3, & \text{if } i \text{ is even} \\ 4, & \text{if } i \text{ is odd.} \end{cases}$$

When $n \equiv 2 \pmod 4$, the color sums of the vertices in C_n are as follows:

$$\sigma_{c_2}(v_i) = \begin{cases} 2, & \text{if } i = n-2 \\ 3, & \text{if } i \text{ is even and } i \neq n-2 \\ 4, & \text{if } i \text{ is odd.} \end{cases}$$

In either case, adjacent vertices in C_n have unequal color sums. Thus, c_2 is a sigma coloring of C_n. Note that $c_2(C_n) = \{1, 2\}$ and c_2 maximizes the number of times that the largest color, 2, is assigned to the vertices of C_n. Thus, c_2 will yield the highest possible value of $S(n)$ using the colors 1 and 2.

Now, let $T(n)$ and $M(n)$ denote the sum of colors of all the vertices of C_n in a minimal and a maximal sigma 2-coloring of C_n, respectively, using the colors 1 and 2. Then the values of $T(n)$ and $M(n)$ are given as follows:

$$T(n) = \begin{cases} 5\left(\frac{n}{4}\right), & \text{if } n \equiv 0 \pmod 4 \\ 5\left\lfloor\frac{n}{4}\right\rfloor + 3, & \text{if } n \equiv 2 \pmod 4, \end{cases} \tag{1}$$

and

$$M(n) = \begin{cases} 7\left(\frac{n}{4}\right), & \text{if } n \equiv 0 \pmod 4 \\ 7\left\lfloor\frac{n}{4}\right\rfloor + 3, & \text{if } n \equiv 2 \pmod 4. \end{cases} \tag{2}$$

4. Main Results

We will use the definitions of $S(n)$, $M(n)$, and $T(n)$ for a cycle C_n, as given in Section 3. The first result shows that for every integer k between the sum $T(n)$ obtained from a minimal coloring and the sum $M(n)$ obtained from a maximal coloring, there exists a sigma 2-coloring of C_n using the colors 1 and 2 such that the corresponding sum of colors is equal to k.

Theorem 4.1. *Let $n \geq 4$ be even. For every integer k such that $T(n) \leq k \leq M(n)$, there exists a sigma 2-coloring c of the vertices of C_n using the colors 1 and 2 such that $S(n) = k$.*

Proof. Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$, $E(C_n) = \{v_iv_{i+1} | 1 \le i \le n-1\} \cup \{v_1v_n\}$, and let $s = S(n) - T(n)$. If $s = 0$, then we use the coloring c_1 presented in Section 3 to color the vertices of C_n. Suppose $s \ge 1$. We have the following cases:

Case 1: Suppose s is an even number and $s \ne M(n) - T(n)$. Define a coloring c_3 of C_n by

$$c_3(v_i) = \begin{cases} 2, & \text{if } i \equiv 1, 2 \pmod 4\text{, where } 2 \le i \le 2s+2 \text{ and } i \ne 2s+1 \\ c_1(v_i), & \text{otherwise,} \end{cases}$$

where c_1 is the minimal sigma coloring of C_n given in Section 3.

Subcase 1.1: If $n \equiv 0 \pmod 4$, then the color sums of the vertices of C_n are given by

$$\sigma_{c_3}(v_i) = \begin{cases} 2, & \text{if } i = 2, \text{ or } i \text{ is even and } 2s \le i \le n \\ 3, & \text{if } i \text{ is even for } 4 \le i \le 2s-2 \text{ or } i \text{ is odd for } 2s+5 \le i \le n \\ 4, & \text{if } i \text{ is odd for } 1 \le i \le 2s+3. \end{cases}$$

Subcase 1.2: If $n \equiv 2 \pmod 4$, then the color sums of the vertices of C_n are given as follows:

$$\sigma_{c_3}(v_i) = \begin{cases} 2, & \text{if } i = 2, \text{ or } i \text{ is even and } 2s \le i \le n \\ 3, & \text{if } i \text{ is even and } 4 \le i \le 2s-2, \text{ or } i \text{ is odd and } 2s+5 \le i \le n-3 \\ 4, & \text{if } i \text{ is odd and } 1 \le i \le 2s+3, \text{ or } i = n-1. \end{cases}$$

