Author:
Jonnadula, L

Title:
Moments and correlations of random matrices and symmetric function theory

University of Bristol

Moments and correlations of random matrices and symmetric function theory

Bhargavi Jonnadula

July 29, 2021

A dissertation submitted to the University of Bristol in accordance with the requirements for award of the degree of Doctor of Philosophy in the Faculty of Science, School of Mathematics.

ABSTRACT

The central theme of this thesis is random matrices and their connections to combinatorics and probability theory. We present the results on correlations of eigenvalues for unitary invariant Hermitian ensembles, also called the $\beta = 2$ Hermitian ensembles, using symmetric functions.

Classical compact groups such as the unitary group, the orthogonal group and the symplectic group have always been the representatives of $\beta = 2$ ensembles. These groups are computationally simple compared to other ensembles due to the compactness of support of the eigenvalues and the underlying representation theory. The group characters are symmetric functions in the eigenvalues. Many quantities relating to the correlations of eigenvalues, the notable ones being the joint moments of traces and joint moments of characteristic polynomials, can be effectively studied using the symmetric function theory and the representation theory of compact groups. Such a combinatorial approach to computing correlations is highly successful as it enables calculating the exact formulae and provides a route to compute large matrix asymptotics.

We develop a parallel theory for Hermitian ensembles, in particular for the Gaussian, Laguerre and Jacobi ensembles. We provide exact formulae for joint moments of traces and joint moments of characteristic polynomials in terms of appropriately defined symmetric functions. As an example of an application, for the joint moments of the traces, we derive explicit asymptotic formulae for the rate of convergence of the moments of polynomial functions of Gaussian unitary matrices to those of a standard normal distribution when the matrix size tends to infinity.

We also calculate the asymptotics of the moments of characteristic polynomials of Hermitian ensembles, specifically the Gaussian unitary ensemble, as the matrix size tends to infinity. Our approach reveals that the even and odd dimensional Gaussian unitary matrices contribute differently to the moments and combine in a unique way to produce the semi-circle law.

ACKNOWLEDGEMENTS

I want to thank my supervisors Prof. Jon Keating and Prof. Francesco Mezzadri, for their guidance and constant encouragement. I am incredibly fortunate to have advisers who gave me the freedom to explore independently and, at the same time, helped me recover when I seemed to waver. Special thanks go to my family, without whose support this dissertation wouldn't have been possible.

I would also like to thank the University of Bristol for generously supporting my studies. I am grateful to my academic family and friends for helping and being supportive in various ways.

DECLARATION

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

July 29, 2021
Bhargavi Jonnadula

Contents

List of Figures

List of Tables

Notation

The following notation will be consistent throughout the document, unless otherwise stated.

$\mathrm{GL}(N, \mathbb{F})$	General linear group of matrices of size $N \times N$ with entries in the field $\mathbb{F}$
M	Matrix $M = (m_{ij})$
M^T	Transpose of M
$\overline{M}$	Complex conjugate of M
$M^\dagger$	Conjugate transpose of M
$\mathcal{M}$	Rescaled matrix with rescaling parameter clearly indicated
$\mathcal{N}(\mu, \sigma^2)$	The gaussian random variable with mean μ and variance σ^2
$\Delta(x_1, \ldots, x_N)$	The $N \times N$ Vandermonde determinant
$\mathcal{S}_n$	Symmetric group of size n
$f(x) = O(g(x))$	There exists some constants c and x_0 such that $f(x) \leq cg(x)$ for $x \geq x_0$.
$f(x) = o(g(x))$	There exists some constants c and x_0 such that $f(x) < cg(x)$ for $x \geq x_0$.
$f(x) = \omega(g(x))$	There exists some constants c and x_0 such that $f(x) > cg(x)$ for $x \geq x_0$.
$f(x) \sim g(x)$	The limit $f(x)/g(x) \to 1$ as $x \to \infty$.

Chapter 1

Introduction

Random matrices are ubiquitous, having numerous applications in a variety of scientific disciplines. To mention a few, they can be used to model biological networks [180, 181], the stock market [204], and quantum systems [245, 246]. Random matrix theory (RMT) is useful to analyse and understand the universal properties of these systems. All the results covered in chapters 3–5 concern random matrices. In this chapter, we introduce relevant topics in RMT and contextualise the present work.

In Sec. 1.1, we define different classes of random matrix ensembles. The history and development of RMT and a few notable applications are discussed in Sec. 1.2. In Sec. 1.3 and Sec. 1.4, we discuss the limiting behaviour and universal properties of matrix ensembles. In Sec. 1.5, we discuss the connection between combinatorics and RMT by emphasising the role of symmetric functions. Surprisingly, RMT is closely connected to number theory [194]. In Sec. 1.6, we discuss a few number theoretic functions to which our results are connected. One of the main results of this work is to evaluate the correlations of characteristic polynomials and traces of powers of matrices. In Sec. 1.8, we discuss some of the applications of these correlators and mention the results known so far. With the knowledge of these correlations, we comment in Sec. 1.9 on what can be said about the limiting distributions of certain random variables.

1.1 Random matrix ensembles

We begin with the definitions of groups that we frequently use.

Definition 1.1.1. *Denoted by $O(N)$, the orthogonal group of size N is defined as*

$$O(N) := \{A \in GL(N, \mathbb{R}) : AA^T = A^T A = I\}. \tag{1.1.1}$$

Here A^T denotes the transpose of A.

Definition 1.1.2. *Denoted by $U(N)$, the unitary group of size N is defined as*

$$U(N) := \{A \in GL(N, \mathbb{C}) : AA^\dagger = A^\dagger A = I\}. \tag{1.1.2}$$

Here $A^\dagger = \overline{A}^T$ denotes the conjugate transpose of A.

Definition 1.1.3. *Denoted by $Sp(2N)$, the symplectic group of size $2N$ is a subgroup of $U(2N)$ defined as*

$$Sp(2N) := \{A \in U(2N) : A\Omega A^T = A^T \Omega A = \Omega\}, \tag{1.1.3}$$

where

$$\Omega = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}. \tag{1.1.4}$$

Definition 1.1.4 (Ensemble)**.** *A random matrix ensemble is a space of matrices endowed with a probability measure.*

The Lebesgue measure on the space of Hermitian random matrices M is the product of Lebesgue measures on the independent entries of M:

$$dM = \prod_{i<j} d(\operatorname{Re} M_{ij})\, d(\operatorname{Im} M_{ij}) \prod_j dM_{jj}. \tag{1.1.5}$$

In the context of this thesis, we are interested in studying the spectral properties of Hermitian random matrices, which we introduce next.

1.1.1 Gaussian random matrices

Definition 1.1.5 (Gaussian unitary ensemble)**.** *Abbreviated as GUE, Gaussian unitary matrices have independent complex normal random variables as matrix entries: (i) M_{jj} are i.i.d. real Gaussian random variables with mean 0 and variance 1, $\mathcal{N}(0,1)$, and (ii) the real and imaginary parts of M_{ij}, $i<j$, are i.i.d. real Gaussians with mean 0 and variance $1/2$, $\mathcal{N}(0,1/2)$.*

The probability measure on the GUE of size N is

$$\begin{aligned} P(M)\, dM &= \prod_{1\le j<k\le N} \frac{1}{\sqrt{\pi}} e^{-(\operatorname{Re} M_{jk})^2} \frac{1}{\sqrt{\pi}} e^{-(\operatorname{Im} M_{jk})^2} \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}M_{jj}^2}\, dM \\ &= \frac{1}{(2\pi^N)^{\frac{N}{2}}} e^{-\frac{1}{2}\operatorname{Tr} M^2}\, dM. \end{aligned} \tag{1.1.6}$$

One of the most important properties of a GUE matrix is that the probability distribution of M remains invariant under unitary transformations. Specifically, for an $N \times N$ unitary matrix U,

$$P(UMU^\dagger)\, d(UMU^\dagger) = P(M)\, dM. \tag{1.1.7}$$

To see this, note that the square of the Hilbert-Schmidt norm of M is $\operatorname{Tr} M^2$ which is invariant under conjugation by U,

$$\operatorname{Tr}(UMU^\dagger)^2 = \operatorname{Tr}(UM^2U^\dagger) = \operatorname{Tr} M^2. \tag{1.1.8}$$

Therefore, the map $M \to UMU^\dagger$ is an isometry and so the Jacobian determinant is 1.

Usually, we are interested in studying the spectral properties of an ensemble. Thus, it is convenient to express the probability density in terms of eigenvalues rather than matrix elements as given in (1.1.6). To do so, denote by $x_1, \ldots, x_N$ the eigenvalues of M. Consider a

function f that depends only on the eigenvalues $x_1, \ldots, x_N$ and is symmetric in x_j:

$$f(x_1, \ldots, x_N) = f(x_{\sigma(1)}, \ldots, x_{\sigma(N)}), \tag{1.1.9}$$

where σ is an element of the symmetric group $\mathcal{S}_N$. The following theorem gives the average of f over the GUE.

Proposition 1.1.6. *The expected value of $f(x_1, \ldots, x_N)$ over the GUE is the following multiple integral*

$$\mathbb{E}_N^{(H)}[f(x_1, \ldots, x_N)] = c_{2,N}^{(H)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_N) \prod_{1 \le j < k \le N} |x_j - x_k|^2 \prod_{j=1}^{N} e^{-\frac{x_j^2}{2}} \, dx_j, \tag{1.1.10}$$

where $c_{2,N}^{(H)}$ is a normalisation constant.

The superscript $^{(H)}$ stands for Hermite because of the Gaussian factor $e^{-x^2/2}$, and Hermite polynomials are orthogonal with respect to the Gaussian weight. This notation becomes useful when we introduce other ensembles corresponding to different weight functions.

To get an insight into the proof, notice that any Hermitian matrix M can be diagonalised by a unitary transformation U:

$$M = U \begin{bmatrix} x_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & x_N \end{bmatrix} U^{\dagger}. \tag{1.1.11}$$

The matrix elements of M can be expressed in terms of the entries of U and eigenvalues $x_1, \ldots, x_N$. Now the unitary invariance property (1.1.7) can be used to integrate the elements of U, leaving just the eigenvalues.

It is often useful to rewrite the above integral in the *Vandermonde determinant* notation. The $N \times N$ Vandermonde matrix is

$$V(x_1, \ldots, x_N) := \begin{bmatrix} x_1^{N-1} & x_2^{N-1} & \ldots & x_N^{N-1} \\ x_1^{N-2} & x_2^{N-2} & \ldots & x_N^{N-2} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}, \tag{1.1.12}$$

whose determinant is the Vandermonde determinant $\Delta(x_1, \ldots, x_N)$:

$$\Delta(x_1, \ldots, x_N) := \det[x_j^{N-k}]_{1 \le j,k \le N} = \prod_{1 \le j < k \le N} (x_j - x_k). \tag{1.1.13}$$

Therefore, (1.1.10) can be re-written as

$$\mathbb{E}_N^{(H)}[f(x_1, \ldots, x_N)] = c_{2,N}^{(H)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_N) |\Delta(x_1, \ldots, x_N)|^2 \prod_{j=1}^{N} e^{-\frac{x_j^2}{2}} \, dx_j. \tag{1.1.14}$$

Different powers of the Vandermonde determinant correspond to different classes of matrix ensembles. These families have probability densities of the form

$$\rho^{(H)}_{\beta,N}(x_1,\dots,x_N) = c^{(H)}_{\beta,N}|\Delta(x_1,\dots,x_N)|^\beta \prod_{j=1}^{N} e^{-\frac{x_j^2}{2}}, \tag{1.1.15}$$

where $\beta = 1, 2, 4$ is called the *Dyson index* [186]. The Dyson index is equal to the dimension of the division algebra over $\mathbb{R}$ of the matrix entries. For example, β is equal to 2 for the GUE because each matrix entry is a complex number with independent real and imaginary parts. If $\beta = 1$, the ensemble is called the *Gaussian orthogonal ensemble* (GOE) with just real matrix entries. The GOE consists of real symmetric matrices and remains invariant under orthogonal transformations. Finally, $\beta = 4$ for the *Gaussian symplectic ensemble* (GSE), whose elements are Hermitian quaternionic matrices. As the name suggests, the GSE is invariant under symplectic transformations. These are the only special values of β for which all the finite dimensional correlation functions can be explicitly computed in terms of orthogonal polynomials. These three ensembles are special cases of a much broader class of matrices called *Wigner* random matrices[1].

Matrix models obtained by generalising β to non-integers were first studied by Dumitriu and Edelman [76,77]. This model, indexed by β, is called the $\beta-$matrix model and consists of $N \times N$ tridiagonal matrices

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \mathcal{N}(0,2) & \chi_\beta \\ & & & \chi_\beta & \mathcal{N}(0,2) \end{bmatrix}, \tag{1.1.16}$$

where $\mathcal{N}(0,2)$ are independent Gaussian random variables with mean 0 and variance 2, and $\chi_{j\beta}$ are chi-distributed with $j\beta$ degrees of freedom. The probability density of the chi-distribution is

$$p(x;r) = \frac{1}{2^{\frac{r-2}{2}}\Gamma(r/2)} x^{r-1} e^{-\frac{x^2}{2}} \mathbb{1}_{x\geq 0}. \tag{1.1.17}$$

The eigenvalues of $\beta-$matrix model have the same joint density given by (1.1.15), which is why they are also referred as $\beta-$Hermite ensembles.

1.1.2 Wishart random matrices

Definition 1.1.7. *A complex Wishart matrix is an $N \times N$ Hermitian matrix of the form*

$$M = XX^\dagger, \tag{1.1.18}$$

[1] A (real, complex, or quaternionic) Wigner matrix is a random matrix M such that (i) M_{jk}, $j < k$, are i.i.d. random variables with mean $\mathbb{E}[M_{jk}] = 0$ and variance $\mathbb{E}[|M_{jk}|^2] = 1$, (ii) M_{jj} are i.i.d. real random variables with mean $\mathbb{E}[M_{jj}] = 0$ and $\mathbb{E}[M_{jk}^2] < \infty$. The distribution of the diagonal entries of a Wigner matrix can be different from the distribution of the off-diagonal entries.

where X is a random matrix of size $N \times m$ ($m \geq N$) containing i.i.d. complex Gaussian entries.

The joint probability density function (j.p.d.f.) of the entries of the Wishart ensemble is given by

$$p(M) \propto e^{-\frac{1}{2}\operatorname{Tr} M}(\det M)^{m-N}, \tag{1.1.19}$$

which can be expressed in terms of the eigenvalues of M, namely $x_1, \ldots, x_N$, as

$$\rho_{2,N}^{(L)}(x_1, \ldots, x_N) = c_{2,N}^{(L)} |\Delta(x_1, \ldots, x_N)|^2 \prod_{j=1}^{N} e^{-\frac{x_j}{2}} x_j^{m-N}. \tag{1.1.20}$$

The Wishart ensemble is also known as the Laguerre ensemble due to the presence of the Laguerre weight $x^{\gamma}e^{-x/2}$. Similar to the Gaussian case, (1.1.20) is invariant under unitary transformations, so we call the complex Wishart ensemble as the *Laguerre unitary ensemble* (LUE). The Laguerre ensemble extended to other values of β is called the $\beta-$Laguerre ensemble, with j.p.d.f. given by

$$c_{\beta,N}^{(L)} |\Delta(x_1, \ldots, x_N)|^{\beta} \prod_{j=1}^{N} e^{-\frac{x_j}{2}} x_j^{\frac{\beta}{2}(m-N+1)-1}, \tag{1.1.21}$$

where the superscript $^{(L)}$ stands for Laguerre. Since we choose $m \geq N$, the above joint density is also well defined and normalisable for $\beta = 1$, and $\beta = 4$ ensembles, namely the *Laguerre orthogonal ensemble* (LOE) and the *Laguerre symplectic ensemble* (LSE). In fact, the joint law in (1.1.21) is sensible for any $m > N - 1$ and $\beta > 0$ so that the exponent $\frac{\beta}{2}(m - N + 1 - 2/\beta) > 0$ ensuring that the j.p.d.f. is normalisable. Similar to $\beta-$Hermite ensembles, there is a tridiagonal representation for $\beta-$Laguerre ensembles. Consider the $N \times N$ bidiagonal matrix

$$X = \begin{bmatrix} \tilde{\chi}_{m\beta} & & & & & \\ \chi_{(N-1)\beta} & \tilde{\chi}_{(m-1)\beta} & & & & \\ & \ddots & \ddots & & & \\ & & \chi_{2\beta} & \tilde{\chi}_{(m-N+2)\beta} & & \\ & & & \chi_{\beta} & \tilde{\chi}_{(m-N+1)\beta} & \end{bmatrix}, \tag{1.1.22}$$

where both χ and $\tilde{\chi}$ are chi-distributed random variables with the indicated degrees of freedom. The eigenvalues of $XX^{\dagger}$ have the same joint density as given in (1.1.21).

Since first introduced by Wishart in 1928 [249], these matrices have found numerous applications in different areas of science and engineering. They arise in statistics, image analysis, mathematical finance, quantum systems, quantum gravity and many more, see for example [4, 11, 38, 104, 117, 158, 162, 163, 210, 247, 249].

1.1.3 Jacobi random matrices

To complete the triad, we introduce Jacobi ensembles.

Definition 1.1.8 (Jacobi unitary ensemble)**.** *Abbreviated as JUE, this ensemble consists of $N \times N$ Hermitian matrices M constructed as follows. Choose $N, m_1, m_2 \in \mathbb{N}$ such that $m_1 \geq N$ and $m_2 \geq N$. Let A and B be two matrices of size $N \times m_1$ and $N \times m_2$, respectively with standard i.i.d. complex Gaussian normal random variables. Then,*

$$M = AA^\dagger/(AA^\dagger + BB^\dagger). \tag{1.1.23}$$

The joint eigenvalue density is given by

$$c_{2,N}^{(J)} |\Delta(x_1, \ldots, x_N)|^2 \prod_{j=1}^{N} x_j^{m_1-N}(1-x_j)^{m_2-N}. \tag{1.1.24}$$

For general values of β, the j.p.d.f. has the form

$$c_{\beta,N}^{(J)} |\Delta(x_1, \ldots, x_N)|^\beta \prod_{j=1}^{N} x_j^{\frac{\beta}{2}(m_1-N+1)-1}(1-x_j)^{\frac{\beta}{2}(m_2-N+1)-1}, \tag{1.1.25}$$

where the superscript $^{(J)}$ stands for Jacobi. The tridiagonal matrix model for the $\beta-$Jacobi ensemble was introduced in [173]. A $\beta-$Jacobi ensemble is also referred as $\beta-$MANOVA ensemble because of its connections to multivariate analysis of variance (MANOVA) model.

The explicit matrix construction of $\beta-$ensembles play a key role in one-dimensional stochastic differential equations in the large N limit [83, 207, 208]. Furthermore, $\beta-$ensembles have important connections to quantum many body systems [21, 22], Selberg-type integrals [186], the theory of Jack polynomials [182], and lattice gas theory. For example, the spectral joint density of $\beta-$ensembles can be written as

$$\rho(x_1, \ldots, x_N) = c_{\beta,N} e^{-\beta W}, \tag{1.1.26}$$

where

$$W = \sum_{j=1}^{N} v(x_j) - \sum_{j<k} \ln |x_j - x_k|, \tag{1.1.27}$$

for some function $v(x)$. The density in (1.1.26) can be interpreted as the probability density of a system of N particles confined by the potential $v(x)$ and which repel each other by a logarithmic Coulomb interaction.

In this work, our goal is to study spectral correlations of $\beta = 2$ Hermitian ensembles by developing a theory similar to the one for classical compact groups. As a representative of the compact groups, we introduce the unitary ensemble below.

1.1.4 Unitary matrices

Definition 1.1.9 (Circular unitary ensemble)**.** *Abbreviated as CUE, this ensemble consists of unitary matrices $U(N)$ endowed with a uniform probability measure.*

The uniform probability measure on the space of $U(N)$ is called the *Haar* measure denoted as μ_{Haar}. One way to express the Haar measure on $U(N)$ is via the Weyl integration formula

[242]. The eigenvalues of a unitary matrix A lie on the unit circle in the complex plane which we denote as $e^{i\theta_1}, \ldots, e^{i\theta_N}$. Consider a function $f(A) := f(\theta_1, \ldots, \theta_N)$ that is symmetric under the permutation of eigenangles θ_j. Then,

$$\int_{U(N)} f(A)\, d\mu_{\text{Haar}}(A) = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \ldots, \theta_N) |\Delta(e^{i\theta_1}, \ldots, e^{i\theta_N})|^2\, d\theta_1 \ldots d\theta_N. \tag{1.1.28}$$

Therefore, the j.p.d.f of the CUE has the form

$$\rho(e^{i\theta_1}, \ldots, e^{i\theta_N}) \propto |\Delta(e^{i\theta_1}, \ldots, e^{i\theta_N})|^2. \tag{1.1.29}$$

Similar to the other $\beta = 2$ ensembles, the Haar measure on $U(N)$ remains invariant under unitary transformations,

$$\mathrm{d}\mu_{\text{Haar}}(A) = \mathrm{d}\mu_{\text{Haar}}(UAU^\dagger). \tag{1.1.30}$$

Similar arguments outlined in proving Prop. 1.1.6 can be used to prove (1.1.28). For notational simplicity, we use

$$\mathbb{E}_{U(N)}[f(A)] \tag{1.1.31}$$

to denote the expectation of f with respect to the Haar measure on the unitary group. When (1.1.29) is generalised to other values, the ensemble is called the *circular $\beta-$ensemble* with j.p.d.f. proportional to

$$|\Delta(e^{i\theta_1}, \ldots, e^{i\theta_N})|^\beta. \tag{1.1.32}$$

The cases when $\beta = 1$ and $\beta = 4$ are called the *circular orthogonal ensemble* (COE) and the *circular symplectic ensemble* (CSE), respectively.

1.2 History and applications of random matrices

Although the origins of RMT can be traced back to work by Wishart [249] and James [150, 151, 153] in mathematical statistics, the field gained prominence due to the seminal work of Wigner [244] in the 1950s. Wigner's idea is to describe the energy levels of highly excited heavy nuclei using random matrices. A complex nuclear system is characterised by a Hamiltonian operator H, which can be regarded as a matrix in an infinite-dimensional Hilbert space. Since the details of H are unknown due to the complexity of the system, Wigner argued that the Hamiltonian could be regarded as a large random matrix from an ensemble satisfying the prescribed symmetries of H. He showed that the energy levels of the nuclei (given by the eigenvalues of H) and the eigenvalues of large random matrices have the same statistical distribution. In addition to demonstrating the semi-circle law (see (1.3.5)) for the mean eigenvalue density, Wigner also provided insights into the nearest neighbour spacing distribution of eigenvalues, namely the *Wigner Surmise*. Later, Gaudin and Mehta provided rigorous analysis for the spacing distribution. They also developed the orthogonal polynomial technique which is one of the powerful techniques in RMT [123, 185, 187].

In a series of papers [81, 82] in the 1960s, Dyson introduced the three-fold classification of Hamiltonians describing the three symmetry classes of random matrices, namely orthogonal

($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) ensembles. The Hamiltonian under (i) $\beta = 1$ symmetry class is time-reversal invariant, (ii) $\beta = 2$ symmetry class is not invariant under time-reversal symmetry and (iii) $\beta = 4$ symmetry class is typically associated with quantum Hamiltonians with half-integer spin particles and without time-reversal symmetry. Dyson also introduced the circular ensembles and established a link to exactly soluble systems, such as Calogero-Sutherland models, by developing a model of Brownian motion in the random matrix ensembles [80].

Since the 1960s, the theory of random matrices has undergone a surge of development and found numerous applications in mathematical physics. After initial applications to nuclear physics, RMT was further advanced due to the connections to quantum chaos. Bohigas, Giannoni, and Schmit [34] conjectured that the the energy levels of highly excited quantum systems, whose classical counterparts are chaotic, show the same statistical behaviour as the spectra of random matrices. In the 1970s, random matrix theory also unified with the theory of disordered systems. In [84], Edwards and Anderson introduced the replica trick, which along with the work of Wegner [240], led to a new paradigm in the theory of Anderson localization. The theory of Anderson localisation is further substantiated by the supermatrix approach due to Efetov [85] and its further adaptation to RMT by Weidenmueller, Verbaarschot, and Zirnbauer [233]. Random matrices also proved to be very successful in studying the statistics of electronic transport in quantum-coherent (mesoscopic) samples [23, 188, 189] and the statistics of level curvatures [116, 122, 237, 238, 251]. The Dyson three-fold way of classifying the Hamiltonian is also broadened to make the theory applicable to quantum chromodynamics [5, 134, 200, 222], scattering theory [112, 117, 133, 217, 218], disordered superconducting structures [9, 252] etc.

Random matrix theory also had a profound impact on quantum field theory. The seminal work of 't Hooft [143, 228] suggests that the partition function in field theory is dominated by planar diagrams, also called planar Feynmann diagrams. Brézin, Itzykson, Parisi and Zuber [42] showed that a similar expansion also holds for random matrix ensembles when the matrix size is large. The combinatorial factors that appear in quantum field theory also arise in random matrices, which in the RMT context can be studied using the loop equation method [10, 91]. Due to the connection between random matrices and integrable systems, the theory of 2d quantum gravity has links to Painlevé transcendents and Toda/KdV hierarchies [1, 2, 97, 124, 167, 230].

In mathematics, random matrix theory went through advancements independent of those in theoretical physics. Some of the important results are with regards to integration measures of random matrix ensembles [144]. Harish-Chandra [139] evaluated a unitary matrix integral which is well known as the Harish-Chandra-Itzykson-Zuber integral [139, 149]. Selberg [212] considered the N dimensional generalisation of the Euler integral, which is now famously regarded as the Selberg integral. Other crucial quantities are zonal polynomials and Jack polynomials. Since James [152] introduced zonal polynomials, they found numerous applications in mathematical statistics. Jack polynomials relate the integrand of Selberg integral to the eigenvalue density function of circular β ensembles. In a series of papers, Voiculescu [235, 236] discovered that random matrices and operator algebras are strongly related to each other resulting in a big breakthrough in the development of free probability theory.

In 1973, Dyson and Montgomery [194] discovered that the asymptotic limit of the two-point

correlation function of the zeros of the Riemann zeta function on the critical line is same as the two-point correlation function of the GUE. Since then, a considerable volume of literature concerns the connections between number theory and random matrix theory. An important result in this direction is due to Keating and Snaith [171]. They provided powerful pieces of evidence favouring a very intimate connection between the moments of the Riemann zeta function (and other L-functions) along the critical line and the properties of characteristic polynomials of random matrices.

In terms of the statistics of the eigenvalues, Marchenko and Pastur [183] described the spectra of large random covariance matrices while Wigner focused on those of Gaussian matrices. Tracy and Widom [231] studied the statistics of the largest eigenvalue in Gaussian ensembles, which is infamously regarded as the Tracy-Widom distribution. The Tracy-Widom distribution arises in combinatorial problems such as the distribution of the length of the longest increasing subsequence of random permutations [17] and polynuclear growth models [205].

One of the reasons for the success of RMT is universality. The universality conjecture states that the fluctuations of the eigenvalues of large random matrices are independent of the choice of the distribution of matrix elements for general RMT ensembles. The universality results are first verified using the orthogonal polynomial technique [202] which resulted in development of the Riemann-Hilbert approach to the asymptotics of orthogonal polynomials [33, 64]. More recently, the universality results are extended to Wigner matrices [90, 161, 229]. Erdős [89] gave an excellent historical account on the development of various proofs of the universality conjecture.

Enormous progress has been made in random matrix ensembles of non-Hermitian matrices, extending the work of Wigner, Ginibre and Girko. In addition, sparse random matrices, random band matrices, heavy-tailed random matrices, Euclidean random matrices, and random matrices with external sources have also been considered. In this short review, we have seen the importance and applications of random matrices to wide-ranging fields. We would also like to mention that the given set of references are a selected few that influenced the area. The scope of RMT is not limited to the indicated. Some of the fascinating examples of RMT include the Airline boarding problem [14]; waiting times for buses in Cuernavaca, Mexico [16]; and the distances between parked cars in London [211].

1.3 Limiting distributions

In this section, we shall focus on the most notable distributions in RMT, the limiting eigenvalue distributions of $\beta = 2$ Gaussian, Laguerre and Jacobi ensembles as the matrix size $N \to \infty$.

We begin with a Wigner matrix M of size N. The correct scaling to compute the limiting eigenvalue distribution can be easily fixed by the following heuristic arguments. We start with

the order of magnitude of the mean and the second moment of the eigenvalues:

$$\frac{1}{N}\sum_{j=1}^{N} x_j = \frac{1}{N}\operatorname{Tr} M = \frac{1}{N}\sum_{j=1}^{N} M_{jj}, \tag{1.3.1}$$

$$\frac{1}{N}\sum_{j=1}^{N} x_j^2 = \frac{1}{N}\operatorname{Tr} M^2 = \frac{1}{N}\sum_{i,j=1}^{N} M_{ij}^2. \tag{1.3.2}$$

Since M_{jj} are i.i.d. Gaussians with mean 0, the first moment converges to zero by the strong law of large numbers. As there are approximately $N^2/2$ independent terms in $\operatorname{Tr} M^2$, we have $\operatorname{Tr} M^2 = O(N^2)$. Due to an additional factor of $1/N$ in (1.3.2), the order of magnitude of eigenvalues is $O(\sqrt{N})$. Therefore the eigenvalues should be rescaled by $1/\sqrt{N}$ in order to see a deterministic limiting behaviour.

To study the limiting distribution, consider the normalised counting function of the eigenvalues $x_1, \ldots, x_N$,

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{x_j}. \tag{1.3.3}$$

The measure in (1.3.3) is also called the *empirical spectral measure.*

Theorem 1.3.1 (Semi-circle law)**.** *Let M be a random $N \times N$ Wigner matrix with eigenvalues $x_1, \ldots, x_N$, and let*

$$\mu(M) = \frac{1}{N}\sum_{j=1}^{N} \delta_{\frac{x_j}{\sqrt{N}}}. \tag{1.3.4}$$

Then as $N \to \infty$, $\mu(M)$ converges (in mean and almost surely) to a deterministic limit $\varrho_{sc}(x)\,dx$, where ϱ_{sc} is given by

$$\varrho_{sc}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2}, & \text{if } -2 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.5}$$

This is the *Wigner's semi-circle law.* The essence of Thm. 1.3.1 is that the sequence of random measures $\mu(M)$ converge to a deterministic measure. Standard techniques such as the moment method, Stieltjes transform method, or tools from the free probability theory can be used to prove Thm. 1.3.1, see for example [93]. In Fig. 1.1a, we numerically illustrate the semi-circle law for GUE matrices. From Fig. 1.1a, it is clear that there is a non-zero probability of finding the eigenvalues outside the support $[-2, 2]$ for any finite matrix size N. Thus, we sometimes call the edges of the semi-circle as *soft edges.*

Theorem 1.3.2 (Marchenko-Pastur law)**.** *For a random Wishart matrix of size N as given by (1.1.18), as $m \to \infty$ and $N \to \infty$ such that $c = N/m \in (0, 1]$, the empirical spectral distribution converges (in mean and almost surely) to a deterministic limit $\rho_{mp}^c(x)\,dx$,*

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{\frac{x_j}{2N}} \to \rho_{mp}^c(x)\,dx. \tag{1.3.6}$$

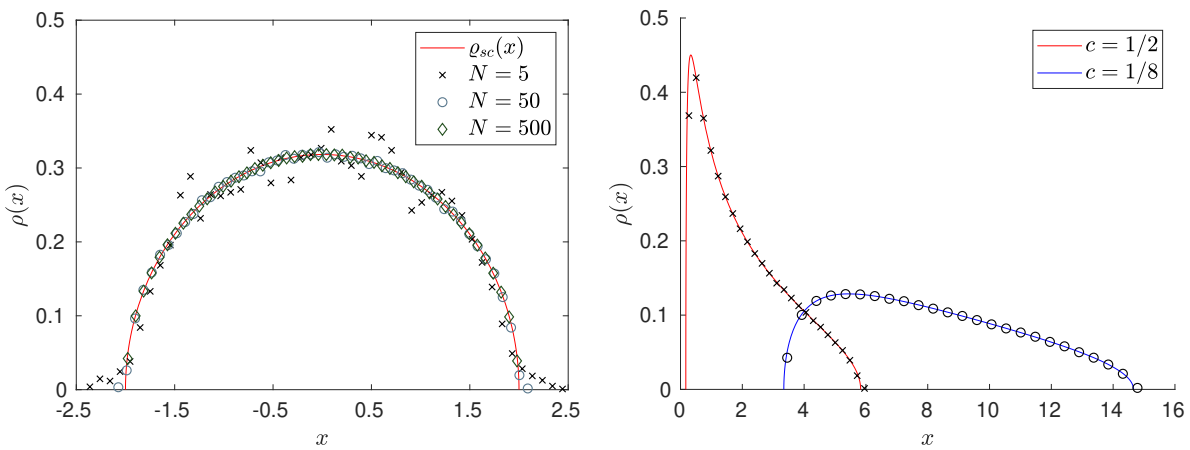
Here ρ^c_{mp} is the Marchenko-Pastur distribution [183] parametrised by c,

$$\rho^c_{mp}(x) = \begin{cases} \frac{1}{2\pi x}\sqrt{(x-c_-)(c_+-x)} & \text{if } c_- \le x \le c_+, \\ 0, & \text{otherwise}, \end{cases} \tag{1.3.7}$$

and $c_\pm = (1 \pm c^{-1/2})^2$.

In Fig. 1.1b, we show that the eigenvalue distribution of the LUE converges to the Marchenko-Pastur distribution for two values of c which are less than 1. For any value of $c < 1$, all the eigenvalues are positive real numbers. For $c < 1$, the edges of the distribution are soft, i.e. there is a non-zero probability of finding the eigenvalues outside the support $[c_-, c_+]$. But as $c \to 1$, more eigenvalues accumulate near 0. At $c = 1$, the origin becomes a hard edge, i.e., no eigenvalues are present to the left of the origin.

It is worth noting that the scaling of the eigenvalues of Wishart matrices is $O(N)$, unlike the scale $O(\sqrt{N})$ for Gaussian ensembles. The Marchenko-Pastur law is the analogue for Wishart random matrices of the Wigner semi-circle law for Hermitian matrices.



(a) Convergence of the density of eigenvalues of GUE matrices to the semi-circle law as matrix size N increases.

(b) Convergence of the density of eigenvalues of LUE matrices to the Marchenko-Pastur law for two different ratios of $N/m = c$. In both cases $N = 100$.

Figure 1.1: Numerical check of the semi-circle law and the Marchenko-Pastur law for $\beta = 2$ Gaussian and Laguerre ensembles in (a) and (b), respectively. In both (a) and (b), 2000 matrices are sampled.

For the Jacobi ensemble, several limiting distributions are possible depending on the relative sizes of parameters m_1 and m_2 to N [51, 78, 239]. Here we state the results for $\beta = 2$.

Theorem 1.3.3. *For a JUE matrix of size N, we have the following limits for the empirical spectral distribution as $N, m_1, m_2 \to \infty$.*

— *If $m_1 + m_2 - 2N = o(N)$, then [78]*

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{x_j} \to \frac{1}{\pi\sqrt{x(1-x)}}\, dx. \tag{1.3.8}$$

— *If $m_1/N \to p_1 > 1$ and $m_2/N \to p_2 > 1$ such that $p_1 + p_2 > 2$, then [78]*

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j} \to \frac{p_1+p_2}{2\pi}\frac{\sqrt{(\xi_+ - x)(x-\xi_-)}}{x(1-x)}\mathbb{1}_{[\xi_-,\xi_+]}\,dx, \tag{1.3.9}$$

where

$$\xi_\pm = \left(\sqrt{\frac{p_1}{p_1+p_2}\left(1-\frac{1}{p_1+p_2}\right)} \pm \sqrt{\frac{1}{p_1+p_2}\left(1-\frac{p_1}{p_1+p_2}\right)}\right)^2. \tag{1.3.10}$$

— *If $m_1 + m_2 - 2N = \omega(N)$ and $\frac{m_1-N}{m_1+m_2-2N} \to \lambda$, then [78]*

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j} \to \delta_\lambda, \tag{1.3.11}$$

where δ_λ is a deterministic measure that depends on λ. Here $f(x) = \omega(g(x))$ indicates that there exists some constants c and x_0 such that $f(x) > cg(x)$ for $x \geq x_0$ (In Prop. 1.3.4, we give an explicit expression for δ_λ for certain ratios of parameters m_1, m_2 and N).

The mode of convergence in (1.3.8), (1.3.9) and (1.3.11) is convergence in probability.

In (1.3.8), both m_1 and m_2 grow sub-linearly in N. In (1.3.9), m_1 and m_2 grow linearly with N and the limiting distribution in this case is clearly different from the sub-linear case. In (1.3.11), m_1 and m_2 grow much faster than N. In this super-linear case one can recover the Marchenko-Pastur law with a proper scaling.

Proposition 1.3.4. *If $N/m_1 \to c_1 \in (0,1]$ and $m_2 = \omega(N^2)$, then [155]*

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{\frac{m_2}{N}x_j} \to \rho_{mp}^{c_1}(x)\,dx, \tag{1.3.12}$$

as $N, m_1, m_2 \to \infty$. Here $\rho_{mp}^{c_1}$ is the Marchenko-Pastur density with parameter c_1.

In Fig. 1.2, all three cases in (1.3.8), (1.3.9), and (1.3.12) are illustrated for the JUE matrices.

The limiting distributions such as the semi-circle law and the Marchenko-Pastur law are not specific to $\beta = 2$ ensembles. For example, all $\beta-$Hermite ensembles have the same deterministic limit for the spectral density when properly normalised:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\delta_{\sqrt{\frac{2}{\beta N}}x_j} \to \varrho_{sc}(x)\,dx. \tag{1.3.13}$$

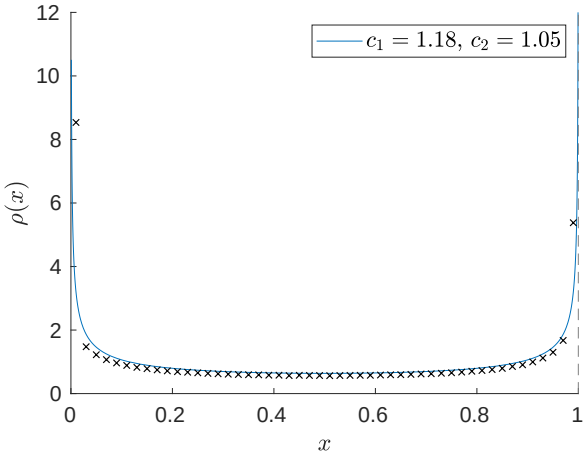
Similarly for the $\beta-$Laguerre,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\delta_{\frac{x_j}{\beta N}} \to \varrho_{mp}^{c}(x)\,dx, \tag{1.3.14}$$

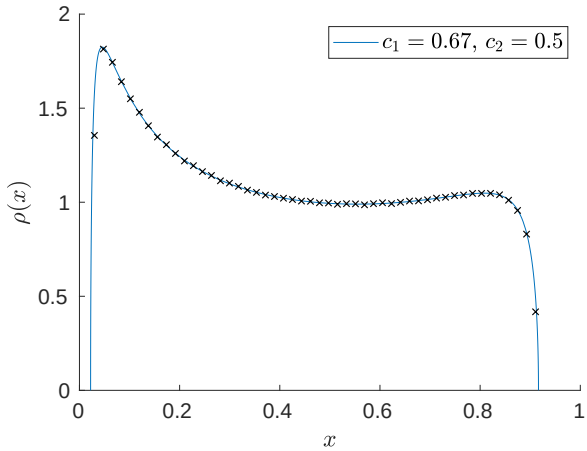
where $c = N/m$. For $\beta-$Jacobi ensembles, Prop. 1.3.4 can be rephrased as follows. If $N\beta/2m_1 \to c_1 \in (0,1]$ and $m_2 = \omega(N^2)$, then as $N, m_1, m_2 \to \infty$,

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{\frac{m_2}{N}x_j} \to \frac{2}{\beta}\,\rho_{mp}^{c_1}(2x/\beta)\,dx, \tag{1.3.15}$$

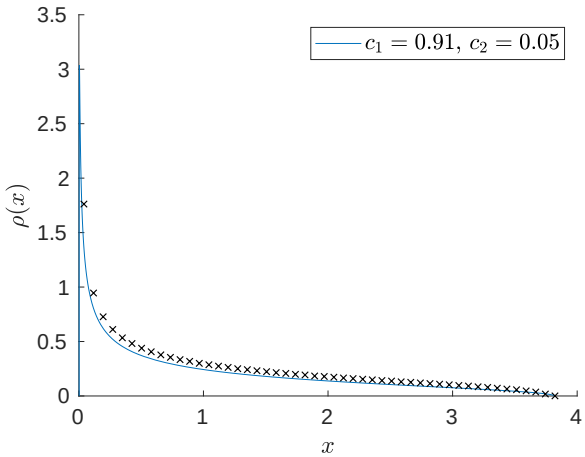
where $\rho_{mp}^{c_1}$ is the Marchenko-Pastur density with parameter c_1.



(a) Here 1000 matrices are sampled with parameters $N = 500$, $m_1 = 425$ and $m_2 = 475$. The solid line represents the density in the R.H.S. of (1.3.8).



(b) Here 1000 matrices are sampled with parameters $N = 500$, $m_1 = 750$ and $m_2 = 1000$. The solid line represents the density in the R.H.S. of (1.3.9).



(c) Here 2000 matrices are sampled with parameters $N = 400$, $m_1 = 440$, and $m_2 = 8000$. The solid line represents the density in the R.H.S. of (1.3.12).

Figure 1.2: Comparison between the asymptotic densities in (1.3.8), (1.3.9), and (1.3.12). The data is obtained by numerical diagonalisation of JUE matrices. In all the figures, $c_1 = m_1/N$ and $c_2 = m_2/N$ where N is the size of the JUE matrix, see (1.1.24).

1.4 Universality

There are several notions of universality that one can think of in random matrix theory. In the previous section we saw that the density of eigenvalues of a particular ensemble (Gaussian, Laguerre) converges to a limiting distribution (semi-circle law, Marchenko-Pastur law). This can be interpreted as a universal behaviour as it is observed in all of the matrices from a given

ensemble.

The other notion of universality, which is the one that we are interested in, is specific to β. The explicit j.p.d.f. of $\beta-$ensembles has the form

$$\rho(x_1, \ldots, x_N) = c_{\beta,N} \prod_{1 \le j < k \le N} |x_j - x_k|^\beta \prod_{j=1}^{N} e^{-v(x_j)}, \tag{1.4.1}$$

where $v(x)$ is a real valued function that grows sufficiently fast as $|x| \to \infty$. The eigenvalues are confined by $v(x)$ but repel each other because of the Vandermonde determinant. This repulsion increases with β and different values of β give rise to different universality classes. For instance, the spacings between the eigenvalues of all $\beta = 2$ ensembles are described by the same law when the matrix size is large [186]. In other words, as long as the matrices have a certain symmetry, all the details on how the matrix is constructed are washed out for sufficiently large matrices.

To formulate the notion of universality precisely, we will focus on $\beta = 2$ universality class. Consider the j.p.d.f.

$$\rho(x_1, \ldots, x_N) = c_{2,N} \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{j=1}^{N} w(x_j), \tag{1.4.2}$$

where $w(x)$ is a weight function (for example, it can be chosen to be one of the Gaussian, Laguerre or Jacobi weights). Define the $k-$point correlation function $R_{N,k}$ to be the marginal

$$R_{N,k}(x_1, \ldots, x_k) = \frac{N!}{(N-k)!} \int \rho(x_1, \ldots, x_N)\, dx_{k+1} \ldots dx_N. \tag{1.4.3}$$

For all the random matrix ensembles discussed so far, $R_{N,k}$ can be written in terms of a kernel $K_N(x, y)$ defined as

$$K_N(x, y) := (w(x)w(y))^{\frac{1}{2}} \sum_{j=0}^{N-1} \frac{\varphi_j(x)\varphi_j(y)}{(\varphi_j, \varphi_j)}, \tag{1.4.4}$$

where $\varphi_j(x)$ are the orthogonal polynomials of degree j with respect to $w(x)$ under the inner product

$$(\varphi_j(x), \varphi_k(x)) = \int \varphi_j(x)\varphi_k(x)w(x)\, dx = c_j \delta_{jk} \tag{1.4.5}$$

for some constant c_j. The kernel $K_N(x, y)$ can also be expressed via the Christoffel-Darboux formula as

$$K_N(x, y) = (w(x)w(y))^{\frac{1}{2}} \frac{A_{N-1}}{A_N} \frac{1}{(\varphi_{N-1}, \varphi_{N-1})} \frac{\varphi_N(x)\varphi_{N-1}(y) - \varphi_{N-1}(x)\varphi_N(y)}{x - y}, \tag{1.4.6}$$

where A_j is the leading coefficient of φ_j. The $k-$point correlation function can be expressed in terms of K_N by the relation [186]

$$R_{N,k}(x_1, \ldots, x_k) = \det[K_N(x_i, x_j)]_{1 \le i,j \le k}. \tag{1.4.7}$$

We now give the statement for universality.