Case 2: Suppose s is an odd number and $s \ne M(n) - T(n)$. Define a coloring c_4 of the vertices of C_n by

$$c_4(v_i) = \begin{cases} 2, & \text{if } i \equiv 1, 2 \pmod 4\text{, where } 2 \le i \le 2s \\ c_1(v_i), & \text{otherwise,} \end{cases}$$

where c_1 is the minimal sigma coloring of C_n given in Section 3.

Subcase 2.1: If $n \equiv 0 \pmod 4$, then the color sums of the vertices of C_n is given by

$$\sigma_{c_4}(v_i) = \begin{cases} 2, & \text{if } i = 2, \text{ or } i \text{ is even and } 2s+4 \le i \le n \\ 3, & \text{if } i \text{ is even and } 4 \le i \le 2s+2, \text{ or } i \text{ is odd and } 2s+5 \le i \le n \\ 4, & \text{if } i \text{ is odd and } 1 \le i \le 2s+3. \end{cases}$$

Subcase 2.2: If $n \equiv 2 \pmod 4$, then the color sums of the vertices of C_n are given as follows:

$$\sigma_{c_4}(v_i) = \begin{cases} 2, & \text{if } i = 2, \text{ or } i \text{ is even and } 2s+2 \le i \le n \\ 3, & \text{if } i \text{ is even and } 4 \le i \le 2s, \text{ or } i \text{ is odd and } 2s+3 \le i \le n-3 \\ 4, & \text{if } i = n-1, \text{ or } i \text{ is odd and } 1 \le i \le 2s+1. \end{cases}$$

Case 3: If $s = M(n) - T(n)$, then $k = M(n)$. We color the vertices of C_n using the maximal sigma coloring c_2 given in Section 2.

In each of the cases above, we have shown that no two adjacent vertices of C_n have equal color sums. Thus, the colorings are sigma 2-colorings of C_n using the colors 1 and 2. In addition, the coloring strategy ensures that the number of vertices of C_n whose colors change from color 1 in c_1 to color 2 in c_3 (or c_4) is exactly s and $S(n) = T(n) + s = k$. ■

Lemma 4.2. *Let n be an even integer, $n \geq 4$, and let c be any sigma 2-coloring of C_n using the colors 1 and 2. If $S(n) \neq T(n)$, then there exists a vertex $v \in V(C_n)$ such that $\sigma(v) = 4$. If $S(n) \neq M(n)$, then there exists a vertex $v \in V(C_n)$ such that $\sigma(v) = 2$.*

Proof. Suppose c is a sigma 2-coloring of C_n using the colors 1 and 2 such that $S(n) \neq T(n)$. Then, at least one block, say $B : u_1, u_2, u_3, u_4$, of four consecutive vertices has two vertices with color 2. Since c is a sigma 2-coloring of C_n, the color pattern of the vertices in B is either $1, 2, 1, 2$ or $2, 1, 2, 1$, or B has exactly three vertices with color 2. In any case, at least one of the vertices of B must have a color sum equal to 4. Hence, the result follows.

In the case that $S(n) \neq M(n)$, a similar proof will show that there exists a vertex $v \in V(C_n)$ such that $\sigma(v) = 2$. ■

If the colors used in a 2-coloring of C_n alternate among the vertices, then we say that the coloring is *alternating.*

Lemma 4.3. *Let c be a sigma 2-coloring of C_n using the colors 1 and 2, and let $CS(C_n) = \{\sigma(u) : u \in V(C_n)\}$.*

(i) If c is alternating, then $CS(C_n) = \{2, 4\}$.