Theorem 1.4.1. *Let s be a point in the bulk*[2] *of the spectral density of the scaled eigenvalues. Then, as $N \to \infty$, [33, 64, 202],*

$$\frac{1}{(K_N(0,0))^k} R_{N,k}\left(s + \frac{x_1}{K_N(0,0)}, \ldots, s + \frac{x_k}{K_N(0,0)}\right) \to \det[K_{sin}(x_i, x_j)]_{1 \le i,j \le k}, \tag{1.4.8}$$

where

$$K_{sin}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{1.4.9}$$

is the sine kernel.

Brézin and Zee [43] used the orthogonal polynomial method and provided heuristic arguments for the universality of the sine kernel. The first rigorous proof was given by Pastur and Shcherbina [202] using the orthogonal polynomials technique. Later Bleher and Its [33] and Deift *et al.* [64] took the Riemann-Hilbert approach to study universality.

The scaling in (1.4.8) is justified as follows. Note that

$$R_{N,1}(x) = N\rho(x) = K_N(x,x) \tag{1.4.10}$$

is nothing but the mean eigenvalue density. This is because for any interval $B = [a,b]$ on the real line,

$$\begin{aligned}
\int_B R_{N,1}(x_1)\, dx_1 &= \int \mathbb{1}_B(x_1) R_{N,1}(x_1)\, dx_1 \\
&= N \int \mathbb{1}_B(x_1) \rho(x_1, \ldots, x_N)\, dx_1 \ldots dx_N \\
&= \int \left(\sum_{j=1}^{N} \mathbb{1}_B(x_j)\right) \rho(x_1, \ldots, x_N)\, dx_1 \ldots dx_N \\
&= \text{Expected number of points in } B.
\end{aligned} \tag{1.4.11}$$

Hence $K_N(0,0)$ gives the density of expected number of eigenvalues at the origin. The scaling $x/K_N(0,0)$ ensures that the expected number of eigenvalues per unit interval is 1, and that the large N limit in (1.4.8) is finite. Since the kernel (1.4.4) involves orthogonal polynomials, it can be expressed as an integral by exploiting the integral representation of orthogonal polynomials. Then, standard tools such as the steepest-descent method can be used to prove the universality in (1.4.8).

The universality result states that in the large N limit, the limiting behaviour of the scaled eigenvalues is independent of the weight w and depends only on the invariance properties of the ensemble. We now give examples for two different ensembles, namely the GUE and the

[2] Here 'bulk' indicates that we choose s well within the interior of the spectrum, and not close to the edges. For a point close to the edge, the scaling is different from $K_N(0,0)$.

CUE to illustrate the universal behaviour. For the GUE,

$$\begin{aligned} K_N(x,x) &= w(x)\sum_{j=0}^{N-1}\frac{H_j^2(x)}{(H_j,H_j)} \\ &= w(x)\frac{1}{(H_{N-1},H_{N-1})}[H_N'(x)H_{N-1}(x)-H_N(x)H_{N-1}'(x)], \end{aligned} \tag{1.4.12}$$

where $H_n(x)$ are classical Hermite polynomials that satisfy

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}n!\delta_{nm}. \tag{1.4.13}$$

By using

$$\begin{aligned} H_n'(x) &= xH_n(x)-H_{n+1}(x), \\ H_n(0) &= \begin{cases} (-1)^{\frac{n}{2}}\frac{n!}{2^{\frac{n}{2}}\frac{n}{2}!}, & \text{if } n \text{ is even}, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{1.4.14}$$

and Stirling's approximation for the factorial

$$n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}, \tag{1.4.15}$$

it can be readily seen that

$$K_N(0,0) = \frac{\sqrt{N}}{\pi} + O\left(\frac{1}{\sqrt{N}}\right). \tag{1.4.16}$$

Therefore, the correct scaling is $O(\sqrt{N})$ which we saw previously in (1.3.4). When the eigenvalue statistics are considered at the origin, as $N\to\infty$, (1.4.8) becomes

$$\left(\frac{\pi}{\sqrt{N}}\right)^k R_{N,k}^{(H)}\left(\frac{\pi x_1}{\sqrt{N}},\ldots,\frac{\pi x_k}{\sqrt{N}}\right) \to \det[K_{\sin}(x_i,x_j)]_{1\le i,j\le k}. \tag{1.4.17}$$

For the CUE, all the eigenvalues lie on the unit circle. The weight $w(\theta)=1$ and polynomials that are orthogonal on the unit circle are $\varphi_j(\theta)=e^{ij\theta}$. When the order of the magnitude of scaling is $O(N)$, as $N\to\infty$, (1.4.8) reads to be

$$\left(\frac{2\pi}{N}\right)^k R_{N,k}^{U(N)}\left(\frac{2\pi x_1}{N},\ldots,\frac{2\pi x_N}{N}\right) \to \det[K_{\sin}(x_i,x_j)]_{1\le i,j\le k}. \tag{1.4.18}$$

If the statistics are studied at a point close to the edge of the limiting spectrum, a different type of universality is observed. In this case, the scaling is different from $K_N(0,0)$. When s is close to the soft edge (such as near 2 or -2 in the semi-circle law (1.3.5)), the limiting kernel is called the *Airy kernel* which depends on the Airy function. If the edge is hard (such as the left edge of the Marchenko-Pastur law when $c=1$), then the limiting kernel is the *Bessel kernel*. Similarly, $\beta=1$ and $\beta=4$ ensembles have a different limiting kernel that involve Pfaffians instead of determinants as given in (1.4.8). For complete details, see [98,186]. In the rest of the thesis, we restrict our attention to $\beta=2$ ensembles unless otherwise stated, omitting explicit

dependence on β.

1.5 Symmetric function theory and random matrix theory

A function f in n variables $x_1, \ldots, x_n$ is symmetric if it remains the same after interchanging x_i with x_j for $i \neq j$:

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \tag{1.5.1}$$

where σ is an element in the symmetric group $\mathcal{S}_n$. Symmetric functions appear naturally in random matrix theory because the joint density in (1.4.1) remains invariant under the permutation of eigenvalues. In this section, we will discuss a few applications of symmetric functions in random matrix theory.

Naturally, permutations and partitions of integers are inseparable from symmetric functions. A partition λ of $n \in \mathbb{N}$ is a non-increasing sequence of integers

$$\lambda_1 \geq \cdots \geq \lambda_l > 0, \quad l \in \mathbb{N}, \tag{1.5.2}$$

such that

$$\lambda_1 + \cdots + \lambda_l = n. \tag{1.5.3}$$

Of many symmetric functions, the Schur functions defined by

$$S_\lambda(x_1, \ldots, x_n) := \frac{\det[x_j^{\lambda_k + n - k}]_{1 \leq j,k \leq n}}{\det[x_j^{n-k}]_{1 \leq j,k \leq n}}, \tag{1.5.4}$$

for

$$\lambda = (\lambda_1, \ldots, \lambda_l, \underbrace{0, \ldots, 0}_{n-l}), \quad l \leq n, \tag{1.5.5}$$

play a distinguished role. The application of symmetric functions to random matrices is best illustrated using the results from the unitary ensemble. The Schur functions arise as the irreducible characters of the unitary group. For two partitions λ and μ,

$$\mathbb{E}_{U(N)}[S_\lambda(U) S_\mu(U^\dagger)] = \mathbb{E}_{U(N)}[S_\lambda(e^{i\theta_1}, \ldots, e^{i\theta_N}) S_\mu(e^{-i\theta_1}, \ldots, e^{-i\theta_N})] = \delta_{\lambda\mu}, \tag{1.5.6}$$

where $e^{i\theta_j}$, $j = 1, \ldots, N$, are the eigenvalues of U. This orthogonality relation for the Schur functions is proven to be a powerful tool in computing integrals with respect to the Haar measure on the unitary group. For example, the expectation

$$\mathbb{E}_{U(N)}\big[f(U)\big], \tag{1.5.7}$$

for any $f(U)$ that is symmetric in the eigenvalues of U, can be readily computed by first expressing f in terms of the Schur functions and then using the orthogonality relation (1.5.6). It is not always easy to find such an expansion for arbitrary f but is possible for sufficiently nice functions. When $f(U)$ is $\operatorname{Tr} U^j$, the Frobenius-Schur duality can be used to express the traces of powers of U in terms of the Schur functions via the characters of the symmetric group.

For a proof of this result and a modern introduction to Frobenius-Schur duality, see [44] (also see (2.1.82) in Ch. 2). Diaconis and Shashahani [72] precisely used this change of basis trick to prove

$$\mathbb{E}_{U(N)}\big[\prod_{j=1}^{k}(\operatorname{Tr}U^j)^{a_j}(\overline{\operatorname{Tr}U^j})^{b_j}\big] = \mathbb{E}\big[\prod_{j=1}^{k}(\sqrt{j}Z_j)^{a_j}(\overline{\sqrt{j}Z_j})^{b_j}\big], \tag{1.5.8}$$

where a_j, b_j are integers and Z_j are independent complex Gaussian normal random variables. The details of the proof are explained in Sec. 3.3 of Ch. 3. Diaconis and Shashahani also computed the correlations of traces for orthogonal and symplectic groups using symmetric functions. Dehaye [62] provided an alternative proof for correlations for symplectic and special orthogonal groups using symmetric functions and gave a combinatorial description of these results.

As a consequence of (1.5.8), we have that the consecutive powers of traces have a limiting Gaussian distribution:

$$\lim_{N\to\infty} P(\operatorname{Tr}U \in B_1, \ldots, \operatorname{Tr}U^k \in B_k) = \prod_{j=1}^{k} P(\sqrt{j}Z \in B_j), \tag{1.5.9}$$

where Z is a standard complex normal, and $B_1, \ldots, B_k$ are any Borel sets.

For a different form of f such as

$$f(U) = \prod_{j=1}^{N} f(e^{i\theta_j}), \tag{1.5.10}$$

an alternate approach to find the expected value of f is via the Weyl integration formula,

$$\mathbb{E}_{U(N)}[f(U)] = \frac{1}{(2\pi)^N N!}\int_0^{2\pi}\cdots\int_0^{2\pi}|\Delta(e^{i\theta_1},\ldots,e^{i\theta_N})|^2\prod_{j=1}^{N} f(e^{i\theta_j})d\theta_j = D_N(f), \tag{1.5.11}$$

where $D_N(f)$ is a Toeplitz determinant with symbol f,

$$D_N(f) = \det[\hat{f}_{j-k}]_{1\le j,k\le N}, \tag{1.5.12}$$

and $\hat{f}_j$ are the Fourier coefficients

$$\hat{f}_j := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ij\theta}\,d\theta, \quad j = 0, \pm1, \pm2, \ldots \tag{1.5.13}$$

Johansson [156] proved (1.5.9) and gave a sharp estimate to the rate of convergence by using (1.5.11) and the Strong Szegő limit theorem [227]. Alternatively, and surprisingly, the result in (1.5.9) provides a new proof of the Strong Szegő's theorem using the theory of symmetric functions [45].

Due to a natural connection to partitions and permutations, the symmetric function theory unifies several theorems in combinatorics; and establishes a bridge between random matrix theory and combinatorics. For example, consider a permutation $\sigma \in \mathcal{S}_n$. An increasing subse-

quence of σ is a sequence $1 \leq j_1 < \cdots < j_k \leq n$ such that

$$\sigma(j_1) < \cdots < \sigma(j_k). \tag{1.5.14}$$

The length of the longest increasing subsequence of a random permutation, denoted by L_n, and the largest eigenvalue of a GUE matrix have the same asymptotic probability distribution, namely the Tracy-Widom distribution [17]. If $P(L_n \leq l)$ denotes the probability that $L_n \leq l$, then Gessel [125] showed that the generating function for $P(L_n \leq l)$ is a Toeplitz determinant. A series of papers [17, 18, 35, 159, 199] show that under the Plancherel measure, the first k parts $(\lambda_1, \ldots, \lambda_k)$ of a random partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ behave statistically as the first k eigenvalues of a GUE matrix in the large matrix size limit. All these classic results can be proved by exploiting the combinatorial nature of the problem and using symmetric functions.

This thesis closely studies quantities in (1.5.8) for Hermitian ensembles using the symmetric function theory. Different symmetric functions and their properties are discussed in detail in Ch. 2.

1.6 Number theory and random matrix theory

In 1973, Montgomery [194] proved that the form factor (the Fourier transform of the two-point correlation function of the eigenvalues) statistic $F(\tau)$ of the GUE and the form factor statistic of the zeros of the Riemann zeta function high up the critical line was identical for a certain range of τ. This was the beginning of a new study of the zeta function through the lens of random matrices. The properties of the Riemann zeros, difficult to study when approached from a number-theoretic perspective, can be studied effectively using random matrices.

For $s \in \mathbb{C}$, the Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.6.1}$$

For $\mathrm{Re}(s) > 1$, $\zeta(s)$ is absolutely convergent. In this regime, $\zeta(s)$ can be expressed as a product over primes p, known as the *Euler product*:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}. \tag{1.6.2}$$

The equivalence between the sum and the product formulas is a manifestation of the Fundamental Theorem of Arithmetic.

Riemann showed that $\zeta(s)$ can be analytically continued to the whole complex plane except for the simple pole at $s = 1$. As a consequence, we have a functional equation for the zeta function

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \tag{1.6.3}$$

which relates $\zeta(s)$ for $\mathrm{Re}(s) > 1/2$ to its values in the other half plane. Here $\Gamma(z)$ is the standard analytical continuation of the factorial.

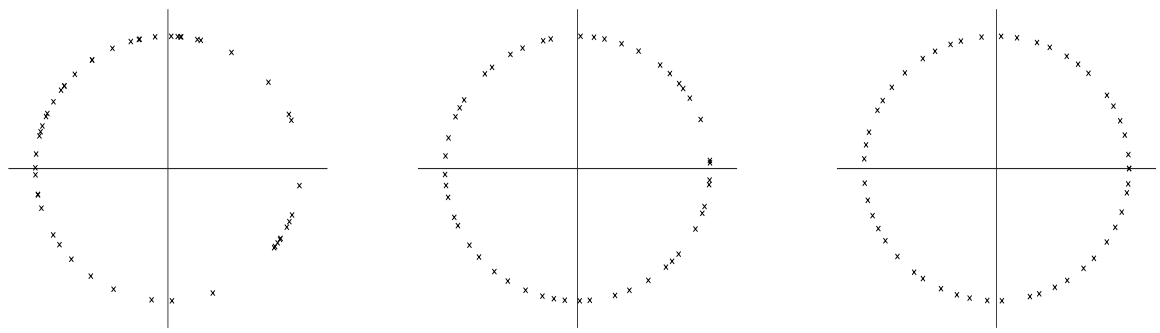
From the functional equation and using other symmetries that the zeta function enjoys,

the Riemann hypothesis asserts that the zeros of the zeta function lie on the line $\mathrm{Re}(s) = 1/2$. Under this assumption, we denote the zeros with $\zeta(\frac{1}{2} + it)$ where $t \in \mathbb{R}$. The number of t such that $0 < t \leq T$ scale as

$$N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T), \quad \text{as } T \to \infty, \tag{1.6.4}$$

which implies that the mean spacing between one zero at $1/2 + it$ and the next one is of order $2\pi/\log t$.

Studies suggest that the correlations of the Riemann zeros and the CUE (or GUE, as the statistics are the same for these two ensembles in the large matrix limit that we deal with, see Sec. 1.4) are identical. In Fig. 1.3, we compare the eigenvalues of a 50×50 Haar distributed unitary matrix and 50 consecutive zeros of the zeta function. The eigenvalues and zeros show similar repulsion in contrast to uniformly picked random points on the unit circle.



(a) 50 points picked uniformly at random on the unit circle.

(b) Eigenvalues of a random 50×50 Haar distributed unitary matrix.

(c) 50 consecutive zeros of the zeta function starting from the 251st zero. The data is taken from [179] and the zeros are scaled to lie on the unit circle.

Figure 1.3: Comparison of random points on the unit circle (a) with the eigenvalues of the CUE (b) and the Riemann zeros high up the critical line (c).

Let U be a unitary matrix of size N with eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_N}$. Rescale the eigenangles θ_j, as done in (1.4.18), so that the mean spacing is 1,

$$\phi_j = \frac{N}{2\pi}\theta_j. \tag{1.6.5}$$

Dyson's results [81] on pair correlation indicate that

$$\lim_{N\to\infty} \mathbb{E}_{U(N)} \left[\frac{1}{N} \left| \{\phi_l, \phi_m : a \leq \phi_m - \phi_l \leq b\} \right| \right] = \int_a^b \left(\delta(x) + 1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2 \right) dx. \tag{1.6.6}$$

As with the unitary case, rescale the zeros of the zeta function so that the mean spacing is 1,

$$u_n = \frac{t_n}{2\pi} \log \frac{t_n}{2\pi}. \tag{1.6.7}$$

Montgomery's conjecture on pair correlation is that

$$\lim_{T\to\infty}\frac{1}{T}|\{u_l,u_m\in[0,T]:a\le u_m-u_l\le b\}|=\int_a^b\left(\delta(x)+1-\left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)dx. \tag{1.6.8}$$

Since the non-trivial zeros are correlated as the eigenvalues of unitary matrices, we model the zeta function with a function whose zeros are these eigenvalues, namely the characteristic polynomial. Studies by Katz and Sarnak [168,168] show that orthogonal and symplectic groups can model other families of $L-$functions. The characteristic polynomial of $U\in U(N)$ is given by

$$Z(U,\theta)=\det(I-Ue^{-i\theta})=\prod_{j=1}^{N}(1-e^{i(\theta_j-\theta)}). \tag{1.6.9}$$

Theorem 1.6.1. *[Keating-Snaith [172]] Let A be a Haar distributed unitary matrix and* $\mathrm{Re}(n)>-1/2$. *Then for any N,*

$$\mathbb{E}_{U(N)}\left[|Z(U,\theta)|^{2n}\right]=\prod_{j=1}^{N}\frac{\Gamma(j)\Gamma(2n+j)}{(\Gamma(n+j))^2}. \tag{1.6.10}$$

In the limit $N\to\infty$, (1.6.10) simplifies to

$$\lim_{N\to\infty}\mathbb{E}_{U(N)}\left[|Z(U,\theta)|^{2n}\right]=\gamma_U(n)N^{n^2} \tag{1.6.11}$$

with

$$\gamma_U(n)=\frac{\mathcal{G}^2(n+1)}{\mathcal{G}(2n+1)}, \tag{1.6.12}$$

where $\mathcal{G}(z)$ is the Barnes G-function that satisfies the following functional equation with the normalisation $\mathcal{G}(1)=1$,

$$\mathcal{G}(z+1)=\Gamma(z)\mathcal{G}(z). \tag{1.6.13}$$

Conjecture 1.6.2. *The moments of the zeta function given by*

$$\frac{1}{T}\int_0^T|\zeta(\tfrac{1}{2}+it)|^{2n}\,dt. \tag{1.6.14}$$

are conjectured to behave as

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|\zeta(\tfrac{1}{2}+it)|^{2n}\,dt=a_\zeta(n)\gamma_\zeta(n)\left(\log\frac{T}{2\pi}\right)^{n^2}, \tag{1.6.15}$$

where $a_\zeta(n)$ is given by the following product over primes p,

$$a_\zeta(n)=\prod_p\left(1-p^{-1}\right)^{n^2}\left[\sum_{j=0}^{\infty}\frac{1}{p^j}\left(\frac{\Gamma(n+j)}{j!\Gamma(n)}\right)^2\right], \tag{1.6.16}$$

and $\gamma_\zeta(n)$ is a function that depends on n.

Except for the first two non-trivial values of n ($n=1,2$) [136,148], $\gamma_\zeta(n)$ remained unknown

for a long time. There are conjectures [54,55] and bounds [140,141,206,220] available for $\gamma_\zeta(n)$ on the assumption of Riemann hypothesis, but a guess on its precise form came into light using the results from RMT. By modelling the zeta function with the characteristic polynomial of a unitary matrix and identifying $N = \log \frac{T}{2\pi}$, Keating and Snaith [172] used their moment results to conjecture that

$$\gamma_\zeta(n) = \gamma_U(n) \tag{1.6.17}$$

for $\mathrm{Re}(n) > -1/2$.

When n is an integer (1.6.12) simplifies to

$$\gamma_U(n) = \prod_{j=0}^{n-1} \frac{j!}{(n+j)!}. \tag{1.6.18}$$

Under the conjecture in (1.6.17), for $n \in \mathbb{N}$, (1.6.15) and (1.6.11) simplifies to

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2n}\, dt = a_\zeta(n) \left(\log \frac{T}{2\pi}\right)^{n^2} \prod_{j=0}^{n-1} \frac{j!}{(n+j)!}, \tag{1.6.19}$$

$$\lim_{N\to\infty} \mathbb{E}_{U(N)}\left[|Z(U,\theta)|^{2n}\right] = N^{n^2} \prod_{j=0}^{n-1} \frac{j!}{(n+j)!}. \tag{1.6.20}$$

Similar results hold for the characteristic polynomials of Hermitian ensembles. For a rescaled GUE matrix $\mathcal{M} = M/\sqrt{N}$, Brezin and Hikami [40] proved that, as $N \to \infty$,

$$e^{nN} e^{-nN\frac{t^2}{2}} \mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^{2n}\right] = (2\pi\varrho_{sc}(t))^{n^2} N^{n^2} \prod_{j=0}^{n-1} \frac{j!}{(n+j)!}. \tag{1.6.21}$$

Here t is a point in the bulk of the spectrum. In addition to N^{n^2} and the common factor $\prod_{j=0}^{n-1} \frac{j!}{(n+j)!}$, the asymptotic moments depend on the limiting distribution, namely the semi-circle law given in (1.3.5). For the unitary case, the limiting distribution is the uniform measure on the unit circle, $1/(2\pi)$, which is cancelled by the factor 2π in (1.6.20). Other Hermitian ensembles, such as the LUE and JUE, also have a structure similar to (1.6.21). For the LUE, up to a factor, the Laguerre weight replaces the Gaussian weight and the Marchenko-Pastur law replaces the semi-circle law.

When t is near the edge of the spectrum, a scaling different from $\sqrt{N}$ is required to obtain finite results in the limit $N \to \infty$. For asymptotics of characteristic polynomials of the GUE in this domain, see [41].

1.7 Characteristic polynomials

Characteristic polynomials of random matrices are of independent interest for several reasons. They have connections to number theory as discussed in Sec. 1.6, combinatorics [71, 223], quantum chaos [12], and many more. We will review some of these connections in this section.

When the characteristic polynomial of a matrix M of size N is expanded in the spectral

variable,

$$\det(t - M) = \sum_{j=0}^{N} (-1)^j \mathrm{Sc}_j(M) t^{N-j}, \tag{1.7.1}$$

the coefficient $\mathrm{Sc}_j(M)$ is called the jth *secular coefficient*. When M is a unitary matrix, the moments of secular coefficients count the number of *magic squares*. Magic squares of order n are squares matrices of size n, whose entries are non-negative integers $(1, \ldots, n^2)$ and whose rows and columns add up to the same number. We have [71]

$$\begin{aligned} \mathbb{E}_{U(N)}[|\mathrm{Sc}_j(M)|^{2n}] = &\text{ Number of magic squares of size } n \\ &\text{ whose rows and columns sum to } j. \end{aligned} \tag{1.7.2}$$

To compute the higher correlations, consider two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_k)$. We have [71]

$$\mathbb{E}_{U(N)}\left[\prod_{j=1}^{k} \mathrm{Sc}_{\lambda_j}(M)\mathrm{Sc}_{\tilde{\lambda}_j}(M^\dagger)\right] = N_{\lambda\tilde{\lambda}}, \tag{1.7.3}$$

where $N_{\lambda\tilde{\lambda}}$ counts the number of non-negative integer matrices of size k such that the jth row and the jth column sum to λ_j and $\tilde{\lambda}_j$, respectively. Similar combinatorial results hold for the other two compact groups $O(N)$ and $Sp(2N)$ [71].

For the GUE, the expectation of the characteristic polynomial is

$$\mathbb{E}_N^{(H)}[\det(t - M)] = \sum_{j=0}^{N} (-1)^j \mathbb{E}_N^{(H)}[\mathrm{Sc}_j(M)] t^{N-j} = H_N(t), \tag{1.7.4}$$

where $H_N(t)$ is the Hermite polynomial of degree N. Due to the symmetries involved, the odd secular coefficients are zero, and the even secular coefficients are given by

$$\mathbb{E}_N^{(H)}[\mathrm{Sc}_{2j}(M)] = (-1)^j \frac{N!}{2^j j!(N-2j)!}. \tag{1.7.5}$$

Hermite polynomials are combinatorial in nature: $H_N(x)$ is the *matching polynomial* of a complete graph K_N with N vertices. In other words, $\mathbb{E}_N^{(H)}[|\mathrm{Sc}_{2j}(M)|]$ counts the number of $2j$ matchings in the complete graph K_N. In [71], Diaconis and Gamburd gave a combinatorial interpretation for higher moments of secular coefficients and characteristic polynomials. For the unitary group, Diaconis and Gamburd also computed the limiting distribution of the secular coefficient Sc_j for a fixed j. Recently, Najnudel, Paquette and Simm [196] studied the limiting distribution of secular coefficients Sc_j for circular $\beta-$ensembles as both j and the size matrix N tend to infinity. They also proved the long standing conjecture that the middle secular coefficient of the CUE of size N tends to zero as $N \to \infty$. For the Gaussian, Laguerre, and Jacobi $\beta-$ensembles, secular coefficients are studied in [190].

The probability distribution of a characteristic polynomial can be fully described by the moments

$$\mathbb{E}_N[\det(t - M)^n], \tag{1.7.6}$$

or more generally by the correlations

$$\mathbb{E}_N\left[\prod_{j=1}^{k}\det(t_j-M)\right]. \tag{1.7.7}$$

Here M can be a GUE, LUE, or JUE matrix, and $\mathbb{E}_N[\cdots]$ denotes the ensemble average. Results in [19,36,40,224] show that these correlations can be written in terms of a determinant of orthogonal polynomials for $\beta=2$ Hermitian ensembles. In its precise form,

$$\mathbb{E}_N\left[\prod_{j=1}^{k}\det(t_j-M)\right]=\frac{1}{\Delta(t_1,\ldots,t_k)}\det[\varphi_{N+k-l}(t_m)]_{1\le l,m\le k}, \tag{1.7.8}$$

where $\varphi_n(x)$ are monic Hermite, Laguerre and Jacobi polynomials for the GUE, LUE and JUE, respectively. The moments can be recovered from (1.7.8) by letting $t_j\to t$ for all j. For a broad class of $\beta=2$ ensembles, Brezin and Hikami [40] calculated the large N asymptotics of the moments. For the GUE, using the integral representation for orthogonal polynomials, Brezin and Hikami showed that in the Dyson limit, $N\to\infty$, $t_l-t_m\to 0$ and $N(t_l-t_m)$ is finite, the moments are equal to (1.6.21).

In Sec. 1.6, we compared the positive moments of random matrices to the positive moments of the zeta-function and noticed the universal features in both cases. Similarly, it is equally interesting to compare the negative moments of characteristic polynomials with the negative moments of the zeta function. The negative correlations are given by

$$\mathbb{E}_N\left[\prod_{j=1}^{k}\frac{1}{\det(T_j-M)}\right]. \tag{1.7.9}$$

For (1.7.9) to be well defined we require $\mathrm{Im}(T_j)\neq 0$. Strahov and Fyodorov [224] computed the negative moments and compared with the negative moments of the zeta function conjectured by Gonek [129]. For example, for a rescaled GUE matrix $\mathcal{M}=M/\sqrt{N}$, the negative moments of the characteristic polynomial behave asymptotically as

$$\begin{aligned}&\mathbb{E}_N^{(H)}\left[\det\left(t+\frac{i\alpha}{4\pi N\varrho_{sc}(t)}-\mathcal{M}\right)^{-n}\det\left(t-\frac{i\alpha}{4\pi N\varrho_{sc}(t)}-\mathcal{M}\right)^{-n}\right]\\ =&(2\pi)^n e^{nN}e^{-nN\frac{t^2}{2}}\left(\frac{2\pi N\varrho_{sc}(t)}{\alpha}\right)^{n^2},\end{aligned} \tag{1.7.10}$$

which should be compared to

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T\left|\zeta\left(\tfrac{1}{2}+\tfrac{\alpha}{\log T}+it\right)\right|^{-2n}dt\sim\left(\frac{\log\frac{T}{2\pi}}{\alpha}\right)^{n^2}. \tag{1.7.11}$$

Both (1.7.10) and (1.7.11) are similar when $2\pi N\varrho_{sc}$ is identified with $\log\frac{T}{2\pi}$.

Another motivation to study negative moments is due to an observation by Berry and Keating [31]. We have seen that the eigenvalues of random matrices repel from each, and

clusters of eigenvalues are very unlikely. Then, it is natural to ask to what extent the clusters are dominant. Berry and Keating addressed this question by showing that the degeneracies in the spectrum can determine the divergences of the negative moments. The asymptotics of the negative moments for $\beta = 1, 2, 4$ ensembles are studied in [31] with a specific focus on the GOE in [110]. The universality results for $\beta = 2$ Hermitian ensembles can be found in [224]. For arbitrary values of β, the large N limits for circular β and $\beta-$Jacobi ensembles are computed in [96].

Yet another important class of correlations involve ratios of characteristic polynomials such as

$$\mathbb{E}_N \left[\prod_{j=1}^{p} \prod_{k=1}^{q} \frac{\det(t_j - M)}{\det(T_k - M)} \right]. \tag{1.7.12}$$

To understand why objects in (1.7.12) are useful, consider the resolvent matrix $(x - M)^{-1}$. The eigenvalue density can be recovered from the resolvent as follows:

$$\rho(x) = \frac{1}{\pi} \lim_{\mathrm{Im}(x) \to 0^-} \mathrm{Im}\, \mathrm{Tr} \frac{1}{x - M}. \tag{1.7.13}$$

The trace of the resolvent can be written as

$$\mathrm{Tr} \frac{1}{x - M} = -\frac{\partial}{\partial y} \frac{\det(x - M)}{\det(y - M)} \Big|_{y=x}. \tag{1.7.14}$$

Clearly, the ratios of characteristic polynomials play a key role in extracting the density of eigenvalues, and correlations of the form in (1.7.12) are useful to extract the multi-point correlation functions. The general correlations in (1.7.12) also found applications in quantum chaos [12, 86, 132]. For example, (1.7.12) can be used in modelling scattering matrices in quantum systems. They are also used for extracting generating functions for the local density of states and the level curvatures.

The correlation functions of both ratios and products of characteristic polynomials of $\beta = 2$ Hermitian ensembles can be expressed in terms of determinants involving orthogonal polynomials and their Cauchy transforms [19, 120] as follows:

$$\begin{aligned} \mathbb{E}_N \left[\prod_{j=1}^{p} \prod_{k=1}^{q} \frac{\det(t_j - M)}{\det(T_k - M)} \right] &= (-2\pi i)^{M-1} \frac{\prod_{j=N-q}^{N-1} c_j^{-1}}{\Delta(\mathbf{t})\Delta(\mathbf{q})} \\ &\times \begin{vmatrix} \vartheta_{N-q}(T_1) & \vartheta_{N-q+1}(T_1) & \dots & \vartheta_{N+p-1}(T_1) \\ \vdots & & & \vdots \\ \vartheta_{N-q}(T_q) & \vartheta_{N-q+1}(T_q) & \dots & \vartheta_{N+p-1}(T_q) \\ \varphi_{N-q}(t_1) & \varphi_{N-q+1}(t_1) & \dots & \varphi_{N+p-1}(t_1) \\ \vdots & & & \vdots \\ \varphi_{N-q}(t_p) & \varphi_{N-q+1}(t_p) & \dots & \varphi_{N+p-1}(t_p) \end{vmatrix}. \end{aligned} \tag{1.7.15}$$

The polynomials $\varphi_n(x)$ are monic orthogonal polynomials with respect to a weight $w(x)$ that satisfy

$$\int \varphi_n(x)\varphi_m(x)w(x)\, dx = c_n \delta_{nm}, \tag{1.7.16}$$

and $\vartheta_n(x)$ is the Cauchy transform of the monic orthogonal polynomial,

$$\vartheta_n(x) = \frac{1}{2\pi i}\int \frac{\varphi_n(y)w(y)\,dy}{y-x}, \quad \mathrm{Im}(x) \neq 0. \tag{1.7.17}$$

Strahov and Fyodorov [224] studied the universal properties of characteristic polynomials in the bulk of the spectrum for $\beta = 2$ Hermitian ensembles by using the Riemann-Hilbert approach. Akemann and Fyodorov [6] extended these results by analysing the universality properties of $\beta = 2$ ensembles at all the three regimes: the soft edge, the bulk and the hard edge of the spectrum.

For $\beta = 1, 4$, the correlations in (1.7.15) can be expressed in terms of Pfaffians [36]. In addition to the GUE, Borodin and Strahov [36] also considered the bulk scaling asymptotic limits of ratios of characteristic polynomials for the GOE and the GSE. The mixed correlations of characteristic polynomials have also been studied for complex random matrix models [8] and non-Hermitian random matrices with independent entries [3]. Moving away from the Dyson values for β, a duality relation exists for products and inverse products of characteristic polynomials [65]. The second order correlation of characteristic polynomials for $\beta-$Hermite ensembles are considered in [225]. Desrosiers and Liu calculated the correlations in (1.7.7) and computed the scaling limits for Gaussian, Laguerre and Jacobi ensembles for arbitrary β in [66]. They also studied the mixed correlations for $\beta-$ensmebles in [67].

The correlations in (1.7.12) can be studied using several methods. Some of the standard techniques are the Riemann-Hilbert method, the super-symmetric technique and its modifications, the orthogonal polynomial method, and the integrals of Selberg type. In the present work, we study them in a new approach using the theory of symmetric functions on par with the results from classical compact groups.

For the classical compact groups such as $U(N)$, $O(N)$, and $Sp(2N)$, the correlations of characteristic polynomials,

$$\mathbb{E}_N\left[\frac{\prod_{j=1}^{p}\det(I+a_j^{-1}A^\dagger)\prod_{k=1}^{q}\det(I+a_{p+k}A)}{\prod_{n=1}^{r}\det(I-b_nA)\prod_{m=1}^{s}\det(I-b_mA^\dagger)}\right], \quad A\in\{U(N),O(N),Sp(2N)\}, \tag{1.7.18}$$

can be studied by taking advantage of the representation-theoretic properties of the groups. It turns out that the characters of these groups are symmetric polynomials in the eigenvalues. By realising that the characteristic polynomial is also a symmetric polynomial in the eigenvalues and the spectral variable, expressing it in terms of the group's characters and using the orthogonality of characters gives a concise way of computing them. In [46], Bump and Gamburd precisely used this technique to study correlations of characteristic polynomials of $U(N)$, $O(N)$, $Sp(2N)$ and provided a beautiful combinatorial interpretation of these moments. The results for the unitary group are discussed in detail in Ch. 3.

One of the main goals of this thesis is to develop a parallel theory for $\beta = 2$ Hermitian ensembles. Though the group structure is not available for Hermitian matrices, we will show how to use the theory of symmetric functions to study characteristic polynomials in Ch. 3. These results then permit a combinatorial approach to study large N limits of characteristic polynomials. Much of Ch. 5 focuses on computing these limits and demonstrating under what

circumstances we recover the semi-circle law in (1.6.21).

1.7.1 Mixed moments

To complete our representative but not exhaustive review of moments, we consider moments of characteristic polynomials along with their derivatives. To keep it simple, we consider the results for the unitary group. Recall that the characteristic polynomial of a unitary matrix U of size N is

$$Z(U,\theta) = \det(I - Ue^{-i\theta}) = \prod_{j=1}^{N}(1 - e^{i(\theta_j - \theta)}), \tag{1.7.19}$$

where $e^{i\theta_1}, \ldots, e^{i\theta_N}$ are the eigenvalues of U. We are interested in the mixed moments involving both moments and derivatives of characteristic polynomials,

$$F_N(k,n) = \mathbb{E}_{U(N)}\left[|Z(U,0)|^{2n-2k}|Z'(U,0)|^{2k}\right]. \tag{1.7.20}$$

For our purpose, it turns out to be beneficial to work with

$$\mathcal{Z}_U(\theta) = e^{\frac{iN}{2}(\pi+\theta)}e^{-\frac{i}{2}\sum_{j=1}^{N}\theta_j}Z(U,\theta) \tag{1.7.21}$$

than the characteristic polynomial $Z(U,\theta)$. For $\mathrm{Re}(k) > -1/2$ and $\mathrm{Re}(n) > \mathrm{Re}(k) - 1/2$, define

$$\tilde{F}_N(k,n) := \mathbb{E}_{U(N)}\left[|\mathcal{Z}_U(0)|^{2n-2k}|\mathcal{Z}_U'(0)|^{2k}\right]. \tag{1.7.22}$$

Note that $\tilde{F}(0,n)$ is precisely the $2n^{th}$ moment of the characteristic polynomial considered in Thm. 1.6.1. For integer values of n and k, Hughes [146] computed $\tilde{F}_N(k,n)$ and computed its large N limit. Additionally, Hughes also shows that the limit

$$\tilde{F}(k,n) = \lim_{N\to\infty}\frac{1}{N^{n^2+2k}}\tilde{F}_N(k,n) \tag{1.7.23}$$

exists and is analytic in n whenever $\mathrm{Re}(n) > k - 1/2$ for $k \in \mathbb{N}$. Dehaye [63] later used the symmetric function theory to derive formulae for $\tilde{F}(k,n)$ for $n,k \in \mathbb{N}$ in terms of sums over partitions. The main interest to study mixed moments in (1.7.22) is due to the connection with number theory to study the Riemann zeta function. Assuming Riemann hypothesis, a series of conjectures [53, 135, 146] indicate that $\tilde{F}(k,n)$ is related to

$$\lim_{T\to\infty}\frac{1}{T}\frac{1}{\left(\log\frac{T}{2\pi}\right)^{n^2+2k}}\int_0^T |\xi(\tfrac{1}{2}+it)|^{2n-2k}|\xi'(\tfrac{1}{2}+it)|^{2k}\,dt, \tag{1.7.24}$$

where

$$\xi(t) = e^{i\nu(t)}\zeta(\tfrac{1}{2}+it), \tag{1.7.25}$$

with

$$\nu(t) := \mathrm{Im}\left[\log\left(\pi^{-\frac{it}{2}}\Gamma(\tfrac{1}{4}+\tfrac{it}{2})\right)\right]. \tag{1.7.26}$$

In general, there is a substantial interest to extend the results of Hughes and Dehaye to non-integer values of k. As a first step in this direction, Winn [248] considered the mixed moments

for half-integer values of k, $k = (2m-1)/2$ for $m \in \mathbb{N}$. Note that the power of the integrand in (1.7.22) is still an integer, but an odd integer for the first time. Basor *et al.* investigated the limiting quantities of (1.7.22) using Riemann Hilbert methods, and gave an epxression for $\tilde{F}(k,n)$ in terms of Painlevé transcendents. In a very recent paper, Assiostis, Keating and Warren [13] extended the results for arbitrary real values of n and positive real values of k for $n > -1/2$, $k < n + 1/2$ and gave a probabilistic interpretation of $\tilde{F}(k,n)$.

Correlation functions involving products and ratios of half-integer powers,

$$\mathbb{E}_N \left[\frac{\det(t_1 - M) \dots \det(t_p - M)}{\det^{1/2}(T_1 - M) \dots \det^{1/2}(T_q - M)} \right] \tag{1.7.27}$$

arise in applications of RMT to quantum chaotic systems. The correlations involving only the half-integer powers of product of characteristic polynomials can be reduced to (1.7.27) by multiplying and dividing the numerator and denominator with corresponding factors. Fyodorov and Nock [114] studied the quantity in (1.7.27) for the GOE and evaluated the large N limits using the formalism of supersymmetry. For the simplest case $p = 1$ and $q = 1$, they recovered the determinantal structure similar to correlations involving only integer powers. For any p and q, the underlying structure of (1.7.27) still remains to be explored. Correlations involving half-integer powers, or more generally any real powers, is a very new and active area of research. In this thesis, we are concerned only with the positive and negative integer powers of characteristic polynomials.

1.8 Spectral moments

The spectral properties of a matrix M can be extracted from the traces $\operatorname{Tr} M^j$ of powers of M. For example, one way of proving the semi-circle law is the moment method, which requires computing the moments $\mathbb{E}_N[\operatorname{Tr} M^j]$ and comparing them with the moments of the semi-circle law, known as the *Catalan numbers*. These numbers play a key role in combinatorics and count a variety of objects ranging from lattice paths to the number of polygon triangulations [221]. This suggests that the traces $\operatorname{Tr} M^j$ are also combinatorial objects. The moment $\mathbb{E}_N[\operatorname{Tr} M^j]$ can be written as a sum over paths of a graph, and the large N limit of moments of these traces can be obtained by carefully counting the number of paths that gives a non-zero contribution, for more details see [79].

Moments of the form

$$\mathbb{E}_N[\operatorname{Tr} M^j] \tag{1.8.1}$$

are well studied for different Hermitian ensembles. For the GUE, they count graphs of certain genus g embedded on surfaces: the $2g^{th}$ coefficient of the large N expansion of (1.8.1) counts the number of pairings of $2g$ vertices in a regular polygon, see for example [154]. Building on the ideas of 't Hooft [143], Brézin, Itzykson, Parisi, and Zuber [42] initiated the connection between random matrices and enumeration problems which led to the start of the random matrix theory of 2d quantum gravity. The averages in (1.8.1) with respect to the GOE and GSE also admit a combinatorial interpretation with corresponding coefficients related to certain maps [195].

The Laguerre and Jacobi moments in (1.8.1) have applications to quantum cavities and

study electrical conductance properties [59, 60, 178, 191, 192, 198, 234]. For the GUE, LUE and JUE, explicit expressions of (1.8.1) in terms of hypergeometric polynomials can be found in [58]. For the time being, we return to our paradigmatic $\beta = 2$ Gaussian ensemble. Towards the end of this section, we will come back to the Laguerre and Jacobi ensembles.

Moments involving more than one trace are of interest, the simplest example being

$$\mathbb{E}_N[\operatorname{Tr} M^l \operatorname{Tr} M^m]. \tag{1.8.2}$$

One can recursively find these moments using the Harer-Zagier recursion [138]. But to the writer's knowledge, no explicit expressions valid for all l and m are available. More generally, one can consider moments of the form

$$\mathbb{E}_N\big[\prod_j (\operatorname{Tr} M^j)^{b_j}\big] \tag{1.8.3}$$

for any sequence of integers b_j. For the GUE, these mixed moments count the number of ribbon graphs on two-dimensional oriented surfaces [32, 138, 143, 228], see details in App. A. These ribbon graphs, also called *Feynmann graphs*, are indispensable in quantum field theory. For the special case when all b_j's are the same, the moments $\mathbb{E}_N^{(H)}[(TrM^j)^n]$ for $j \geq 3$ count polygon numbers on Riemann surfaces [32, 87]. These combinatorial connections are one of the main motivations of studying GUE correlators; see [25, 49, 68, 69, 75, 88, 169, 203].

In similar spirit to the GUE, LUE and JUE correlators are also combinatorial in nature, and can be expressed in terms of double and triple monotone Hurwitz numbers [57, 126, 127]. Hurwitz numbers count factorisations in the symmetric group and can be computed using the character theory of the symmetric group. This problem is equivalent to Hurwitz's original motivation to count branched coverings of the Riemann sphere with specified ramification data [147]. Monotone Hurwitz numbers count a restricted subset of these coverings [130].

Having discussed why the joint moments of traces are important, we shall now explore a method to compute these moments. A convenient way to compute the correlations is by studying the generating functions of $\operatorname{Tr} M^j$. The resolvent can be defined as

$$W_1(x) := \mathbb{E}_N\left[\operatorname{Tr}\frac{1}{x-M}\right] = \sum_{j=0}^{\infty}\frac{1}{x^{j+1}}\mathbb{E}_N[\operatorname{Tr} M^j]. \tag{1.8.4}$$

The moments in (1.8.3) can be obtained from the correlations of resolvents[3]

$$\begin{aligned} W_k(x_1,\dots,x_k) &= \mathbb{E}_N\left[\operatorname{Tr}\frac{1}{x_1-M}\dots\operatorname{Tr}\frac{1}{x_k-M}\right] \\ &= \sum_{j_1,\dots,j_k}\frac{1}{x_1^{j_1+1}\dots x_k^{j_k+1}}\mathbb{E}_N\left[\operatorname{Tr} M^{j_1}\dots\operatorname{Tr} M^{j_k}\right] \end{aligned} \tag{1.8.5}$$

using the loop equations for W_k. For more details and introduction to the loop equation method, see [92]. Also see [101, 250].

[3] We are not concerned with convergence issues while interchanging the summation and integration as we treat the expansion in (1.8.5) as a formal series in variables.

1.9 Central limit theorem

We have seen in Sec. 1.3 that with a proper rescaling, the empirical spectral measure converges to a non-random measure. This result can be viewed as an analogue of the Law of large numbers of probability theory. An alternative way to describe this limit law is as follows. Consider any bounded continuous function f, then,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(x_j) = \int f(x)\rho(x)\, dx, \tag{1.9.1}$$

where x_j's are appropriately scaled eigenvalues and ρ is the limiting distribution. An interesting question to ask is what are the fluctuations of the linear statistic $\sum_j f(x_j)$ around ρ. This can be thought as analogue of the central limit theorem (CLT) of probability theory. This is far from a trivial question as the eigenvalues of random matrices are highly correlated. A variety of studies show that these fluctuations are asymptotically Gaussian [15, 24, 56, 70, 131, 157, 174, 176, 197, 216, 219].

When f in (1.9.1) is a monomial, we get $\operatorname{Tr} M^j$ whose correlations are discussed in the previous section. Often it is convenient to replace the monomials with a better structured entity, such as polynomials orthogonal with respect to the asymptotic eigenvalues density. For the GUE, the limiting spectral density is the semi-circle law, and Chebyshev polynomials are orthogonal with respect to this density.

Johansson [157] showed that for a wide class of Hermitian matrices, the random variable

$$X_k = \operatorname{Tr} T_k(M) - \mathbb{E}_N[\operatorname{Tr} T_k(M)], \quad k \in \mathbb{N}, \tag{1.9.2}$$

converges in distribution to a Gaussian random variable as $N \to \infty$. Here T_k is the Chebyshev polynomial of the first kind of degree k. For the (rescaled) GUE in particular,

$$(X_1, \ldots, X_k) \overset{d}{\Rightarrow} \left(\tfrac{\sqrt{1}}{2} r_1, \ldots, \tfrac{\sqrt{k}}{2} r_k\right), \tag{1.9.3}$$

where $\overset{d}{\Rightarrow}$ denotes convergence in distribution and r_j's are independent real Gaussian random variables with mean 0 and variance 1. Similar results hold for Laguerre and Jacobi ensembles. Much of Ch. 4 involves computing the moments of X_k and estimating the error between these moments and the moments of scaled Gaussians for a finite matrix size N.

1.10 Overview

To prove the original results in this work, we heavily rely on ideas and tools from the theory of symmetric functions. It transpires that the correlations of quantities we study can be formulated in terms of the symmetric functions. We exploit this connection in all the subsequent chapters. In Ch. 2, we introduce various symmetric functions and review the relevant properties.

In Ch. 3, we attempt to provide a new and a concise way of computing correlations of various fundamental quantities. We apply the results from Ch. 2 to study joint moments of

traces and correlations of characteristic polynomials of $\beta = 2$ Hermitian matrices.

In Ch. 4, we study the moments of variables X_k for $\beta = 2$ Gaussian, Laguerre, and Jacobi ensembles. Various approaches, such as the Riemann Hilbert method [30], the Stein's method [176] etc., have already been taken to prove the CLT. In this chapter, we use the moment method with the information about the joint moments of traces computed in 3 to comment on the CLT.

The whole of Ch. 5 is devoted to the large N limits of moments of characteristic polynomials. We do this by using the properties of symmetric polynomials and provide a combinatorial method to study asymptotics. The last chapter summarises the work.

1.11 Authorship

Original research can be found within all subsequent chapters, and here we emphasise where such results can be found. Where the results have appeared in papers (either published or submitted), we give the relevant reference. Additional details are given within the introduction to the respective chapters regarding authorship and, for those based on existing papers, how the chapters differ from the respective manuscripts.

1. In Ch. 2, we introduce symmetric functions along with their properties. Most of the material is taken from the book *Symmetric functions and Hall polynomials* by Macdonald. Relevant references are given at places where we deviate from the book. This chapter also has some new results, which we indicate appropriately.

2. Theorems 3.2.2 and 3.2.3 are the main results in Ch. 3 which appear in a paper *Symmetric function theory and unitary invariant ensembles* with Prof. Jon Keating and Prof. Francesco Mezzadri, submitted.

3. The main result of Ch. 4 is Thm. 4.1.5 which is also a part of the paper *Symmetric function theory and unitary invariant ensembles.*

4. The asymptotic results in Ch. 5 are also a joint work with Prof. Jon Keating and Prof. Francesco Mezzadri. The results appear in the paper *On the moments of characteristic polynomials*, submitted.

Chapter 2

Symmetric function theory

2.1 Symmetric functions and their properties

Symmetric functions appear naturally in random matrix theory since the joint eigenvalue density (1.1.26) is invariant under the permutation of eigenvalues. They are useful whenever an algebraic or combinatorial structure of the ensemble needs to be studied. Symmetric functions are indispensable in this work. Here we give a thorough review of these functions and provide all necessary tools to prove results in the upcoming chapters.

2.1.1 Partitions

The objects we study are parametrised by partitions. Here we introduce the notation and terminology of partitions and recall some of their properties.

Definition 2.1.1. *A partition λ is an ordered sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ satisfying*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l. \tag{2.1.1}$$

Here $l \equiv l(\lambda)$ is the length of the partition. We denote the weight of the partition as

$$|\lambda| = \lambda_1 + \cdots + \lambda_l. \tag{2.1.2}$$

We do not distinguish partitions that only differ by a string of zeros at the end,

$$\lambda = (\lambda_1, \ldots, \lambda_l) = (\lambda_1, \ldots, \lambda_l, 0, \ldots, 0). \tag{2.1.3}$$

For example, $(4, 2, 1, 0, 0)$ and $(4, 2, 1)$ are the same partitions with length 3 and weight 7.

An alternate way of representing a partition λ is by indicating the frequency of an integer that appears in λ:

$$\lambda = (1^{b_1} 2^{b_2} \ldots k^{b_k}) = (\underbrace{k, \ldots, k}_{b_k}, \ldots, \underbrace{2, \ldots, 2}_{b_2}, \underbrace{1, \ldots, 1}_{b_1}), \tag{2.1.4}$$

which means that exactly b_j parts of λ are equal to j for $1 \leq j \leq k$. In this notation, length $l(\lambda) = b_1 + \cdots + b_k$ and weight $|\lambda| = b_1 + 2b_2 + \cdots + kb_k$. For example, 4 has a total of 5

partitions listed below:

$$\begin{aligned}
&(4)\\
&(3,1)\\
&(2,2) \equiv (2^2)\\
&(2,1,1) \equiv (1^2 2)\\
&(1,1,1,1) \equiv (1^4)
\end{aligned} \tag{2.1.5}$$

We use the notation $\lambda \vdash n$ to denote that λ is a partition of n.

Definition 2.1.2 (Dominance order)**.** *Given two partitions $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_l)$, we say that λ dominates μ, $\mu \preceq \lambda$, if*

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \quad \forall i \geq 1, \tag{2.1.6}$$

where it is understood that $\lambda_i = 0$ if $i > l(\lambda)$ (similarly for μ).

Dominance order is only a partial order because there exists partitions λ and μ so that neither $\lambda \preceq \mu$ nor $\mu \prec \lambda$. For example, $(4,1,1)$ and $(3,3)$ cannot be compared under dominance order. Therefore, a total order on partitions, such as defined below, is useful sometimes. Note that $\mu \preceq \lambda$ implies that $|\mu| \leq |\lambda|$ which further indicates that λ and μ can be partitions corresponding to different integers.

Definition 2.1.3 (Lexicographic order)**.** *A total ordering on the set of partitions is defined by saying that $\lambda > \mu$ if for some j we have $\lambda_k = \mu_k$ for all $k < j$, and $\lambda_j > \mu_j$.*

By comparing the two orderings on the set of partitions, it is clear that if $\mu \preceq \lambda$ then $\lambda > \mu$, but the other way is not necessarily true.

Proposition 2.1.4. *Let $\#\operatorname{par}(n)$ to be the total number of partitions of n. The generating function for partition number is*

$$\sum_{n \geq 0} \#\operatorname{par}(n)\, t^n = \prod_{i=1}^{\infty} \frac{1}{1-t^i}. \tag{2.1.7}$$

Proof. The right hand side of (2.1.7) can be expanded as

$$\begin{aligned}
\prod_{i=1} \frac{1}{1-t^i} &= \prod_{i=1} (1 + t^i + t^{2i} + \dots)\\
&= (1 + t + t^2 + \dots)(1 + t^2 + t^4 + \dots)(1 + t^3 + t^6 + \dots)\dots
\end{aligned} \tag{2.1.8}$$

The partition number $\#\operatorname{par}(n)$ is the coefficient of t^n because a term that has n^{th} power in the above expansion is obtained by selecting t^{b_1} from the first factor, t^{2b_2} from the second factor, and so on such that $b_1 + 2b_2 + \cdots = n$. Hence every t^n in the expansion (2.1.8) comes from a partition of n, with the exponent of t resulted by summing the integers smaller than or equal to n. ■

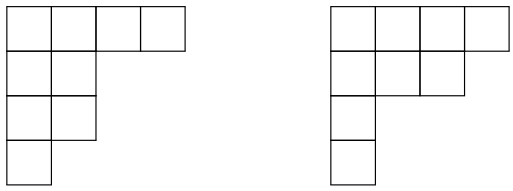
No closed-form expressions for partition numbers are known but there are recursive relations which can be used to compute $\#\operatorname{par}(n)$ exactly. The precise estimate for asymptotics of $\#\operatorname{par}(n)$ is first obtained by Hardy and Ramanujan [137],

$$\#\operatorname{par}(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}, \quad n \to \infty. \tag{2.1.9}$$

Definition 2.1.5 (Young diagram)**.** *A partition can be represented with a Young diagram or Ferrers' diagram which is a left adjusted table of $|\lambda|$ boxes and $l(\lambda)$ rows such that the first row contains λ_1 boxes, the second row contains λ_2 boxes, and so on.*

In other words, a Young diagram of λ is the set $(i,j) \in \mathbb{Z}^2$ such that $i \geq 0$ and $\lambda_i \geq j \geq 0$. With each pair (i,j) we associate a box at the ith row and the jth column where the row index i increases from top to bottom and the column index j increases from left to right.

Definition 2.1.6 (Conjugate partition)**.** *The conjugate partition λ' is defined by transposing the Young diagram of λ along the main diagonal.*



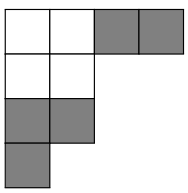
Young diagram of λ Young diagram of λ' (2.1.10)

In the above example $\lambda = (4,2,2,1)$, $|\lambda| = 9$ and $l(\lambda) = 4$. Clearly $l(\lambda') = \lambda_1$, $l(\lambda) = \lambda'_1$ and $\lambda'' = \lambda$. Suppose $\nu = \lambda'$, then $(i,j) \in \lambda$ iff $(j,i) \in \nu$. Hence,

$$j \leq \lambda_i \iff i \leq \nu_j. \tag{2.1.11}$$

Definition 2.1.7 (Sub-partition)**.** *We denote a sub-partition μ of λ by $\mu \subseteq \lambda$ if the Young diagram of μ is contained in the Young diagram of λ. The set-theoretic difference between λ and μ denoted by $\lambda - \mu$ is called a skew-diagram.*

For example, if $\lambda = (4,2,2,1)$ and $\mu = (2,2)$, then μ is a sub-partition of λ and the skew-diagram $\lambda - \mu$ is the shaded region in the picture below:



(2.1.12)

Definition 2.1.8 (Young tableau)**.** *Given a totally ordered finite alphabet of symbols and a partition λ, a Young tableau is a Young diagram of shape λ with each cell of the diagram filled with a symbol from the alphabet.*

Definition 2.1.9. *A standard Young tableau (SYT) of shape λ is a filling of the Young diagram with numbers such that each row and each column forms an increasing sequence. This implies that the alphabet should have at least* $\max(l(\lambda), \lambda_1)$ *numbers.*

Definition 2.1.10. *A semi standard Young tableau (SSYT) is a filling with entries weakly increasing in each row and strictly increasing in each column.*

In Young tableau, SYT and SSYT some numbers can be repeated provided they obey the required condition. The weight of the tableau is the sequence obtained by recording the number of times each number appears. Given below are the examples of standard and semi standard tableau of shape $\lambda = (4, 2, 2, 1)$. We chose the alphabet of n to be the first n natural numbers $\{1, \ldots, n\}$.

$$\begin{array}{|c|c|c|c|}\hline 5 & 1 & 2 & 3 \\ \hline 1 & 2 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 5 & 6 & 7 \\ \hline 2 & 8 \\ \cline{1-2} 3 & 9 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 3 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \tag{2.1.13}$$

A Young tableau of shape λ with the alphabet $\{1, 2, 3, 4, 5\}$. A SYT of shape λ, weight: $(1, \ldots, 1)$ A SSYT of shape λ, weight: $(2, 1, 2, 1, 1, 1, 1)$

Definition 2.1.11 (Content and Hook-length)**.** *A cell (i, j) in λ is located at the ith row and the jth column and its content is given by $j - i$. The hook $\mathfrak{H}_\lambda(i, j)$ is the set of cells (a, b) such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. The hook-length $\mathfrak{h}_\lambda(i, j)$ is the size of the set $\mathfrak{H}_\lambda(i, j)$.*

As an example, the following figures give the content and the hook length of each cell in λ for $\lambda = (4, 2, 2, 1)$.

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 \\ \cline{1-2} -2 & -1 \\ \cline{1-2} -3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 7 & 6 & 2 & 1 \\ \hline 4 & 2 \\ \cline{1-2} 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \tag{2.1.14}$$

Contents of λ Hook lengths of λ

The hook length of λ at (i, j) is given by

$$\mathfrak{h}_\lambda(i, j) = \lambda_i + \lambda'_j - i - j + 1. \tag{2.1.15}$$

The hook length formula expresses the number of standard Young tableaux of shape λ, denoted by f^λ, as

$$f^\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} \mathfrak{h}_{\lambda(i,j)}}. \tag{2.1.16}$$

Proposition 2.1.12. *Let λ be a partition such that $\lambda_1 \leq q$ and $\lambda'_1 \leq p$. Then the $p + q$*

numbers

$$\lambda_i + p - i \, (1 \le i \le p), \quad p - 1 + j - \lambda'_j \, (1 \le j \le q), \tag{2.1.17}$$

are a permutation of $\{0, \ldots, p+q-1\}$ *[182].*

Proof. It can be seen that all the numbers lie between 0 and $p+q-1$. For $1 \le i \le p$, we have $q \ge \lambda_1 \ge \lambda_i \ge 0$. Thus,

$$0 \le \lambda_i + p - i < p + q. \tag{2.1.18}$$

For $1 \le j \le q$, we have $p \ge \lambda'_1 \ge \lambda'_i \ge 0$. Thus,

$$0 \le p - 1 + j - \lambda'_j < p + q. \tag{2.1.19}$$

Now it is sufficient to show that there are no repetitions among these numbers. The sequence $\lambda_i + p - i$ is strictly decreasing by definition as i runs from 1 to p. So there can be no repetitions. Similarly $p - 1 + j - \lambda'_j$ is strictly increasing as j runs from 1 to q. Now it remains to show that $\lambda_i + p - i \ne p - 1 + j - \lambda'_j$ for $1 \le i \le p$ and $1 \le j \le q$. In other words, we require

$$\lambda_i + \lambda'_j + 1 \ne i + j. \tag{2.1.20}$$

There are two cases: If $j \le \lambda_i$ then using (2.1.11) $i \le \lambda'_j$. So, $\lambda_i + \lambda'_j + 1 > \lambda_i + \lambda'_j \ge i + j$. On the other hand, if $j > \lambda_i$ then $i > \lambda'_j$ which implies that $\lambda_i + \lambda'_j + 1 < \lambda_i + \lambda'_j \le i + j$. Thus, (2.1.20) is always satisfied. ■

Proposition 2.1.13. *It follows that [182]*

$$\prod_{(i,j)\in\lambda} (1 - t^{\mathfrak{h}_\lambda(i,j)}) = \frac{\prod_{j\ge1} \prod_{k=1}^{\lambda_j+p-j} (1 - t^k)}{\prod_{j<k} (1 - t^{\lambda_j - \lambda_k - j + k})}. \tag{2.1.21}$$

Proof. Let

$$f_{\lambda,p} = \sum_{j=1}^{p} t^{\lambda_j + p - j}. \tag{2.1.22}$$

Then,

$$\begin{aligned} f_{\lambda,p} + t^{p+q-1} f_{\lambda',q}(t^{-1}) &= \sum_{j=1}^{p} t^{\lambda_j+p-j} + \sum_{j=1}^{q} t^{p-1+j-\lambda'_j} \\ &= 1 + t + \cdots + t^{p+q-1} \\ &= \frac{1 - t^{p+q}}{1 - t}. \end{aligned} \tag{2.1.23}$$

By interchanging λ with λ' in (2.1.23) and setting $q = l(\lambda') = \lambda_1$, we get

$$f_{\lambda',\lambda_1} + t^{p+q-1} f_{\lambda,p}(t^{-1}) = \sum_{j=1}^{\lambda_1} t^{\lambda'_j + \lambda_1 - j} + \sum_{j=1}^{p} t^{\lambda_1 - \lambda_j - 1 + j} = \sum_{j=0}^{\lambda_1+p-1} t^j. \tag{2.1.24}$$

In other words,

$$\sum_{j=1}^{\lambda_1} t^{\mathfrak{h}_\lambda(1,j)} + \sum_{j=2}^{p} t^{\lambda_1-\lambda_j-1+j} = \sum_{j=1}^{\lambda_1+p-1} t^j, \tag{2.1.25}$$

where $\mathfrak{h}_\lambda(1,j)$ are the hook lengths corresponding to the first row of λ. Applying this identity for partitions $(\lambda_j, \lambda_{j+1}, \dots)$, and then summing over $j = 1, \dots, l(\lambda)$ leads to

$$\sum_{(i,j)\in\lambda} t^{\mathfrak{h}_\lambda(i,j)} + \sum_{j<k} t^{\lambda_j-\lambda_k-j+k} = \sum_{j\geq 1} \sum_{k=1}^{\lambda_j+p-j} t^k, \tag{2.1.26}$$

from which it follows that

$$\prod_{(i,j)\in\lambda} (1 - t^{\mathfrak{h}_\lambda(i,j)}) \prod_{j<k} (1 - t^{\lambda_j-\lambda_k-j+k}) = \prod_{j\geq 1} \prod_{k=1}^{\lambda_j+p-j} (1 - t^k). \tag{2.1.27}$$

■

Corollary 2.1.14. *The product of the hook lengths of λ is*

$$\prod_{(i,j)\in\lambda} \mathfrak{h}_\lambda(i,j) = \frac{\prod_{j\geq 1}(\lambda_j + p - j)!}{\prod_{j<k}(\lambda_j - \lambda_k - j + k)}. \tag{2.1.28}$$

Proof. Multiplying both sides of (2.1.21) with $(1-t)^{-|\lambda|}$ and setting $t = 1$ proves the result. ■

Proposition 2.1.15. *Consider a partition λ such that $l(\lambda) \leq n$. Denote the content of each cell by $c_\lambda(i,j)$. Then [182]*

$$\prod_{(i,j)\in\lambda} (1 - t^{n+c_\lambda(i,j)}) = \frac{\prod_{j\geq 1} \prod_{k=1}^{\lambda_j+n-j} (1 - t^k)}{\prod_{j\geq 1} \prod_{k=1}^{n-j} (1 - t^k)}. \tag{2.1.29}$$

Proof. The numbers $n + c_\lambda(i,j)$ in the ith row of λ are $n - i + 1, \dots, n - i + \lambda_i$. Thus we have the above equality. ■

The given results are just a few among several other interesting properties of partitions. For more results, readers can refer to [182].

2.1.2 Symmetric functions

A function f is symmetric if it is invariant under the permutation of its arguments. For $\lambda = (\lambda_1, \lambda_2, \dots)$, let

$$X^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots. \tag{2.1.30}$$

A *homogeneous symmetric function* of degree n is the formal power series

$$f(x_1, x_2, \dots) = \sum_\lambda c_\lambda X^\lambda, \tag{2.1.31}$$

such that $c_\lambda \in \mathbb{R}$ and every term in the sum has the same degree. If the number of variables is finite, we have symmetric polynomials instead of symmetric functions.

More formally, consider the ring of polynomials $\mathbb{Q}[x_1, \ldots, x_n]$ in n independent variables with rational coefficients. The polynomial is symmetric if it is invariant under the action of the symmetric group $\mathcal{S}_n$. Symmetric polynomials form a graded subring

$$\Lambda_n = \mathbb{Q}[x_1, \ldots, x_n]^{\mathcal{S}_n}, \quad \Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k, \tag{2.1.32}$$

where Λ_n^k consists of homogeneous symmetric polynomials of degree k including the zero polynomial. The number of variables is usually irrelevant and it is convenient to work with symmetric functions instead of polynomials. Define the graded ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$ to be

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k. \tag{2.1.33}$$

The elements in Λ are no longer polynomials but are formal sums of monomials. The space of symmetric functions has several important bases usually indexed by partitions. Here we list some of these bases.

The *monomial symmetric functions* are

$$M_\lambda = \sum_\alpha X^\alpha, \tag{2.1.34}$$

where α is summed over all distinct permutations of λ. For instance,

$$\begin{aligned} M_{(1,1)}(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\ M_{(3,1,1)}(x_1, x_2, x_3) &= x_1^3x_2x_3 + x_1x_2^3x_3 + x_1x_2x_3^3. \end{aligned} \tag{2.1.35}$$

Polynomials M_λ such that $l(\lambda) \leq n$ and $|\lambda| = k$ forms a $\mathbb{Z}-$basis of Λ_n^k. The M_λ when λ runs over all partitions of length $\leq n$ forms a $\mathbb{Z}-$basis of Λ_n.

Next are the *complete symmetric functions* h_r,

$$h_r = \begin{cases} \sum_{|\mu|=r} M_\mu, & r \geq 0, \\ 0, & r < 0. \end{cases} \tag{2.1.36}$$

Equivalently, if the number of variables is n,

$$h_r(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n} x_{i_1} x_{i_2} \ldots x_{i_r}, \tag{2.1.37}$$

and $h_r = 0$ for $r < 0$. The generating function for the h_r is

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{j \geq 1} \frac{1}{1 - x_j t}. \tag{2.1.38}$$

We define

$$H_\lambda = h_{\lambda_1} h_{\lambda_2} \dots \tag{2.1.39}$$

which form a $\mathbb{Z}-$basis of Λ. A few examples of H_λ are

$$\begin{aligned} H_{(1,1)}(x_1, x_2, x_3) &= (x_1 + x_2 + x_3)^2, \\ H_{(3,1,1)}(x_1, x_2, x_3) &= \left(x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 \right. \\ &\qquad \left. + x_2^2 x_3 + x_2 x_3^2 + x_1 x_2 x_3\right)(x_1 + x_2 + x_3)^2. \end{aligned} \tag{2.1.40}$$

For each $r \geq 0$, the rth *elementary symmetric function* e_r is

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \dots x_{i_r}, \tag{2.1.41}$$

and $e_r = 0$ for $r < 0$. The generating function for the e_r is

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{j \geq 1} (1 + x_j t). \tag{2.1.42}$$

Similar to the complete symmetric functions,

$$E_\lambda = e_{\lambda_1} e_{\lambda_2} \dots \tag{2.1.43}$$

forms a $\mathbb{Z}-$basis of Λ. Examples of E_λ are

$$\begin{aligned} E_{(1,1)}(x_1, x_2, x_3) &= (x_1 + x_2 + x_3)^2, \\ E_{(3,1,1)}(x_1, x_2, x_3) &= x_1 x_2 x_3 (x_1 + x_2 + x_3)^2. \end{aligned} \tag{2.1.44}$$

Clearly, from (2.1.38) and (2.1.42), one has

$$H(t)E(-t) = 1. \tag{2.1.45}$$

Equivalently, for all $n \geq 1$, we see that

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0. \tag{2.1.46}$$

Since $\Lambda = \mathbb{Z}(e_1, e_2, \dots)$, the e_r are algebraically independent over $\mathbb{Z}$. Therefore, we can define a homomorphism ω of graded rings,

$$\omega : \Lambda \to \Lambda \tag{2.1.47}$$

by

$$\omega(e_r) = h_r. \tag{2.1.48}$$

Using (2.1.46), it can be readily seen that $\omega^2 = 1$, i.e. ω is an involution. Solving (2.1.46) for e_r, we obtain

$$e_r = \det(h_{1-j+k})_{1 \leq j,k \leq r}. \tag{2.1.49}$$

Dually, we obtain

$$h_r = \det(e_{1-j+k})_{1\le j,k\le r}. \tag{2.1.50}$$

Next, we define the rth *power sum function* as

$$p_r = \sum_j x_j^r = m_{(r)}. \tag{2.1.51}$$

for $r \ge 1$. The generating function for the p_r is

$$\begin{aligned} P(t) &= \sum_{r\ge1} p_r t^{r-1} = \sum_{j\ge1}\sum_{r\ge1} x_j^r t^{r-1} \\ &= \sum_{j\ge1} \frac{x_j}{1-x_j t} = \sum_{j\ge1} \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{1}{1-x_j t}, \end{aligned} \tag{2.1.52}$$

so that

$$P(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log \prod_{j\ge1} \frac{1}{1-x_j t} = \frac{\mathrm{d}}{\mathrm{d}t} \log H(t) = \frac{H'(t)}{H(t)}. \tag{2.1.53}$$

Similarly, one can show that

$$P(-t) = \frac{E'(t)}{E(t)}. \tag{2.1.54}$$

From (2.1.53) and (2.1.54), we obtain

$$re_r = \sum_{j=1}^{r} (-1)^{j-1} p_j e_{r-j}, \tag{2.1.55}$$

$$rh_r = \sum_{j=1}^{r} p_j h_{r-j}. \tag{2.1.56}$$

The above equations are called the *Newton's identities*, and they can be used to express the p's in terms of the h's and the e's, and vice versa. Newton's formulae can also be written in a determinant. By treating the e's to be the known functions in (2.1.55) and solving for the p's gives

$$p_r = \begin{vmatrix} e_1 & 1 & 0 & \dots & 0 \\ 2e_2 & e_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ re_r & e_{r-1} & e_{r-2} & \dots & e_1 \end{vmatrix}. \tag{2.1.57}$$

Likewise, by treating the e's to be unknown functions and the p's to be known, we obtain

$$r!\, e_r = \begin{vmatrix} p_1 & 1 & 0 & \dots & 0 \\ p_2 & p_1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{r-1} & p_{r-2} & p_{r-3} & \dots & r-1 \\ p_r & p_{r-1} & p_{r-2} & \dots & p_1 \end{vmatrix}. \tag{2.1.58}$$

The dual relations of (2.1.57) and (2.1.58) are

$$
\begin{aligned}
(-1)^{r-1}p_r &= \begin{vmatrix} h_1 & 1 & 0 & \dots & 0 \\ 2h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ rh_r & h_{r-1} & h_{r-2} & \dots & h_1 \end{vmatrix}, \\
r!\,h_r &= \begin{vmatrix} p_1 & -1 & 0 & \dots & 0 \\ p_2 & p_1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{r-1} & p_{r-2} & p_{r-3} & \dots & -r+1 \\ p_r & p_{r-1} & p_{r-2} & \dots & p_1 \end{vmatrix}.
\end{aligned} \tag{2.1.59}
$$

For example,

$$e_2 = \frac{1}{2}(p_1^2 - p_2), \quad h_2 = \frac{1}{2}(p_1^2 + p_2), \tag{2.1.60}$$

and

$$p_2 = e_1^2 - 2e_2 = -h_1^2 + 2h_2. \tag{2.1.61}$$

If we define

$$P_\lambda = p_{\lambda_1} p_{\lambda_2} \dots \tag{2.1.62}$$

then from the Newton identities, it is clear that the P_λ forms a $\mathbb{Q}-$basis of Λ. Since the involution ω interchanges the e's and the h's, from (2.1.55) and (2.1.56) we have

$$\omega(p_r) = (-1)^{r-1} p_r, \tag{2.1.63}$$

for $r \geq 1$. For any partition λ,

$$\omega(P_\lambda) = (-1)^{|\lambda| - l(\lambda)} P_\lambda. \tag{2.1.64}$$

Similar to (2.1.53) and (2.1.54), the generating functions for the e_r and the h_r can be written in terms of the generating function for the p_r.

Proposition 2.1.16. *Consider a partition $\lambda = (1^{b_1} 2^{b_2} \dots)$ and let*

$$z_\lambda = \prod_{j \geq 1} j^{b_j} b_j!. \tag{2.1.65}$$

Then [182],

$$E(t) = \sum_\lambda (-1)^{|\lambda| - l(\lambda)} \frac{1}{z_\lambda} P_\lambda t^{|\lambda|}, \tag{2.1.66}$$

$$H(t) = \sum_\lambda \frac{1}{z_\lambda} P_\lambda t^{|\lambda|}. \tag{2.1.67}$$

Proof. From (2.1.53),

$$\begin{aligned} H(t) &= \prod_{j\geq 1} \exp\left(\frac{p_j}{j} t^j\right) \\ &= \prod_{j\geq 1} \sum_{b_j=0}^{\infty} \frac{1}{j^{b_j} b_j!} p_j^{b_j} t^{jb_j} \\ &= \sum_{\lambda} \frac{1}{z_\lambda} P_\lambda t^{|\lambda|}. \end{aligned} \tag{2.1.68}$$

Now applying the involution ω proves the first line of the proposition. ■

Next in the list are *Schur polynomials* S_λ given by

$$\begin{aligned} S_\lambda(x_1, \ldots, x_n) &= \frac{\det(x_j^{\lambda_k+n-k})_{1\leq j,k\leq n}}{\det(x_j^{n-k})_{1\leq j,k\leq n}} \\ &= \frac{1}{\Delta(\mathbf{x})} \begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \ldots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \ldots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \ldots & x_n^{\lambda_n} \end{vmatrix}, \end{aligned} \tag{2.1.69}$$

where

$$\Delta(\mathbf{x}) = \det(x_j^{n-k})_{1\leq j,k\leq n} = \prod_{1\leq j<k\leq n} (x_j - x_k) \tag{2.1.70}$$

is the Vandermode determinant. We define $S_\lambda(\mathbf{x}) = 0$ when $l(\lambda) > n$. If $l(\lambda) < n$, we append $n-l(\lambda)$ zeros at the end of λ so that $\lambda_{l+1} = \cdots = \lambda_n = 0$. The Vandermonde determinant is an alternating polynomial i.e. a polynomial which changes sign under the permutation of variables. Since $\Delta(\mathbf{x})$ is an alternating polynomial of the lowest possible degree, it is a factor of every other alternating polynomial. Clearly, the determinant in the numerator is also alternating, and hence divisible by the Vandermonde determinant. Thus, S_λ is a symmetric polynomial in the variables $x_1, \ldots, x_n$. The S_λ form a $\mathbb{Z}-$basis of Λ, and S_λ such that $|\lambda| = k \geq 0$ forms a $\mathbb{Z}-$basis of Λ^k.

A Schur polynomial may be defined combinatorially as a sum of monomials,

$$S_\lambda = \sum_{T\in \mathrm{SSYT}(\lambda)} X^T = \sum_{T\in \mathrm{SSYT}(\lambda)} x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n}, \tag{2.1.71}$$

where the summation is over all semi standard Young tableau T of shape λ. Here t_j counts the occurrences of j in T.

For example,

$$S_{(1,1)}(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1x_3. \tag{2.1.72}$$

The monomials in the R.H.S. arise from the SSYT

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \tag{2.1.73}$$

As another example,

$$S_{(3,1,1)}(x_1,x_2,x_3) = x_1^3x_2x_3 + x_1x_2^3x_3 + x_1x_2x_3^3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2, \tag{2.1.74}$$

and these summands arise from the SSYT

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&3&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$$
$$\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \tag{2.1.75}$$

The combinatorial way of computing the Schur polynomials (2.1.71) becomes cumbersome quickly. On the other hand, the determinantal formula is more reliable for explicit polynomial expressions.

The Schur functions S_λ can be expressed as a polynomial in the complete symmetric functions h_r and the elementary symmetric functions e_r. These expansions are called the *Jacobi-Trudi identities*. Here we state these results and standard proofs can be found in [105, 182]. For the first Jacobi-Trudi identity, we have

$$S_\lambda = \det(h_{\lambda_i - i + j})_{1\le j,k\le l(\lambda)} = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+l(\lambda)-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+l(\lambda)-2} \\ \vdots & \vdots & & \vdots \\ h_{\lambda_l-l(\lambda)+1} & h_{\lambda_l-l(\lambda)+2} & \cdots & h_{\lambda_l} \end{vmatrix}. \tag{2.1.76}$$

Dually, the second Jacobi-Trudi identity is

$$S_\lambda = \det(e_{\lambda'_i - i + j})_{1\le j,k\le l(\lambda')} = \begin{vmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & \cdots & e_{\lambda'_1+l(\lambda')-1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} & \cdots & e_{\lambda'_2+l(\lambda')-2} \\ \vdots & \vdots & & \vdots \\ e_{\lambda'_l-l(\lambda')+1} & e_{\lambda'_l-l(\lambda')+2} & \cdots & e_{\lambda'_l} \end{vmatrix}. \tag{2.1.77}$$

Clearly,

$$S_{(r)} = h_r, \text{ and } S_{(1^r)} = e_r. \tag{2.1.78}$$

From (2.1.76) and (2.1.77), it follows that

$$\omega(S_\lambda) = S_{\lambda'}. \tag{2.1.79}$$

The Schur polynomials can also be expressed as a linear combination of the monomial symmetric polynomials,

$$S_\lambda = \sum_\mu K_{\lambda\mu} M_\mu, \tag{2.1.80}$$

where μ is a partition of $|\lambda|$. Here $K_{\lambda\mu}$ are *Kostka numbers*: non-negative integers that count the number of SSYT of shape λ and weight μ. The values of $K_{\lambda\mu}$ for $|\lambda| = |\mu| = 3$ are listed

below:

$$
\begin{aligned}
&K_{(3)(3)} = K_{(3)(2,1)} = K_{(3)(1,1,1)} = 1, \\
&K_{(2,1)(3)} = 0,\ K_{(2,1)(2,1)} = 1,\ K_{(2,1)(1,1,1)} = 2, \\
&K_{(1,1,1)(3)} = K_{(1,1,1)(2,1)} = 0,\ K_{(1,1,1)(1,1,1)} = 1.
\end{aligned}
\tag{2.1.81}
$$

Like other symmetric polynomials, the power sum polynomials are also related to the Schur polynomials. Consider $\mu = (1^{b_1} 2^{b_2} \dots k^{b_k})$, we have

$$
\begin{gathered}
P_\mu = \sum_\lambda \chi^\lambda_\mu S_\lambda, \quad S_\lambda = \sum_\mu \frac{\chi^\lambda_\mu}{z_\mu} P_\mu, \\
z_\mu = \prod_j j^{b_j} b_j!,
\end{gathered}
\tag{2.1.82}
$$

where χ^λ_μ are the characters of the symmetric group $\mathcal{S}_m$, $m = |\lambda| = |\mu|$. Here λ denotes the irreducible representation and μ denotes the conjugacy class of $\mathcal{S}_m$. The constant z_μ is the size of the *centraliser* of an element in the conjugacy class μ. The centraliser, also called the commutant, of an element g in a group G is the set of elements of G that commute with g. The above equation is an equivalent way of writing the Frobenius formula for the characters of the symmetric group. The orthogonality relation for the characters is

$$
\begin{gathered}
\sum_\lambda \chi^\lambda_\mu \chi^\lambda_\nu = z_\mu \delta_{\mu\nu}, \\
\sum_\mu \frac{1}{z_\mu} \chi^\lambda_\mu \chi^\nu_\mu = \delta_{\lambda\nu},
\end{gathered}
\tag{2.1.83}
$$

Proposition 2.1.17. *For any λ such that $l(\lambda) \leq n$, where n is the number of variables [182],*

$$
S_\lambda(1, \dots, 1) = \prod_{1 \leq j < k \leq n} \frac{\lambda_j - \lambda_k - j + k}{k - j} = \prod_{(j,k) \in \lambda} \frac{n + c_\lambda(j,k)}{\mathfrak{h}_\lambda(j,k)}. \tag{2.1.84}
$$

Proof. First consider the Schur polynomial evaluated at $(1, x, x^2, \dots, x^{n-1})$,

$$
S_\lambda(1, x, \dots, x^{n-1}) = \frac{\det(x^{(j-1)(\lambda_k + n - k)})}{\det(x^{(j-1)(n-k)})}. \tag{2.1.85}
$$

The numerator and the denominator are Vandermonde determinants in the variables $x^{\lambda_k + n - k}$ and x^{n-k}, respectively. Hence,

$$
S_\lambda(1, x, \dots, x^{n-1}) = \frac{\prod_{k<j} (x^{\lambda_j + n - j} - x^{\lambda_k + n - k})}{\prod_{k<j} (x^{n-j} - x^{n-k})}. \tag{2.1.86}
$$

By using the L'Hôpital's rule and taking the limit $x \to 1$, we prove the first equality in (2.1.84).

To prove the second equality, consider

$$\begin{aligned}
\det(x^{(j-1)(\lambda_k+n-k)}) &= \prod_{k<j}(x^{\lambda_j+n-j} - x^{\lambda_k+n-k}) \\
&= x^{\sum_{k<j}\lambda_j+n-j}\prod_{k<j}(1-x^{\lambda_k-\lambda_j-k+j}) \\
&= x^{\sum_j(j-1)\lambda_j+\frac{1}{6}n(n-1)(n-2)}\prod_{k<j}(1-x^{\lambda_k-\lambda_j-k+j}) \\
&= x^{\sum_j(j-1)\lambda_j+\frac{1}{6}n(n-1)(n-2)}\frac{\prod_{j\geq 1}\prod_{k=1}^{\lambda_j+n-j}(1-x^k)}{\prod_{(i,j)\in\lambda}(1-x^{\mathfrak{h}_\lambda(i,j)})} \\
&= x^{\sum_j(j-1)\lambda_j+\frac{1}{6}n(n-1)(n-2)}\prod_{j=1}^{n-1}\prod_{k=1}^{n-j}(1-x^k)\prod_{(i,j)\in\lambda}\frac{1-x^{n+c_\lambda(i,j)}}{1-x^{\mathfrak{h}_\lambda(i,j)}}.
\end{aligned} \tag{2.1.87}$$

We used (2.1.21) in the last but one line, and we used (2.1.29) in the last line. The Vandermonde evaluated at $(1, x, \dots, x^{n-1})$ is

$$\Delta(1, x, \dots, x^{n-1}) = x^{\frac{1}{6}n(n-1)(n-2)}\prod_{j=1}^{n-1}\prod_{k=1}^{n-j}(1-x^k). \tag{2.1.88}$$

Hence,

$$S_\lambda(1, x, \dots, x^{n-1}) = x^{\sum_j(j-1)\lambda_j}\prod_{(i,j)\in\lambda}\frac{1-x^{n+c_\lambda(i,j)}}{1-x^{\mathfrak{h}_\lambda(i,j)}}. \tag{2.1.89}$$

Now setting $x = 1$ proves the proposition. ■

The Schur polynomial evaluated at all 1's counts the number of SSYT of shape λ. Next we have the *Pieri* formula which provides a way to multiply a Schur function S_λ with another Schur function of the form $S_{(r)}$,

$$S_\lambda S_{(r)} = S_\lambda h_r = \sum_\mu S_\mu, \tag{2.1.90}$$

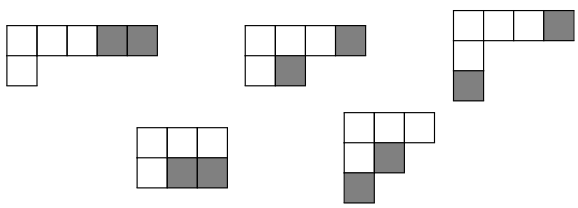
where the sum μ is over all partitions obtained from λ by adding a total of r boxes to the rows, but with no two boxes in the same column. That is those $\mu = (\mu_1, \mu_2, \dots)$ such that

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq 0. \tag{2.1.91}$$

For example,

$$S_{(3,1)}S_{(2)} = S_{(5,1)} + S_{(4,2)} + S_{(4,1,1)} + S_{(3,3)} + S_{(3,2,1)}, \tag{2.1.92}$$

which can be diagrammatically seen as



(2.1.93)

Likewise,

$$S_\lambda S_{(1^r)} = \sum_\nu S_\nu, \tag{2.1.94}$$

where partitions ν are obtained from λ by adding r boxes such that no two boxes are added in the same row.

The *Littlewood-Richardson rule* tells how to multiply two Schur functions S_λ and S_μ for arbitrary λ and μ. More precisely,

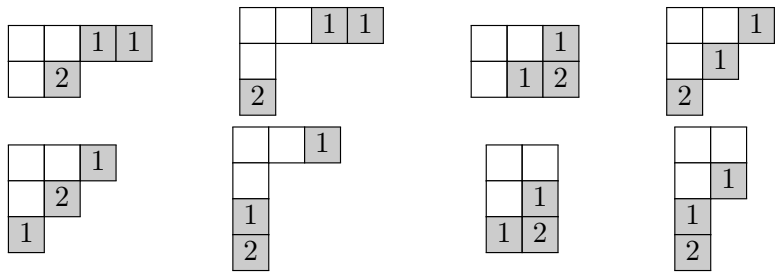
$$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu, \tag{2.1.95}$$

where $c^\nu_{\lambda\mu}$ are non-negative integers and $c^\nu_{\lambda\mu} = 0$ unless $|\nu| = |\lambda| + |\mu|$ with $\mu, \lambda \subseteq \nu$. The coefficients $c^\nu_{\lambda\mu}$ count the number of ways of expanding the Young diagram of λ to the Young diagram of ν by a *strict μ expansion*. A μ expansion of a Young diagram is obtained by first adding μ_1 boxes according to Piere's description, and putting 1 in each of these μ_1 boxes; then adding μ_2 boxes by putting 2 and so on. The expansion is complete when the last entry of μ, say μ_k, is added with integer k. The expansion is *strict* when the following condition is satisfied: If the integers in the boxes are listed from left to right starting from the top row and going down, then each integer a between 1 and $k-1$ appears as many times as the the next integer $a+1$ among the first b entries for any b between 1 and $|\mu|$.

For example,

$$S_{(2,1)}S_{(2,1)} = S_{(4,2)} + S_{(4,1,1)} + S_{(3^2)} + 2S_{(3,2,1)} + S_{(3,1,1,1)} + S_{(2^3)} + S_{(2,2,1,1)}. \tag{2.1.96}$$

The above product $S_{(2,1)}S_{(2,1)}$ can be computed by listing the $(2,1)-$expansion of the Young diagram of $(2,1)$.



(2.1.97)

The coefficients $c^\nu_{\lambda\mu}$ also have the symmetries

$$c^\nu_{\lambda\mu} = c^\nu_{\mu\lambda} = c^{\nu'}_{\lambda'\mu'}. \tag{2.1.98}$$

The functions M_λ, E_λ, H_λ, P_λ, and S_λ are the most important and useful functions for our purposes. In the next section, we give some properties satisfied by these polynomials.

2.1.3 Orthogonality

Define a scalar product on Λ by imposing the condition that the bases (M_λ) and (H_λ) should be orthogonal with respect to each other:

$$\langle M_\lambda, H_\mu \rangle = \delta_{\lambda\mu}, \tag{2.1.99}$$

where λ and μ are any two partitions and $\delta_{\lambda\mu}$ is a Kronecker delta.

Proposition 2.1.18. *Let x_i and y_j, $j = 1, \ldots, n$, be any two sequences of independent variables. Let (v_λ) and (w_λ), indexed by partitions, be a $\mathbb{Q}-$basis of Λ. Then the following conditions are equivalent [182]:*

$$(i) \quad \langle v_\lambda, w_\mu \rangle = \delta_{\lambda\mu}, \ \forall \lambda, \mu. \tag{2.1.100}$$

$$(ii) \quad \sum_\lambda v_\lambda(\boldsymbol{x}) w_\lambda(\boldsymbol{y}) = \prod_{j,k} \frac{1}{1 - x_j y_k}. \tag{2.1.101}$$

Before proving Prop. 2.1.18, we pause to gather the required identities.