(ii) Suppose c is not alternating and $n \equiv 0$ (mod 4).
- *If $S(n) = T(n)$, then $CS(C_n) = \{2, 3\}$.*
- *If $S(n) = M(n)$, then $CS(C_n) = \{3, 4\}$.*
- *If $T(n) < S(n) < M(n)$, then $CS(C_n) = \{2, 3, 4\}$.*

(iii) Suppose c is not alternating and $n \equiv 2$ (mod 4). Then, $CS(C_n) = \{2, 3, 4\}$.

Proof. Suppose c is a sigma 2-coloring of C_n using the colors 1 and 2. Clearly, if c is alternating, then $CS(C_n) = \{2, 4\}$ regardless of the congruence class of n.

Suppose c is not alternating. If $n \equiv 0 \pmod 4$ and $S(n) = T(n)$, then by definition of minimal coloring and by recalling the values of $\sigma(c_1(v))$ given in Section 3, we have $CS(C_n) = \{2, 3\}$. Likewise, if $S(n) = M(n)$, then by definition of maximal coloring and by recalling the values of $\sigma(c_2(v))$ given in Section 3, we have $CS(C_n) = \{3, 4\}$. On the other hand, if $T(n) < S(n) < M(n)$, then by Lemma 4.2, the color sums 2 and 4 are in $CS(C_n)$. Furthermore, since c is not alternating, C_n contains a block of four consecutive vertices having three vertices with color 1 or with color 2. Hence, one of these vertices must have a color sum equal to 3. Thus, $CS(C_n) = \{2, 3, 4\}$ and this is true regardless of the congruence class of n. In the case that $n \equiv 2 \pmod 4$ and $S(n) = T(n)$ or $S(n) = M(n)$, we have $CS(C_n) = \{2, 3, 4\}$ as shown in Section 3. ■

The next result considers sigma colorings of the join of two even cycles C_n.

Lemma 4.4. *Let $G = 2C_n = C_n + C_n$, where n is even and $n \geq 8$. Suppose c is a coloring of G using the colors 1 and 2 such that the restriction of c to each copy of C_n in G is a sigma 2-coloring of C_n. Suppose $S_i(n)$ denotes the sum of colors of the vertices restricted to the ith copy of C_n, where $i \in \{1, 2\}$, and assume without loss of generality that $S_2(n) \geq S_1(n)$. Then, c is a sigma 2-coloring of G if and only if $S_2(n) \geq S_1(n) + 3$.*

Proof. For notation purposes, denote the ith copy of C_n in $G = C_n + C_n$ by C_{n_i} and let $c|_{C_{n_i}}$ be the restriction of c to C_{n_i}. From the assumption, $S_i(n) = \sum_{v \in V(C_{n_i})} c(v)$ and $S_1(n) \leq S_2(n)$. In the following, we list all possible conditions between $S_2(n)$ and $S_1(n)$ and show that only the condition $S_2(n) \geq S_1(n) + 3$ will give a sigma 2-coloring of G using the colors 1 and 2.

Case 1: Suppose $S_2(n) = S_1(n)$.

Subcase 1.1: If $S_1(n) \neq M(n_1)$, then since $n = n_1 = n_2$, we have $S_2(n) \neq M(n_2)$. By Lemma 4.2, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 2 + S_2(n) = 2 + S_1(n) = \sigma(v)$.

Subcase 1.2: If $S_1(n) = M(n_1)$, then $S_2(n) = M(n_2)$. By Lemma 4.2, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 4 + S_2(n) = 4 + S_1(n) = \sigma(v)$.

Case 2: Suppose $S_2(n) = S_1(n) + 1$.

Subcase 2.1: If $c|_{C_{n_2}}$ is alternating, then $c|_{C_{n_1}}$ is not. By Lemma 4.3, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 3 + S_2(n) = 4 + S_1(n) = \sigma(v)$.

Subcase 2.2: If $c|_{C_{n_2}}$ is not alternating, then $c|_{C_{n_1}}$ is alternating. By Lemma 4.3, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 2 + S_2(n) = 3 + S_1(n) = \sigma(v)$.