Proposition 2.1.19 (Cauchy determinant)**.** *We have*

$$\det \left(\frac{1}{1 - x_j y_k} \right)_{j,k=1,\ldots,n} = \frac{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})}{\prod_{j,k=1}^n (1 - x_j y_k)}. \tag{2.1.102}$$

Proof. We begin with the determinant in the L.H.S. Subtract the first row from the remaining $n-1$ rows by noting that

$$\frac{1}{1 - x_j y_k} - \frac{1}{1 - x_1 y_k} = \frac{x_j - x_1}{1 - x_1 y_k} \frac{y_k}{1 - x_j y_k}. \tag{2.1.103}$$

Factor out the common terms, and subtract the first column from the remaining columns by noting that

$$\frac{y_k}{1 - x_j y_k} - \frac{y_1}{1 - x_j y_1} = \frac{y_k - y_1}{1 - x_j y_1} \frac{1}{1 - x_j y_k}. \tag{2.1.104}$$

After factoring out the common terms, we are left with a determinant whose first row is $(1, 0, \ldots, 0)$ and the entries in the lower right corner being same as the original matrix entries. The proposition can be proved inductively by repeating this process. ■

Proposition 2.1.20 (Cauchy Identity)**.** *We have*

$$(i) \quad \prod_{j,k} \frac{1}{1 - x_j y_k} = \sum_\lambda \frac{1}{z_\lambda} P_\lambda(\boldsymbol{x}) P_\lambda(\boldsymbol{y}) \tag{2.1.105}$$

$$(ii) \quad \prod_{j,k} \frac{1}{1 - x_j y_k} = \sum_\lambda H_\lambda(\boldsymbol{x}) M_\lambda(\boldsymbol{y}) \tag{2.1.106}$$

$$(iii) \quad \prod_{j,k} \frac{1}{1 - x_j y_k} = \sum_\lambda S_\lambda(\boldsymbol{x}) S_\lambda(\boldsymbol{y}) \tag{2.1.107}$$

Proof. *(i)* Denote $\mathbf{xy}$ to be the sequence of variables $x_i y_j$, $i, j = 1, 2 \ldots$. Note that

$$p_r(\mathbf{xy}) = \sum_{i,j} (x_i y_j)^r = p_r(\mathbf{x}) p_r(\mathbf{y}), \tag{2.1.108}$$

which implies

$$P_\lambda(\mathbf{xy}) = P_\lambda(\mathbf{x}) P_\lambda(\mathbf{y}). \tag{2.1.109}$$

Recall

$$h_r = \sum_{\lambda \vdash r} z_\lambda^{-1} P_\lambda, \tag{2.1.110}$$

and

$$H(t) = \sum_r h_r t^r = \prod_j \frac{1}{1 - x_j t}. \tag{2.1.111}$$

By setting $t = 1$,

$$\begin{aligned} \prod_{j,k} \frac{1}{1 - x_j y_k} &= \sum_r h_r(\mathbf{xy}) \\ &= \sum_\lambda z_\lambda^{-1} P_\lambda(\mathbf{xy}) \\ &\sum_\lambda z_\lambda^{-1} P_\lambda(\mathbf{x}) P_\lambda(\mathbf{y}). \end{aligned} \tag{2.1.112}$$

(ii) Using (2.1.38), we have

$$\begin{aligned} \prod_{j,k} \frac{1}{1 - x_j y_k} = \prod_k H(y_k) &= \prod_k \sum_j h_j(\mathbf{x}) y_k^j \\ &= \sum_\lambda H_\lambda(\mathbf{x}) M_\lambda(\mathbf{y}) \\ &= \sum_\lambda M_\lambda(\mathbf{x}) H_\lambda(\mathbf{y}) \end{aligned} \tag{2.1.113}$$

(iii) This is the most useful and important result among the listed identities. Here we give three different proofs of (2.1.107) by using some of the definitions and tools introduced so far. *Representation-theoretic-proof:* Using the orthogonality of the characters of the symmetric group and (2.1.82),

$$\begin{aligned} \prod_{j,k} \frac{1}{1 - x_j y_k} &= \sum_\lambda \frac{1}{z_\lambda} P_\lambda(\mathbf{x}) P_\lambda(\mathbf{y}) \\ &= \sum_\lambda \sum_{\mu,\nu} \frac{1}{z_\lambda} \chi_\lambda^\mu \chi_\lambda^\nu S_\mu(\mathbf{x}) S_\nu(\mathbf{y}) \\ &= \sum_\mu S_\mu(\mathbf{x}) S_\mu(\mathbf{y}). \end{aligned} \tag{2.1.114}$$

Symmetric-function-theoretic-proof: First consider the case when the number of variables is

finite, say n. Let $\delta = (n-1, n-2, \ldots, 0)$, and for any partition λ let

$$a_\lambda = \det(x_j^{\lambda_j}). \tag{2.1.115}$$

Clearly

$$S_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}, \tag{2.1.116}$$

where $\lambda + \delta$ is a partition whose parts are obtained by summing the individual parts of λ and δ,

$$\lambda + \delta = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n). \tag{2.1.117}$$

For an element $\sigma \in \mathcal{S}_n$, denote $\sigma \cdot \lambda$ to be the permutation of the entries of λ. We have,

$$a_{\sigma\cdot\lambda} = \operatorname{sgn}(\sigma) a_\lambda, \tag{2.1.118}$$

$$a_\lambda(\mathbf{y}) = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) Y^{\sigma\cdot\lambda}, \tag{2.1.119}$$

where Y^λ is given in (2.1.30). Recall the first Jacobi-Trudi identity

$$S_\lambda = \frac{a_{\lambda+\delta}}{a_\delta} = \det[h_{\lambda_i - i + j}]. \tag{2.1.120}$$

Therefore for $\alpha = \lambda + \delta$,

$$a_\alpha = a_\delta \det[h_{\alpha_i - n + j}] = a_\delta \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) H_{\alpha-\sigma\cdot\delta}. \tag{2.1.121}$$

By using (2.1.106), (2.1.119) and (2.1.121), we obtain

$$\begin{aligned} a_\delta(\mathbf{x}) a_\delta(\mathbf{y}) \prod_{i,j=1}^n \frac{1}{1 - x_j y_k} &= a_\delta(\mathbf{x}) a_\delta(\mathbf{y}) \sum_\mu H_\mu(\mathbf{x}) M_\mu(\mathbf{y}) \\ &= a_\delta(\mathbf{x}) \sum_{\sigma\in\mathcal{S}_n} \sum_\mu \operatorname{sgn}(\sigma) H_\mu(\mathbf{x}) Y^{\mu+\sigma\cdot\delta} \\ &= a_\delta(\mathbf{x}) \sum_{\sigma\in\mathcal{S}_n} \sum_\nu \operatorname{sgn}(\sigma) H_{\nu-\sigma\cdot\delta}(\mathbf{x}) Y^\nu \\ &= \sum_\nu a_\nu(\mathbf{x}) Y^\nu. \end{aligned} \tag{2.1.122}$$

Using (2.1.118), the sum in the last line is equal to

$$\sum_\alpha a_\alpha(\mathbf{x}) a_\alpha(\mathbf{y}) \tag{2.1.123}$$

for some α such that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. By writing $\alpha = \lambda + \delta$, we arrive at

$$a_\delta(\mathbf{x}) a_\delta(\mathbf{y}) \prod_{j,k=1}^n \frac{1}{1 - x_j y_k} = \sum_\lambda a_{\lambda+\delta}(\mathbf{x}) a_{\lambda+\delta}(\mathbf{y}). \tag{2.1.124}$$

Now letting $n \to \infty$ proves the statement.

Algebraic proof: Start with the identity (2.1.102) and expand the entries of the determinant in the L.H.S. as a formal series in x and y variables:

$$(1 - x_j y_k)^{-1} = 1 + x_j y_k + x_j^2 y_k^2 + \dots . \tag{2.1.125}$$

The coefficient of Y^γ, for some γ such that $\gamma_1 > \gamma_2 > \dots$, in the determinant is $a_\gamma(\mathbf{x})$ where a_γ is the same as in (2.1.115). By the symmetry of the x and y variables we have

$$\begin{aligned} \det\left(\frac{1}{1 - x_j y_k}\right) &= \det(1 + x_j y_k + x_j^2 y_k^2 + \dots) \\ &= \sum_\gamma \det(x_k^{\gamma_j}) \det(y_k^{\gamma_j}) \\ &= \sum_\gamma a_\gamma(\mathbf{x}) a_\gamma(\mathbf{y}), \end{aligned} \tag{2.1.126}$$

Therefore,

$$\det\left(\frac{1}{1 - x_j y_k}\right)_{j,k=1,\dots,n} = \frac{a_\delta(\mathbf{x}) a_\delta(\mathbf{y})}{\prod_{j,k=1}^n (1 - x_j y_k)} = \sum_\gamma a_\gamma(\mathbf{x}) a_\gamma(\mathbf{y}), \tag{2.1.127}$$

Writing $\gamma = \lambda + \delta$ for some partition λ and rearranging the above equation proves the proposition. ■

We are now ready to prove Prop. 2.1.18.

Proof of Prop. 2.1.18. Let

$$v_\lambda = \sum_\alpha a_{\lambda\alpha} M_\alpha, \quad w_\mu = \sum_\beta b_{\mu\beta} H_\beta. \tag{2.1.128}$$

Using (2.1.99), the inner product of v_λ and w_μ is

$$\langle v_\lambda, w_\mu \rangle = \sum_\alpha a_{\lambda\alpha} b_{\mu\alpha}, \tag{2.1.129}$$

Therefore (2.1.100) is equivalent to

$$\sum_\alpha a_{\lambda\alpha} b_{\mu\alpha} = \delta_{\lambda\mu}. \tag{2.1.130}$$

Using (2.1.106), the identity in *(ii)* is equal to

$$\sum_\lambda v_\lambda(\mathbf{x}) w_\lambda(\mathbf{y}) = \sum_\alpha H_\alpha(\mathbf{x}) M_\alpha(\mathbf{y}). \tag{2.1.131}$$

But, we have that

$$\sum_\lambda v_\lambda(\mathbf{x}) w_\lambda(\mathbf{y}) = \sum_\lambda \sum_{\alpha\beta} a_{\lambda\alpha} b_{\lambda\beta} H_\alpha(\mathbf{x}) M_\beta(\mathbf{y}) \tag{2.1.132}$$

By comparing (2.1.131) and (2.1.132),

$$\sum_{\lambda} a_{\lambda\alpha} b_{\lambda\beta} = \delta_{\alpha\beta}, \tag{2.1.133}$$

which is equivalent to (2.1.130). Therefore, (2.1.100) is equivalent to (2.1.101). ■

Proposition 2.1.21 (Dual Cauchy identity)**.** *By applying the involution ω to the symmetric functions in x variables in* (2.1.105)*,* (2.1.106) *and* (2.1.107)*,*

$$(i) \quad \prod_{j,k}(1+x_j y_k) = \sum_{\lambda} \frac{1}{z_\lambda}(-1)^{|\lambda|-l(\lambda)} P_\lambda(\boldsymbol{x}) P_\lambda(\boldsymbol{y}) \tag{2.1.134}$$

$$(ii) \quad \prod_{j,k}(1+x_j y_k) = \sum_{\lambda} E_\lambda(\boldsymbol{x}) M_\lambda(\boldsymbol{y}) = \sum_{\lambda} M_\lambda(\boldsymbol{x}) E_\lambda(\boldsymbol{y}) \tag{2.1.135}$$

$$(iii) \quad \prod_{j,k}(1+x_j y_k) = \sum_{\lambda} S_\lambda(\boldsymbol{x}) S_{\lambda'}(\boldsymbol{y}) \tag{2.1.136}$$

Next, we prove a version of dual Cauchy identity, an essential tool in studying characteristic polynomials.

Proposition 2.1.22. *We have [182]*

$$\prod_{i=1}^{p}\prod_{j=1}^{q}(t_i - x_j) = \sum_{\lambda \subseteq (q^p)} (-1)^{|\tilde{\lambda}|} S_\lambda(\boldsymbol{t}) S_{\tilde{\lambda}}(\boldsymbol{x}), \tag{2.1.137}$$

where $\tilde{\lambda} = (p - \lambda'_q, \ldots, p - \lambda'_1)$.

Proof. For the finite set of variables $t_1, \ldots, t_p$ and $y_1, \ldots, y_q$, the identity in (2.1.136) becomes

$$\prod_{i=1}^{p}\prod_{j=1}^{q}(1 + t_i y_j) = \sum_{\lambda \subseteq (q^p)} S_\lambda(\mathbf{t}) S_{\lambda'}(\mathbf{y}). \tag{2.1.138}$$

By replacing y_j with $-1/x_j$ and denoting

$$\frac{\mathbf{1}}{\mathbf{x}} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_q}\right), \tag{2.1.139}$$

we obtain

$$\prod_{i=1}^{p}\prod_{j=1}^{q}(t_i - x_j) = (-1)^{pq} \prod_{j=1}^{q} x_j^p \sum_{\lambda} S_\lambda(\mathbf{t}) S_{\lambda'}(\mathbf{-1}/\mathbf{x}). \tag{2.1.140}$$

Moreover,

$$\Delta(\mathbf{-1}/\mathbf{x}) = \frac{1}{\prod_{j=1}^{q} x_j^{q-1}} \Delta(\mathbf{x}). \tag{2.1.141}$$

Thus,

$$
\begin{aligned}
(-1)^{pq}\prod_{j=1}^{q} x_j^p S_{\lambda'}(\mathbf{-1}/\mathbf{x}) &= \frac{\prod_{j=1}^{q} x_j^p}{\Delta(\mathbf{-1}/\mathbf{x})}\det\left(\left(\frac{-1}{x_l}\right)^{\lambda'_m+q-m}\right)_{l,m=1,\ldots,q} \\
&= \frac{(-1)^{|\lambda|+\frac{q(q-1)}{2}+pq}}{\Delta(\mathbf{x})}\begin{vmatrix} x_1^{p-\lambda'_1} & \ldots & x_q^{p-\lambda'_1} \\ \vdots & & \vdots \\ x_1^{p+q-\lambda'_q-1} & \ldots & x_q^{p+q-\lambda'_q-1}\end{vmatrix} \\
&= (-1)^{|\tilde{\lambda}|}S_{\tilde{\lambda}}(\mathbf{x}).
\end{aligned}
\tag{2.1.142}
$$

Combining (2.1.140), (2.1.141) and (2.1.142) proves the proposition. ■

The stated results are just a few among several other properties of symmetric functions. Interested readers can refer to [182] for more details.

2.1.4 Multivariate orthogonal polynomials

The Schur polynomials can be generalised by replacing the monomials in the matrix entries with polynomials. Define

$$
\Phi_\mu(\mathbf{x}) := \frac{1}{\Delta(\mathbf{x})}\begin{vmatrix} \varphi_{\mu_1+n-1}(x_1) & \varphi_{\mu_1+n-1}(x_2) & \ldots & \varphi_{\mu_1+n-1}(x_n) \\ \varphi_{\mu_2+n-2}(x_1) & \varphi_{\mu_2+n-2}(x_2) & \ldots & \varphi_{\mu_2+n-2}(x_n) \\ \vdots & \vdots & & \vdots \\ \varphi_{\mu_n}(x_1) & \varphi_{\mu_n}(x_2) & \ldots & \varphi_{\mu_n}(x_n)\end{vmatrix}, \tag{2.1.143}
$$

where $l(\mu) \leq n$ and $\varphi_i, i = 0, 1, \ldots,$ are a sequence of polynomials. If $l(\mu) < n$, we append a sequence of zeros to μ such that $\mu_j = 0$ for $j = l(\mu)+1, \ldots, n$. When $l(\mu) > n$,

$$
\Phi_\mu(\mathbf{x}) := 0. \tag{2.1.144}
$$

These Φ_μ are called *generalised Schur polynomials* [213]. For example, Choose $\varphi_j(x)$ to be

$$
\varphi_j(x) = \sum_{k=0}^{j} a_{jk}x^k \tag{2.1.145}
$$

for some coefficients a_{jk}. Then,

$$
\begin{aligned}
\Phi_{(2)}(x_1,x_2,x_3) &= a_{11}a_{00}\left[a_{44}(x_1^2+x_2^2+x_3^2+x_1x_2+x_2x_3+x_1x_3)\right. \\
&\qquad \left.+a_{43}(x_1+x_2+x_3)+a_{42}\right], \\
\Phi_{(1,1)}(x_1,x_2,x_3) &= a_{00}\left[a_{33}a_{22}(x_1x_2+x_2x_3+x_1x_3)+a_{33}a_{21}(x_1+x_2+x_3)\right. \\
&\qquad \left.+a_{21}a_{32}-a_{22}a_{31}\right].
\end{aligned}
\tag{2.1.146}
$$

When $\varphi_j(x)$ are chosen to be a sequence of polynomials orthogonal with respect to a weight $w(x)$, then the Φ_μ are also called *multivariate orthogonal polynomials* (MOPs) [21,22]. Several properties such as the Cauchy identity and the dual Cauchy identity can also be generalised to MOPs.

We begin with the generalised Jacobi-Trudi identity satisfied by the Φ_μ. Let $\varphi_j(x)$ satisfy a three-term recurrence relation

$$x\varphi_j(x) = \varphi_{j+1}(x) + a_j\varphi_j(x) + b_j\varphi_{j-1}(x) \tag{2.1.147}$$

with $\varphi_0 = 1$ and $\varphi_{-j} = 0$ for $j \in \mathbb{N}$. For the sequence of coefficients a_j and b_j, define the polynomials $h_r^{(j)}(x_1, \ldots, x_n)$ recursively by

$$h_r^{(j+1)} = h_{r+1}^{(j)} + a_{r+n-1}h_r^{(j)} + b_{r+n-1}h_{r-1}^{(j)} \tag{2.1.148}$$

with the initial data

$$h_r^{(0)} = \Phi_r(\mathbf{x}) = \frac{1}{\Delta(\mathbf{x})}\begin{vmatrix} \varphi_{r+n-1}(x_1) & \varphi_{r+n-1}(x_2) & \ldots & \varphi_{r+n-1}(x_n) \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{vmatrix}. \tag{2.1.149}$$

Note that the integers j and r can be less than zero, but it can be checked that $h_r^{(j)} = 0$ if $r + j < 0$. For the recursive relation (2.1.148) to be well-defined, extend the coefficients a_i and b_i arbitrarily to negative i. Whenever $j \leq r + 2n - 2$, the $h_r^{(j)}$ does not depend on the coefficients a_i and b_i extended to the negative i.

Lemma 2.1.23. *We have [213]*

$$h_r^{(j)}(x_1, \ldots, x_n) - x_1h_r^{(j-1)}(x_1, \ldots, x_n) = h_{r+1}^{(j-1)}(x_2, \ldots, x_n) \tag{2.1.150}$$

Proof. First set $j = 1$. By using (2.1.147) and (2.1.148), the L.H.S. of (2.1.150) is

$$\begin{aligned} & h_r^{(1)}(x_1, \ldots, x_n) - x_1h_r^{(0)}(x_1, \ldots, x_n) \\ =& h_{r+1}^{(0)}(x_1, \ldots, x_n) + a_{r+n-1}h_r^{(0)}(x_1, \ldots, x_n) + b_{r+n-1}h_{r-1}^{(0)}(x_1, \ldots, x_n) \\ =& \frac{1}{\Delta(\mathbf{x})}\begin{vmatrix} 0 & (x_2 - x_1)\varphi_{r+n-1}(x_2) & \ldots & (x_n - x_1)\varphi_{r+n-1}(x_n) \\ \varphi_{n-2}(x_1) & \varphi_{n-2}(x_2) & \ldots & \varphi_{n-2}(x_n) \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{vmatrix} \end{aligned} \tag{2.1.151}$$

Subtracting the first column from the remaining columns give

$$\begin{aligned}
&\frac{1}{\Delta(x_1,\dots,x_n)}\begin{vmatrix} 0 & (x_2-x_1)\varphi_{r+n-1}(x_2) & \dots & (x_n-x_1)\varphi_{r+n-1}(x_n) \\ \varphi_{n-2}(x_1) & \varphi_{n-2}(x_2)-\varphi_{n-2}(x_1) & \dots & \varphi_{n-2}(x_n)-\varphi_{n-2}(x_1) \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{vmatrix} \\
=&\frac{1}{\Delta(x_2,\dots,x_n)}\begin{vmatrix} \varphi_{r+n-1}(x_2) & \dots & \varphi_{r+n-1}(x_n) \\ \frac{\varphi_{n-2}(x_2)-\varphi_{n-2}(x_1)}{x_2-x_1} & \dots & \frac{\varphi_{n-2}(x_n)-\varphi_{n-2}(x_1)}{x_n-x_1} \\ \vdots & & \vdots \\ \frac{\varphi_1(x_2)-\varphi_1(x_1)}{x_2-x_1} & \dots & \frac{\varphi_1(x_n)-\varphi_1(x_1)}{x_n-x_1} \end{vmatrix}.
\end{aligned} \tag{2.1.152}$$

The determinant in the last can be simplified further. The entry

$$\frac{\varphi_k(x_l)-\varphi_k(x_m)}{x_l-x_m} \tag{2.1.153}$$

is a polynomial of degree $k-1$ in x_l and x_m. Doing the rows operations on the last $n-2$ rows simplifies the determinant to

$$h^{(0)}_{r+1}(x_2,\dots,x_n). \tag{2.1.154}$$

Using (2.1.148), the lemma can be proved by induction in j. ■

Proposition 2.1.24. *The generalised Jacobi-Trudi formula is [213]*

$$\Phi_\mu = \begin{vmatrix} h^{(0)}_{\mu_1} & h^{(1)}_{\mu_1} & \dots & h^{(l-1)}_{\mu_1} \\ h^{(0)}_{\mu_2-1} & h^{(1)}_{\mu_2-1} & \dots & h^{(l-1)}_{\mu_2-1} \\ \vdots & \vdots & & \vdots \\ h^{(0)}_{\mu_l-l+1} & h^{(1)}_{\mu_l-l+1} & \dots & h^{(l-1)}_{\mu_l-l+1} \end{vmatrix}, \tag{2.1.155}$$

where $l = l(\mu)$.

Proof. To make the proof more readable, we use the following notation to denote the permutation of variables x_j:

$$\{f(x_1,\dots,x_n)\} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) f(x_{\sigma(1)},\dots,x_{\sigma(n)}). \tag{2.1.156}$$

By the definition of $h^{(0)}_r$ we have

$$h^{(0)}_r(x_1,\dots,x_n)\Delta(x_1,\dots,x_n) = \{h^{(0)}_r x_1^{n-1}\dots x_n^0\} = \{\phi_{r+n-1}(x_1)x_2^{n-2}\dots x_n^0\}. \tag{2.1.157}$$

Using (2.1.147) and (2.1.148), it can be shown by induction in j that

$$\{h^{(j)}_r x_1^{n-1}\dots x_n^0\} = \{x_1^j \phi_{r+n-1}(x_1)x_2^{n-2}\dots x_n^0\} \tag{2.1.158}$$

for $j \leq r + 2n - 2$. Now, consider

$$
\begin{aligned}
&\Delta(x_1, \dots, x_n) \det(h^{(k)}_{\mu_j - j + 1})_{j,k=0,\dots,l(\mu)-1} \\
&= \left\{ \begin{vmatrix} h^{(0)}_{\mu_1} & h^{(1)}_{\mu_1} & \dots & h^{(l-1)}_{\mu_1} \\ h^{(0)}_{\mu_2 - 1} & h^{(1)}_{\mu_2 - 1} & \dots & h^{(l-1)}_{\mu_2 - 1} \\ \vdots & \vdots & & \vdots \\ h^{(0)}_{\mu_l - l + 1} & h^{(1)}_{\mu_l - l + 1} & \dots & h^{(l-1)}_{\mu_l - l + 1}, \end{vmatrix} x_1^{n-1} \dots x_n^0 \right\} \\
&= \left\{ \begin{vmatrix} \phi_{\mu_1 + n - 1}(x_1) & x_1 \phi_{\mu_1 + n - 1}(x_1) & \dots & x_1^{l-1} \phi_{\mu_1 + n - 1}(x_1) \\ h^{(0)}_{\mu_2 - 1}(x_1, \dots, x_n) & h^{(1)}_{\mu_2 - 1}(x_1, \dots, x_n) & \dots & h^{(l-1)}_{\mu_2 - 1}(x_1, \dots, x_n) \\ \vdots & \vdots & & \vdots \\ h^{(0)}_{\mu_l - l + 1}(x_1, \dots, x_n) & h^{(1)}_{\mu_l - l + 1}(x_1, \dots, x_n) & \dots & h^{(l-1)}_{\mu_l - l + 1}(x_1, \dots, x_n), \end{vmatrix} x_2^{n-2} \dots x_n^0 \right\}.
\end{aligned}
\tag{2.1.159}
$$

Except for the last column, multiply all the other column with x_1 and subtract it from the next column. Then, applying Lemma. 2.1.23 to all the rows except for the first row gives

$$
\left\{ \phi_{\mu_1 + n - 1}(x_1) \begin{vmatrix} h^{(0)}_{\mu_2}(x_2, \dots, x_n) & \dots & h^{(l-2)}_{\mu_2}(x_2, \dots, x_n) \\ \vdots & & \vdots \\ h^{(0)}_{\mu_l - l + 2}(x_2, \dots, x_n) & \dots & h^{(l-1)}_{\mu_l - l + 2}(x_2, \dots, x_n) \end{vmatrix} x_2^{n-2} \dots x_n^0 \right\}. \tag{2.1.160}
$$

By induction this simplifies to

$$
\{\phi_{\mu_1 + n - 1}(x_1) \phi_{\mu_2 + n - 2}(x_2) \dots \phi_{\mu_n}(x_n)\}. \tag{2.1.161}
$$

■

Proposition 2.1.25 (Laplace Expansion)**.** *Let $\Xi_{p,q}$ consist of all permutations $\sigma \in \mathcal{S}_{p+q}$ such that*

$$
\sigma(1) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q). \tag{2.1.162}
$$

Let $A = a_{ij}$ be a $(p+q) \times (p+q)$ matrix, then the Laplace expansion in the first p rows can be written as

$$
\det[a_{ij}] = \sum_{\sigma \in \Xi_{p,q}} sgn(\sigma) \begin{vmatrix} a_{1,\sigma(1)} & \dots & a_{1,\sigma(p)} \\ \vdots & & \vdots \\ a_{p,\sigma(1)} & \dots & a_{p,\sigma(p)} \end{vmatrix} \times \begin{vmatrix} a_{p+1,\sigma(p+1)} & \dots & a_{p+1,\sigma(p+q)} \\ \vdots & & \vdots \\ a_{p+q,\sigma(p+1)} & \dots & a_{p+q,\sigma(p+q)} \end{vmatrix}. \tag{2.1.163}
$$

Proposition 2.1.26 (Generalised dual Cauchy identity)**.** *We have [165]*

$$
\prod_{i=1}^{p} \prod_{j=1}^{q} (t_i - x_j) = \sum_{\lambda \subseteq (q^p)} (-1)^{|\tilde{\lambda}|} \Phi_\lambda(\boldsymbol{t}) \Phi_{\tilde{\lambda}}(\boldsymbol{x}). \tag{2.1.164}
$$

Here $\tilde{\lambda} = (p - \lambda'_q, \dots, p - \lambda'_1)$.

Proof. Assume that φ_j are monic. Using the definition of generalised polynomials, Proposi-

tion 2.1.25 and Proposition 2.1.12, the right-hand side of (2.1.164) can be written as

$$\frac{1}{\Delta_p(\mathbf{t})}\frac{1}{\Delta_q(\mathbf{x})}\begin{vmatrix} \varphi_{p+q-1}(t_1) & \varphi_{p+q-2}(t_1) & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \varphi_{p+q-1}(t_p) & \varphi_{p+q-2}(t_p) & \dots & 1 \\ \varphi_{p+q-1}(x_1) & \varphi_{p+q-2}(x_1) & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \varphi_{p+q-1}(x_q) & \varphi_{p+q-2}(x_q) & \dots & 1 \end{vmatrix}. \tag{2.1.165}$$

Now, using column operations we arrive at

$$\frac{1}{\Delta_p(\mathbf{t})}\frac{1}{\Delta_q(\mathbf{x})}\begin{vmatrix} t_1^{p+q-1} & t_1^{p+q-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ t_p^{p+q-1} & t_p^{p+q-2} & \dots & 1 \\ x_1^{p+q-1} & x_1^{p+q-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ x_q^{p+q-1} & x_q^{p+q-2} & \dots & 1 \end{vmatrix}. \tag{2.1.166}$$

The determinant in (2.1.166) can be evaluated using the formula for the Vandermonde determinant. We have

$$\prod_{1\le i<j\le p}(t_i-t_j)\prod_{1\le i<j\le q}(x_i-x_j)\prod_{i=1}^{p}\prod_{j=1}^{q}(t_i-x_j). \tag{2.1.167}$$

Combining eqs. (2.1.165)–(2.1.167) proves the lemma. ■

If $\varphi_j(-x)=(-1)^j\varphi_j(x)$, as for Hermite polynomials, then

$$\Phi_\mu(-x_1,\dots,-x_N)=(-1)^{|\mu|}\Phi_\mu(x_1,\dots,x_N). \tag{2.1.168}$$

It follows that (2.1.164) becomes

$$\prod_{i=1}^{p}\prod_{j=1}^{q}(t_i+x_j)=\sum_{\lambda\subseteq(q^p)}\Phi_\lambda(t_1,\dots,t_p)\Phi_{\tilde{\lambda}}(x_1,\dots,x_q). \tag{2.1.169}$$

The proof of Prop. 2.1.26 also gives an alternative way to prove the classical dual Cauchy identity when the polynomials in (2.1.165) are replaced with monomials.

Polynomials Φ_μ can be expressed as a linear combination of Schur polynomials and other classical symmetric polynomials. For example

$$\Phi_\mu(\mathbf{x})=\sum_{\nu\subseteq\mu}\kappa_{\mu\nu}S_\nu(\mathbf{x}). \tag{2.1.170}$$

We give the explicit expressions for the coefficients $\kappa_{\mu\nu}$ in Ch. 3, Sec. 3.4. For a well-defined weight $w(x)$, polynomials Φ_μ satisfy several interesting properties similar to their univariate analogues. In Ch. 3, we will mention a few of these properties, such as the orthogonality relations and differential equations satisfied by Φ_μ.

Because the Schur polynomials and the multivariate orthogonal polynomials can be expressed as a ratio of determinants, both S_λ and Φ_λ defined in (2.1.69) and (2.1.143) are specific to $\beta = 2$ ensembles. Schur polynomials generalised to other values of β are called Jack polynomials $C_\lambda^{(\alpha)}$, $\alpha = 2/\beta$, which are homogeneous symmetric polynomials that satisfy the following properties:

— We say that the monomial $x_1^{\lambda_1} \dots x_l^{\lambda_l}$ is of higher weight than $x_1^{\mu_1} \dots x_l^{\mu_l}$ if $\lambda > \mu$. The polynomial $C_\lambda^{(\alpha)}(\mathbf{x})$ has the form

$$C_\lambda^{(\alpha)}(\mathbf{x}) = c_\lambda x_1^{\lambda_1} \dots x_l^{\lambda_l} + \text{monomials with lower weight}, \tag{2.1.171}$$

where c_λ is a constant and the monomial $x_1^{\lambda_1} \dots x_l^{\lambda_l}$ is of highest weight.

— The normalisation of $C_\lambda^{(\alpha)}$ is fixed by the condition

$$(x_1 + \dots + x_n)^j = \sum_{\substack{\lambda \vdash j \\ l(\lambda) \leq n}} C_\lambda^{(\alpha)}(x_1, \dots, x_n). \tag{2.1.172}$$

— The polynomial $C_\lambda^{(\alpha)}(x_1, \dots, x_n)$ is an eigenfunction of the differential operator

$$\sum_{j=1}^{n} x_j^2 \frac{\partial^2}{\partial x_j^2} + \frac{2}{\alpha} \sum_{j \neq k} \frac{x_j^2}{x_j - x_k} \frac{\partial}{\partial x_j}. \tag{2.1.173}$$

All the above conditions define the Jack polynomial $C_\lambda^{(\alpha)}$ uniquely. The differential operator in (2.1.173) is the Hamiltonian of a Calogero-Sutherland-type quantum system [21]. The Jack polynomials that we defined are referred as 'C' normalised. There are other normalisations for Jack polynomials, namely the 'P' and 'J' normalisations. In the 'J' normalisation, the coefficient of the monomial $x_1 \dots x_n$ in $C_\lambda^{(\alpha)}(\mathbf{x})$ is $n!$ for $|\lambda| = n$. In the 'P' normalisation, the coefficient of the monomial of the the highest weight should be 1. For a detailed description of different normalisations and their uses, the reader is encouraged to refer to [77].

Similar to the Schur polynomials, multivariate orthogonal polynomials can also be defined for arbitrary β. Throughout this work, we are interested when $\varphi_n(x)$ in (2.1.143) is one of the Hermite, Laguerre or Jacobi polynomials. The classical Hermite $H_n(x)$, Laguerre $L_n^{(\gamma)}(x)$ and Jacobi polynomials $J_n^{(\gamma_1,\gamma_2)}(x)$ satisfy the differential equations

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} H_n(x) - x \frac{\mathrm{d}}{\mathrm{d}x} H_n(x) = -n H_n(x), \tag{2.1.174}$$

$$x \frac{\mathrm{d}^2}{\mathrm{d}x^2} L_n^{(\gamma)}(x) + (1 + \gamma - x) L_n^{(\gamma)}(x) = -n L_n^{(\gamma)}(x), \tag{2.1.175}$$

$$\begin{aligned} &x(1-x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} J_n^{(\gamma_1,\gamma_2)}(x) + (\gamma_1 + 1 - x(\gamma_1 + \gamma_2 + 2)) \frac{\mathrm{d}}{\mathrm{d}x} J_n^{(\gamma_1,\gamma_2)}(x) \\ &= -n(n + \gamma_1 + \gamma_2 + 1) J_n^{(\gamma_1,\gamma_2)}(x). \end{aligned} \tag{2.1.176}$$

Likewise, the multivariate Hermite, Laguerre and Jacobi polynomials defined for any β are the

polynomial part of the eigenfunctions of the operators

$$H^{(H)} = \sum_{j=1}^{N} \left(\frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j} \right) + 2 \sum_{\substack{j,k=1 \\ k \neq j}}^{N} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}, \tag{2.1.177}$$

$$H^{(L)} = \sum_{j=1}^{N} \left(x_j \frac{\partial^2}{\partial x_j^2} + (\gamma - x_j + 1) \frac{\partial}{\partial x_j} \right) + 2 \sum_{\substack{j,k=1 \\ k \neq j}}^{N} \frac{x_j}{x_j - x_k} \frac{\partial}{\partial x_j}, \tag{2.1.178}$$

$$H^{(J)} = \sum_{j=1}^{N} \left(x_j(1 - x_j) \frac{\partial^2}{\partial x_j^2} + (\gamma_1 + 1 - x_j(\gamma_1 + \gamma_2 + 2)) \frac{\partial}{\partial x_j} \right) + 2 \sum_{\substack{j,k=1 \\ k \neq j}}^{N} \frac{x_j(1 - x_j)}{x_j - x_k} \frac{\partial}{\partial x_j}. \tag{2.1.179}$$

No explicit expressions for MOPs are available for $\beta \neq 2$, but they can be defined using recursive relations as indicated in [21, 22]. In [77], Dumitriu, Edelman and Shuman developed a Maple package to compute the multivariate orthogonal polynomials for any β.

Chapter 3

Mixed Moments of Hermitian ensembles

The basis of this chapter is the paper *Symmetric function theory and unitary invariant ensembles* [165] which is a joint work with J. P. Keating and F. Mezzadri. The present author entirely carried the project with the advisement from J. P. Keating and F. Mezzadri.

Most of the material in this chapter closely follows [165] except for a few changes. This chapter is expanded, and more examples are included for better readability. An additional section, Sec. 3.3, is added in the thesis by the present author for contextualising and a better understanding of the original results. The background section in [165] is relocated to Ch. 2 where all the necessary tools are introduced. The last section in [165] is moved to Ch. 4 as it is an application of the results given in this chapter and involves a slightly different topic. The current text also incorporates one of the appendices in [165].

3.1 Introduction

Many important quantities in random matrix theory, such as the joint moments of traces and the joint moments of characteristic polynomials, can be calculated exactly for matrices drawn from the CUE and the other circular ensembles related to the classical compact groups using representation theory and the theory of symmetric polynomials. In the case of joint moments of the traces, this approach has proved highly successful, as in, the work of Diaconis and Shahshahani [72]. For example, for the unitary group we have the following theorem.

Theorem 3.1.1. *Consider two sets of positive integers* $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)$, $\boldsymbol{b} = (b_1, b_2, \ldots, b_k)$, *and let* $Z_1, \ldots, Z_k$ *be independent standard complex normal random variables. For a Haar distributed unitary matrix* M *of size* N, *and for* $N \geq \max(\sum_j j a_j, \sum_j j b_j)$ *[72],*

$$\mathbb{E}_{U(N)}\left[\prod_{j=1}^{k}(\operatorname{Tr} M^j)^{a_j}\overline{(\operatorname{Tr} M^j)}^{b_j}\right] = \prod_{j=1}^{k} j^{a_j} a_j!\,\delta_{\boldsymbol{ab}} = \mathbb{E}\left[\prod_{j=1}^{k}(\sqrt{j}Z_j)^{a_j}(\sqrt{j}\,\overline{Z_j})^{b_j}\right]. \tag{3.1.1}$$

It is quite remarkable that the mixed moments of traces are exactly equal to that of complex Gaussians for any finite matrix size N. Similarly, the joint moments of characteristic polynomials were calculated exactly in terms of Schur polynomials by Bump and Gamburd [46],

leading to expressions equivalent to those obtained using the Selberg integral and related techniques [20, 52, 171, 172].

Theorem 3.1.2 (Bump and Gamburd [46]). *If $K, L \in \mathbb{N}$ and $a_1, \ldots, a_{K+L} \in \mathbb{C}$, then*

$$\mathbb{E}_{U(N)}\left[\prod_{j=1}^{L}\det(I + a_j^{-1}M^\dagger)\prod_{l=1}^{K}\det(I + a_{L+l}M)\right] = \frac{S_{(N^L)}(a_1, \ldots, a_{K+L})}{\prod_{l=1}^{L} a_l^N}. \tag{3.1.2}$$

To prove the above theorem, Bump and Gamburd expressed the products of characteristic polynomials, which are symmetric polynomials in the eigenvalues and variables a_j, in the Schur basis and used the orthogonality of Schur polynomials. Our aim here is to develop a parallel theory for the classical unitary invariant Hermitian ensembles of random matrices, in particular for the GUE, LUE, and JUE.

Characteristic polynomials and their asymptotics have been well studied for Hermitian matrices using orthogonal polynomials, super-symmetric techniques, Selberg and Itzykson-Zuber integrals, see, for example, [19, 36, 102, 118–120]. Other properties including universality [39, 224], and ensembles with external sources [99, 108] have also been considered. Here we give a symmetric-function-theoretic approach similar to that established by Bump and Gamburd [46], using multivariate orthogonal polynomials [21, 22] introduced in Ch. 2, to compute the correlation functions of characteristic polynomials for $\beta = 2$ ensembles.

Diaconis and Shashahani [72] used group-theoretic arguments and symmetric functions to calculate the joint moments of traces of matrices for classical compact groups. Here, using multivariate orthogonal polynomials, we develop a similar approach to calculate joint moments of traces for Hermitian ensembles, leading to closed form expressions using combinatorial and symmetric-function-theoretic methods.

Moments of Hermitian ensembles and their correlators have recently received considerable attention as discussed in Sec. 1.8 of Ch. 1. Cunden *et al.* [58] showed that as a function of their order, the moments are hypergeometric orthogonal polynomials. Cunden, Dahlqvist and O'Connell [57] showed that the cumulants of the Laguerre ensemble admit an asymptotic expansion in inverse powers of N of whose coefficients are the Hurwitz numbers. Dubrovin and Yang [75] computed the cumulant generating function for the GUE, while Gisonni, Grava and Ruzza calculated the generating function of the cumulants for the LUE in [127] and the JUE in [126].

This chapter is structured as follows. In Sec. 3.2, we state the results for Hermitian ensembles. Since the inspiration behind this work are the results from the unitary group, we discuss them in Sec. 3.3. After introducing the relevant symmetric polynomials in Sec. 3.4, we discuss the change of basis among different symmetric functions in Sec. 3.5. Finally, we calculate the correlations of characteristic polynomials in Sec. 3.6, and the joint moments of traces in Sec. 3.7.

3.2 Statements and results for Hermitian matrices

For a partition μ and a weight function $w(x)$, the multivariate symmetric polynomials defined in (2.1.143) satisfy the orthogonality relation

$$\int \Phi_\mu(x_1,\dots,x_N)\Phi_\nu(x_1,\dots,x_N) \prod_{1\le i<j\le N}(x_i-x_j)^2 \prod_{j=1}^N w(x_j)\,dx_j = \delta_{\mu\nu}C_\mu. \tag{3.2.1}$$

Here the lengths of the partitions μ and ν are less than or equal to the number of variables N, and C_μ is a constant which depends on N. These Φ_μ satisfy Prop. 2.1.26 which we recall below.

Lemma 3.2.1. *Let Φ_μ be multivariate polynomials given in (2.1.143) with leading coefficient equal to 1. For $\lambda \subseteq (q^p)$, let $\tilde{\lambda} = (p-\lambda'_q,\dots,p-\lambda'_1)$. Then, for $p,q\in\mathbb{N}$,*

$$\prod_{i=1}^p\prod_{j=1}^q (t_i-x_j) = \sum_{\lambda\subseteq(q^p)} (-1)^{|\tilde{\lambda}|}\Phi_\lambda(t_1,\dots,t_p)\Phi_{\tilde{\lambda}}(x_1,\dots,x_q). \tag{3.2.2}$$

This lemma appears in [98, p.625] for the Jacobi multivariate polynomials for arbitrary β. In Prop. 2.1.26, we present a different proof for $\beta=2$, which holds for the Hermite and Laguerre polynomials, too. A key difference in our approach is that we have closed-form expressions for multivariate polynomials as determinants of univariate classical orthogonal polynomials, while in the previous literature their construction was based on recurrence relations. This means that in this thesis formula (2.1.164) becomes a powerful tool and plays a role analogous to that of the classical dual Cauchy identity for $U(N)$.

We focus in particular on when $w(x)$ in (3.2.1) is a Gaussian, Laguerre and Jacobi weight:

$$w(x) = \begin{cases} e^{-\frac{x^2}{2}}, & x\in\mathbb{R}, & & \text{Gaussian},\\ x^\gamma e^{-x}, & x\in\mathbb{R}_+, & \gamma>-1, & \text{Laguerre},\\ x^{\gamma_1}(1-x)^{\gamma_2}, & x\in[0,1], & \gamma_1,\gamma_2>-1, & \text{Jacobi}. \end{cases} \tag{3.2.3}$$

The classical polynomials orthogonal with respect to these weights satisfy

$$\int_{\mathbb{R}} H_m(x)H_n(x)e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}m!\delta_{mn}, \tag{3.2.4a}$$

$$\int_{\mathbb{R}_+} L_m^{(\gamma)}L_n^{(\gamma)}x^\gamma e^{-x}\,dx = \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)}\delta_{nm}, \tag{3.2.4b}$$

$$\int_0^1 J_n^{(\gamma_1,\gamma_2)}(x)J_m^{(\gamma_1,\gamma_2)}(x)x^{\gamma_1}(1-x)^{\gamma_2}\,dx$$
$$= \frac{1}{(2n+\gamma_1+\gamma_2+1)}\frac{\Gamma(n+\gamma_1+1)\Gamma(n+\gamma_2+1)}{n!\Gamma(n+\gamma_1+\gamma_2+1)}\delta_{mn}. \tag{3.2.4c}$$

The identity in (2.1.164) gives a compact way to calculate the correlation functions and moments of characteristic polynomials of unitary ensembles using symmetric functions. The results are as follows.

Theorem 3.2.2. *Let M be an $N \times N$ GUE, LUE or JUE matrix and $t_1, \dots, t_p \in \mathbb{C}$. Then,*

$$
\begin{aligned}
&(a) \quad \mathbb{E}_N^{(H)}\big[\prod_{j=1}^{p} \det(t_j - M)\big] = \mathcal{H}_{(N^p)}(t_1, \dots, t_p) \\
&(b) \quad \mathbb{E}_N^{(L)}\big[\prod_{j=1}^{p} \det(t_j - M)\big] = \left(\prod_{j=N}^{p+N-1} (-1)^j j!\right) \mathcal{L}_{(N^p)}^{(\gamma)}(t_1, \dots, t_p) \\
&(c) \quad \mathbb{E}_N^{(J)}\big[\prod_{j=1}^{p} \det(t_j - M)\big] = \left(\prod_{j=N}^{p+N-1} (-1)^j j! \frac{\Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+1)}\right) \mathcal{J}_{(N^p)}^{(\gamma_1,\gamma_2)}(t_1, \dots, t_p)
\end{aligned}
\tag{3.2.5}
$$

Here $\mathcal{H}_\lambda$, $\mathcal{L}_\lambda^\gamma$, $\mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}$ are multivariate Hermite, Laguerre and Jacobi polynomials orthogonal with respect to the generalised weights in (3.2.1).

Similar to the case of the classical compact groups, correlations of traces of Hermitian ensembles can be calculated using the theory of symmetric functions. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$, $\sum_j \lambda_j \le N$, define

$$
\begin{aligned}
C_\lambda(N) &:= \prod_{j=1}^{N} \frac{(\lambda_j + N - j)!}{(N-j)!}, \\
G_\lambda(N, \gamma) &:= \prod_{j=1}^{N} \Gamma(\lambda_j + N - j + \gamma + 1).
\end{aligned}
\tag{3.2.6}
$$

The constants $C_\lambda(N)$ and $G_\lambda(N, \gamma)$ have several interesting combinatorial interpretations which are discussed in Sec. 3.5.

Theorem 3.2.3. *Let M be an $N \times N$ GUE, LUE or JUE matrix and let $\mu = (\mu_1, \dots, \mu_l)$ be a partition such that $|\mu| = \sum_{j=1}^{l} \mu_l \le N$. Then,*

(a)

$$
\mathbb{E}_N^{(H)}\big[\prod_{j=1}^{l} \operatorname{Tr} M^{\mu_j}\big] = \begin{cases} \frac{1}{2^{\frac{|\mu|}{2}} \frac{|\mu|}{2}!} \sum_{\lambda \vdash |\mu|} \chi^\lambda_{(2^{|\lambda|/2})} \chi^\lambda_\mu C_\lambda(N), & |\mu| \text{ is even}, \\ 0, & \text{otherwise}, \end{cases}
\tag{3.2.7}
$$

which is a polynomial in N.

(b)

$$
\mathbb{E}_N^{(L)}\big[\prod_{j=1}^{l} \operatorname{Tr} M^{\mu_j}\big] = \frac{1}{|\mu|!} \sum_{\lambda \vdash |\mu|} \frac{G_\lambda(N, \gamma)}{G_0(N, \gamma)} C_\lambda(N) \chi^\lambda_{(1^{|\lambda|})} \chi^\lambda_\mu,
\tag{3.2.8}
$$

which is a polynomial in N.

(c)

$$
\mathbb{E}_N^{(J)}\big[\prod_{j=1}^{l} \operatorname{Tr} M^{\mu_j}\big] = \sum_{\lambda \vdash |\mu|} \frac{G_\lambda(N, \gamma_1)}{G_0(N, \gamma_1)} C_\lambda(N) \chi^\lambda_\mu D^{(J)}_{\lambda 0},
\tag{3.2.9}
$$

where

$$D_{\lambda 0}^{(J)} = \det\left[\mathbb{1}_{\lambda_i - i + j \geq 0} \frac{1}{(\lambda_i - i + j)!} \frac{\Gamma(2N - 2j + \gamma_1 + \gamma_2 + 2)}{\Gamma(2N + \lambda_i - i - j + \gamma_1 + \gamma_2 + 2)}\right]_{1 \leq i,j \leq N}. \quad (3.2.10)$$

In the above equations, χ_μ^λ are the characters of the symmetric group $\mathcal{S}_m$, $m = |\lambda| = |\mu|$, associated to the λth irreducible representation on the μth conjugacy class.

3.3 Results for the Unitary group

The aim of this section is two-fold. Firstly, we highlight the role of symmetric functions to study correlations and moments of random matrices. Secondly, to understand how the theory can be generalised to Hermitian ensembles.

Here we review the results of the unitary group by Diaconis and Shashahani; Bump and Gamburd. They take a representation theoretic approach to calculate the correlations of traces and characteristic polynomials, and provide a combinatorial interpretation of these results. All the tools required to prove Thm. 3.1.1 and Thm. 3.1.2 are already introduced in Ch. 2. Any additional results required are stated within this section.

Both Thm. 3.1.1 and Thm. 3.1.2 can be proved elegantly using the properties of Schur polynomials. Let $e^{i\theta_1}, \ldots, e^{i\theta_N}$ be the eigenvalues of $M \in U(N)$. Recall the definition of Schur polynomials indexed by a partition λ,

$$S_\lambda(e^{i\theta_1}, \ldots, e^{i\theta_N}) = \frac{\det[e^{i(\lambda_k + N - k)\theta_j}]_{1 \leq j,k \leq N}}{\det[e^{i(N-k)\theta_j}]_{1 \leq j,k \leq N}}, \quad (3.3.1)$$

for $l(\lambda) \leq N$. If $l(\lambda) < N$, we append a sequence of zeros to the tail of λ so that

$$\lambda = (\lambda_1, \ldots, \lambda_l, \underbrace{0, \ldots, 0}_{N-l}). \quad (3.3.2)$$

The crucial property of the Schur functions is that they are the characters of the unitary group. We have the following proposition as a particular case of the Weyl character formula [246].

Proposition 3.3.1. *Consider $N \in \mathbb{N}$ and a partition λ such that $l(\lambda) \leq N$. For $A \in GL(N, \mathbb{C})$ with eigenvalues $x_1, \ldots, x_N$, define $\chi_\lambda(A) = S_\lambda(x_1, \ldots, x_N)$. Then, the function χ_λ is the character of the irreducible analytic representation of $GL(N, \mathbb{C})$, and its restriction to $U(N)$ is also irreducible.*

The $S_\lambda(x_1, \ldots, x_N)$ is equal to 0 whenever $l(\lambda) > N$. Therefore, as long as λ runs over partitions with N or fewer parts, we recover all the characters of $U(N)$. Using the Weyl

integration formula [241,242], the orthogonality relation for the S_λ is

$$\begin{aligned}
&\mathbb{E}_{U(N)}\left[S_\lambda(e^{i\theta_1},\dots,e^{i\theta_N})S_\mu(e^{-i\theta_1},\dots,e^{-i\theta_N})\right]\\
=&\frac{1}{(2\pi)^N N!}\int_{[0,2\pi]^N} S_\lambda(e^{i\theta_1},\dots,e^{i\theta_N})S_\mu(e^{-i\theta_1},\dots,e^{-i\theta_N})\prod_{1\le j<k\le N}|e^{i\theta_j}-e^{i\theta_k}|^2\,d\theta_1\dots d\theta_N\\
=&\delta_{\lambda\mu}.
\end{aligned} \tag{3.3.3}$$

Next, we comment on two different ways of proving Thm. 3.1.1 to illustrate the versatility of symmetric functions.

Proof of Thm. 3.1.1 (Method 1.) : Recall that the power sum polynomial in the eigenvalues is

$$p_j(M)=p_j(e^{i\theta_1},\dots,e^{i\theta_N})=\sum_{k=1}^N e^{ij\theta_k}=\operatorname{Tr}M^j. \tag{3.3.4}$$

Thus, one has

$$\prod_{j=1}^k(\operatorname{Tr}M^j)^{a_j}\overline{(\operatorname{Tr}M^j)}^{b_j}=P_\mu(M)\overline{P_\nu(M)}, \tag{3.3.5}$$

for partitions $\mu=(1^{a_1}\dots k^{a_k})$ and $\nu=(1^{b_1}\dots k^{b_k})$. Recall that the P_μ can be expanded in the Schur basis as

$$P_\mu=\sum_\lambda \chi_\mu^\lambda S_\lambda, \tag{3.3.6}$$

where χ_μ^λ are the characters of the symmetric group that satisfy

$$\begin{aligned}
\sum_\lambda \chi_\mu^\lambda\chi_\nu^\lambda&=z_\mu\delta_{\mu\nu},\\
\sum_\mu \frac{1}{z_\mu}\chi_\mu^\lambda\chi_\mu^\nu&=\delta_{\lambda\nu},
\end{aligned} \tag{3.3.7}$$

with

$$z_\mu=\prod_j j^{a_j}a_j!. \tag{3.3.8}$$

Expanding P_μ and P_ν in the Schur basis and using the orthogonality relation (3.3.3) gives zero unless $\mu=\nu$. This is because

$$\begin{aligned}
\mathbb{E}_{U(N)}\left[\prod_{j=1}^k(\operatorname{Tr}M^j)^{a_j}\overline{(\operatorname{Tr}M^j)}^{b_j}\right]&=\mathbb{E}_{U(N)}\left[P_\mu(M)\overline{P_\nu(M)}\right]\\
&=\sum_\alpha\sum_\beta \chi_\mu^\alpha\chi_\nu^\beta\mathbb{E}_{U(N)}\left[S_\alpha(M)\overline{S_\beta(M)}\right]\\
&=\sum_\alpha \chi_\mu^\alpha\chi_\nu^\alpha=\prod_{j=1}^k j^{a_j}a_j!.
\end{aligned} \tag{3.3.9}$$

■

In the next method, we re-derive (3.1.1) from the identities for Toeplitz determinants. A Toeplitz determinant of size N with symbol f is defined by

$$D_N(f) := \det[\hat{f}_{j-k}]_{1\le j,k\le N}, \tag{3.3.10}$$

where f is an integrable function on the unit circle with Fourier coefficients

$$\hat{f}_j := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ij\theta}\,d\theta, \quad j = 0, \pm 1, \pm 2, \dots. \tag{3.3.11}$$

Toeplitz determinants are intimately connected to random matrices, and are closely related to polynomials orthogonal with respect to the weight f on the unit circle [227]. This connection is best described by the following identity, sometimes called the Heine's identity [186],

$$D_N(f) = \mathbb{E}_{U(N)}\left[\prod_{j=1}^N f(e^{i\theta_j})\right] = \frac{1}{(2\pi)^N N!}\int_{[0,2\pi]^N}\prod_{1\le j<k\le N}|e^{i\theta_j}-e^{i\theta_k}|^2\prod_{j=1}^N f(e^{i\theta_j})\,d\theta_j. \tag{3.3.12}$$

Proposition 3.3.2. *Let X_l and Y_m be two complex polynomials,*

$$\begin{aligned} X_l(z) &= \prod_{i=1}^l (1-a_i z), \\ Y_m(z) &= \prod_{j=1}^m (1-b_j z), \end{aligned} \tag{3.3.13}$$

with $|a_i| < 1$ and $|b_j| < 1$ for every i, j. If $l \le N$ or $m \le N$, then [26]

$$D_N\left(\frac{1}{X_l(e^{i\theta})Y_m(e^{-i\theta})}\right) = \prod_{i=1}^l\prod_{j=1}^m \frac{1}{1-a_i b_j}. \tag{3.3.14}$$

Proof. The Toeplitz structure of the determinant in (3.3.14) can be manipulated to prove the proposition. Alternatively, it can be proved using symmetric functions and the representation theory of $U(N)$. Recall (2.1.107) from Prop. 2.1.20 in Ch. 2,

$$\prod_{j,k}\frac{1}{1-x_j y_k} = \sum_\lambda S_\lambda(\mathbf{x})S_\lambda(\mathbf{y}). \tag{3.3.15}$$

Therefore, we see that

$$
\begin{aligned}
D_N\left(\frac{1}{X_l(e^{i\theta})Y_m(e^{-i\theta})}\right) &= \mathbb{E}_{U(N)}\left[\prod_{p=1}^{l}\prod_{q=1}^{m}\prod_{r=1}^{N}\frac{1}{(1-a_pe^{i\theta_r})(1-b_qe^{-i\theta_r})}\right] \\
&= \sum_{\mu,\lambda} S_\lambda(\mathbf{a})S_\mu(\mathbf{b})\mathbb{E}_{U(N)}[S_\lambda(M)\overline{S_\mu(M)}] \\
&= \sum_{\mu,\lambda} S_\lambda(\mathbf{a})S_\mu(\mathbf{b})\delta_{\lambda\mu} \\
&= \prod_{i=1}^{l}\prod_{j=1}^{m}\frac{1}{1-a_ib_j}.
\end{aligned}
\tag{3.3.16}
$$

■

Proof of Thm. 3.1.1 (Method 2.) : From (2.1.105) in Prop. 2.1.20, one has

$$
\prod_j \frac{1}{1-x_jt} = \sum_n t^n \sum_{\lambda\vdash n}\frac{1}{z_\lambda}P_\lambda(\mathbf{x}). \tag{3.3.17}
$$

Choose any two sets of variables $\mathbf{x} = (x_1,\ldots,x_N)$ and $\mathbf{y} = (y_1,\ldots,y_N)$. If $\mathbf{xy}$ denotes the sequence of N^2 variables x_py_q, $1 \le p,q \le N$, then

$$
p_j(\mathbf{xy}) = \sum_{p,q}(x_py_q)^j = p_j(\mathbf{x})p_j(\mathbf{y}). \tag{3.3.18}
$$

Consider $t,s \in \mathbb{C}$ such that $|t| < 1$ and $|s| < 1$. Therefore,

$$
\prod_{j,k=1}^{N}\frac{1}{1-x_jy_kt} = \sum_n t^n \sum_{\lambda\vdash n}\frac{1}{z_\lambda}P_\lambda(\mathbf{x})P_\lambda(\mathbf{y}). \tag{3.3.19}
$$

If we let y_k to be the eigenvalues of M, $y_k = e^{i\theta_k}$, and x_j to be the variables a_j or b_j, then

$$
\prod_{j,k=1}^{N}\frac{1}{1-a_je^{i\theta_k}t} = \sum_n t^n \sum_{\lambda\vdash n}\frac{1}{z_\lambda}P_\lambda(\mathbf{a})P_\lambda(M), \tag{3.3.20}
$$

$$
\prod_{j,k=1}^{N}\frac{1}{1-b_je^{-i\theta_k}s} = \sum_n s^n \sum_{\lambda\vdash n}\frac{1}{z_\lambda}P_\lambda(\mathbf{b})\overline{P_\lambda(M)}. \tag{3.3.21}
$$

According to Prop. 3.3.2, for the Toeplitz symbol

$$
f(e^{i\theta}) = \prod_{j=1}^{N}\frac{1}{(1-a_je^{i\theta}t)(1-b_je^{-i\theta}s)}, \tag{3.3.22}
$$

the Toeplitz determinant is

$$
D_N(f) = \prod_{j,k=1}^{N}\frac{1}{1-a_jb_kts} = \sum_n (ts)^n \sum_{\lambda\vdash n}\frac{1}{z_\lambda}P_\lambda(\mathbf{a})P_\lambda(\mathbf{b}). \tag{3.3.23}
$$

After (i) multiplying the identities in (3.3.20) and (3.3.21), and integrating over the Haar measure of the unitary group, and (ii) subtracting the result from (3.3.23) gives

$$\sum_{n,m\geq 0} t^n s^m \sum_{\lambda\vdash n}\sum_{\mu\vdash m}\left(z_\lambda^{-1}z_\mu^{-1}\mathbb{E}_{U(N)}\left[P_\lambda(M)\overline{P_\mu(M)}\right]-\delta_{\lambda\mu}z_\lambda^{-1}\right)P_\lambda(\mathbf{a})P_\mu(\mathbf{b})=0. \tag{3.3.24}$$

The above relation is valid as long as the P_λ forms a basis for symmetric polynomials of degree $|\lambda|$ in N variables. Therefore, we end up with

$$\mathbb{E}_{U(N)}\left[P_\lambda(M)\overline{P_\mu(M)}\right]=\delta_{\lambda\mu}z_\lambda \tag{3.3.25}$$

whenever $|\lambda|\leq N$ and $|\mu|\leq N$. ■

In the above proof, Method 1 involves only the representation theory of $U(N)$. On the other hand, Method 2 connects the algebraic properties (Toeplitz determinants) and representation-theoretic-properties (symmetric functions) of the unitary group. Next, we prove the results for characteristic polynomials.

Proof of Thm. 3.1.2. We begin the proof by rewriting the *L.H.S.* of (3.1.2) as

$$\begin{aligned}&\mathbb{E}_{U(N)}\left[\prod_{j=1}^{L}\det(I+a_j^{-1}M^\dagger)\prod_{l=1}^{K}\det(I+a_{L+l}M)\right]\\&=\left(\prod_{j=1}^{L}a_j^{-N}\right)\mathbb{E}_{U(N)}\left[\overline{\det(M)^L}\prod_{k=1}^{L+K}\det(I+a_kM)\right].\end{aligned} \tag{3.3.26}$$

Recall the dual Cauchy identity from Prop. 2.1.21 in Ch. 2,

$$\prod_{j,k}(1+x_jy_k)=\sum_\lambda S_\lambda(\mathbf{x})S_{\lambda'}(\mathbf{y}). \tag{3.3.27}$$

If x_j are the eigenvalues of M and y_k are the variables a_k,

$$\det(I+a_kM)=\sum_\lambda S_\lambda(a_1,\ldots,a_{L+K})S_{\lambda'}(M), \tag{3.3.28}$$

where λ runs through all partitions such that $l(\lambda)\leq L+K$ and $l(\lambda')\leq N$. According to the Jacobi-Trudi identity (2.1.77),

$$\det M^L=S_\mu(M), \tag{3.3.29}$$

where $\mu=(L^N)$ is a rectangular partition with L rows and N columns. An example of a rectangular partition is shown in Fig. 3.1. Inserting (3.3.28) and (3.3.29) in (3.3.26), and using

(3.3.3) results in

$$\begin{aligned}\mathbb{E}_{U(N)}\left[\prod_{j=1}^{L}\det(I+a_j^{-1}M^\dagger)\prod_{l=1}^{K}\det(I+a_{L+l}M)\right] &= \left(\prod_{j=1}^{L}a_j^{-N}\right)\mathbb{E}_{U(N)}\left[\overline{\det M^L}\prod_{k=1}^{L+K}\det(I+a_kM)\right]\\ &= \left(\prod_{j=1}^{L}a_j^{-N}\right)\sum_{\lambda}S_\lambda(\mathbf{a})\mathbb{E}_{U(N)}[S_{\lambda'}(M)S_{(L^N)}(M)]\\ &= \left(\prod_{j=1}^{L}a_j^{-N}\right)S_{(N^L)}(\mathbf{a}).\end{aligned} \tag{3.3.30}$$

In the last line, we used the fact that $\lambda=(N^L)$ if $\lambda'=(L^N)$. ■

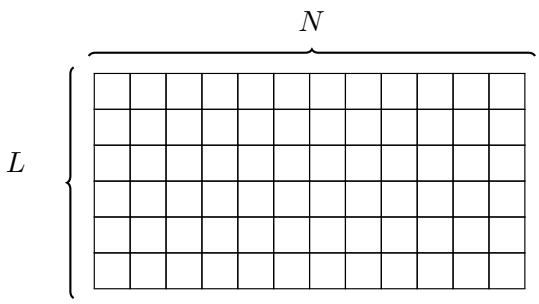


Figure 3.1: A rectangular partition of shape $\lambda=(N^L)$ with L rows and N columns.

Corollary 3.3.3. *When $L=K=p$ and $a_j=1$, $j=1,\dots,2p$, in Thm. 3.1.2, we recover the moments of the characteristic polynomial:*

$$\mathbb{E}_{U(N)}\left[|\det(I-M)|^{2p}\right]=\prod_{j=0}^{N-1}\frac{j!(j+2p)!}{(j+p)!^2}. \tag{3.3.31}$$

Proof. From Thm. 3.1.2, one sees that

$$\mathbb{E}_{U(N)}\left[|\det(I-M)|^{2p}\right]=S_{(N^{2p})}(1,\dots,1). \tag{3.3.32}$$

Recall (2.1.84),

$$S_\lambda(\underbrace{1,\dots,1}_{2p})=\prod_{1\le j<k\le 2p}\frac{\lambda_j-\lambda_k-j+k}{k-j}. \tag{3.3.33}$$

The result in (3.3.31) can be immediately recovered by computing $S_\lambda(1,\dots,1)$ for $\lambda=(N^{2p})$. ■

In addition to the positive correlations, negative correlations

$$\mathbb{E}_{U(N)}\left[\frac{1}{\prod_{n=1}^{r}\det(I-b_nM)\prod_{m=1}^{s}\det(I-b_mM^\dagger)}\right], \tag{3.3.34}$$

or more generally, mixed correlations

$$\mathbb{E}_{U(N)}\left[\frac{\prod_{j=1}^{p}\det(I+a_j^{-1}M^\dagger)\prod_{k=1}^{q}\det(I+a_{p+k}M)}{\prod_{n=1}^{r}\det(I-b_nM)\prod_{m=1}^{s}\det(I-b_mM^\dagger)}\right], \tag{3.3.35}$$

are very useful. For example, we have already encountered an application of (3.3.34) in calculating the mixed moments of traces in Thm. 3.1.1. The Toeplitz determinant in Prop. 3.3.2 is precisely the expected value of the inverse of characteristic polynomials. Bump and Gamburd [46] calculated these mixed correlations of characteristic polynomials in terms of the Littlewood-Schur symmetric functions [182], also called the hook Schur functions [29].

Similar results hold for other compact groups. For the orthogonal and symplectic groups, suitable symmetric functions replace the Schur functions. For more details, the reader is encouraged to refer to [46, 72].

For the classical compact groups, Schur polynomials and their generalisations are the characters of $U(N)$, $O(N)$ and $Sp(2N)$. As discussed, they have been used extensively to calculate correlation functions of characteristic polynomials and joint moments of the traces. Although group theoretic tools are not available for the set of Hermitian matrices, multivariate orthogonal polynomials play the role of Schur functions for the GUE, LUE and JUE. In the following sections, we will present the results for correlations of traces and characteristic polynomials for the Hermitian ensembles.

3.4 Multivariate Hermite, Laguerre, and Jacobi polynomials

Multivariate orthogonal polynomials Φ_μ are defined by the determinantal formula in (2.1.143). One can check by straightforward substitution that up to a constant, the Φ_μ coincide with those in (3.2.1). When φ_j in (2.1.143) are the Hermite, Laguerre and Jacobi polynomials we have the multivariate generalizations $\mathcal{H}_\mu$, $\mathcal{L}_\mu^{(\gamma)}$ and $\mathcal{J}_\mu^{(\gamma_1,\gamma_2)}$. These polynomials can be expressed as a linear combination of Schur polynomials,

$$\Phi_\mu(\mathbf{x}) = \sum_{\nu\subseteq\mu}\kappa_{\mu\nu}S_\nu(\mathbf{x}). \tag{3.4.1}$$

For the Hermite, Laguerre and Jacobi multivariate polynomials we set the leading coefficient $\kappa_{\mu\mu}$ in consistency with the definitions (3.2.4) and (2.1.143),

$$\begin{aligned}
&\kappa_{\mu\mu}^{(H)} = 1, \quad \kappa_{\mu\mu}^{(L)} = \frac{(-1)^{|\lambda|+\frac{1}{2}N(N-1)}}{G_\lambda(N,0)},\\
&\kappa_{\mu\mu}^{(J)} = \frac{(-1)^{|\lambda|+\frac{1}{2}N(N-1)}}{G_\lambda(N,\gamma_1+\gamma_2)G_\lambda(N,0)}\prod_{j=1}^{N}\Gamma(2N+2\lambda_j-2j+\gamma_1+\gamma_2+1).
\end{aligned} \tag{3.4.2}$$

The Hermite polynomials in (3.2.4) are monic. This fact is reflected in the multivariate Hermite polynomial as $\kappa_{\mu\mu}^{(H)} = 1$. On the other hand, we chose non-monic Laguerre and Jacobi polynomials in (3.2.4) to be consistent with the literature. As a result, the coefficients $\kappa_{\mu\mu}^{(L)}$ and $\kappa_{\mu\mu}^{(J)}$ are different from 1.

The analogy between multivariate orthogonal polynomials and Schur functions becomes

apparent by comparing definitions (2.1.69) with (2.1.143). The classical Hermite, Laguerre and Jacobi polynomials satisfy second order Sturm Liouville problems. Similarly, their multivariate generalizations are eigenfunctions of second-order partial differential operators, known as Calogero–Sutherland Hamiltonians,

$$
\begin{aligned}
H^{(H)} &= \sum_{j=1}^{N}\left(\frac{\partial^2}{\partial x_j^2} - x_j\frac{\partial}{\partial x_j}\right) + 2\sum_{\substack{j,k=1\\k\neq j}}^{N}\frac{1}{x_j-x_k}\frac{\partial}{\partial x_j},\\
H^{(L)} &= \sum_{j=1}^{N}\left(x_j\frac{\partial^2}{\partial x_j^2} + (\gamma - x_j + 1)\frac{\partial}{\partial x_j}\right) + 2\sum_{\substack{j,k=1\\k\neq j}}^{N}\frac{x_j}{x_j-x_k}\frac{\partial}{\partial x_j},\\
H^{(J)} &= \sum_{j=1}^{N}\left(x_j(1-x_j)\frac{\partial^2}{\partial x_j^2} + (\gamma_1 + 1 - x_j(\gamma_1+\gamma_2+2))\frac{\partial}{\partial x_j}\right) + 2\sum_{\substack{j,k=1\\k\neq j}}^{N}\frac{x_j(1-x_j)}{x_j-x_k}\frac{\partial}{\partial x_j}.
\end{aligned}
\tag{3.4.3}
$$

These multivariate polynomials are orthogonal with respect to the measures

$$d\mu^{(H)}(\mathbf{x}) = \prod_{j=1}^{N} e^{-\frac{x_j^2}{2}} \prod_{1\le j<k\le N}|x_j-x_k|^2 \tag{3.4.4}$$

$$d\mu^{(L)}(\mathbf{x}) = \prod_{j=1}^{N} x_j^{\gamma} e^{-x_j} \prod_{1\le j<k\le N}|x_j-x_k|^2 \tag{3.4.5}$$

$$d\mu^{(J)}(\mathbf{x}) = \prod_{j=1}^{N} x_j^{\gamma_1}(1-x_j)^{\gamma_2} \prod_{1\le j<k\le N}|x_j-x_k|^2 \tag{3.4.6}$$

These generalised orthogonal polynomials obey similar properties to their univariate counterparts [21]. The differential equations in (3.4.3) are also related to the Dyson Brownian motion.

3.5 Change of basis between symmetric functions

Before proceeding to proving the main results stated in Sec. 3.2, it is important to understand how the multivariate polynomials can be expressed in the basis of other symmetric polynomials. In this section, we give expressions for the change of basis between multivariate polynomials and Schur polynomials. The identities introduced in Ch. 2 can be used to further express multivariate polynomials in terms of other symmetric polynomials.

We begin with the following proposition, which is a very useful tool involving polynomials as matrix entries in the determinant.

Proposition 3.5.1. *If $\phi_j(x)$, $0 \le j \le N-1$, is a sequence of monic polynomials of degree j, then*

$$\det[\phi_{N-j}(x_k)]_{1\le j,k\le N} = \det[x_k^{N-j}]_{1\le j,k\le N} = \prod_{1\le j<k\le N}(x_j - x_k). \tag{3.5.1}$$

Proof. Let

$$\phi_j(x) = x^j + a_{j-1}x^{j-1} + \cdots + a_0 \tag{3.5.2}$$

for some sequence a_i. The coefficients a_i can be different for each polynomial $\phi_j(x)$. Since the determinant is unchanged by performing row or column operations, we do the following trick. We start with the Vandermonde determinant

$$\det[x_k^{N-j}]_{1\le j,k\le N} = \begin{vmatrix} x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \dots & x_N \\ 1 & 1 & \dots & 1 \end{vmatrix}. \tag{3.5.3}$$

If $\phi_{N-1}(x) = x^{N-1} + \sum_{j=0}^{N-2} a_j x^j$, multiply the last row by a_0, the last but one row by a_1 and so on. Now add them to the first row so that the first row becomes

$$(\phi_{N-1}(x_1), \dots, \phi_{N-1}(x_N)). \tag{3.5.4}$$

Repeat this process for other rows going from top to bottom. Since ϕ_j are monic, $\phi_0 = 1$. Therefore, no operations are required for the last row. ■

Proposition 3.5.2. *If $\phi_j(x)$, $0 \le j \le N-1$ is a sequence of polynomials with leading coefficient A_j, then*

$$\det[\phi_{N-j}(x_k)]_{1\le j,k\le N} = \left(\prod_{j=0}^{N-1} A_j\right) \det[x_k^{N-j}]_{1\le j,k\le N}. \tag{3.5.5}$$

In the rest of the section, we mainly focus on the GUE but the same approach can be used for the LUE and the JUE.

Gaussian Ensemble. Let M be an $N \times N$ GUE matrix. The j.p.d.f. of the eigenvalues is

$$\begin{aligned} \rho^{(H)}(x_1, \dots, x_N) &= \frac{1}{Z_N^{(H)}} \Delta^2(\mathbf{x}) \prod_{i=1}^{N} e^{-\frac{x_i^2}{2}}, \\ Z_N^{(H)} &= (2\pi)^{\frac{N}{2}} \prod_{j=1}^{N} j!. \end{aligned} \tag{3.5.6}$$

Denote by $H_n(x)$ the Hermite polynomials normalised according to (3.2.4a). Given a partition λ with $l(\lambda) \le N$, the multivariate Hermite polynomials are given by

$$\mathcal{H}_\lambda(\mathbf{x}) = \frac{1}{\Delta(\mathbf{x})} \begin{vmatrix} H_{\lambda_1+N-1}(x_1) & H_{\lambda_1+N-1}(x_2) & \dots & H_{\lambda_1+N-1}(x_N) \\ H_{\lambda_2+N-2}(x_1) & H_{\lambda_2+N-2}(x_2) & \dots & H_{\lambda_2+N-2}(x_N) \\ \vdots & \vdots & & \vdots \\ H_{\lambda_N}(x_1) & H_{\lambda_N}(x_2) & \dots & H_{\lambda_N}(x_N) \end{vmatrix}, \tag{3.5.7}$$

and satisfy the orthogonality relation

$$\begin{aligned}\langle \mathcal{H}_\lambda, \mathcal{H}_\mu \rangle :=& \frac{1}{Z_N^{(H)}} \int_{\mathbb{R}^N} \mathcal{H}_\lambda(\mathbf{x}) \mathcal{H}_\mu(\mathbf{x}) \Delta^2(\mathbf{x}) \prod_i e^{-\frac{x_i^2}{2}}\, dx_i = C_\lambda(N) \delta_{\mu\lambda}, \\ C_\lambda(N) =& \prod_{i=1}^N \frac{(\lambda_i + N - i)!}{(N-i)!}.\end{aligned} \tag{3.5.8}$$

Since $\lambda_i \geq 0$, the constant $C_\lambda(N)$ is a polynomial in N of degree $|\lambda|$. It turns out that it has a nice interpretation in terms of the characters of the symmetric group. Let $(i,j) \in \lambda$, $1 \leq j \leq \lambda_i$, denote a node in the Young diagram of λ. The roots of $C_\lambda(N)$ are $i-j$, where i runs across the rows from top to bottom and j across the columns from left to right of the Young diagram. For example, if $\lambda = (4,3,3,1)$, the roots of $C_\lambda(N)$ are

0	−1	−2	−3
1	0	−1	
2	1	0	
3			

(3.5.9)

Proposition 3.5.3. *We have [170]*

$$\begin{aligned}C_\lambda(N) &= \prod_{j=1}^{l(\lambda)} \frac{(\lambda_j + N - j)!}{(N-j)!} = \prod_{(i,j)\in\lambda} (N - i + j) \\ &= \frac{|\lambda|!}{\dim V_\lambda} \sum_{\mu \vdash |\lambda|} \frac{\chi_\mu^\lambda}{z_\mu} N^{l(\mu)} = |\lambda|! \frac{S_\lambda(1^N)}{\dim V_\lambda}.\end{aligned} \tag{3.5.10}$$

The constant z_λ is given in (2.1.82) and $\dim V_\lambda$ is the dimension of the irreducible representation, labelled by λ, of the symmetric group $\mathcal{S}_{|\lambda|}$,

$$\dim V_\lambda = |\lambda|! \frac{\prod_{1\leq j<k\leq l(\lambda)} (\lambda_j - \lambda_k - j + k)}{\prod_{j=1}^{l(\lambda)} (\lambda_j + l(\lambda) - j)!}. \tag{3.5.11}$$

Proof. Recall

$$S_\lambda(\underbrace{1,\ldots,1}_{N}) = \prod_{1\leq j<k\leq N} \frac{\lambda_j - \lambda_k - j + k}{k-j}. \tag{3.5.12}$$

Combining the above result with (3.5.11) proves that

$$C_\lambda(N) = |\lambda|! \frac{S_\lambda(1^N)}{\dim\, V_\lambda}. \tag{3.5.13}$$

Since $P_\lambda(1^N) = N^{l(\lambda)}$, using (2.1.82) gives

$$\begin{aligned}C_\lambda(N) &= \frac{|\lambda|!}{\dim\, V_\lambda} \sum_\mu \frac{\chi_\mu^\lambda}{z_\mu} p_\mu(1^N) \\ &= \frac{|\lambda|!}{\dim\, V_\lambda} \sum_\mu \frac{\chi_\mu^\lambda}{z_\mu} N^{l(\lambda)}.\end{aligned} \tag{3.5.14}$$

■

Schur polynomials can be expressed in terms of multivariate Hermite polynomials,

$$S_\lambda = \sum_{\nu\subseteq\lambda} \psi^{(H)}_{\lambda\nu}\mathcal{H}_\nu = \sum_{j=0}^{\lfloor\frac{|\lambda|}{2}\rfloor} \sum_{\nu\vdash g(j)} \psi^{(H)}_{\lambda\nu}\mathcal{H}_\nu, \quad g(j) = \begin{cases} 2j, & |\lambda| \text{ is even,} \\ 2j+1, & |\lambda| \text{ is odd.} \end{cases} \tag{3.5.15}$$

The function $g(j)$ takes care of the fact that polynomials of odd and even degree do not mix similar to the one variable case. The first summation in (3.5.15) running over all lower order partitions takes care of the fact that $\mathcal{H}_\lambda$ are, unlike S_λ, not homogeneous polynomials. For example, when $|\lambda|$ is even, the only partitions that appear in (3.5.15) are those with weight $|\nu| = |\lambda| - 2k$, $k = 0, \dots, \frac{|\lambda|}{2}$, and $\nu \subseteq \lambda$. The following proposition gives an explicit expression for the coefficients $\psi^{(H)}_{\lambda\nu}$.

Proposition 3.5.4. *If λ is a partition of length L and ν is a sub-partition of λ such that $|\lambda| - |\nu| = 0$ mod 2 and $N \geq L$, then $\psi^{(H)}_{\lambda\nu}$ is a polynomial in N given by*

$$\psi^{(H)}_{\lambda\nu} = \frac{1}{2^{\frac{|\lambda|-|\nu|}{2}}} D^{(H)}_{\lambda\nu} \prod_{j=1}^{L} \frac{(\lambda_j + N - j)!}{(\nu_j + N - j)!}, \tag{3.5.16}$$

where

$$D^{(H)}_{\lambda\nu} = \det\left[\mathbb{1}_{\lambda_j - \nu_k - j + k = 0 \text{ mod } 2} \left(\left(\frac{\lambda_j - \nu_k - j + k}{2} \right)! \right)^{-1} \right]_{j,k=1,\dots,L}. \tag{3.5.17}$$

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_L, 0, \dots, 0)$ and $\nu = (\nu_1, \dots, \nu_l, 0, \dots, 0)$. Here l is the length of ν and $N - l$ is the length of the sequence of zeros added to ν. From (3.5.8) and the fact that $\nu \subseteq \lambda$, $l \leq L$, it follows that

$$\begin{aligned}
\psi^{(H)}_{\lambda\nu} =& \frac{\langle S_\lambda, \mathcal{H}_\nu\rangle}{\langle \mathcal{H}_\nu, \mathcal{H}_\nu\rangle} = \frac{1}{Z^{(H)}_N \langle \mathcal{H}_\nu, \mathcal{H}_\nu\rangle} \int_{\mathbb{R}^N} S_\lambda(\mathbf{x})\mathcal{H}_\nu(\mathbf{x})\Delta_N^2(\mathbf{x}) \prod_{i=1}^{N} e^{-\frac{x_i^2}{2}}\, dx_i \\
=& \frac{1}{Z^{(H)}_N \langle \mathcal{H}_\nu, \mathcal{H}_\nu\rangle} \int_{\mathbb{R}^N} \prod_{i=1}^{N} e^{-\frac{x_i^2}{2}}\, dx_i \\
&\times \begin{vmatrix} x_1^{\lambda_1+N-1} & \dots & x_1^{\lambda_L+N-L} & H_{N-L-1}(x_1) & \dots & 1 \\ x_2^{\lambda_1+N-1} & \dots & x_2^{\lambda_L+N-L} & H_{N-L-1}(x_2) & \dots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ x_N^{\lambda_1+N-1} & \dots & x_N^{\lambda_L+N-L} & H_{N-L-1}(x_N) & \dots & 1 \end{vmatrix} \\
&\times \begin{vmatrix} H_{\nu_1+N-1}(x_1) & \dots & H_{\nu_l+N-l}(x_1) & H_{N-l-1}(x_1) & \dots & 1 \\ H_{\nu_1+N-1}(x_2) & \dots & H_{\nu_l+N-l}(x_2) & H_{N-l-1}(x_2) & \dots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ H_{\nu_1+N-1}(x_N) & \dots & H_{\nu_l+N-l}(x_N) & H_{N-l-1}(x_N) & \dots & 1 \end{vmatrix}.
\end{aligned} \tag{3.5.18}$$

The last $N - L$ and $N - l$ columns in S_λ and in $\mathcal{H}_\nu$, respectively, are written in terms of the Hermite polynomials using column operations, see Prop. 3.5.1. In addition, $\psi^{(H)}_{\lambda\nu}$ can be

expanded as a sum over the permutations of N:

$$\psi^{(H)}_{\lambda\nu} = \frac{1}{Z^{(H)}_N \langle \mathcal{H}_\nu, \mathcal{H}_\nu \rangle} \times \sum_{\sigma \in \mathcal{S}_N} \operatorname{sgn}(\sigma) \int_{\mathbb{R}^N} \prod_{i=1}^{N} e^{-\frac{x_i^2}{2}} dx_i \left(x_{\sigma(1)}^{\lambda_1+N-1} \dots x_{\sigma(L)}^{\lambda_L+N-L} H_{N-L-1}(x_{\sigma(N-L-1)}) \dots H_0(x_{\sigma(0)}) \right) \times \begin{vmatrix} H_{\nu_1+N-1}(x_1) & \dots & H_{\nu_l+N-l}(x_1) & H_{N-l-1}(x_1) & \dots & 1 \\ H_{\nu_1+N-1}(x_2) & \dots & H_{\nu_l+N-l}(x_2) & H_{N-l-1}(x_2) & \dots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ H_{\nu_1+N-1}(x_N) & \dots & H_{\nu_l+N-l}(x_N) & H_{N-l-1}(x_N) & \dots & 1 \end{vmatrix}. \tag{3.5.19}$$

Since the integrand is symmetric in x_i, every term in the above sum gives the same contribution. Therefore, it is sufficient to consider only the identity permutation. All the factors can be absorbed into the determinant by multiplying the jth row with $x_j^{\lambda_j+N-j}$ if $j \leq L$, and with $H_{N-j}(x_{N-j})$ if $N \geq j > L$. Then, using the orthogonality of Hermite polynomials (3.2.4a) for the last $N-L$ rows results in

$$\psi^{(H)}_{\lambda\nu} = \frac{N!}{Z^{(H)}_N \langle \mathcal{H}_\nu, \mathcal{H}_\nu \rangle} (2\pi)^{\frac{N-L}{2}} \prod_{i=L+1}^{N} (N-i)! \det \left[\int_{\mathbb{R}} x_j^{\lambda_j+N-j} H_{\nu_k+N-k}(x_j) e^{-\frac{x_j^2}{2}} dx_j \right]_{1 \leq j,k \leq L}. \tag{3.5.20}$$

Expanding monomials in terms of Hermite polynomials with the formula

$$x^n = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^m m!(n-2m)!} H_{n-2m}(x), \tag{3.5.21}$$

and using orthogonality leads to (3.5.16). The determinant $D^{(H)}_{\lambda\nu}$ is independent of N and $\psi^{(H)}_{\lambda\nu}$ is a polynomial in N, since $\nu \subseteq \lambda$. ■

Corollary 3.5.5. *The roots of coefficients the $\psi^{(H)}_{\lambda\nu}$ are integers given by the content of the skew diagram $\lambda - \nu$.*

Proof. The skew diagram $\lambda - \nu$ is the set-theoretic difference of the Young diagrams of λ and ν: the set of squares that belong to the diagram of λ but not to that of ν. Using (3.5.10),

$$\psi^{(H)}_{\lambda\nu} = \frac{1}{2^{\frac{|\lambda|-|\nu|}{2}}} \frac{C_\lambda(N)}{C_\nu(N)} D^{(H)}_{\lambda\nu}. \tag{3.5.22}$$

Since $\nu \subseteq \lambda$, the roots of $\psi^{(H)}_{\lambda\nu}$ are integers and can be read from the skew diagram $\lambda - \nu$ whenever $D^{(H)}_{\lambda\nu} \neq 0$. For example, if $\lambda = (4,1,1)$ and $\nu = (2)$, then the roots of $\psi^{(H)}_{\lambda\nu}$ are $\{-3,-2,1,2\}$:

0	−1	−2	−3
1			
2			

(3.5.23)

■

Corollary 3.5.6. *The coefficient* $\psi_{\lambda\lambda}^{(H)} = 1$.

Proof. If $\nu = \lambda$,

$$\psi_{\lambda\lambda}^{(H)} = \frac{N!}{Z_N^{(H)} \langle \mathcal{H}_\lambda, \mathcal{H}_\lambda \rangle} \det \left[\int_{\mathbb{R}} x_j^{\lambda_j+N-j} H_{\lambda_k+N-k}(x_j) e^{-\frac{x_j^2}{2}} \, dx_j \right]_{j=1,\dots,N}. \tag{3.5.24}$$

By expanding the monomials in terms of the Hermite polynomials, only the diagonal terms survive. ■

The coefficient $\psi_{\lambda\nu}^{(H)}$ for $\nu = 0$ simplifies further and can be expressed in terms of a character of the symmetric group. One way to compute the characters of the symmetric group is using the Frobenius formula.

Proposition 3.5.7 (Frobenius formula [105])**.** *Let* χ_μ^λ *be a character of the symmetric group on the* λ^{th} *irreducible representation and the* μ^{th} *conjugacy class. Then the value of* χ_μ^λ *is the coefficient of the monomial*

$$x_1^{\lambda_1+l(\lambda)-1} x_2^{\lambda_2+l(\lambda)-2} \dots x_l^{\lambda_l} \tag{3.5.25}$$

in the product

$$P_\mu(x_1, \dots, x_{l(\lambda)}) \prod_{1 \le j < k \le l(\lambda)} (x_j - x_k). \tag{3.5.26}$$

Proposition 3.5.8. *The coefficient*

$$\psi_{\lambda 0}^{(H)} = \begin{cases} \frac{C_\lambda(N)}{2^{\frac{|\lambda|}{2}} \frac{|\lambda|}{2}!} \chi_{(2^{|\lambda|/2})}^\lambda, & |\lambda| \text{ is even,} \\ 0, & |\lambda| \text{ is odd,} \end{cases} \tag{3.5.27}$$

where $\chi_{(2^{|\lambda|/2})}^\lambda$ *is the character of the* λ^{th} *irreducible representation evaluated on the elements of cycle-type* $(2^{|\lambda|/2})$.