Case 3: Suppose $S_2(n) = S_1(n) + 2$. Then, $S_1(n) \neq M(n_1)$ and $S_2(n) \neq T(n_2)$. By Lemma 4.2, there exist two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, such that $\sigma(u) = 2 + S_2(n) = 4 + S_1(n) = \sigma(v)$.

Case 4: Suppose $S_2(n) \geq S_1(n) + 3$. Then, for any two adjacent vertices, $u \in V(C_{n_1})$ and $v \in V(C_{n_2})$, we have $\sigma(u) \geq 2 + S_2(n) \geq 5 + S_1(n) > 4 + S_1(n) \geq \sigma(v)$.

We note that in each of Cases 1 to 3, there are two vertices u and v which are adjacent in the join $G = 2C_n$ such that $\sigma(u) = \sigma(v)$. This means that c is not a sigma coloring in each of these cases. Finally, we should note that only Case 4 yields a sigma 2-coloring of C_n. This proves the lemma. ■

The result below deals with the main problem of this study which is to determine the sigma value and sigma range of the join of a finite number of even cycles of the same order.

Theorem 4.5. *Let $G = kC_n$, where $n \geq 8$ is even and $k \geq 1$. Then,*

$$\nu(G) = 2$$

if and only if

$$k \leq \begin{cases} \lfloor \frac{n}{6} \rfloor + 1, & \text{if } n \equiv 0 \pmod 4 \\ \lfloor \frac{n-2}{6} \rfloor + 1, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Since n is even, we have either $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. Considering the values of $T(n)$ and $M(n)$ as given in equations (1) and(2) in Section 3, it follows that

$$\left\lfloor \frac{M(n) - T(n)}{3} \right\rfloor + 1 = \begin{cases} \lfloor \frac{n}{6} \rfloor + 1, & \text{if } n \equiv 0 \pmod 4 \\ \lfloor \frac{n-2}{6} \rfloor + 1, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Hence, we will prove that $\nu(G) = 2$ if and only if $k \leq \left\lfloor \frac{M(n)-T(n)}{3} \right\rfloor + 1$. As in the proof of Lemma 4.4, we denote the ith copy of C_n in G by C_{n_i}, $c|_{C_{n_i}}$ the restriction of c to C_{n_i}, and $S_i(n) = \sum_{v \in V(C_{n_i})} c(v)$.

First, suppose $\nu(G) = 2$. Then, by definition, there exists a sigma 2-coloring c of kC_n using colors 1 and 2. Now, $c|_{C_{n_i}}$ as well as $c|_{C_{n_i}+C_{n_j}}$ are also sigma colorings for $1 \leq i, j, \leq k$, by Theorem 2.2. By permuting the position of the cycles, if necessary, we can assume without loss of generality that the values of $S_i(n)$, for $1 \leq i \leq k$, are nondecreasing. By Lemma 4.4, we must have $S_{i+1}(n) \geq S_i(n) + 3$ for each $1 \leq i \leq k-1$. As a consequence, $S_k(n) - S_1(n) \geq 3(k-1)$. Since $S_k(n) \leq M(n)$ and $S_1(n) \geq T(n)$, it follows that $S_k(n) - S_1(n) \leq M(n) - T(n)$. Combining the inequalities, we obtain $3(k-1) \leq M(n) - T(n)$, and so, $k \leq \frac{M(n)-T(n)}{3} + 1$. Since k is an integer, we have $k \leq \left\lfloor \frac{M(n)-T(n)}{3} \right\rfloor + 1$.