Proof. Since Hermite polynomials of odd and even degree do not mix, $\psi_{\lambda 0}^{(H)} = 0$ when $|\lambda|$ is odd. When $|\lambda|$ is even,

$$D_{\lambda 0}^{(H)} = \det \left[\mathbb{1}_{\lambda_j - j + k = 0 \bmod 2} \frac{1}{\left(\frac{\lambda_j - j + k}{2}\right)!} \right]. \tag{3.5.28}$$

Denote $n = |\lambda|/2$, $L = l(\lambda)$, $g(x_1, \dots, x_L)$ to be a formal power series in variables x_i, and $(k_1, \dots, k_L)$ to be a partition constructed from λ such that $k_j = \lambda_j + L - j$, $j = 1, \dots, L$. Let

$$[g(x_1, \dots, x_L)]_{(k_1,\dots,k_L)} = \text{coefficient of } x_1^{k_1} \dots x_L^{k_L}. \tag{3.5.29}$$

By using the Frobenius formula for the characters of the symmetric group,

$$\begin{aligned} \chi_{(2^n)}^\lambda &= \left[\Delta(x_1, \dots, x_L)(x_1^2 + \dots + x_L^2)^n \right]_{(k_1,\dots,k_L)} \\ &= \sum_{n_1+\dots+n_L=n} \frac{n!}{n_1! \dots n_L!} \left[\det \left[x_i^{L-j} \right] x_1^{2n_1} x_2^{2n_2} \dots x_L^{2n_L} \right]_{(\lambda_1+L-1, \lambda_2+L-2, \dots, \lambda_L)}. \end{aligned} \tag{3.5.30}$$

After absorbing $x_i^{2n_i}$ into the ith row of the determinant, for each n_i at most one term in the ith row has the exponent λ_i+L-i, say the $(i,j)^{th}$ element $x_i^{2n_i+L-j}$, which implies $2n_i = \lambda_i-i+j$. For L-tuples $\{n_1,\dots,n_L\}$ such that there is exactly one term in each row that has the required exponent, the non-zero summands are given by $n!\,\mathrm{sgn}(\sigma)\prod_i((\lambda_i-i+\sigma(i)/2)!)^{-1}$ where $\sigma\in\mathcal{S}_L$. Considering all such L-tuples and using the Laplace expansion for determinants proves the proposition. ■

Therefore, the expansion of Schur polynomials in terms of multivariate Hermite polynomials can be written as

$$S_\lambda(x_1,\dots,x_N) = C_\lambda(N)\sum_{\nu\subseteq\lambda}\frac{1}{2^{\frac{|\lambda|-|\nu|}{2}}}\frac{1}{C_\nu(\lambda)}D^{(H)}_{\lambda\nu}\mathcal{H}_\nu(x_1,\dots,x_N). \tag{3.5.31}$$

In a similar way, by expanding Hermite polynomials in terms of monomials in the definition of $\mathcal{H}_\lambda$, multivariate Hermite polynomials can be written in the Schur basis as

$$\mathcal{H}_\lambda = \sum_{\nu\subseteq\lambda}\kappa^{(H)}_{\lambda\nu}S_\nu = \sum_{j=0}^{\lfloor\frac{|\lambda|}{2}\rfloor}\sum_{\nu\vdash g(j)}\kappa^{(H)}_{\lambda\nu}S_\nu,\quad g(j)=\begin{cases}2j, & |\lambda|\text{ is even},\\ 2j+1, & |\lambda|\text{ is odd},\end{cases} \tag{3.5.32}$$

where

$$\kappa^{(H)}_{\lambda\nu} = \left(\frac{-1}{2}\right)^{\frac{|\lambda|-|\nu|}{2}} D^{(H)}_{\lambda\nu}\prod_{j=1}^{L}\frac{(\lambda_j+N-j)!}{(\nu_j+N-j)!}. \tag{3.5.33}$$

Alternatively,

$$\mathcal{H}_\lambda(x_1,\dots,x_N) = C_\lambda(N)\sum_{\nu\subseteq\lambda}\left(\frac{-1}{2}\right)^{\frac{|\lambda|-|\nu|}{2}}\frac{1}{C_\nu(N)}D^{(H)}_{\lambda\nu}S_\nu(x_1,\dots,x_N), \tag{3.5.34}$$

where $|\lambda|-|\nu| = 0 \bmod 2$. This expansion should be compared with the classical Hermite polynomial expansion

$$H_n(x) = n!\sum_{j=0}^{n}\mathbb{1}_{n-j=0\bmod 2}\frac{1}{\left(\frac{n-j}{2}\right)!}\frac{(-1)^{\frac{n-j}{2}}}{2^{\frac{n-j}{2}}j!}x^j, \tag{3.5.35}$$

and (3.5.31) should be compared with

$$x^n = n!\sum_{j=0}^{n}\mathbb{1}_{n-j=0\bmod 2}\frac{1}{\left(\frac{n-j}{2}\right)!}\frac{1}{2^{\frac{n-j}{2}}j!}H_j(x), \tag{3.5.36}$$

which is an alternate way of writing (3.5.21). Clearly, we see the analogies between classical Hermite polynomials and their multivariate counterparts: the sum over j is replaced by the sum over partitions; the role of monomials is played by Schur polynomials; the factorials are replaced with $C_\lambda(N)$.

Proposition 3.5.9. *Let $x_1,\dots,x_N$ and $t_1,\dots,t_N$ be two sets of variables. The multivariate*

Hermite polynomials defined in (3.5.7) have the following generating function [21]:

$$\sum_{\lambda} \frac{\mathcal{H}_\lambda(\boldsymbol{x})}{C_\lambda(N)} S_\lambda(\boldsymbol{t}) = \left(\sum_{\mu} \frac{S_\mu(\boldsymbol{x}) S_\mu(\boldsymbol{t})}{C_\mu(N)} \right) \prod_j \exp\left(-\frac{t_j^2}{2} \right). \tag{3.5.37}$$

Several other analogues of the properties of the classical Hermite polynomials, including an integral representation, summation, integration and differentiation formulae, are given for $\beta-$ensembles in [21]. Note that in [21], C_μ^α ($\alpha \in \mathbb{R}$) is used to denote Schur polynomials with a specific normalisation, where as in this work $C_\mu(N)$ is a constant in N given in (3.5.10).

A few examples of (3.5.34) and (3.5.31) are given below. These expansions are given for partitions of 4 with N variables.

$$\begin{aligned}
\mathcal{H}_{(4)} &= S_{(4)} - \frac{1}{2}\frac{(N+3)!}{(N+1)!} S_{(2)} + \frac{1}{8}\frac{(N+3)!}{(N-1)!} \\
\mathcal{H}_{(3,1)} &= S_{(3,1)} - \frac{1}{2}\frac{(N+2)!}{N!} S_{(1^2)} - \frac{1}{8}\frac{(N+2)!}{(N-2)!} \\
\mathcal{H}_{(2^2)} &= S_{(2^2)} - \frac{1}{2}\frac{N!}{(N-2)!} S_{(2)} + \frac{1}{2}\frac{(N+1)!}{(N-1)!} S_{(1^2)} + \frac{1}{4}\frac{N!(N+1)!}{(N-2)!(N-1)!} \\
\mathcal{H}_{(2,1^2)} &= S_{(2,1^2)} + \frac{1}{2}\frac{(N-1)!}{(N-3)!} S_{(2)} - \frac{1}{8}\frac{(N+1)!}{(N-3)!} \\
\mathcal{H}_{(1^4)} &= S_{(1^4)} + \frac{1}{2}\frac{(N-2)!}{(N-4)!} S_{(1^2)} + \frac{1}{8}\frac{N!}{(N-4)!}
\end{aligned} \tag{3.5.38}$$

$$\begin{aligned}
S_{(4)} &= \mathcal{H}_{(4)} + \frac{1}{2}\frac{(N+3)!}{(N+1)!} \mathcal{H}_{(2)} + \frac{1}{8}\frac{(N+3)!}{(N-1)!} \\
S_{(3,1)} &= \mathcal{H}_{(3,1)} + \frac{1}{2}\frac{(N+2)!}{N!} \mathcal{H}_{(1^2)} - \frac{1}{8}\frac{(N+2)!}{(N-2)!} \\
S_{(2^2)} &= \mathcal{H}_{(2^2)} + \frac{1}{2}\frac{N!}{(N-2)!} \mathcal{H}_{(2)} - \frac{1}{2}\frac{(N+1)!}{(N-1)!} \mathcal{H}_{(1^2)} + \frac{1}{4}\frac{N!(N+1)!}{(N-2)!(N-1)!} \\
S_{(2,1^2)} &= \mathcal{H}_{(2,1^2)} - \frac{1}{2}\frac{(N-1)!}{(N-3)!} \mathcal{H}_{(2)} - \frac{1}{8}\frac{(N+1)!}{(N-3)!} \\
S_{(1^4)} &= \mathcal{H}_{(1^4)} - \frac{1}{2}\frac{(N-2)!}{(N-4)!} \mathcal{H}_{(1^2)} + \frac{1}{8}\frac{N!}{(N-4)!}
\end{aligned} \tag{3.5.39}$$

Laguerre ensemble. Let M be an $N \times N$ LUE matrix with eigenvalues $x_1, \ldots, x_N$. For $\gamma > -1$, the j.p.d.f. of eigenvalues is

$$\begin{aligned}
\rho^{(L)}(x_1, \ldots, x_N) &= \frac{1}{Z_N^{(L)}} \Delta^2(\mathbf{x}) \prod_{i=1}^N x_i^\gamma e^{-x_i}, \\
Z_N^{(L)} &= N! G_0(N, \gamma) G_0(N, 0),
\end{aligned} \tag{3.5.40}$$

where $G_\lambda(N,\gamma)$ is given in (3.2.6). The multivariate Laguerre polynomials defined by

$$\mathcal{L}_\lambda^{(\gamma)}(\mathbf{x}) = \frac{1}{\Delta_N}\begin{vmatrix} L_{\lambda_1+N-1}^{(\gamma)}(x_1) & L_{\lambda_1+N-1}^{(\gamma)}(x_2) & \dots & L_{\lambda_1+N-1}^{(\gamma)}(x_N) \\ L_{\lambda_2+N-2}^{(\gamma)}(x_1) & L_{\lambda_2+N-2}^{(\gamma)}(x_2) & \dots & L_{\lambda_2+N-2}^{(\gamma)}(x_N) \\ \vdots & \vdots & & \vdots \\ L_{\lambda_N}^{(\gamma)}(x_1) & L_{\lambda_N}^{(\gamma)}(x_2) & \dots & L_{\lambda_N}^{(\gamma)}(x_N) \end{vmatrix}, \tag{3.5.41}$$

$l(\lambda) \le N$, satisfy the orthogonality relation

$$\begin{aligned} \left\langle \mathcal{L}_\lambda^{(\gamma)}, \mathcal{L}_\mu^{(\gamma)} \right\rangle :=& \frac{1}{Z_N^{(L)}} \int_{\mathbb{R}_+^N} \mathcal{L}_\lambda^{(\gamma)}(\mathbf{x}) \mathcal{L}_\mu^{(\gamma)}(\mathbf{x}) \Delta^2(\mathbf{x}) \prod_{i=1}^N x_i^\gamma e^{-x}\, dx_i \\ =& \frac{G_\lambda(N,\gamma)}{G_0(N,\gamma)} \frac{1}{G_\lambda(N,0)} \frac{1}{G_0(N,0)} \delta_{\lambda\mu}. \end{aligned} \tag{3.5.42}$$

The polynomials in the determinant (3.5.41) are normalized according to (3.2.4b).

The Schur polynomials can be expanded in terms of multivariate Laguerre polynomials as

$$S_\lambda = \sum_{\nu \subseteq \lambda} \psi_{\lambda\nu}^{(L)} \mathcal{L}_\nu^{(\gamma)}, \tag{3.5.43}$$

where

$$\psi_{\lambda\nu}^{(L)} = (-1)^{|\nu| + \frac{1}{2}N(N-1)} \frac{G_\lambda(N,\gamma)}{G_\nu(N,\gamma)} G_\lambda(N,0) D_{\lambda\nu}^{(L)}, \tag{3.5.44}$$

$$D_{\lambda\nu}^{(L)} = \det \left[\mathbb{1}_{\lambda_i - \nu_j - i + j \ge 0} \tfrac{1}{(\lambda_i - \nu_j - i + j)!} \right]_{i,j=1,\dots,l(\lambda)}. \tag{3.5.45}$$

The coefficients $\psi_{\lambda\nu}^{(L)}$ in (3.5.44) can be computed in a similar way as in Prop. 3.5.4. It is interesting to note that the quantity $|\lambda/\nu|! D_{\lambda\nu}^{(L)}$ gives the number of standard Young tableaux of shape λ/ν [221, p.344]. Multivariate Laguerre polynomials can also be expanded in the Schur basis:

$$\begin{aligned} \mathcal{L}_\lambda^{(\gamma)} &= \sum_{\nu \subseteq \lambda} \kappa_{\lambda\nu}^{(L)} S_\nu, \\ \kappa_{\lambda\nu}^{(L)} &= (-1)^{|\nu| + \frac{1}{2}N(N-1)} \frac{G_\lambda(N,\gamma)}{G_\nu(N,\gamma)} \frac{1}{G_\nu(N,0)} D_{\lambda\nu}^{(L)}. \end{aligned} \tag{3.5.46}$$

Similar to the Hermite case, $D_{\lambda 0}^{(L)}$ turns out to be a character of the symmetric group.

Proposition 3.5.10. *We have*

$$D_{\lambda 0}^{(L)} = \frac{\chi_{(1^{|\lambda|})}^\lambda}{|\lambda|!} = \frac{\dim V_\lambda}{|\lambda|!}. \tag{3.5.47}$$

Proof. Same as Prop. 3.5.8. Note that $|\lambda|! D_{\lambda 0}^{(L)}$ gives the number of standard Young tableaux of shape λ. ■

Expansions in (3.5.46) and (3.5.43) should be compared with the results of classical La-

guerre polynomials:

$$
\begin{aligned}
L_n^{(\gamma)} &= \sum_{j=0}^{n} (-1)^j \frac{\Gamma(n+\gamma+1)}{\Gamma(j+\gamma+1)(n-j)!} \frac{x^j}{j!}, \\
x^n &= n! \sum_{j=0}^{n} \frac{(-1)^j}{(n-j)!} \frac{\Gamma(n+\gamma+1)}{\Gamma(j+\gamma+1)} L_j^{(\gamma)}(x).
\end{aligned}
\tag{3.5.48}
$$

The G_λ replaces the Gamma-function, the summation over j is replaced by the summation over partitions, the Schur polynomials replace the monomials, and the $\mathcal{L}_\lambda^{(\gamma)}$ replace the $L_n^{(\gamma)}$.

The generating function for multivariate Laguerre polynomials [21] is

$$
\sum_\nu \frac{1}{G_\nu(N,\gamma)} \mathcal{L}_\nu^{(\gamma)}(\mathbf{x}) S_\nu(\mathbf{t}) = (-1)^{\frac{N(N-1)}{2}} \left(\sum_\lambda S_\lambda(\mathbf{t}) D_{\lambda 0}^{(L)} \right) \left(\sum_\mu \frac{(-1)^{|\mu|}}{G_\mu(N,\gamma)} \frac{S_\mu(\mathbf{x}) S_\mu(\mathbf{t})}{G_\mu(N,0)} \right),
\tag{3.5.49}
$$

or equivalently using (3.5.47),

$$
\sum_\nu \frac{1}{G_\nu(N,\gamma)} \mathcal{L}_\nu^{(\gamma)}(\mathbf{x}) S_\nu(\mathbf{t}) = (-1)^{\frac{N(N-1)}{2}} \left(\sum_\mu \frac{(-1)^{|\mu|}}{G_\mu(N,\gamma)} \frac{S_\mu(\mathbf{x}) S_\mu(\mathbf{t})}{G_\mu(N,0)} \right) \prod_{j=1}^{N} e^{t_j}.
\tag{3.5.50}
$$

Below, we give a few examples for explicit expansions of the $\mathcal{L}_\lambda^{(\gamma)}$ in terms of the S_μ, and vice versa.

$$
\begin{aligned}
\mathcal{L}^{(\gamma)}_{(4)} = \prod_{j=0}^{N-1} \frac{(-1)^j}{j!} \Bigg[& \frac{(N-1)!}{(N+3)!} S_{(4)} - \frac{\Gamma(N+\gamma+4)}{\Gamma(N+\gamma+3)} \frac{(N-1)!}{(N+2)!} S_{(3)} + \frac{1}{2} \frac{\Gamma(N+\gamma+4)}{\Gamma(N+\gamma+2)} \frac{(N-1)!}{(N+1)!} S_{(2)} \\
& - \frac{1}{6} \frac{\Gamma(N+\gamma+4)}{\Gamma(N+\gamma+1)} \frac{(N-1)!}{N!} S_{(1)} + \frac{1}{24} \frac{\Gamma(N+\gamma+4)}{\Gamma(N+\gamma)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{(\gamma)}_{(3,1)} = \prod_{j=0}^{N-1} \frac{(-1)^j}{j!} \Bigg[& \frac{(N-2)!}{(N+2)!} S_{(3,1)} - \frac{\Gamma(N+\gamma)}{\Gamma(N+\gamma-1)} \frac{(N-1)!}{(N+2)!} S_{(3)} - \frac{\Gamma(N+\gamma+3)}{\Gamma(N+\gamma+2)} \frac{(N-2)!}{(N+1)!} S_{(2,1)} \\
& + \frac{\Gamma(N+\gamma+3)\Gamma(N+\gamma)}{\Gamma(N+\gamma+2)\Gamma(N+\gamma-1)} \frac{(N-1)!}{(N+1)!} S_{(2)} + \frac{1}{2} \frac{\Gamma(N+\gamma+3)}{\Gamma(N+\gamma+1)} \frac{(N-2)!}{N!} S_{(1^2)} \\
& - \frac{1}{2} \frac{\Gamma(N+\gamma+3)\Gamma(N+\gamma)}{\Gamma(N+\gamma+1)\Gamma(N+\gamma-1)} \frac{(N-1)!}{(N+1)!} S_{(1)} + \frac{1}{8} \frac{\Gamma(N+\gamma+3)}{\Gamma(N+\gamma-1)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{(\gamma)}_{(2^2)} = \prod_{j=0}^{N-1} \frac{(-1)^j}{j!} \Bigg[& \frac{(N-1)!(N-2)}{(N+1)!N!} S_{(2^2)} - \frac{\Gamma(N+\gamma+1)}{\Gamma(N+\gamma)} \frac{(N-2)!}{(N+1)!} S_{(2,1)} \\
& + \frac{1}{2} \frac{\Gamma(N+\gamma+1)}{\Gamma(N+\gamma-1)} \frac{(N-1)!}{(N+1)!} S_{(2)} + \frac{1}{2} \frac{\Gamma(N+\gamma+2)}{\Gamma(N+\gamma)} \frac{(N-2)!}{N!} S_{(1^2)} \\
& - \frac{1}{3} \frac{\Gamma(N+\gamma+2)}{\Gamma(N+\gamma-1)} \frac{(N-1)!}{N!} S_{(1)} + \frac{1}{12} \frac{\Gamma(N+\gamma+2)\Gamma(N+\gamma+1)}{\Gamma(N+\gamma)\Gamma(N+\gamma-1)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{(\gamma)}_{(2,1,1)} = \prod_{j=0}^{N-1} \frac{(-1)^j}{j!} \Bigg[& \frac{(N-3)!}{(N+1)!} S_{(2,1,1)} - \frac{\Gamma(N+\gamma-1)}{\Gamma(N+\gamma-2)} \frac{(N-2)!}{(N+1)!} S_{(2,1)} - \frac{\Gamma(N+\gamma+2)}{\Gamma(N+\gamma+1)} \frac{(N-3)!}{N!} S_{(1^3)} \\
& + \frac{1}{2} \frac{\Gamma(N+\gamma)}{\Gamma(N+\gamma-2)} \frac{(N-1)!}{(N+1)!} S_{(2)} + \frac{\Gamma(N+\gamma+2)\Gamma(N+\gamma-1)}{\Gamma(N+\gamma+1)\Gamma(N+\gamma-2)} \frac{(N-2)!}{N!} S_{(1^2)} \\
& - \frac{1}{2} \frac{\Gamma(N+\gamma+2)\Gamma(N+\gamma)}{\Gamma(N+\gamma+1)\Gamma(N+\gamma-2)} \frac{(N-1)!}{N!} S_{(1)} + \frac{1}{8} \frac{\Gamma(N+\gamma+2)}{\Gamma(N+\gamma-2)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{(\gamma)}_{(1^4)} = \prod_{j=0}^{N-1} \frac{(-1)^j}{j!} \Bigg[& \frac{(N-4)!}{N!} S_{(1^4)} - \frac{\Gamma(N+\gamma-2)}{\Gamma(N+\gamma-3)} \frac{(N-3)!}{N!} S_{(1^3)} \\
& + \frac{1}{2} \frac{\Gamma(N+\gamma-1)}{\Gamma(N+\gamma-3)} \frac{(N-2)!}{N!} S_{(1^2)} - \frac{1}{6} \frac{\Gamma(N+\gamma)}{\Gamma(N+\gamma-3)} \frac{(N-1)!}{N!} S_{(1)} + \frac{1}{24} \frac{\Gamma(N+\gamma+1)}{\Gamma(N+\gamma-3)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
S_{(4)} &= \prod_{j=0}^{N-1}(-1)^j j!\,\frac{(N+3)!}{(N-1)!}\Big[\mathcal{L}^{(\gamma)}_{(4)} - \frac{\Gamma(N+\gamma+4)}{\Gamma(N+\gamma+3)}\mathcal{L}^{(\gamma)}_{(3)} + \frac{1}{2}\frac{\Gamma(N+\gamma+4)}{\Gamma(N+\gamma+2)}\mathcal{L}^{(\gamma)}_{(2)} \\
&\quad -\frac{1}{6}\frac{\Gamma(N+\gamma+4)}{\Gamma(N+\gamma+1)}\mathcal{L}^{(\gamma)}_{(1)} + \frac{1}{24}\frac{\Gamma(N+\gamma+4)}{\Gamma(N+\gamma)}\mathcal{L}^{(\gamma)}_{0}\Big] \\
S_{(3,1)} &= \prod_{j=0}^{N-1}(-1)^j j!\,\frac{(N+2)!}{(N-2)!}\Big[\mathcal{L}^{(\gamma)}_{(3,1)} - \frac{\Gamma(N+\gamma)}{\Gamma(N+\gamma-1)}\mathcal{L}^{(\gamma)}_{(3)} - \frac{\Gamma(N+\gamma+3)}{\Gamma(N+\gamma+2)}\mathcal{L}^{(\gamma)}_{(2,1)} \\
&\quad +\frac{\Gamma(N+\gamma+3)\Gamma(N+\gamma)}{\Gamma(N+\gamma+2)\Gamma(N+\gamma-1)}\mathcal{L}^{(\gamma)}_{(2)} + \frac{1}{2}\frac{\Gamma(N+\gamma+3)}{\Gamma(N+\gamma+1)}\mathcal{L}^{(\gamma)}_{(1^2)} \\
&\quad -\frac{1}{2}\frac{\Gamma(N+\gamma+3)\Gamma(N+\gamma)}{\Gamma(N+\gamma+1)\Gamma(N+\gamma-1)}\mathcal{L}^{(\gamma)}_{(1)} + \frac{1}{8}\frac{\Gamma(N+\gamma+3)}{\Gamma(N+\gamma-1)}\mathcal{L}^{(\gamma)}_{0}\Big] \\
S_{(2^2)} &= \prod_{j=0}^{N-1}(-1)^j j!\,\frac{(N+1)!N!}{(N-1)!(N-2)}\Big[\mathcal{L}^{(\gamma)}_{(2^2)} - \frac{\Gamma(N+\gamma+1)}{\Gamma(N+\gamma)}\mathcal{L}^{(\gamma)}_{(2,1)} + \frac{1}{2}\frac{\Gamma(N+\gamma+1)}{\Gamma(N+\gamma-1)}\mathcal{L}^{(\gamma)}_{(2)} \\
&\quad +\frac{1}{2}\frac{\Gamma(N+\gamma+2)}{\Gamma(N+\gamma)}\mathcal{L}^{(\gamma)}_{(1^2)} - \frac{1}{3}\frac{\Gamma(N+\gamma+2)}{\Gamma(N+\gamma-1)}\mathcal{L}^{(\gamma)}_{(1)} + \frac{1}{12}\frac{\Gamma(N+\gamma+2)\Gamma(N+\gamma+1)}{\Gamma(N+\gamma)\Gamma(N+\gamma-1)}\mathcal{L}^{(\gamma)}_{0}\Big] \\
S_{(2,1,1)} &= \prod_{j=0}^{N-1}(-1)^j j!\,\frac{(N+1)!}{(N-3)!}\Big[\mathcal{L}^{(\gamma)}_{(2,1,1)} - \frac{\Gamma(N+\gamma-1)}{\Gamma(N+\gamma-2)}\mathcal{L}^{(\gamma)}_{(2,1)} - \frac{\Gamma(N+\gamma+2)}{\Gamma(N+\gamma+1)}\mathcal{L}^{(\gamma)}_{(1^3)} \\
&\quad +\frac{1}{2}\frac{\Gamma(N+\gamma)}{\Gamma(N+\gamma-2)}\mathcal{L}^{(\gamma)}_{(2)} + \frac{\Gamma(N+\gamma+2)\Gamma(N+\gamma-1)}{\Gamma(N+\gamma+1)\Gamma(N+\gamma-2)}\mathcal{L}^{(\gamma)}_{(1^2)} \\
&\quad -\frac{1}{2}\frac{\Gamma(N+\gamma+2)\Gamma(N+\gamma)}{\Gamma(N+\gamma+1)\Gamma(N+\gamma-2)}\mathcal{L}^{(\gamma)}_{(1)} + \frac{1}{8}\frac{\Gamma(N+\gamma+2)}{\Gamma(N+\gamma-2)}\mathcal{L}^{(\gamma)}_{0}\Big] \\
S_{(1^4)} &= \prod_{j=0}^{N-1}(-1)^j j!\,\frac{N!}{(N-4)!}\Big[\mathcal{L}^{(\gamma)}_{(1^4)} - \frac{\Gamma(N+\gamma-2)}{\Gamma(N+\gamma-3)}\mathcal{L}^{(\gamma)}_{(1^3)} + \frac{1}{2}\frac{\Gamma(N+\gamma-1)}{\Gamma(N+\gamma-3)}\mathcal{L}^{(\gamma)}_{(1^2)} \\
&\quad -\frac{1}{6}\frac{\Gamma(N+\gamma)}{\Gamma(N+\gamma-3)}\mathcal{L}^{(\gamma)}_{(1)} + \frac{1}{24}\frac{\Gamma(N+\gamma+1)}{\Gamma(N+\gamma-3)}\mathcal{L}^{(\gamma)}_{0}\Big]
\end{aligned}
$$

Jacobi ensemble. Let M be an $N\times N$ JUE matrix with eigenvalues $x_1,\dots,x_N$. For $\gamma_1,\gamma_2>-1$, the j.p.d.f. of eigenvalues is

$$
\begin{aligned}
\rho^{(J)}(x_1,\dots,x_N) &= \frac{1}{Z_N^{(J)}}\Delta^2(\mathbf{x})\prod_{i=1}^{N} x_i^{\gamma_1}(1-x_i)^{\gamma_2}, \\
Z_N^{(J)} &= N!\prod_{j=0}^{N-1}\frac{j!\,\Gamma(j+\gamma_1+1)\Gamma(j+\gamma_2+1)\Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+2)\Gamma(2j+\gamma_1+\gamma_2+1)}.
\end{aligned}
\tag{3.5.51}
$$

Classical Jacobi polynomials are given by

$$
J_n^{(\gamma_1,\gamma_2)}(x) = \frac{\Gamma(n+\gamma_1+1)}{\Gamma(n+\gamma_1+\gamma_2+1)}\sum_{j=0}^{n}\frac{(-1)^j}{j!(n-j)!}\frac{\Gamma(n+j+\gamma_1+\gamma_2+1)}{\Gamma(j+\gamma_1+1)}x^j, \tag{3.5.52}
$$

and satisfy the orthogonality relation (3.2.4c). The multivariate Jacobi polynomials are

$$
\mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}(\mathbf{x}) = \frac{1}{\Delta_N}\begin{vmatrix} J^{(\gamma_1,\gamma_2)}_{\lambda_1+N-1}(x_1) & J^{(\gamma_1,\gamma_2)}_{\lambda_1+N-1}(x_2) & \dots & J^{(\gamma_1,\gamma_2)}_{\lambda_1+N-1}(x_N) \\ J^{(\gamma_1,\gamma_2)}_{\lambda_2+N-2}(x_1) & J^{(\gamma_1,\gamma_2)}_{\lambda_2+N-2}(x_2) & \dots & J^{(\gamma_1,\gamma_2)}_{\lambda_2+N-2}(x_N) \\ \vdots & \vdots & & \vdots \\ J^{(\gamma_1,\gamma_2)}_{\lambda_N}(x_1) & J^{(\gamma_1,\gamma_2)}_{\lambda_N}(x_2) & \dots & J^{(\gamma_1,\gamma_2)}_{\lambda_N}(x_N) \end{vmatrix}, \tag{3.5.53}
$$

$l(\lambda) \leq N$, and obey the orthogonality relation

$$\begin{aligned}\left\langle \mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}, \mathcal{J}_\mu^{(\gamma_1,\gamma_2)} \right\rangle :=& \frac{1}{Z_N^{(J)}} \int_{[0,1]^N} \mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}(\mathbf{x}) \mathcal{J}_\mu^{(\gamma_1,\gamma_2)}(\mathbf{x}) \Delta^2(\mathbf{x}) \prod_{i=1}^N x_i^{\gamma_1}(1-x_i)^{\gamma_2}\, dx_i \\ =& \frac{N!}{Z_N^{(J)}} \frac{G_\lambda(N,\gamma_1)G_\lambda(N,\gamma_2)}{G_\lambda(N,\gamma_1+\gamma_2)G_\lambda(N,0)} \prod_{j=1}^N (2\lambda_j + 2N - 2j + \gamma_1 + \gamma_2 + 1)^{-1} \delta_{\lambda\mu}.\end{aligned} \tag{3.5.54}$$

The expansion of the Schur polynomials in terms of multivariate Jacobi polynomials is

$$S_\lambda = \sum_{\nu \subseteq \lambda} \psi_{\lambda\nu}^{(J)} \mathcal{J}_\nu^{(\gamma_1,\gamma_2)}, \tag{3.5.55}$$

where

$$\begin{aligned}\psi_{\lambda\nu}^{(J)} =& (-1)^{|\nu| + \frac{1}{2}N(N-1)} \frac{G_\lambda(N,\gamma_1)}{G_\nu(N,\gamma_1)} G_\nu(N,\gamma_1+\gamma_2) G_\lambda(N,0) \\ &\times \mathcal{D}_{\lambda\nu}^{(J)} \prod_{j=1}^N (2\nu_j + 2N - 2j + \gamma_1 + \gamma_2 + 1),\end{aligned} \tag{3.5.56}$$

and

$$\mathcal{D}_{\lambda\nu}^{(J)} = \det\left[\mathbb{1}_{\lambda_j - \nu_k - j + k \geq 0} ((\lambda_j - \nu_k - j + k)!\, \Gamma(2N + \lambda_j + \nu_k - j - k + \gamma_1 + \gamma_2 + 2))^{-1}\right]_{1 \leq j,k \leq N}. \tag{3.5.57}$$

When $N = 1$, (3.5.55) coincides with the one variable analogue

$$x^n = n!\, \Gamma(n + \gamma_1 + 1) \sum_{j=0}^n \frac{(-1)^j}{(n-j)!} \frac{(2j + \gamma_1 + \gamma_2 + 1)\Gamma(j + \gamma_1 + \gamma_2 + 1)}{\Gamma(j + \gamma_1 + 1)\Gamma(n + j + \gamma_1 + \gamma_2 + 2)} J_j^{(\gamma_1,\gamma_2)}(x). \tag{3.5.58}$$

Multivariate Jacobi polynomials can be expanded in Schur polynomials via

$$\mathcal{J}_\lambda^{(\gamma_1,\gamma_2)} = \sum_{\nu \subseteq \lambda} \kappa_{\lambda\nu}^{(J)} S_\nu, \tag{3.5.59}$$

where

$$\kappa_{\lambda\nu}^{(J)} = (-1)^{|\nu| + \frac{1}{2}N(N-1)} \frac{G_\lambda(N,\gamma_1)}{G_\nu(N,\gamma_1)} \frac{1}{G_\lambda(N,\gamma_1+\gamma_2) G_\nu(N,0)} \tilde{\mathcal{D}}_{\lambda\nu}^{(J)}, \tag{3.5.60}$$

$$\tilde{\mathcal{D}}_{\lambda\nu}^{(J)} = \det\left[\mathbb{1}_{\lambda_j - \nu_k - j + k \geq 0} \frac{\Gamma(2N + \lambda_j + \nu_k - j - k + \gamma_1 + \gamma_2 + 1)}{(\lambda_j - \nu_k - j + k)!}\right]_{1 \leq j,k \leq N}. \tag{3.5.61}$$

Below, we give a few examples for explicit expansions of the $\mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}$ in terms of the S_μ, and vice versa.

$$
\begin{aligned}
\mathcal{J}_{(2)}^{(\gamma_1,\gamma_2)} &= \frac{\Gamma(N+\gamma_1+\gamma_2)}{\Gamma(N+\gamma_1+\gamma_2+2)\Gamma(2N+\gamma_1+\gamma_2-1)} \prod_{j=0}^{N-1}(-1)^j \frac{\Gamma(2j+\gamma_1+\gamma_2+1)}{j!\Gamma(j+\gamma_1+\gamma_2+1)} \\
&\times \Bigg[\Gamma(2N+\gamma_1+\gamma_2+3)\frac{(N-1)!}{(N+1)!}S_{(2)} - \Gamma(2N+\gamma_1+\gamma_2+2)\frac{\Gamma(N+\gamma_1+2)}{\Gamma(N+\gamma_1+1)}\frac{(N-1)!}{N!}S_{(1)} \\
&\quad + \frac{1}{2}\Gamma(2N+\gamma_1+\gamma_2+1)\frac{\Gamma(N+\gamma_1+2)}{\Gamma(N+\gamma_1)}\Bigg] \\
\mathcal{J}_{(1^2)}^{(\gamma_1,\gamma_2)} &= \frac{\Gamma(N+\gamma_1+\gamma_2-1)}{\Gamma(N+\gamma_1+\gamma_2+1)}\frac{1}{\Gamma(2N+\gamma_1+\gamma_2-1)\Gamma(2N+\gamma_1+\gamma_2-3)} \prod_{j=0}^{N-1}(-1)^j \frac{\Gamma(2j+\gamma_1+\gamma_2+1)}{j!\Gamma(j+\gamma_1+\gamma_2+1)} \\
&\times \Bigg[\Gamma(2N+\gamma_1+\gamma_2+1)\Gamma(2N+\gamma_1+\gamma_2-1)\frac{(N-2)!}{N!}S_{(1^2)} \\
&\quad - \Gamma(2N+\gamma_1+\gamma_2+1)\Gamma(2N+\gamma_1+\gamma_2-2)\frac{\Gamma(N+\gamma_1)}{\Gamma(N+\gamma_1-1)}\frac{(N-1)!}{N!}S_{(1)} \\
&\quad + \frac{\Gamma(N+\gamma_1+1)}{\Gamma(N+\gamma_1-1)}\left(\Gamma(2N+\gamma_1+\gamma_2)\Gamma(2N+\gamma_1+\gamma_2-2) - \frac{1}{2}\Gamma(2N+\gamma_1+\gamma_2-1)^2\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
S_{(2)} &= (2N+\gamma_1+\gamma_2-1)\frac{(N+1)!}{(N-1)!}\prod_{j=0}^{N-1}(-1)^j\frac{j!\Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+1)} \\
&\times \Bigg[\frac{1}{\Gamma(2N+\gamma_1+\gamma_2+3)}\frac{\Gamma(N+\gamma_1+\gamma_2+2)}{\Gamma(N+\gamma_1+\gamma_2)}\mathcal{J}_{(2)}^{(\gamma_1,\gamma_2)} \\
&\quad - \frac{(2N+\gamma_1+\gamma_2+1)}{\Gamma(2N+\gamma_1+\gamma_2+3)}\frac{\Gamma(N+\gamma_1+\gamma_2+1)}{\Gamma(N+\gamma_1+\gamma_2)}\frac{\Gamma(N+\gamma_1+2)}{\Gamma(N+\gamma_1+1)}\mathcal{J}_{(1)}^{(\gamma_1,\gamma_2)} \\
&\quad + \frac{1}{2}\frac{(2N+\gamma_1+\gamma_2-1)}{\Gamma(2N+\gamma_1+\gamma_2+2)}\frac{\Gamma(N+\gamma_1+2)}{\Gamma(N+\gamma_1)}\mathcal{J}_{0}^{(\gamma_1,\gamma_2)}\Bigg] \\
S_{(1^2)} &= (2N+\gamma_1+\gamma_2-1)(2N+\gamma_1+\gamma_2-3)\frac{N!}{(N-2)!}\prod_{j=0}^{N-1}(-1)^j\frac{j!\Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+1)} \\
&\times \Bigg[\frac{(2N+\gamma_1+\gamma_2+1)(2N+\gamma_1+\gamma_2-1)}{\Gamma(2N+\gamma_1+\gamma_2+2)\Gamma(2N+\gamma_1+\gamma_2)}\frac{\Gamma(N+\gamma_1+\gamma_2+1)}{\Gamma(N+\gamma_1+\gamma_2-1)}\mathcal{J}_{(1^2)}^{(\gamma_1,\gamma_2)} \\
&\quad - \frac{(2N+\gamma_1+\gamma_2+1)(2N+\gamma_1+\gamma_2-3)}{\Gamma(2N+\gamma_1+\gamma_2+2)\Gamma(2N+\gamma_1+\gamma_2-1)}\frac{\Gamma(N+\gamma_1+\gamma_2+1)}{\Gamma(N+\gamma_1+\gamma_2)}\frac{\Gamma(N+\gamma_1)}{\Gamma(N+\gamma_1-1)}\mathcal{J}_{(1)}^{(\gamma_1,\gamma_2)} \\
&\quad + \frac{(2N+\gamma_1+\gamma_2-1)(2N+\gamma_1+\gamma_2-3)}{\Gamma(2N+\gamma_1+\gamma_2+1)\Gamma(2N+\gamma_1+\gamma_2-1)}\frac{\Gamma(N+\gamma_1+1)}{\Gamma(N+\gamma_1-1)}\mathcal{J}_{0}^{(\gamma_1,\gamma_2)} \\
&\quad - \frac{1}{2}\frac{(2N+\gamma_1+\gamma_2-1)(2N+\gamma_1+\gamma_2-3)}{\Gamma(2N+\gamma_1+\gamma_2)^2}\frac{\Gamma(N+\gamma_1+1)}{\Gamma(N+\gamma_1-1)}\mathcal{J}_{0}^{(\gamma_1,\gamma_2)}\Bigg]
\end{aligned}
$$

3.6 Correlation functions of characteristic polynomials

The main tool to compute the correlations of characteristic polynomials and the spectral moments is Lemma. 3.2.1, which was proved in Prop. 2.1.26 of Ch. 2. Recall

$$
\prod_{i=1}^{p}\prod_{j=1}^{q}(t_i - x_j) = \sum_{\lambda\subseteq(q^p)} (-1)^{|\tilde{\lambda}|}\Phi_\lambda(t_1,\ldots,t_p)\Phi_{\tilde{\lambda}}(x_1,\ldots,x_q). \tag{3.6.1}
$$

When the polynomials $\varphi_j(x)$ in (2.1.143) are not monic, we have the following identity for generalised dual Cauchy identity.

Proposition 3.6.1. *If A_j are the leading coefficients of $\varphi_j(x)$, then*

$$\prod_{i=1}^{p}\prod_{j=1}^{q}(t_i - x_j) = \prod_{j=0}^{p+q-1} A_j^{-1} \sum_{\lambda\subseteq(q^p)} (-1)^{|\tilde{\lambda}|}\Phi_\lambda(\boldsymbol{t})\Phi_{\tilde{\lambda}}(\boldsymbol{x}), \tag{3.6.2}$$

where $\tilde{\lambda} = (p - \lambda'_q, \ldots, p - \lambda'_1)$.

The proposition can be proved in a similar way to Prop. 2.1.26 by using (3.5.5) instead of (3.5.1) after (2.1.165).

Proof of Thm. 3.2.2. Unlike Hermite polynomials, the univariate Laguerre and Jacobi polynomials that obey (3.2.4) are not monic. This fact is reflected in the normalisation in (3.4.2) and also in the following formulae,

$$\begin{aligned}
\prod_{i=1}^{p}\prod_{j=1}^{N}(t_i - x_j) &= \sum_{\lambda\subseteq(N^p)} (-1)^{|\tilde{\lambda}|}\mathcal{H}_\lambda(t_1,\ldots,t_p)\mathcal{H}_{\tilde{\lambda}}(x_1,\ldots,x_N)\\
\prod_{i=1}^{p}\prod_{j=1}^{N}(t_i - x_j) &= \left(\prod_{j=0}^{p+N-1}(-1)^j j!\right) \sum_{\lambda\subseteq(N^p)} (-1)^{|\tilde{\lambda}|}\mathcal{L}_\lambda^{(\gamma)}(t_1,\ldots,t_p)\mathcal{L}_{\tilde{\lambda}}^{(\gamma)}(x_1,\ldots,x_N)\\
\prod_{i=1}^{p}\prod_{j=1}^{N}(t_i - x_j) &= \left(\prod_{j=0}^{p+N-1}(-1)^j j!\frac{\Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+1)}\right)\\
&\quad\times \sum_{\lambda\subseteq(N^p)} (-1)^{|\tilde{\lambda}|}\mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}(t_1,\ldots,t_p)\mathcal{J}_{\tilde{\lambda}}^{(\gamma_1,\gamma_2)}(x_1,\ldots,x_N)
\end{aligned} \tag{3.6.3}$$

After taking the expectation value, the non-zero contribution comes from $\tilde{\lambda} = 0$ because of (3.2.1). Therefore, $\lambda' = (p^N)$ which implies $\lambda = (N^p)$. It remains now to evaluate the multivariate polynomials at the zero partition. Since Hermite polynomials are monic, by using Prop. 3.5.1,

$$\mathcal{H}_0(x_1,\ldots,x_N) = \frac{1}{\Delta(\mathbf{x})}\det[H_{N-j}(x_k)]_{1\le j,k\le N} = 1. \tag{3.6.4}$$

On the other hand, the leading coefficients of Laguerre and Jacobi polynomials of degree j are

$$\frac{(-1)^j}{j!}, \text{ and } \frac{(-1)^j}{j!}\frac{\Gamma(2j+\gamma_1+\gamma_2+1)}{\Gamma(j+\gamma_1+\gamma_2+1)}, \tag{3.6.5}$$

respectively. Now, using Prop. 3.5.2 gives

$$\mathcal{L}_0^{(\gamma)}(x_1,\ldots,x_N) = \prod_{j=0}^{N-1}\frac{(-1)^j}{j!}, \tag{3.6.6}$$

$$\mathcal{J}_0^{(\gamma_1,\gamma_2)}(x_1,\ldots,x_N) = \prod_{j=0}^{N-1}\frac{(-1)^j}{j!}\frac{\Gamma(2j+\gamma_1+\gamma_2+1)}{\Gamma(j+\gamma_1+\gamma_2+1)}. \tag{3.6.7}$$

Inserting (3.6.4), (3.6.6) and (3.6.7) in (3.6.3) proves the result. ■

Corollary 3.6.2. *Let $\lambda = (N^p)$. If $t_i = t$ in Thm. 3.2.2, then*

$$\begin{aligned}
\mathbb{E}_N^{(H)}\left[(\det(t-M))^p\right] &= C_\lambda(p)\sum_{\nu\subseteq\lambda}\left(\frac{-1}{2}\right)^{\frac{|\lambda|-|\nu|}{2}}\frac{\dim V_\nu}{|\nu|!}D_{\lambda\nu}^{(H)}t^{|\nu|},\\
\mathbb{E}_N^{(L)}\left[(\det(t-M))^p\right] &= (-1)^{p(p+N-1)}G_\lambda(p,\gamma)\frac{G_\lambda(p,0)}{G_0(p,0)}\sum_{\nu\subseteq\lambda}\frac{(-1)^{|\nu|}}{|\nu|!\,G_\nu(p,\gamma)}\dim V_\nu D_{\lambda\nu}^{(L)}t^{|\nu|},\\
\mathbb{E}_N^{(J)}\left[(\det(t-M))^p\right] &= \left(\prod_{j=N}^{p+N-1}\frac{1}{\Gamma(2j+\gamma_1+\gamma_2+1)}\right)(-1)^{p(p+N-1)}\frac{G_\lambda(p,\gamma_1)G_\lambda(p,0)}{G_0(p,0)}\\
&\quad\times\sum_{\nu\subseteq\lambda}\frac{(-1)^{|\nu|}}{|\nu|!\,G_\nu(p,\gamma_1)}\dim V_\nu\tilde{\mathcal{D}}_{\lambda\nu}^{(J)}t^{|\nu|},
\end{aligned}\tag{3.6.8}$$

where $\dim V_\nu$ *is given in* (3.5.11).

Proof. Since Schur polynomials are homogeneous,

$$S_\nu(\underbrace{t,\dots,t}_{p}) = t^{|\nu|}S_\nu(\underbrace{1,\dots,1}_{p}) = \frac{\dim\,V_\nu}{|\nu|!}C_\nu(p)\,t^{|\nu|}.\tag{3.6.9}$$

First, consider the Gaussian ensemble. We have

$$\mathbb{E}_N^{(H)}\left[(\det(t-M))^p\right] = \mathcal{H}_{(N^p)}(t^p).\tag{3.6.10}$$

Using (3.5.34) and calculating C_λ for $\lambda = (N^p)$,

$$C_{(N^p)}(p) = \prod_{j=1}^{p}\frac{(N+p-j)!}{(p-j)!},\tag{3.6.11}$$

proves the statement. Similarly, the Laguerre and Jacobi cases can be computed in a similar way by using

$$G_\lambda(p,0) = C_\lambda(p)G_0(p,0).\tag{3.6.12}$$

■

The Cauchy identity can be written as

$$\prod_{i=1}^{q}\prod_{j=1}^{N}\frac{1}{(T_i-x_j)} = \frac{1}{\prod_{j=1}^{q}T_j^N}\sum_\lambda\sum_{\mu\subseteq\lambda}\psi_{\lambda\mu}S_\lambda(T_1^{-1},\dots,T_q^{-1})\Phi_\mu(x_1,\dots,x_N),\tag{3.6.13}$$

where Φ_μ is one of the generalised polynomials $\mathcal{H}_\mu$, $\mathcal{L}_\mu^{(\gamma)}$ or $\mathcal{J}_\mu^{(\gamma_1,\gamma_2)}$. By using orthogonality of multivariate polynomials (3.5.8), (3.5.42) and (3.5.54), we have the following proposition.

Proposition 3.6.3. *Let $t_1, \dots, t_p$ and $T_1, \dots, T_q$ be two sets of variables. Then*

$$\begin{aligned}
\prod_{j=1}^{p}\prod_{k=1}^{q}\mathbb{E}_N^{(H)}\left[\frac{\det(t_j-M)}{\det(T_k-M)}\right] &= \prod_{j=1}^{q}\frac{1}{T_j^N}\sum_{\substack{\lambda\subseteq(N^p)\\ s.t.\ \tilde{\lambda}=\nu}}\sum_{\mu}\sum_{\nu\subseteq\mu}\frac{(-1)^{|\nu|}}{2^{\frac{|\mu|-|\nu|}{2}}}C_\mu(N)D_{\mu\nu}^{(H)}\mathcal{H}_\lambda(\boldsymbol{t})S_\mu(\boldsymbol{T}^{-1})\\
\prod_{j=1}^{p}\prod_{k=1}^{q}\mathbb{E}_N^{(L)}\left[\frac{\det(t_j-M)}{\det(T_k-M)}\right] &= \prod_{j=N}^{p+N-1}(-1)^j j!\prod_{k=1}^{q}\frac{1}{T_k^N}\\
&\quad\times\sum_{\substack{\lambda\subseteq(N^p)\\ s.t.\ \tilde{\lambda}=\nu}}\sum_{\mu}\sum_{\nu\subseteq\mu}\frac{G_\mu(N,\gamma)}{G_0(N,\gamma)}\frac{C_\mu(N)}{C_\nu(N)}D_{\mu\nu}^{(L)}\mathcal{L}_\lambda^{(\gamma)}(\boldsymbol{t})S_\mu(\boldsymbol{T}^{-1})\\
\prod_{j=1}^{p}\prod_{k=1}^{q}\mathbb{E}_N^{(J)}\left[\frac{\det(t_j-M)}{\det(T_k-M)}\right] &= \prod_{j=N}^{p+N-1}(-1)^j j!\frac{\Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+1)}\prod_{k=0}^{N-1}\Gamma(2k+\gamma_1+\gamma_2+2)\prod_{l=1}^{q}\frac{1}{T_l^N}\\
&\quad\times\sum_{\substack{\lambda\subseteq(N^p)\\ s.t.\ \tilde{\lambda}=\nu}}\sum_{\mu}\sum_{\nu\subseteq\mu}\frac{G_\mu(N,\gamma_1)}{G_0(N,\gamma_1)}\frac{G_\nu(N,\gamma_2)}{G_0(N,\gamma_2)}\frac{C_\mu(N)}{C_\nu(N)}\mathcal{D}_{\mu\nu}^{(J)}\mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}(\boldsymbol{t})S_\mu(\boldsymbol{T}^{-1})
\end{aligned} \tag{3.6.14}$$

Note that the RHS is a formal power series in the variables T.

3.6.1 Moments of Schur polynomials

Gaussian ensemble. Similar to the moments of monomials with respect to the Gaussian weight,

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}x^{2n}e^{-\frac{x^2}{2}}\,dx = (-1)^n H_{2n}(0) = \frac{2n!}{2^n n!},\\
&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}x^{2n+1}e^{-\frac{x^2}{2}}\,dx = 0,
\end{aligned} \tag{3.6.15}$$

the moments of Schur polynomials associated to a partition λ are given by

$$\mathbb{E}_N^{(H)}[S_\lambda] = \begin{cases}(-1)^{\frac{|\lambda|}{2}}\mathcal{H}_\lambda(0^N), & |\lambda| \text{ is even},\\ 0, & |\lambda| \text{ is odd},\end{cases} \tag{3.6.16}$$

where

$$H_\lambda(0^N) = \frac{(-1)^{\frac{|\lambda|}{2}}}{2^{\frac{|\lambda|}{2}}\frac{|\lambda|}{2}!}C_\lambda(N)\chi^\lambda_{(2^{|\lambda|/2})}. \tag{3.6.17}$$

This can be easily seen from (3.5.15), (3.5.27) and (3.5.34) by observing that $S_\lambda = 1$ for $\lambda = ()$, and $S_\lambda(0^N) = 0$ for any non-empty partition λ. Using (3.5.8), $\mathbb{E}_N^{(H)}[S_\lambda]$ is a polynomial in N with integer roots given by the content of λ whenever $\chi^\lambda_{2^{|\lambda|/2}}$ is non-zero. Below, we give a few

examples of the moments of Schur polynomials corresponding to partitions of 4:

$$\begin{aligned}
\mathbb{E}_N^{(H)}\left[S_{(4)}\right] &= \frac{1}{8}N(N+1)(N+2)(N+3) \\
\mathbb{E}_N^{(H)}\left[S_{(3,1)}\right] &= -\frac{1}{8}(N-1)N(N+1)(N+2) \\
\mathbb{E}_N^{(H)}\left[S_{(2,2)}\right] &= \frac{1}{4}(N-1)N^2(N+1) \\
\mathbb{E}_N^{(H)}\left[S_{(2,1,1)}\right] &= -\frac{1}{8}(N-2)(N-1)N(N+1) \\
\mathbb{E}_N^{(H)}\left[S_{(1^4)}\right] &= \frac{1}{8}(N-3)(N-2)(N-1)N
\end{aligned} \tag{3.6.18}$$

Laguerre ensemble. The univariate moments are

$$\frac{1}{\Gamma(\gamma+1)}\int_0^\infty x^{n+\gamma}e^{-x}\,dx = \frac{\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)} = n!L_n^{(\gamma)}(0). \tag{3.6.19}$$

The moments of the Schur polynomials with respect to the Laguerre weight can be computed using (3.5.43),

$$\begin{aligned}
\mathbb{E}_N^{(L)}[S_\lambda] &= \frac{C_\lambda(N)}{|\lambda|!}\frac{G_\lambda(N,\gamma)}{G_0(N,\gamma)}\chi^\lambda_{(1^{|\lambda|})} \\
&= (-1)^{\frac{N(N-1)}{2}}G_\lambda(N,0)\mathcal{L}_\lambda^{(\gamma)}(0^N).
\end{aligned} \tag{3.6.20}$$

Like in the the Hermite case, $\mathbb{E}_N^{(L)}(S_\lambda)$ are polynomials in N with roots $i-j$ and $i-j-\gamma$, where $(i,j)\in\lambda$ as discussed in Sec. 3.5.

A few examples are

$$\begin{aligned}
\mathbb{E}_N^{(L)}\left[S_{(4)}\right] &= \frac{1}{24}\frac{(N+3)!}{(N-1)!}\frac{\Gamma(N+\gamma+4)}{\Gamma(N+\gamma)} \\
\mathbb{E}_N^{(L)}\left[S_{(3,1)}\right] &= \frac{1}{8}\frac{(N+2)!}{(N-2)!}\frac{\Gamma(N+\gamma+3)}{\Gamma(N+\gamma-1)} \\
\mathbb{E}_N^{(L)}\left[S_{(2,2)}\right] &= \frac{1}{12}\frac{(N+1)!N!}{(N-1)!(N-2)!}\frac{\Gamma(N+\gamma+2)\Gamma(N+\gamma+1)}{\Gamma(N+\gamma)\Gamma(N+\gamma-1)} \\
\mathbb{E}_N^{(L)}\left[S_{(2,1,1)}\right] &= \frac{1}{8}\frac{(N+1)!}{(N-3)!}\frac{\Gamma(N+\gamma+2)}{\Gamma(N+\gamma-2)} \\
\mathbb{E}_N^{(L)}\left[S_{(1^4)}\right] &= \frac{1}{24}\frac{N!}{(N-4)!}\frac{\Gamma(N+\gamma+1)}{\Gamma(N+\gamma-3)}
\end{aligned} \tag{3.6.21}$$

Jacobi ensemble. We have

$$\begin{aligned}
\int_0^1 x^{n+\gamma_1}(1-x)^{\gamma_2}\,dx &= n!\frac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}{\Gamma(n+\gamma_1+\gamma_2+2)}J_n^{(\gamma_1,\gamma_2)}(0) \\
&= \frac{\Gamma(n+\gamma_1+1)\Gamma(\gamma_2+1)}{\Gamma(n+\gamma_1+\gamma_2+2)}.
\end{aligned} \tag{3.6.22}$$

Similarly,

$$\begin{aligned}\mathbb{E}_N^{(J)}[S_\lambda] &= \frac{G_\lambda(N,\gamma_1)}{G_0(N,\gamma_1)} C_\lambda(N) D_{\lambda 0}^{(J)} \\ &= (-1)^{\frac{N(N-1)}{2}} \frac{D_{\lambda 0}^{(J)}}{\tilde{\mathcal{D}}_{\lambda 0}^{(J)}} G_\lambda(N,\gamma_1+\gamma_2) G_\lambda(N,0) \mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}(0^N),\end{aligned} \tag{3.6.23}$$

where

$$\mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}(0^N) = (-1)^{\frac{N(N-1)}{2}} \frac{G_\lambda(N,\gamma_1)}{G_\lambda(N,\gamma_1+\gamma_2) G_0(N,\gamma_1) G_0(N,0)} \tilde{\mathcal{D}}_{\lambda 0}^{(J)}, \tag{3.6.24}$$

and $D_{\lambda 0}^{(J)}$ and $\tilde{\mathcal{D}}_{\lambda 0}^{(J)}$ are given in (3.2.10) and (3.5.60), respectively.

The following are a few examples:

$$\begin{aligned}\mathbb{E}_N^{(J)}\left[S_{(4)}\right] &= \frac{1}{24} \frac{(N+3)!}{(N-1)!} \frac{\Gamma(N+\gamma_1+4)}{\Gamma(N+\gamma_1)} \frac{\Gamma(2N+\gamma_1+\gamma_2)}{\Gamma(2N+\gamma_1+\gamma_2+4)} \\ \mathbb{E}_N^{(J)}\left[S_{(3,1)}\right] &= \frac{1}{8} \frac{(N+2)!}{(N-2)!} \frac{\Gamma(N+\gamma_1+3)}{\Gamma(N+\gamma_1-1)} \frac{\Gamma(2N+\gamma_1+\gamma_2-1)}{\Gamma(2N+\gamma_1+\gamma_2+3)} \\ \mathbb{E}_N^{(J)}\left[S_{(2,2)}\right] &= \frac{1}{12} \frac{(N+1)!N!}{(N-1)!(N-2)!} \frac{\Gamma(N+\gamma_1+2)\Gamma(N+\gamma_1+1)}{\Gamma(N+\gamma_1)\Gamma(N+\gamma_1-1)} \\ &\quad \times \frac{\Gamma(2N+\gamma_1+\gamma_2-2)\Gamma(2N+\gamma_1+\gamma_2-3)}{\Gamma(2N+\gamma_1+\gamma_2+2)\Gamma(2N+\gamma_1+\gamma_2+1)} \\ \mathbb{E}_N^{(J)}\left[S_{(2,1,1)}\right] &= \frac{1}{8} \frac{(N+1)!}{(N-3)!} \frac{\Gamma(N+\gamma_1+2)}{\Gamma(N+\gamma_1-2)} \frac{\Gamma(2N+\gamma_1+\gamma_2-2)}{\Gamma(2N+\gamma_1+\gamma_2+2)} \\ \mathbb{E}_N^{(J)}\left[S_{(1^4)}\right] &= \frac{1}{24} \frac{N!}{(N-4)!} \frac{\Gamma(N+\gamma_1+1)}{\Gamma(N+\gamma_1-3)} \frac{\Gamma(2N+\gamma_1+\gamma_2-3)}{\Gamma(2N+\gamma_1+\gamma_2+1)}\end{aligned} \tag{3.6.25}$$

3.7 Joint moments of traces

Recently, the study of moments and joint moments of Hermitian ensembles have attracted considerable interest [57, 58, 75, 127]. Here we give new and self contained formulae for the joint moments of unitary ensembles in terms of the characters of the symmetric group. We focus on the GUE but exactly the same method applies to the LUE and JUE.

Using (2.1.82) and (3.5.15), power sum symmetric polynomials can be written in terms of multivariate Hermite polynomials

$$P_\mu = \sum_\lambda \sum_{\nu \subseteq \lambda} \chi_\mu^\lambda \psi_{\lambda\nu}^{(H)} \mathcal{H}_\nu. \tag{3.7.1}$$

Proof of Thm. 3.2.3. When $|\mu|$ is odd P_μ is a sum of product of monomials in x_i with the degree of at least one x_i being odd. Since the generalised weight $\Delta_N^2(\mathbf{x}) \prod_{i=1}^N e^{-\frac{x_i^2}{2}}$ is an even function and $P_\mu(\mathbf{x})$ is odd, $\mathbb{E}_N^{(H)}[P_\mu]$ vanishes.

When $|\mu|$ is even, writing P_μ in terms of multivariate Hermite polynomials (3.7.1) and using the orthogonality of $\mathcal{H}_\nu$ along with (3.5.27) proves the first line of (3.2.7). ■

Corollary 3.7.1. *Correlators of traces in the L.H.S. of (3.2.7) are either even or odd polynomials in N. More precisely, we have*

$\mathbb{E}_N^{(H)}[P_\mu]$	$l(\mu)$	$\lvert\mu\rvert/2$
Even polynomial	*even*	*even*
	odd	*odd*
Odd polynomial	*even*	*odd*
	odd	*even*

Proof. Let $|\mu|$ be even. Since $\mathbb{E}_N^{(H)}[S_\mu]$ is a polynomial in N of degree $|\mu|$ and the characters χ_λ^μ are integers, $\mathbb{E}_N^{(H)}[P_\lambda]$ is also a polynomial in N. Now for any partitions λ and μ,

$$\begin{aligned} \chi_\mu^{\lambda'} &= (-1)^{|\mu|-l(\mu)}\chi_\mu^\lambda, \\ C_{\mu'}(N) &= C_\mu(-N). \end{aligned} \tag{3.7.2}$$

Thus,

$$\begin{aligned} \mathbb{E}_N^{(H)}[P_\mu] &= \frac{1}{2}\sum_\lambda \left(\chi_\mu^\lambda \mathbb{E}_N^{(H)}[S_\lambda] + \chi_\mu^{\lambda'}\mathbb{E}_N^{(H)}[S_{\lambda'}]\right) \\ &= \frac{1}{2^{\frac{|\mu|+2}{2}}\frac{|\mu|}{2}!}\sum_\lambda \chi^\lambda_{(2^{|\lambda|/2})}\chi_\mu^\lambda \left(C_\lambda(N) + (-1)^{\frac{|\mu|}{2}-l(\mu)}C_\lambda(-N)\right). \end{aligned} \tag{3.7.3}$$

The corollary is proved by noticing that the symmetric and anti-symmetric combination of $C_\lambda(N)$ and $C_\lambda(-N)$ is an even and odd polynomial in N, respectively. ■

Remark 3.7.2. *When $|\mu|$ is even, the orthogonality of characters indicate that $\mathbb{E}_N^{(H)}[(\operatorname{Tr} M^2)^{\frac{|\mu|}{2}}]$ is a polynomial in N of degree $|\mu|$. The polynomial degree of all other joint moments corresponding to partitions of $|\mu|$ is strictly less than $|\mu|$.*

Since $\mathbb{E}_N^{(H)}[P_\mu]$ are polynomials in N, the domain of N can be analytically continued from integers to the whole complex plane. In [58], it is shown that $\mathbb{E}_N^{(H)}[\operatorname{Tr} M^{2j}]$, $j \in \mathbb{N}$, are Meixner-Pollaczek polynomials,

$$\mathbb{E}_N^{(H)}\left[\operatorname{Tr} M^{2j}\right] = N(2j-1)!!i^{-j}\frac{1}{j+1}P_j^{(1)}\left(iN, \frac{\pi}{2}\right) \tag{3.7.4}$$

$$= N(2j-1)!!{}_2F_1\left(\begin{matrix}-j, 1-N \\ 2\end{matrix}; 2\right), \tag{3.7.5}$$

where $P_k^{(1)}(iN, \pi/2)$ is a Meixner-Pollaczek polynomial that satisfies

$$\int_{-\infty}^{\infty} P_m^{(\lambda)}(x,\phi)\, P_n^{(\lambda)}(x,\phi)\, |\Gamma(\lambda+ix)|^2 e^{(2\phi-\pi)x}\, dx = \frac{2\pi\Gamma(n+2\lambda)}{(2\sin\phi)^{2\lambda}n!}\delta_{nm},\ \lambda>0,\ 0<\phi<\pi. \tag{3.7.6}$$

Here ${}_2F_1(\dots)$ is a terminating hypergeometric series in the standard notation,

$${}_pF_q\left(\begin{matrix}a_1 \dots & a_p \\ b_1 \ \dots & b_q\end{matrix}; x\right) = \sum_{j=0}^{\infty}\frac{(a_1)_j\dots(a_p)_j}{(b_1)_j\dots(b_q)_j}\frac{x^j}{j!}, \tag{3.7.7}$$

where $(q)_n = \Gamma(q+n)/\Gamma(q)$. From (3.7.4) and (3.7.6) it can be seen that the zeros of $\mathbb{E}_N^{(H)}\left[\operatorname{Tr} M^{2j}\right]$ lie on the line $\operatorname{Re}(N) = 0$.

Correlators of traces are combinatorial objects as they are connected to enumeration of ribbon graphs [32, 143, 228]. This connection is briefly discussed in App. A. By counting ribbon graphs, it can be easily seen that

$$\begin{aligned}\mathbb{E}_N^{(H)}\left[\operatorname{Tr} M^{2k-1}\operatorname{Tr} M\right] &= (2k-1)\mathbb{E}_N^{(H)}[\operatorname{Tr} M^{2k-2}] \\ &= N(2k-1)!!i^{-k+1}\frac{1}{k}P_{k-1}^{(1)}\left(iN,\frac{\pi}{2}\right).\end{aligned} \tag{3.7.8}$$

Therefore, $\mathbb{E}_N^{(H)}[P_\mu]$, $\mu = (2k-1,1)$, is also a polynomial in N with roots on the line $\operatorname{Re}(N) = 0$. But in general, this phenomenon is not observed for all partitions μ i.e. the zeros of $\mathbb{E}_N^{(H)}[P_\mu]$ doesn't lie on the imaginary axis for any partition. For example,

$$\mathbb{E}_N^{(H)}[\operatorname{Tr} M^8 \operatorname{Tr} M^6] = 5N(14N^8 + 578N^6 + 6881N^4 + 16170N^2 + 3384) \tag{3.7.9}$$

whose zeros are not on the line $\operatorname{Re}(N) = 0$.

In Table. 3.1, we give the moments of traces of the GUE corresponding to the first 8 partitions. The Fig. 3.2 shows the roots of these polynomials corresponding to partitions of 6. The moments $\mathbb{E}_N^{(L)}[\operatorname{Tr} M^j]$ of the Laguerre ensemble can be expressed as continuous dual Hahn polynomials [58]. For the Jacobi ensemble, the moments $\mathbb{E}_N^{(J)}[\operatorname{Tr} M^j]$ are not polynomials in N. But by treating j as a complex number, $\mathbb{E}_N^{(J)}[\operatorname{Tr} M^j]$ can be written as a Wilson polynomial [58], which is a hypergeometric orthogonal polynomial. In Table. 3.2 and Table. 3.3, we give mixed moments of traces for the LUE and the JUE.

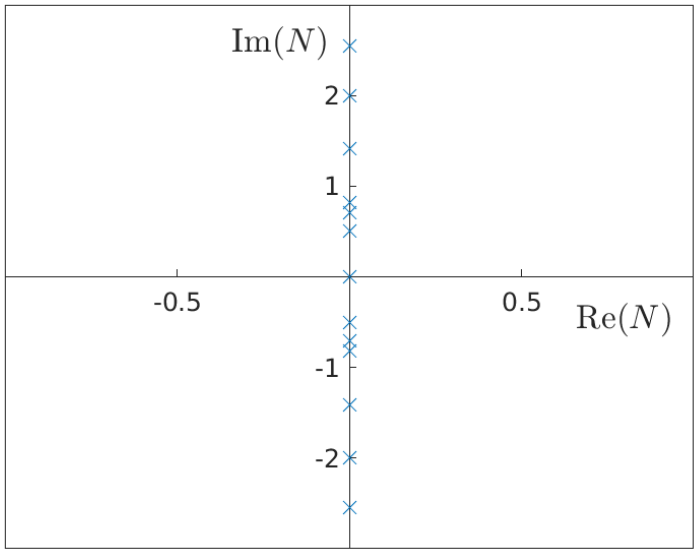
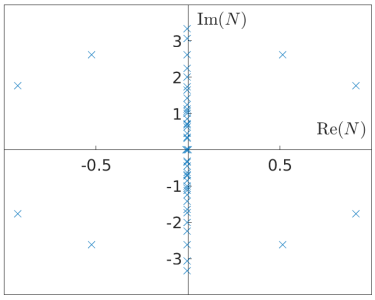
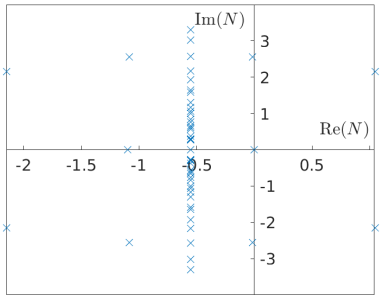


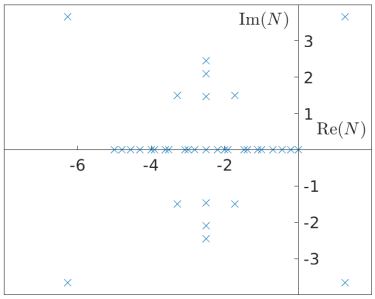
Figure 3.2: Zeros of polynomials $\mathbb{E}_N^{(H)}[P_\mu(M)]$ when μ runs over all partitions of 6. When μ is a partition of a larger integer, the zeros of $\mathbb{E}_N^{(H)}[P_\mu(M)]$ tend to deviate from the line $\operatorname{Re}(N) = 0$.



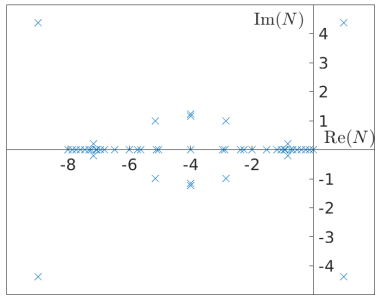
(a) $\gamma = 0.01$



(b) $\gamma = 1.1$



(c) $\gamma = 5$



(d) $\gamma = 8$

Figure 3.3: Zeros of $\mathbb{E}_N^{(L)}[P_\mu(M)]$ for different values of γ when μ runs over all partitions of 6.

P_μ	$\mathbb{E}_N^{(H)}[P_\mu]$
p_2	N^2
p_1^2	N
p_4	$N(2N^2+1)$
p_3p_1	$3N^2$
p_2^2	$N^2(N^2+2)$
$p_2p_1^2$	$N(N^2+2)$
p_1^4	$3N^2$
p_6	$5N^2(N^2+2)$

(To be continued)

P_μ	$\mathbb{E}_N^{(H)}[P_\mu]$
p_5p_1	$5N(2N^2+1)$
p_4p_2	$N(2N^2+1)(N^2+4)$
$p_4p_1^2$	$N^2(2N^2+13)$
p_3^2	$3N(4N^2+1)$
$p_3p_2p_1$	$3N^2(N^2+4)$
$p_3p_1^3$	$3N(3N^2+2)$
p_2^3	$N^2(N^2+2)(N^2+4)$
$p_2^2p_1^2$	$N(N^2+2)(N^2+4)$
$p_2p_1^4$	$3N^2(N^2+4)$
p_1^6	$15N^3$
p_8	$7N(2N^4+10N^2+3)$
p_7p_1	$35N^2(N^2+2)$
p_6p_2	$5N^2(N^2+2)(N^2+6)$
$p_6p_1^2$	$5N(N^4+14N^2+6)$
p_5p_3	$15N^2(3N^2+4)$
$p_5p_2p_1$	$5N(2N^4+13N^2+6)$
$p_5p_1^3$	$15N^2(2N^2+5)$
p_4^2	$N^2(4N^4+40N^2+61)$
$p_4p_3p_1$	$3N(2N^4+25N^2+8)$

(To be continued)

P_μ	$\mathbb{E}_N^{(H)}[P_\mu]$
$p_4p_2^2$	$N(2N^2+1)(N^2+4)(N^2+6)$
$p_4p_2p_1^2$	$N^2(N^2+6)(2N^2+13)$
$p_4p_1^4$	$3N(2N^4+25N^2+8)$
$p_3^2p_2$	$3N(4N^2+1)(N^2+6)$
$p_3^2p_1^2$	$15N^2(2N^2+5)$
$p_3p_2^2p_1$	$3N^2(N^2+4)(N^2+6)$
$p_3p_2p_1^3$	$3N(3N^2+2)(N^2+6)$
$p_3p_1^5$	$15N^2(3N^2+4)$
p_2^4	$N^2(N^2+2)(N^2+4)(N^2+6)$
$p_2^3p_1^2$	$N(N^2+2)(N^2+4)(N^2+6)$
$p_2^2p_1^4$	$3N^2(N^2+4)(N^2+6)$
$p_2p_1^6$	$15N^3(N^2+6)$
p_1^8	$105N^4$

Table 3.1: Mixed moments of traces of the GUE for the first 8 partitions. Only even partitions are listed since $\mathbb{E}_N^{(H)}[P_\mu(M)] = 0$ for odd $|\mu|$.

P_μ	$\mathbb{E}_N^{(L)}[P_\mu]$
p_1	$N(N+\gamma)$
p_2	$N(N+\gamma)(2N+\gamma)$
p_1^2	$N(N+\gamma)(N^2+\gamma N+1)$
p_3	$N(N+\gamma)(5N^2+5\gamma N+\gamma^2+1)$

(To be continued)

P_μ	$\mathbb{E}_N^{(L)}[P_\mu]$
p_2p_1	$N(N+\gamma)(2N+\gamma)(N^2+\gamma N+2)$
p_1^3	$N(N+\gamma)(N^2+\gamma N+1)(N^2+\gamma N+2)$
p_4	$N(N+\gamma)(2N+\gamma)(7N^2+7\gamma N+a^2+5)$
p_3p_1	$N(N+\gamma)(N^2+\gamma N+3)\left(5N^2+5\gamma N+a^2+1\right)$
p_2^2	$N(N+\gamma)\left(4N^4+8\gamma N^3+(5\gamma^2+18)N^2+\gamma(\gamma^2+18)N +4\gamma^2+2\right)$
$p_2p_1^2$	$N(N+\gamma)(2N+\gamma)(N^2+\gamma N+2)(N^2+\gamma N+3)$
p_1^4	$N(N+\gamma)(N^2+\gamma N+1)(N^2+\gamma N+2)(\gamma^2+\gamma N+3)$
p_5	$N(N+\gamma)\left(42N^4+84\gamma N^3+14(4\gamma^2+5)N^2 +14\gamma(\gamma^2+5)N+\gamma^4+15\gamma^2+8\right)$
p_4p_1	$N(N+\gamma)(2N+\gamma)(N^2+\gamma N+4)(7N^2+7\gamma N+\gamma^2+5)$
p_3p_2	$N(N+\gamma)(2N+\gamma)\left(5N^4+10\gamma N^3+(6\gamma^2+37)N^2 +\gamma(\gamma^2+37)N+6(\gamma^2+3)\right)$
$p_3p_1^2$	$N(N+\gamma)(N^2+\gamma N+3)(N^2+\gamma N+4)(5N^2+5\gamma N+\gamma^2+1)$
$p_2^2p_1$	$N(N+\gamma)(N^2+\gamma N+4)\left(4N^4+8\gamma N^3+(5\gamma^2+18)N^2 +\gamma(\gamma^2+18)N+2(2\gamma^2+1)\right)$
$p_2p_1^3$	$N(N+\gamma)(2N+\gamma)(N^2+\gamma N+2)(N^2+\gamma N+3)(N^2+\gamma N+4)$
p_1^5	$N(N+\gamma)(N^2+\gamma N+1)(N^2+\gamma N+2)(N^2+\gamma N+3) \times(N^2+\gamma N+4)$
p_6	$N(N+\gamma)(2N+\gamma)\left(66N^4+132\gamma N^3+42(2\gamma^2+5)N^2 +6\gamma(3\gamma^2+35)N+\gamma^4+35\gamma^2+84\right)$

(To be continued)

P_μ	$\mathbb{E}_N^{(L)}[P_\mu]$
p_5p_1	$N(N+\gamma)(N^2+\gamma N+5)\big(42N^4+84\gamma N^3+14(4\gamma^2+5)N^2$ $+14\gamma(\gamma^2+5)N+\gamma^4+15\gamma^2+8\big)$
p_4p_2	$N(N+\gamma)\big(28N^6+84\gamma N^5+5(19\gamma^2+60)N^4+50\gamma(\gamma^2+12)N^3$ $+3(4\gamma^4+135\gamma^2+120)N^2+\gamma(\gamma^4+105\gamma^2+360)N$ $+8(\gamma^4+10\gamma^2+4)\big)$
$p_4p_1^2$	$N(N+\gamma)(2N+\gamma)(N^2+\gamma N+4)(N^2+\gamma N+5)$ $\times(7N^2+7\gamma N+\gamma^2+5)$
p_3^2	$N(N+\gamma)\big(25N^6+75\gamma N^5+5(17\gamma^2+62)N^4$ $+5\gamma(9\gamma^2+124)N^3+(11\gamma^4+420\gamma^2+349)N^2$ $+\gamma(\gamma^4+110\gamma^2+349)N+3(3\gamma^4+25\gamma^2+12)\big)$
$p_3p_2p_1$	$N(N+\gamma)(2N+\gamma)(N^2+\gamma N+5)\big(5N^4+10\gamma N^3$ $+(6\gamma^2+37)N^2+\gamma(\gamma^2+37)N+6(\gamma^2+3)\big)$
$p_3p_1^3$	$N(N+\gamma)(N^2+\gamma N+3)(N^2+\gamma N+4)(N^2+\gamma N+5)$ $(5N^2+5\gamma N+\gamma^2+1)$
p_2^3	$N(N+\gamma)(2N+\gamma)\big(4N^6+12\gamma N^5+(13\gamma^2+54)N^4$ $+6\gamma(\gamma^2+18)N^3+(\gamma^4+66\gamma^2+222)N^2+6\gamma(2\gamma^2+37)N$ $+40(\gamma^2+2)\big)$
$p_2^2p_1^2$	$N(N+\gamma)(N^2+\gamma N+4)(N^2+\gamma N+5)\big(4N^4+8\gamma N^3$ $+(5\gamma^2+18)N^2+\gamma(\gamma^2+18)N+2(2\gamma^2+1)\big)$
$p_2p_1^4$	$N(N+\gamma)(2N+\gamma)(N^2+\gamma N+2)(N^2+\gamma N+3)$ $(N^2+\gamma N+4)(N^2+\gamma N+5)$
p_1^6	$N(N+\gamma)(N^2+\gamma N+1)(N^2+\gamma N+2)(N^2+\gamma N+3)$ $(N^2+\gamma N+4)(N^2+\gamma N+5)$

Table 3.2: Mixed moments of traces of the LUE for the first 6 partitions.

P_μ	$\mathbb{E}_N^{(J)}[P_\mu]$
p_1	$\dfrac{N(N+\gamma_1)}{2N+\gamma_1+\gamma_2}$
p_2	$N(N+\gamma_1)\dfrac{3N^2+(3\gamma_1+2\gamma_2)N+\gamma_1^2+\gamma_1\gamma_2-1}{(2N+\gamma_1+\gamma_2-1)(2N+\gamma_1+\gamma_2)(2N+\gamma_1+\gamma_2+1)}$
p_1^2	$N(N+\gamma_1)\dfrac{2N^3+(3\gamma_1+\gamma_2)N^2+(\gamma_1^2+\gamma_1\gamma_2)N+\gamma_2}{(2N+\gamma_1+\gamma_2-1)(2N+\gamma_1+\gamma_2)(2N+\gamma_1+\gamma_2+1)}$
p_3	$N(N+\gamma_1)\dfrac{\Gamma(2N+\gamma_1+\gamma_2-2)}{\Gamma(2N+\gamma_1+\gamma_2+3)}$ $\times\big(10N^4+2(10\gamma_1+7\gamma_2)N^3+((\gamma_1+\gamma_2)(16\gamma_1+5\gamma_2)-14)N^2$ $+(\gamma_1((\gamma_1+\gamma_2)(6\gamma_1+5\gamma_2)-14)-8\gamma_2)N+\gamma_1^4+2\gamma_1^3\gamma_2$ $+\gamma_1^2(\gamma_2^2-5)-4\gamma_1\gamma_2+\gamma_2^2+4\big)$
p_2p_1	$N(N+\gamma_1)\dfrac{\Gamma(2N+\gamma_1+\gamma_2-2)}{\Gamma(2N+\gamma_1+\gamma_2+3)}$ $\times\big(6N^5+(15\gamma_1+7\gamma_2)N^4+2(7\gamma_1^2+7\gamma_1\gamma_2+\gamma_2^2-3)N^3$ $+3(2\gamma_1^3+3\gamma_1^2\gamma_2+\gamma_1(\gamma_2^2-3)+\gamma_2)N^2$ $+(\gamma_1^4+2\gamma_1^3\gamma_2+\gamma_1^2(\gamma_2^2-3)+3\gamma_1\gamma_2+4\gamma_2^2)N$ $+2\gamma_2(\gamma_1^2+\gamma_1\gamma_2-2)\big)$
p_1^3	$N(N+\gamma_1)\dfrac{\Gamma(2N+\gamma_1+\gamma_2-2)}{\Gamma(2N+\gamma_1+\gamma_2+3)}$ $\times\big(4N^6+4(3\gamma_1+\gamma_2)N^5+(13\gamma_1^2+10\gamma_1\gamma_2+\gamma_2^2-2)N^4$ $+2(\gamma_1(\gamma_1+\gamma_2)(3\gamma_1+\gamma_2)-2)+3\gamma_2)N^3$ $+(\gamma_1^4+2\gamma_1^3\gamma_2+\gamma_1^2(\gamma_2^2-2)+9\gamma_1\gamma_2+3\gamma_2^2-2)N^2$ $+(\gamma_1(3\gamma_2(\gamma_1+\gamma_2)-2)-4\gamma_2)N+2\gamma_2(\gamma_2-\gamma_1))$

Table 3.3: Mixed moments of traces of the JUE for the first 3 partitions.

Chapter 4

Bounds in central limit theorem

This chapter is a part of the paper *Symmetric function theory and unitary invariant ensembles*, which is a joint work with J. P. Keating and F. Mezzadri. The present author entirely carried the project with the advisement from J. P. Keating and F. Mezzadri. We also thank Tamara Grava and Sergey Berezin for helpful discussions.

The last section of [165] is rephrased into this chapter with more details and additional examples. The notation has also been changed to be consistent with the rest of this thesis. All such changes and inclusions are due to the present author.

4.1 Introduction

In this chapter, we are interested in the global fluctuations of the spectra of Hermitian ensembles. To set it more clearly, we consider the GUE, a paradigmatic ensemble for random matrices. For a GUE matrix M of size N, consider the rescaled matrix $\mathcal{M} = M/\sqrt{4N}$ with j.p.d.f.

$$\frac{(4N)^{\frac{N^2}{2}}}{(2\pi)^{\frac{N}{2}} \prod_{j=1}^{N} j!} \prod_{1\le i<j\le N} (x_i - x_j)^2 \prod_{j=1}^{N} e^{-2Nx_j^2}. \tag{4.1.1}$$

This choice of scaling is to make the eigenvalue support compact in the limit $N \to \infty$. Note that the scaling is different from Ch. 1[1]. As a result, the asymptotic spectral density, namely the semi-circle law, is confined between -1 and 1 instead of -2 and 2:

$$\rho_{sc} = \frac{2}{\pi}\sqrt{1-x^2},\ -1 \le x \le 1. \tag{4.1.2}$$

Therefore, for a well-defined function g,

$$\frac{1}{N}\sum_{j=1}^{N} g(x_j) \to \frac{2}{\pi}\int_{-1}^{1} g(x)\sqrt{1-x^2}\,dx,\ \text{as } N \to \infty. \tag{4.1.3}$$

[1]Different scalings are chosen in each chapter to make our results consistent with the literature.

An interesting object to study is the linear statistic $\sum_j g(x_j)$ and its fluctuations around the semi-circle. In [157], Johansson proved that as $N \to \infty$, the centered random variable

$$\operatorname{Tr} g(\mathcal{M}) - \mathbb{E}_N^{(H)}[\operatorname{Tr} g(\mathcal{M})] \tag{4.1.4}$$

converges in distribution to a normal random variable with mean zero and variance that depends on g. Note the absence of $1/\sqrt{N}$ normalisation typically seen in the CLT in probability theory. This is due to very effective cancellations involved in (4.1.4).

Even for any finite N, normal random variables are closely connected to the GUE in the sense that the real and complex matrix entries are i.i.d. Gaussians. This close representation also appears in the large N limit as discussed above.

In Ch. 3, Sec. 3.3, we have seen that the correlators of traces of a Haar distributed unitary matrix are exactly equal to that of complex Gaussians. For $U \in U(N)$, the random variable TrU^j converges in distribution to $\sqrt{j}Z_j$, where Z_j are independent complex normal random variables. More generally, for a real-valued function g on the unit circle, the linear statistic $\operatorname{Tr} g(U)$ converges in distribution to a normal random variable. Alternatively, the fluctuations of $\operatorname{Tr} g(U)$ can also be studied using Szegő's theorem, which we state below.

Theorem 4.1.1 (Szegő [227])**.** *Let g be a continuous function on the unit circle with Fourier coefficients $\hat{g}_j$, $-\infty < j < \infty$. If $\sum_{j=-\infty}^{\infty} |j||\hat{g}_j|^2 < \infty$, then*

$$\mathbb{E}_{U(N)}[e^{\operatorname{Tr} g(U)}] = \exp\left(N\hat{g}_0 + \sum_{j=1}^{\infty} j\hat{g}_{-j}\hat{g}_j + o(1)\right), \text{ as } N \to \infty. \tag{4.1.5}$$

The above theorem can be proved by using the properties of Toeplitz determinants:

$$\mathbb{E}_{U(N)}\left[e^{\operatorname{Tr} g(U)}\right] = \mathbb{E}_{U(N)}\left[\prod_{j=1}^{N} e^{g(e^{i\theta_j})}\right] \tag{4.1.6}$$

which is a Toeplitz determinant with symbol e^g. For a proof of Szegő's theorem see one of the following [27, 128, 142, 166, 227]. For a proof of Szegő's theorem using symmetric functions, see [45]. The central limit theorem for $\operatorname{Tr} g(U)$ can be recovered from Szegő's theorem as follows.

Corollary 4.1.2. *For each $\xi \in \mathbb{R}$ and a real-valued function g such that $\hat{g}_0 = 0$ with*

$$A(g) = \sum_{j=1}^{\infty} j|\hat{g}_j|^2 < \infty, \tag{4.1.7}$$

we have

$$\mathbb{E}_{U(N)}[e^{i\xi \operatorname{Tr} g(U)}] \to \exp(-\xi^2 A(g)), \tag{4.1.8}$$

i.e. $\operatorname{Tr} g(U)$ converges in distribution to a normal random variable with mean 0 and variance $2A(g)$.

Consider the random variable

$$Z = \sum_{j=1}^{m} \xi_{2j-1}\sqrt{\frac{2}{j}}\,\mathrm{Re}\,\mathrm{Tr}\,U^j + \xi_{2j}\sqrt{\frac{2}{j}}\,\mathrm{Im}\,\mathrm{Tr}\,U^j = \mathrm{Tr}\,g(U) \tag{4.1.9}$$

where

$$g(e^{i\theta}) = \sum_{j=-m}^{m} c_j e^{ij\theta}, \tag{4.1.10}$$

with

$$c_j = \frac{1}{\sqrt{2j}}(\xi_{2j-1} - i\xi_{2j}), \quad c_{-j} = \overline{c_j}, \quad c_0 = 0. \tag{4.1.11}$$

As an immediate consequence of the Szegő's theorem, we have that the random variable e^{iZ} has the limit

$$\mathbb{E}_{U(N)}[e^{iZ}] \to \exp\left(-\frac{\|\xi\|^2}{2}\right) \tag{4.1.12}$$

as $N \to \infty$ for a fixed m. That is, if we denote

$$X_{2j-1} = \sqrt{\frac{2}{j}}\,\mathrm{Re}\,\mathrm{Tr}\,U^j,\ X_{2j} = \sqrt{\frac{2}{j}}\,\mathrm{Im}\,\mathrm{Tr}\,U^j,\ 1 \le j \le m, \tag{4.1.13}$$

then the vector $\mathbf{X} = (X_1, \ldots, X_{2m})$ converges in distribution to independent normal random variables as the matrix size goes to infinity,

$$\mathbf{X} = (X_1, \ldots, X_{2m}) \overset{d}{\Rightarrow} (r_1, \ldots, r_{2m}) = \mathbf{r}. \tag{4.1.14}$$

Here r_j are independent Gaussians with mean zero and variance 1, and $\overset{d}{\Rightarrow}$ means convergence in distribution.

The result in (4.1.14) also follows from the fact that joint moments of $X_1, \ldots, X_{2m}$ are equal to the joint moments of $r_1, \ldots, r_{2m}$ up to very high orders, see Ch. 3, Sec. 3.3. Due to this exact equality, Diaconis and Shashahani predicted that the random variable Z should converge very fast to the normal random variable $\mathcal{N}(0, \|\xi^2\|)$. Johansson showed that the rate of convergence is super-exponential by giving an estimate for the total variation distance,

$$d_{\mathrm{TV}}(Z, \mathcal{N}(0, \|\xi\|^2)) \le CN^{-\delta N} \tag{4.1.15}$$

for some constants C and δ which do not have any explicit expressions. More recently, Johansson and Gaultier [160] studied the multivariate rate of convergence of the vector $\mathbf{X}$ to $\mathbf{r}$ in the total variation distance as the variable m increases with N. In addition to showing that the rate is super-exponential, they also gave explicit expressions for the constants that appear in the CLT.

We now return to the random Hermitian matrices. Similar to the unitary group, for all sufficiently nice functions g, a version of the strong Szegő's theorem holds for Hermitian random matrices [157]. In particular for the GUE, we have the following theorem.

Theorem 4.1.3. *For a locally Hölder continuous function $g : \mathbb{R} \to \mathbb{R}$, $N \to \infty$,*

$$\log \mathbb{E}_N^{(H)} \left[\exp \left(\sum_j g(x_j) \right) \right] - N \int_{\mathbb{R}} g(t) \rho_{sc}(t)\, dt \to \frac{1}{8} \sum_{k=1}^{\infty} k a_k^2, \tag{4.1.16}$$

where

$$a_k = \frac{2}{\pi} \int_0^{\pi} g(\cos\theta) \cos k\theta \, d\theta \tag{4.1.17}$$

are the coefficients in the Chebyshev expansion of $g(t)$.

The proof involves variational formulae, the properties of orthogonal polynomials and tools from analysis. We do not discuss the proof here but interested readers can refer to [157].

If g in (4.1.16) is a real-valued function such that $A(g) = \frac{1}{8} \sum_{k=1}^{\infty} k a_k^2 < \infty$, then

$$\int \exp \left(i\xi \operatorname{Tr} \left(g(\mathcal{M}) - \int_{\mathbb{R}} g(t) \rho(t)\, dt \right) \right) d\mathcal{M} \to \exp(-\xi^2 A(g)) \tag{4.1.18}$$

as $N \to \infty$ for each $\xi \in \mathbb{R}$. Here $d\mathcal{M}$ is the uniform probability measure on the space of $N \times N$ rescaled Hermitian matrices, and $\rho(x)$ is the eigenvalue density. Analogous results holds other $\beta-$ensembles and for weight functions different from the Gaussian weight.

Remark 4.1.4. *It is worth mentioning that the variance $A(g)$ depends on the geometry involved. For the unitary group, we have geometry of the unit circle, and for the Hermitian ensembles we have the geometry of an interval.*

In particular, let the function g in (4.1.16) be Chebyshev polynomial of the first kind T_k of degree k. Chebyshev polynomials appear naturally in the GUE, more generally in Wigner random matrices: Chebyshev polynomials of the second type are orthogonal with respect to the semi-circle law. If

$$X_k = \operatorname{Tr} T_k(\mathcal{M}) - \mathbb{E}_N^{(H)}[\operatorname{Tr} T_k(\mathcal{M})], \quad k = 0, 1, \ldots, \tag{4.1.19}$$

then [157]

$$(X_1, \ldots, X_{2m}) \overset{d}{\Rightarrow} (\tfrac{1}{2} r_1, \ldots, \tfrac{\sqrt{2m}}{2} r_{2m}), \tag{4.1.20}$$

where r_j are independent standard normal random variables.