Conversely, suppose $k \leq \left\lfloor \frac{M(n)-T(n)}{3} \right\rfloor + 1$. Consider the sequence $a_1 = T(n), a_2 = T(n) + 3, \cdots, a_k = T(n) + 3(k-1)$. Note that the last equation yields $k = \left\lfloor \frac{a_k - T(n)}{3} \right\rfloor + 1$, and since $k \leq \left\lfloor \frac{M(n)-T(n)}{3} \right\rfloor + 1$, then we must have $a_k \leq M(n)$. Clearly, a_i is an increasing sequence and $T(n) \leq a_i \leq M(n)$ for $1 \leq i \leq k$. By Theorem 4.1, there exists a sigma 2-coloring of C_{n_i} using colors 1 and 2 such that $S_i(n) = a_i$. Since $a_{i+1} = a_i + 3$ for each $1 \leq i \leq k-1$, then by applying Lemma 4.4 repeatedly, it follows that c is a sigma 2-coloring of kC_n using the colors 1 and 2. Consequently, $\nu(G) = 2$. ■

By Observation 2.1, the sigma chromatic number of a connected graph is 1 if and only if every two adjacent vertices of G have different degrees. Since this is not the case for the join kC_n of k cycles with $n \geq 4$, then we must have $\sigma(kC_n) \geq 2$. Since by Theorem 2.3, $\sigma(kC_n) \leq \rho(kC_n) \leq \nu(kC_n)$, we have the following corollary.

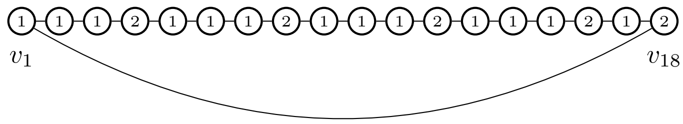
Corollary 4.6. *Let $G = kC_n$, where $n \geq 8$ is even and $k \geq 1$. Then,*

$$\rho(G) = 2$$

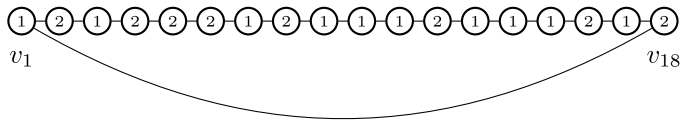
if and only if

$$k \leq \begin{cases} \left\lfloor \frac{n}{6} \right\rfloor + 1, & \text{if } n \equiv 0 \pmod 4 \\ \left\lfloor \frac{n-2}{6} \right\rfloor + 1, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

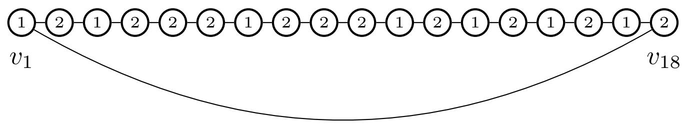
Example 1. *Suppose $n = 18$. By Theorem 4.5, we have $\nu(kC_{18}) = 2$ if and only if $k \leq 3$. Using Theorem 4.1 and the strategy of coloring kC_n in the proof of Theorem 4.5, we give a sigma 2-coloring of $3C_{18}$ using the colors 1 and 2. For simplicity, we omit the edges joining vertices between different copies of C_{18} in Figure 4.*



(a) 1st copy of $C_{18}, S_1(n) = 23$



(b) 2nd copy of $C_{18}, S_2(n) = 26$



(c) 3rd copy of $C_{18}, S_3(n) = 29$

Figure 4. Sigma 2-coloring of $3C_{18}$

Observe that when $k = 4$, then in order to still have a sigma 2-coloring of $4C_{18}$, $S_4(n) \geq S_3(n) + 3 = 32$ by Lemma 4.4. Since $M(n) = 31$ and $M(n)$ is the sum of colors in a maximal coloring of C_{18}, it follows that no sigma coloring of $4C_{18}$ using only the colors 1 and 2 will exist. As a consequence, $\nu(4C_{18}) \neq 2$.

5. Conclusion

In this paper, we considered the sigma value and sigma range in relation to the join of even cycles of the same order. While $\rho(C_n) = \nu(C_n) = 2$ when n is even, we determined necessary and sufficient conditions so that $\rho(kC_n)$ and $\nu(kC_n)$ are still equal to 2.

A problem that can be investigated further is that of determining $\rho(kC_n)$ and $\nu(kC_n)$ when n is odd. One might be interested also to consider other families of graphs whose sigma value and sigma range are yet unknown.

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