The central limit theorem for Hermitian random matrix ensembles has been the focus of numerous studies. For example see [28, 37, 101, 174, 176, 197, 201, 214, 215, 250] and references therein. The main tools used to prove the CLT in the case of Hermitian and Wigner ensembles have been orthogonal polynomial techniques, Riemann-Hilbert methods, the Stein's method etc. In this chapter, as an example of an application of the general approach taken in Ch. 3, we apply our results to establish explicit asymptotic formulae for the rate of convergence of the moments and cumulants of X_k to those of a standard normal distribution when the matrix size tends to infinity.

4.1.1 Statement of results

Define

$$\mathcal{E}_{n,k} := \mathbb{E}_N^{(H)}[X_k^n] - \left(\frac{\sqrt{k}}{2}\right)^n \mathbb{E}[r_k^n]. \tag{4.1.21}$$

The formalism that we developed to study moments of traces in Ch. 3 allow us to derive explicit estimates for the error $\mathcal{E}_{n,k}$ as a function of matrix size N. For the rescaled Gaussian matrices, the correlators of traces are Laurent polynomials in N. This fact can be seen from (3.2.7) when applied to rescaled matrices. Consequently, the moments of polynomial test functions are also Laurent polynomials in N. In particular, for the Chebyshev polynomials, we have the following theorem.

Theorem 4.1.5. *Fix $k \in \mathbb{N}$ and let $kn \leq N$. With the notation introduced above the following statements hold as $N \to \infty$.*

1. *For k odd and $k > 1$,*

$$\mathcal{E}_{n,k} = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ d_1(n,k)\frac{1}{N^2} + O\left(\frac{1}{N^4}\right), & \text{if } n \text{ is even}, \end{cases} \tag{4.1.22}$$

where, when $n \to \infty$ with k fixed,

$$\begin{aligned} d_1(n,k) << \; & A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{7nk}{8} - \frac{13n}{8} + \frac{n}{6k}} k^{\frac{3n}{8}(k+2) + \frac{n}{8} + \frac{n}{4k}} n^{\frac{3n}{8}(k+1) - \frac{k}{4} + \frac{7}{8}} \\ & \times e^{-\frac{n}{8}(k+1) + \frac{9n}{4} + \frac{5n}{8k} + \pi\sqrt{\frac{n}{3}(k+1)}}. \end{aligned} \tag{4.1.23}$$

When $k = 1$, $\mathcal{E}_{n,k} = 0$.

2. *For k even,*

$$\mathcal{E}_{n,k} = \begin{cases} d_2(n,k)\frac{1}{N} + O\left(\frac{1}{N^3}\right), & \text{if } n \text{ is odd}, \\ d_3(n,k)\frac{1}{N^2} + O\left(\frac{1}{N^4}\right), & \text{if } n \text{ is even}, \end{cases} \tag{4.1.24}$$

where, when $n \to \infty$ with k fixed,

$$\begin{aligned} d_2(n,k) << \; & A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{3nk}{8} - 3n + \frac{n}{6k}} k^{\frac{3nk}{8} + \frac{n}{2} + \frac{9n}{4k}} n^{\frac{3nk}{8} + \frac{2n}{k} - \frac{k}{2} - \frac{3}{8}} e^{-\frac{n}{8}(k-18) + \pi\sqrt{\frac{nk}{3}} - \frac{19n}{8k}}, \\ d_3(n,k) << \; & A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{3nk}{8} - 3n + \frac{n}{6k}} k^{\frac{3nk}{8} + \frac{n}{2} + \frac{9n}{4k}} n^{\frac{3nk}{8} + \frac{2n}{k} - \frac{k}{2} + \frac{5}{8}} e^{-\frac{n}{8}(k-18) + \pi\sqrt{\frac{nk}{3}} - \frac{19n}{8k}}. \end{aligned} \tag{4.1.25}$$

Here $A = 1.2824\ldots$ is the Glaisher–Kinkelin constant [50].

Along with the moments, we also give an estimate for the cumulants of random variables X_k. Computing cumulants from (3.2.7) is not straightforward. We instead employ the well established connection between correlators of traces and the enumeration of ribbon graphs to estimate the cumulants. The results are elaborated in Section 4.2.2.

To summarise, for a fixed n and k, we show that the nth moment of X_k converges to the nth moment of independent scaled Gaussian variable as N^{-1} or N^{-2} depending on the parity

of n; and the nth cumulant of X_k converges to 0 as N^{n-2} for $n > 2$. Theorem 4.1.5 provides explicit asymptotic estimates for the rate of convergence of the moments.

4.2 Eigenvalue fluctuations

4.2.1 Moments

Here we focus on the GUE but the Laguerre and Jacobi ensembles can be studied in a similar way.

Proposition 4.2.1. *We have*

$$\mathbb{E}_N^{(H)}\left[(\operatorname{Tr}\mathcal{M})^{2n}\right] = \frac{2n!}{2^{3n}n!}. \tag{4.2.1}$$

Proof. When $\mu = (1^{2n})$ in (3.2.7), using (3.5.10) and the fact that $\chi^{\lambda}_{(1^{2n})} = \dim V_\lambda$,

$$\mathbb{E}_N^{(H)}\left[(\operatorname{Tr}\mathcal{M})^{2n}\right] = \frac{2n!}{2^{3n}n!}\frac{1}{N^n}\sum_{\lambda\vdash 2n}\chi^{\lambda}_{(2^n)}S_\lambda(1^N). \tag{4.2.2}$$

Using (2.1.82) and $P_\nu(1^N) = N^{l(\nu)}$,

$$\mathbb{E}_N^{(H)}\left[(\operatorname{Tr}\mathcal{M})^{2n}\right] = \frac{2n!}{2^{3n}n!}\frac{1}{N^n}P_{(2^n)}(1^N) = \frac{2n!}{2^{3n}n!}. \tag{4.2.3}$$

■

The R.H.S. is the $2n^{th}$ moment of $r_1/2$ where $r_1 \sim \mathcal{N}(0,1)$. This exact equality of moments with the moments of Gaussian normals is special to $\mathbb{E}_N^{(H)}\left[(\operatorname{Tr}\mathcal{M})^{2n}\right]$. In general, one can consider moments of the form $\mathbb{E}_N^{(H)}[(\operatorname{Tr}\, g(\mathcal{M}))^n]$ for a well-defined function g.

Johansson [157] showed that when g is the Chebyshev polynomial of the first kind of degree k, the random variable

$$X_k = \operatorname{Tr} T_k(\mathcal{M}) - \mathbb{E}_N^{(H)}[\operatorname{Tr} T_k(\mathcal{M})], \quad k = 0, 1, \ldots, \tag{4.2.4}$$

converges in distribution to the Gaussian variable $\mathcal{N}(0, k/4)$. In this section we prove Theorem 4.1.5, which implies that

$$\mathbb{E}_N^{(H)}[X_k^n] = \left(\frac{\sqrt{k}}{2}\right)^n \frac{n!}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!}\eta_n + d(n,k)\frac{1}{N^{1+m_{k,n}}} + O(N^{-2}), \tag{4.2.5}$$

where $\eta_n = 1$ if n is even and 0 otherwise, and where $m_{k,n}$ is either 0 or 1, with asymptotic estimates for $d(n,k)$. Results for $k = 1$ are already discussed in Prop. 4.2.1. We first consider X_2 and discuss results for general values of k in Sec. 4.2.1.2.

4.2.1.1 Second degree

One sees that

$$\mathbb{E}_N^{(H)}[(\operatorname{Tr}\mathcal{M}^2)^n] = \frac{1}{(4N)^n}\prod_{j=0}^{n-1}(N^2+2j). \tag{4.2.6}$$

For a fixed n, this can be obtained by substituting in the character values of $\mathcal{S}_{2n}$ in (3.2.7). Alternatively, a proof by counting topologically invariant ribbon graphs is sketched in App. A. Clearly, the nth moment of X_2 is equal to

$$\begin{aligned}\mathbb{E}_N^{(H)}[X_2^n] &= \mathbb{E}_N^{(H)}\left[\left(2\operatorname{Tr}\mathcal{M}^2 - \frac{N}{2}\right)^n\right] \\ &= \sum_{j=0}^{n}\binom{n}{j}\left(-\frac{N}{2}\right)^{n-j}\mathbb{E}_N^{(H)}[(2\operatorname{Tr}\mathcal{M}^2)^j] \\ &= \frac{N^n}{2^{n+1}}\sum_{j=0}^{n}(-1)^{n-j}2^jN^{2-2j}\binom{n}{j}\frac{\Gamma\left(\frac{N^2}{2}+j\right)}{\Gamma\left(\frac{N^2}{2}+1\right)}.\end{aligned} \tag{4.2.7}$$

The asymptotic expansion for the ratio of Gamma functions is [103]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}\sum_{l=0}^{\infty}\frac{1}{z^l}\binom{a-b}{l}B_l^{(a-b+1)}(a), \quad a,b\in\mathbb{C}, \quad z\to\infty, \tag{4.2.8}$$

where $B_j^{(l)}$ are generalised Bernoulli polynomials. Hence,

$$\mathbb{E}_N^{(H)}[X_2^n] = \frac{N^n}{2^n}\sum_{j=1}^{n}\sum_{l=0}^{j-1}(-1)^{n-j+l}\frac{2^l}{N^{2l}}\binom{n}{j}\binom{j-1}{l}B_l^{(j)}(0). \tag{4.2.9}$$

In arriving at (4.2.9) we used

$$B_l^{(j)}(j) = (-1)^l B_l^{(j)}(0). \tag{4.2.10}$$

Here $B_l^{(j)}(0)$ are generalised Bernoulli numbers and the first few numbers are given below:

$$\begin{aligned}B_0^{(j)}(0) &= 1 \\ B_1^{(j)}(0) &= -\frac{j}{2} \\ B_2^{(j)}(0) &= \frac{j^2}{4} - \frac{j}{12} \\ B_3^{(j)}(0) &= -\frac{j^3}{8} + \frac{j^2}{8}.\end{aligned} \tag{4.2.11}$$

By inserting (4.2.11) into (4.2.9),

$$\begin{aligned}
\text{Coef. of } N^n &: \frac{1}{2^n}\sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j} = 0 \\
\text{Coef. of } N^{n-2} &: \frac{1}{2^n}\sum_{j=2}^{n}(-1)^{n-j}\binom{n}{j}j(j-1) \\
&= \frac{n}{2^n}\sum_{j=2}^{n}(-1)^{n-j}(j-1)\binom{n-1}{j-1} = 0.
\end{aligned} \tag{4.2.12}$$

Calculating the coefficient of N^{n-2l} for arbitrary n and l is not straightforward because there are no simple expressions for generalised Bernoulli numbers. Though these numbers can be written in terms of Stirling's numbers of first kind, the coefficients can be explicitly computed only for small values of l. It can be shown for a given n that

$$\begin{aligned}
\text{Coef. of } N^{n-2k} &= 0, \quad \text{for } 0 \le k < \lfloor n/2 \rfloor, \\
\text{Coef. of } N^0 &= \frac{n!}{2^n\left(\frac{n}{2}\right)!}\eta_n,
\end{aligned} \tag{4.2.13}$$

where $\eta_n = 1$ if n is even and 0 otherwise. Our goal is not to compute these coefficients more generally, but rather to give an estimate for the sub-leading term in (4.2.5). Since the Chebyshev polynomials of even and odd degree do not mix, the moments of X_k show a similar behaviour as in Corollary. 3.7.1 in Ch. 3:

$\mathbb{E}_N^{(H)}[P_\mu(\mathcal{M})]$	$l(\mu)$
Odd Laurent polynomial	odd
Even Laurent polynomial	even

Table 4.1: Parity dependence of $\mathbb{E}_N^{(H)}[P_\mu(\mathcal{M})]$.

Therefore,

$$\mathbb{E}_N^{(H)}[X_2^n] = \begin{cases} d_2(n,2)\frac{1}{N} + O(N^{-3}), & \text{if } n \text{ is odd}, \\ \frac{n!}{2^n\left(\frac{n}{2}\right)!} + d_3(n,2)\frac{1}{N^2} + O(N^{-4}), & \text{if } n \text{ is even}. \end{cases} \tag{4.2.14}$$

Coefficients $d_2(n,2)$ and $d_3(n,2)$ can be estimated using (4.2.9). In Table. 4.2, we give the results for $\mathbb{E}_N^{(H)}[X_2^n]$ along with the coefficients $d_2(n,2)$ and $d_3(n,2)$. In the next section, we give an estimate of $d_2(n,k)$ and $d_3(n,k)$ for arbitrary values of n and k.

4.2.1.2 General degree

The explicit expressions for the joint moments of eigenvalues in Thm. 3.2.3 in Ch. 3 allows us to obtain Thm. 4.1.5. Consequently, X_k converges to a normal random variable. For a fixed k and n,

$$X_k \to \frac{\sqrt{k}}{2}\mathcal{N}(0,1) \quad \text{as } N \to \infty. \tag{4.2.15}$$

n	$\mathbb{E}_N^{(H)}[X_2^n]$
1	0
2	$\frac{1}{2}$
3	$\frac{1}{N}$
4	$\frac{3}{4}+\frac{3}{N^2}$
5	$\frac{5}{N}+\frac{12}{N^3}$
6	$\frac{15}{8}+\frac{65}{2N^2}+\frac{60}{N^4}$
7	$\frac{105}{4N}+\frac{231}{N^3}+\frac{360}{N^5}$
8	$\frac{105}{16}+\frac{595}{2N^2}+\frac{1827}{N^4}+\frac{2520}{N^6}$
9	$\frac{315}{2N}+\frac{3304}{N^3}+\frac{16056}{N^5}+\frac{20160}{N^7}$
10	$\frac{945}{32}+\frac{11025}{4N^2}+\frac{75915}{2N^4}+\frac{155844}{N^6}+\frac{181440}{N^8}$

Table 4.2: The first 10 moments of $X_2 = \operatorname{Tr} T_2(\mathcal{M}) - \mathbb{E}_N^{(H)}[\operatorname{Tr} T_2(\mathcal{M})]$.

In reality, the correct bounds in Thm. 4.1.5 are much more smaller than given. This is due to sequential cancellations in the sum

$$\sum_{\lambda} \chi_\mu^\lambda \chi_{(2^{|\mu|/2})}^\lambda C_\lambda(N) \tag{4.2.16}$$

and in the Chebyshev expansion

$$\operatorname{Tr}\, T_k(\mathcal{M}) = \frac{k}{2}\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^j \frac{(k-j-1)!}{j!(k-2j)!} 2^{k-2j}\mathcal{M}^{k-2j}. \tag{4.2.17}$$

The bounds in Thm. 4.1.5 are better for smaller moments.

To prove Thm. 4.1.5, we first need to estimate the coefficient of the $1/N$ term in the Laurent series of $\mathbb{E}_N^{(H)}[P_\mu]$ of rescaled matrices, which leads to estimating the characters of the symmetric group. All the characters of the symmetric group are integers and satisfy

$$\frac{|\chi_\mu^\lambda|}{\chi_{(1^{|\mu|})}^\lambda} < 1. \tag{4.2.18}$$

It turns out that under suitable assumptions, the ratios $|\chi_\mu^\lambda|/\chi_{(1^{|\mu|})}^\lambda$ are very small, sometimes exponentially and super-exponentially small [94, 209]. Particularly useful bounds are of the form

$$|\chi_\mu^\lambda| \leq (\chi_{(1^{|\mu|})}^\lambda)^{a_\mu}, \tag{4.2.19}$$

where a_μ depends on μ.

The frequency representation of a partition $\mu = (1^{b_1}2^{b_2}\dots k^{b_k})$ also represents a permutation cycle of an element in $\mathcal{S}_{|\mu|}$. The number b_j gives the number of $j-$cycles in μ, $1 \leq j \leq k$. For example, if $b_1 = 0$ then are no 1-cycles. In other words, there are no fixed points when $b_1 = 0$. The only obstruction to the small character values of $|\chi_\mu^\lambda|$ is when μ has many short cycles [177]. With this information,

Proposition 4.2.2.

(a) Given any $\lambda \in Irr(\mathcal{S}_n)$, let $\mu = (m^{n/m})$, then [94]

$$|\chi^\lambda_\mu| \leq c\, n^{\frac{1}{2}(1-\frac{1}{m})} \left(\chi^\lambda_{(1^{|\mu|})}\right)^{\frac{1}{m}}, \tag{4.2.20}$$

where c is an absolute constant.

(b) If $\mu \in \mathcal{S}_n$ is fixed-point-free, or has $n^{o(1)}$ fixed points, then for any $\lambda \in Irr(\mathcal{S}_n)$ [177],

$$|\chi^\lambda_\mu| \leq \left(\chi^\lambda_{1^{|\mu|}}\right)^{\frac{1}{2}+o(1)}. \tag{4.2.21}$$

(c) Fix $a \leq 1$ and let $\mu \in \mathcal{S}_n$ with at most n^a cycles. Then for any $\lambda \in Irr(\mathcal{S}_n)$ [177],

$$|\chi^\lambda_\mu| \leq \left(\chi^\lambda_{(1^{|\mu|})}\right)^{a+o(1)}. \tag{4.2.22}$$

Proposition 4.2.3. *For a given μ, $\mathbb{E}^{(H)}_N[P_\mu]$ is a Laurent polynomial in N with*

$$\text{Coefficient of } 1/N^q \text{ in } \mathbb{E}^{(H)}_N[P_\mu] << 2^{-\frac{|\mu|}{2}-q-\frac{3}{2}} |\mu|^{\frac{3|\mu|}{4}-\frac{11}{8}+q} e^{-\frac{|\mu|}{4}+\pi\sqrt{\frac{2}{3}|\mu|}}. \tag{4.2.23}$$

Here q is a positive even (odd) integer when $l(\mu)$ is even (odd).

Proof. For rescaled matrices, the expected value of P_μ is

$$\mathbb{E}^{(H)}_N\Big[\prod_{j=1}^{l} \operatorname{Tr} \mathcal{M}^{\mu_j}\Big] = \begin{cases} \frac{1}{(8N)^{\frac{|\mu|}{2}} \frac{|\mu|}{2}!} \sum_{\lambda \vdash |\mu|} \chi^\lambda_{(2^{|\lambda|/2})} \chi^\lambda_\mu C_\lambda(N), & |\mu| \text{ is even}, \\ 0, & \textit{otherwise}, \end{cases} \tag{4.2.24}$$

Using (3.5.10), we obtain

$$\frac{\Gamma(N+1)}{\Gamma(N-|\lambda|+1)} \leq C_\lambda(N) \leq \frac{\Gamma(N+|\lambda|)}{\Gamma(N)}. \tag{4.2.25}$$

Using the asymptotics of Gamma functions, as $N \to \infty$,

$$\frac{\Gamma(N+|\lambda|)}{\Gamma(N)} \sim N^{|\lambda|} \sum_{l=0}^{\infty} \frac{1}{N^l} \binom{|\lambda|}{l} B_l^{(|\lambda|+1)}(|\lambda|), \tag{4.2.26}$$

where $B_l^{(j)}(x)$ are generalised Bernoulli polynomials of degree l. Thus, the coefficient of $1/N^q$ in (3.2.7) is bounded by

$$\text{Coefficient of } 1/N^q \text{ in } \mathbb{E}^{(H)}_N[P_\mu] \leq \frac{1}{8^{\frac{|\mu|}{2}} \frac{|\mu|}{2}!} \binom{|\mu|}{\frac{|\mu|}{2}+q} B^{(|\mu|+1)}_{\frac{|\mu|}{2}+q}(|\mu|) \sum_\lambda |\chi^\lambda_\mu| |\chi^\lambda_{2^{|\mu|/2}}|. \tag{4.2.27}$$

Using (4.2.19) and (4.2.20), the R.H.S. of (4.2.27) is bounded from above by

$$\frac{c}{8^{\frac{|\mu|}{2}} \frac{|\mu|}{2}!} \binom{|\mu|}{\frac{|\mu|}{2}+q} |\mu|^{\frac{1}{4}} (\chi^\lambda_{1^{|\mu|}})^{a_\mu+\frac{1}{2}}_{\max} \# \operatorname{par}(|\mu|)\, B^{(|\mu|+1)}_{\frac{|\mu|}{2}+q}(|\mu|). \tag{4.2.28}$$

The maximum of the dimension of the irreducible representation is [184]

$$(\chi^{\lambda}_{1^{|\mu|}})_{\max} \leq (2\pi)^{\frac{1}{4}} |\mu|^{\frac{|\mu|}{2}+\frac{1}{4}} e^{-\frac{|\mu|}{2}}, \tag{4.2.29}$$

and number of partitions grow as [137, 232]

$$\#\,\mathrm{par}(|\mu|) \sim \frac{1}{4\sqrt{3}|\mu|} \exp\left(\pi\sqrt{\frac{2|\mu|}{3}}\right), \quad \text{as } |\mu| \to \infty. \tag{4.2.30}$$

Polynomials $B_l^{(j)}(x)$ satisfy

$$B_l^{(j+1)}(x) = \left(1 - \frac{l}{j}\right) B_l^{(j)}(x) + l\left(\frac{x}{j} - 1\right) B_{l-1}^{(j)}(x). \tag{4.2.31}$$

Hence,

$$B^{(|\mu|+1)}_{\frac{|\mu|}{2}+q}(|\mu|) = \left(\frac{1}{2} - \frac{q}{|\mu|}\right) B^{(|\mu|)}_{\frac{|\mu|}{2}+q}(|\mu|) \sim \frac{1}{2^{\frac{|\mu|}{2}+q+1}} |\mu|^{\frac{|\mu|}{2}+q}. \tag{4.2.32}$$

Inserting $a_\mu = 1$, (4.2.29) and (4.2.32) in (4.2.28), the coefficient of $1/N^q$ in $\mathbb{E}_N^{(H)}[P_\mu]$ is bounded from above by

$$\frac{1}{2^{2|\mu|+q+3}} \frac{1}{\frac{|\mu|}{2}!} \binom{|\mu|}{\frac{|\mu|}{2}+q} |\mu|^{\frac{5|\mu|}{4}-\frac{3}{8}+q} e^{-3\frac{|\mu|}{4}+\pi\sqrt{\frac{2}{3}|\mu|}}. \tag{4.2.33}$$

The bound in (4.2.33) is much larger than the original value due to cancellations involved in (4.2.24). Now using Stirling's approximation,

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}, \; n \to \infty, \tag{4.2.34}$$

proves the proposition. ■

Proof of Thm. 4.1.5. Using (4.2.24), it can be seen that the joint moments of traces of rescaled matrices are Laurent polynomial in N with rational coefficients. Thus the central moments of traces of Chebyshev polynomials are also Laurent polynomials. Since $X_k(\mathcal{M})$ converges in distribution to a normal random variable as $N \to \infty$, $\mathbb{E}_N^{(H)}[X_k^n]$ is a polynomial in $1/N$ with constant term given in (4.1.22) and (4.1.24) depending on whether k is odd and even, respectively. To estimate the sub-leading term in $\mathbb{E}_N^{(H)}[X_k^n]$, we consider k even and odd cases separately.

(1) For k odd, $\mathbb{E}_N^{(H)}[\mathrm{Tr}\, T_k(\mathcal{M})] = 0$. Using the expansion of Chebyshev polynomials of the first

kind,

$$
\begin{aligned}
&\mathbb{E}_N^{(H)}[X_k^n] \\
=&\mathbb{E}_N^{(H)}\left[(\mathrm{Tr}\ T_k(\mathcal{M}))^n\right] \\
=&\mathbb{E}_N^{(H)}\left[\left(k\sum_{j=0}^{\frac{k-1}{2}}(-1)^{\frac{k-1}{2}-j}\frac{(\frac{k-1}{2}+j)!}{(\frac{k-1}{2}-j)!(2j+1)!}2^{2j}\ \mathrm{Tr}\ \mathcal{M}^{2j+1}\right)^n\right] \\
=&k^n\sum_{n_0+\cdots+n_{\frac{k-1}{2}}=n}\binom{n}{n_0,\ldots,n_{\frac{k-1}{2}}}\prod_{j=0}^{\frac{k-1}{2}}(-1)^{\frac{k-1}{2}n_j-jn_j}\left(\frac{(\frac{k-1}{2}+j)!}{(\frac{k-1}{2}-j)!(2j+1)!}\right)^{n_j}2^{2jn_j} \\
&\times\mathbb{E}_N^{(H)}\left[P_\mu\right],
\end{aligned}
\tag{4.2.35}
$$

where

$$
\mathbb{E}_N^{(H)}[P_\mu]=\mathbb{E}_N^{(H)}\left[\prod_{l=0}^{\frac{k-1}{2}}(\mathrm{Tr}\ \mathcal{M}^{2l+1})^{n_l}\right],\quad \mu=(1^{n_0}3^{n_1}\ldots k^{n_{\frac{k-1}{2}}}). \tag{4.2.36}
$$

The odd moments of $\mathrm{Tr}\ T_k(\mathcal{M})$ are identically zero because $\mathbb{E}_N^{(H)}[P_\mu]=0$ when $|\mu|$ is odd, see (4.2.24). When n is even, the leading term is given by the nth moment of $\sqrt{k}r_k/2$, $r_k\sim\mathcal{N}(0,1)$, according to Szegő's theorem. For n even, $l(\mu)$ is always even. Hence the sub-leading term in (4.2.35) is $O(N^{-2})$ (see Table. 4.1).

The maximum possible degree of μ is $|\mu|=nk$ when $n_{\frac{k-1}{2}}=n$, $n_j=0$ for $j=0,\ldots,\frac{k-3}{2}$, and the minimum degree is $|\mu|=n$ when $n_0=n$, $n_j=0$ for $j=1,\ldots,\frac{k-1}{2}$. The coefficient of $1/N^2$ in (4.2.35) is estimated using (4.2.23) by choosing an appropriate μ. Note that the multinomial coefficient is maximum when all n_j's are approximately equal. In this case $\mu=(1^{\frac{2n}{k+1}}3^{\frac{2n}{k+1}}\ldots k^{\frac{2n}{k+1}})$ and $|\mu|=n(k+1)/2$. For a fixed k as n increases, the number of short cycles in μ increases. Using (4.2.22),

$$
|\chi_\mu^\lambda|\le\chi_{(1^{|\mu|})}^\lambda, \tag{4.2.37}
$$

which implies $a_\mu=1$ in (4.2.23).

Let

$$
d_1(n,k)=\left[\mathbb{E}_N^{(H)}[(\mathrm{Tr}\ T_k(\mathcal{M}))^n]\right]_{1/N^2} \tag{4.2.38}
$$

denote the coefficient of $1/N^2$ in $\mathbb{E}_N^{(H)}\left[(\mathrm{Tr}\ T_k(\mathcal{M}))^n\right]$. Putting $q=2$ in (4.2.23),

$$
d_1(n,k)\le k^n\frac{n!}{\left(\frac{2n}{k+1}!\right)^{\frac{k+1}{2}}}\left(\prod_{j=0}^{\frac{k-1}{2}}\frac{(\frac{k-1}{2}+j)!}{(\frac{k-1}{2}-j)!(2j+1)!}\right)^{\frac{2n}{k+1}}2^{2|\mu|}\left[\mathbb{E}_N^{(H)}\left[P_\mu\right]\right]_{1/N^2}. \tag{4.2.39}
$$

Now,

$$
\prod_{j=0}^{\frac{k-1}{2}}\frac{(\frac{k-1}{2}+j)!}{(\frac{k-1}{2}-j)!(2j+1)!}=2^{-\frac{5}{24}-\frac{1}{4}k(k+2)}e^{\frac{1}{8}}\pi^{\frac{1}{4}(k+2)}\frac{1}{A^{\frac{3}{2}}}\frac{\mathcal{G}(k+1)}{\mathcal{G}\left(\frac{k}{2}+2\right)\mathcal{G}\left(\frac{k+1}{2}\right)\left(\mathcal{G}\left(\frac{k+3}{2}\right)\right)^2}, \tag{4.2.40}
$$

where we have used the relations

$$\begin{aligned}
\prod_{j=0}^{\frac{k-1}{2}} \frac{(\frac{k-1}{2}+j)!}{(\frac{k-1}{2}-j)!} &= \frac{\mathcal{G}(k+1)}{\mathcal{G}\left(\frac{k+1}{2}\right)\left(\mathcal{G}\left(\frac{k+3}{2}\right)\right)}, \\
\prod_{j=0}^{\frac{k-1}{2}} \frac{1}{(2j+1)!} &= 2^{-\frac{5}{24}-\frac{1}{4}k(k+2)} e^{\frac{1}{8}} \pi^{\frac{1}{4}(k+2)} \frac{1}{A^{\frac{3}{2}}} \frac{1}{\left(\mathcal{G}\left(\frac{k}{2}+2\right)\right)\mathcal{G}\left(\frac{k+3}{2}\right)}.
\end{aligned} \tag{4.2.41}$$

Here $\mathcal{G}(x)$ is Barnes-G function and $A = 1.2824\ldots$ is the Glaisher-Kinkelin constant.

Using the asymptotics of Barnes-G functions,

$$\mathcal{G}(x+1) \sim \frac{1}{A}(2\pi)^{\frac{x}{2}} x^{\frac{x^2}{2}-\frac{1}{12}} e^{-\frac{3x^2}{4}}, \text{ as } x \to \infty, \tag{4.2.42}$$

and Stirling's approximation, we obtain

$$\begin{aligned}
\left(\prod_{j=0}^{\frac{k-1}{2}} \frac{(\frac{k-1}{2}+j)!}{(\frac{k-1}{2}-j)!(2j+1)!}\right)^{\frac{2n}{k+1}} &\sim A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{nk}{2}-n+\frac{n}{6k}} k^{-\frac{3n}{2}+\frac{n}{4k}} e^{\frac{9n}{4}+\frac{5n}{8k}}, \\
\frac{n!}{\left(\frac{2n}{k+1}!\right)^{\frac{k+1}{2}}} &\sim \frac{1}{\pi^{\frac{k-1}{4}}} \frac{1}{2^{n+\frac{k}{2}}} n^{-\frac{k-1}{4}} (k+1)^{n+\frac{k+1}{4}}.
\end{aligned} \tag{4.2.43}$$

By combining all the results that came along,

$$\begin{aligned}
d_1(n,k) << {} & A^{\frac{3n}{k}} \pi^{-\frac{1}{4}(2n+k)} 2^{\frac{7nk}{8}-\frac{13n}{8}+\frac{n}{6k}-\frac{k}{2}} n^{\frac{3n}{8}(k+1)-\frac{k}{4}+\frac{7}{8}} k^{\frac{3n}{8}(k+2)+\frac{n}{8}+\frac{n}{4k}+\frac{k}{4}+\frac{7}{8}} \\
& \times e^{-\frac{n}{8}(k+1)+\frac{9n}{4}+\frac{5n}{8k}+\pi\sqrt{\frac{n}{3}(k+1)}}.
\end{aligned} \tag{4.2.44}$$

We are interested to find the order of the coefficient of $1/N$ as n increases for a fixed k. To capture the right behaviour, it is sufficient to approximate (4.2.44) to

$$d_1(n,k) << A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{7nk}{8}-\frac{13n}{8}+\frac{n}{6k}} k^{\frac{3n}{8}(k+2)+\frac{n}{8}+\frac{n}{4k}} n^{\frac{3n}{8}(k+1)-\frac{k}{4}+\frac{7}{8}} e^{-\frac{n}{8}(k+1)+\frac{9n}{4}+\frac{5n}{8k}+\pi\sqrt{\frac{n}{3}(k+1)}} \tag{4.2.45}$$

(2) When k is even, let

$$c_k = \frac{1}{N}\mathbb{E}_N^{(H)}[\mathrm{Tr}\ T_k(\mathcal{M})]. \tag{4.2.46}$$

We have,

$$
\begin{aligned}
\mathbb{E}_N^{(H)}[X_k^n] &= \mathbb{E}_N^{(H)}\left[(\mathrm{Tr}\ T_k(\mathcal{M}) - Nc_k)^n\right] \\
&= \mathbb{E}_N^{(H)}\left[\left(N\left((-1)^{\frac{k}{2}} - c_k\right) + k\sum_{j=1}^{\frac{k}{2}}(-1)^{\frac{k}{2}-j}\frac{(\frac{k}{2}+j-1)!}{(\frac{k}{2}-j)!(2j)!}2^{2j-1}\,\mathrm{Tr}\ \mathcal{M}^{2j}\right)^n\right] \\
&= \sum_{n_0+\cdots+n_{\frac{k}{2}}=n}\binom{n}{n_0,\ldots,n_{\frac{k}{2}}}N^{n_0}\left((-1)^{\frac{k}{2}} - c_k\right)^{n_0} \\
&\quad\times\prod_{j=1}^{\frac{k}{2}}(-1)^{\frac{k}{2}n_j - jn_j}k^{n_j}\left(\frac{(\frac{k}{2}+j-1)!}{(\frac{k}{2}-j)!(2j)!}\right)^{n_j}2^{(2j-1)n_j}\mathbb{E}_N^{(H)}\left[P_\mu\right],
\end{aligned}
\tag{4.2.47}
$$

where

$$
\mathbb{E}_N^{(H)}[P_\mu] = \mathbb{E}_N\left[\prod_{l=0}^{\frac{k}{2}}(\mathrm{Tr}\ \mathcal{M}^{2l})^{n_l}\right], \quad \mu = (2^{n_1}4^{n_2}\ldots k^{n_{\frac{k}{2}}}). \tag{4.2.48}
$$

According to Szegő's theorem, when n is even the leading order term in the R.H.S. of (4.2.47) is given by $\mathbb{E}[(\sqrt{k}r_k/2)^n]$, $r_k \sim \mathcal{N}(0,1)$. The sub-leading term is $d_3(n,k)N^{-2}$. When n is odd, the leading term in the R.H.S. is given by $d_2(n,k)N^{-1}$. Next we compute the coefficients $d_2(n,k)$ and $d_3(n,k)$.

Coefficient $d_2(n,k)$: c_k decays as $1/N^2$ for $k > 2$, so we neglect it in (4.2.47). Note that μ in (4.2.48) doesn't have any 1-cycles. So μ is fixed-point-free and (4.2.21) can also be used to estimate characters χ_μ^λ in Prop. 4.2.3. Here we just use (4.2.23) for $q = 2$ and follow the exact same calculation as k odd. This leads to

$$
d_2(n,k) << A^{\frac{3n}{k}}\pi^{-\frac{n}{2}}2^{\frac{3nk}{8}-3n+\frac{n}{6k}}k^{\frac{3nk}{8}+\frac{n}{2}+\frac{9n}{4k}}n^{\frac{3nk}{8}+\frac{2n}{k}-\frac{k}{2}-\frac{3}{8}}e^{-\frac{n}{8}(k-18)+\pi\sqrt{\frac{nk}{3}}-\frac{19n}{8k}}. \tag{4.2.49}
$$

Similarly, $d_3(n,k)$ can be approximated as

$$
d_3(n,k) << A^{\frac{3n}{k}}\pi^{-\frac{n}{2}}2^{\frac{3nk}{8}-3n+\frac{n}{6k}}k^{\frac{3nk}{8}+\frac{n}{2}+\frac{9n}{4k}}n^{\frac{3nk}{8}+\frac{2n}{k}-\frac{k}{2}+\frac{5}{8}}e^{-\frac{n}{8}(k-18)+\pi\sqrt{\frac{nk}{3}}-\frac{19n}{8k}}. \tag{4.2.50}
$$

■

4.2.2 Cumulants

In general, the moments and the cumulants are related by the recurrence relation

$$
\kappa_n = m_n - \sum_{j=1}^{n-1}\binom{n-1}{j-1}\kappa_j m_{n-j}. \tag{4.2.51}
$$

Cumulants and moments can also be expressed in terms of each other through a more elegant formula. For a partition $\lambda = (1^{b_1} 2^{b_2} \dots r^{b_r})$, define

$$\kappa_\lambda := \prod_{j=1}^{r} \kappa_j^{b_j}, \quad m_\lambda := \prod_{j=1}^{r} m_j^{b_j}. \tag{4.2.52}$$

One has that

$$\begin{aligned} m_n &= \sum_\lambda d_\lambda \kappa_\lambda, \\ \kappa_n &= \sum_\lambda (-1)^{l(\lambda)-1} (l(\lambda)-1)! d_\lambda m_\lambda, \end{aligned} \tag{4.2.53}$$

where

$$d_\lambda = \frac{n!}{(1!)^{b_1} b_1! \dots (r!)^{b_r} b_r!} \tag{4.2.54}$$

is the number of decompositions of a set of n elements into disjoint subsets containing $\lambda_1, \dots, \lambda_l$ elements.

In this section, we give an estimate on the cumulants of random variables X_k and to do so we rely on the well studied connection between GUE correlators and enumerating ribbon graphs which has been briefly discussed in App. A.

4.2.2.1 Connections to Ribbon graphs

The mixed moments of traces are combinatorial objects and count the ribbon graphs, which are graphs that can be drawn on surfaces. This connection is captured by the following theorem due to Brézin-Itzykson-Parisi-Zuber [42].

Theorem 4.2.4. *The joint moments of traces of the Gaussian matrix model are the sums of weighted ribbon graphs,*

$$\mathbb{E}_N^{(H)} \left[\prod_{j=1}^{n} \frac{1}{n_j!} \left(\frac{N}{j} \operatorname{Tr} \mathcal{M}^j \right)^{n_j} \right] = \sum_{Ribbon\ Graphs\ G} \frac{1}{\#Aut(G)} 4^{-\#edges} N^{\chi(G)}, \tag{4.2.55}$$

where the sum is over non-topologically equivalent ribbon graphs G with n_j vertices, each with j valencies, and $\#Aut(G)$ is the number of automorphisms of G.

Each graph G in (4.2.55) is either connected or disconnected. To study the cumulants, we require to count only the connected graphs. When the summation over graphs in (4.2.55) is restricted to connected ribbon graphs, we obtain the connected components of $\mathbb{E}_N^{(H)} \left[\prod_{j=1}^{n} \left(\operatorname{Tr} \mathcal{M}^j \right)^{n_j} \right]$ indicated by $\mathbb{E}_N^{(H)} \left[\prod_{j=1}^{n} \left(\operatorname{Tr} \mathcal{M}^j \right)^{n_j} \right]_c$. That is,

$$\mathbb{E}_N^{(H)} \left[\prod_{j=1}^{n} \left(\operatorname{Tr} \mathcal{M}^j \right)^{n_j} \right]_c : \text{counts only connected ribbon graphs.} \tag{4.2.56}$$

A few examples of connected components of traces are listed below:

$$\begin{aligned}
\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_1}]_c &= \mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_1}] \\
\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_1}\operatorname{Tr}\mathcal{M}^{l_2}]_c &= \mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_1}\operatorname{Tr}\mathcal{M}^{l_2}] - \mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_1}]\,\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_2}] \\
\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_1}\operatorname{Tr}\mathcal{M}^{l_2}\operatorname{Tr}\mathcal{M}^{l_3}]_c &= \mathbb{E}_N^{(H)}[\mathcal{M}^{l_1}\operatorname{Tr}\mathcal{M}^{l_2}\operatorname{Tr}\mathcal{M}^{l_3}] - \mathbb{E}_N^{(H)}[\mathcal{M}^{l_1}]\,\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_2}\operatorname{Tr}\mathcal{M}^{l_3}] \\
&\quad - \mathbb{E}_N^{(H)}[\mathcal{M}^{l_2}]\,\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_1}\operatorname{Tr}\mathcal{M}^{l_3}] - \mathbb{E}_N^{(H)}[\mathcal{M}^{l_3}]\,\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_1}\operatorname{Tr}\mathcal{M}^{l_2}] \\
&\quad + 2\mathbb{E}_N^{(H)}[\mathcal{M}^{l_1}]\,\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_2}]\,\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{l_3}]
\end{aligned} \tag{4.2.57}$$

Corollary 4.2.5. *Connected integrals in the Gaussian matrix model are the sum of weighted connected ribbon graphs,*

$$\mathbb{E}_N^{(H)}\left[\prod_{j=1}^{n}\frac{1}{n_j!}\left(\frac{N}{j}\operatorname{Tr}\mathcal{M}^j\right)^{n_j}\right]_c = \sum_{\textit{Connected ribbon Graphs } G}\frac{1}{\#Aut(G)}4^{-\#edges}N^{\chi(G)}. \tag{4.2.58}$$

Alternatively for $\mu = (\mu_1, \ldots, \mu_l) = (1^{n_1}\ldots k^{n_k})$, such that $|\mu|$ is even,

$$\mathbb{E}_N^{(H)}\left[\prod_{j=1}^{l}\operatorname{Tr}\,\mathcal{M}^{\mu_j}\right]_c \equiv \mathbb{E}_N^{(H)}\left[\prod_{j=1}^{k}(\operatorname{Tr}\,\mathcal{M}^j)^{n_j}\right]_c = \sum_{0\le g\le\frac{|\mu|}{4}-\frac{l}{2}+\frac{1}{2}}\frac{1}{2^{|\mu|}}a_g(\mu_1,\ldots,\mu_l)N^{2-2g-l}. \tag{4.2.59}$$

Here g is the genus of the graph, and

$$\begin{aligned}
a_g(\mu_1,\ldots,\mu_l) &= \#\{\text{connected oriented labelled ribbon graphs} \\
&\qquad \text{of genus } g \text{ with } l \text{ vertices of valencies } \mu_1,\ldots,\mu_l\} \\
&= l!\sum_{\Gamma}\frac{1}{\#\mathrm{Sym}(\Gamma)},
\end{aligned} \tag{4.2.60}$$

where Γ is a connected (unlabelled) ribbon graph of genus g with l vertices of valencies $\mu_1,\ldots,\mu_l$, $\#\mathrm{Sym}(\Gamma)$ is the order of the symmetry group of Γ, and the last summation is taken over all such Γ. Table. 4.3 gives a few examples of connected components of traces corresponding to partitions of 6. For more examples and explicit results for expectations of connected traces, see [75]. Special cases of connected correlators such as

$$\mathbb{E}_N^{(H)}[(\operatorname{Tr}M^j)^k]_c, \quad j\ge 3,\ k\ge 1, \tag{4.2.61}$$

count polygon numbers on Riemann surfaces and can be determined exactly by certain algorithms.

We now return to the cumulants of random variables X_k. Consider the formal matrix integral over the space of $N\times N$ rescaled GUE matrices,

$$Z_N(\mathbf{s},\xi) = e^{s_0 N\xi}\int e^{-2N\operatorname{Tr}\,M^2}e^{\xi\operatorname{Tr}\,V(M)}\,dM. \tag{4.2.62}$$

P_μ	$\mathbb{E}_N^{(H)}[P_\mu]_c$
p_6	$\frac{5N}{64}+\frac{5}{32N}$
p_5p_1	$\frac{5}{32}+\frac{5}{64N^2}$
p_4p_2	$\frac{1}{8}+\frac{1}{16N^2}$
$p_4p_1^2$	$\frac{3}{16N}$
p_3^2	$\frac{3}{16}+\frac{3}{64N^2}$
$p_3p_2p_1$	$\frac{3}{16N}$
$p_3p_1^3$	$\frac{3}{32N^2}$
p_2^3	$\frac{1}{8N}$
$p_2^2p_1^2$	$\frac{1}{8N^2}$
$p_2p_1^4$	$\frac{3}{16N}$
p_1^6	$\frac{15}{64}$

Table 4.3: $\mathbb{E}_N^{(H)}[P_\mu(\mathcal{M})]_c$ when μ is a partition of 6.

Here the formal series $V(M)$ depends on the parameters $\mathbf{s}=\{s_1,\dots,s_k\}$, and has the form

$$V(M)=\sum_{j=1}^{k} s_j M^j. \tag{4.2.63}$$

The integral in (4.2.62) can be considered as a formal expansion in the set of parameters s_j and ξ. Now,

$$\begin{aligned}\frac{Z(\mathbf{s},\xi)}{Z(0,\xi)} &= \sum_{n_0,n_1,\dots,n_k} \xi^{\sum n_j}\frac{(s_0N)^{n_0}}{n_0!}\frac{s_1^{n_1}}{n_1!}\cdots\frac{s_k^{n_k}}{n_k!}\mathbb{E}_N^{(H)}\left[\prod_{j=1}^{k}(\mathrm{Tr}\ \mathcal{M}^j)^{n_j}\right]\\ &= \sum_{n\geq 0}\xi^n \sum_{n_0+\cdots+n_k=n}\frac{(s_0N)^{n_0}}{n_0!}\frac{s_1^{n_1}}{n_1!}\cdots\frac{s_k^{n_k}}{n_k!}\mathbb{E}_N^{(H)}\left[\prod_{j=1}^{k}(\mathrm{Tr}\ \mathcal{M}^j)^{n_j}\right].\end{aligned} \tag{4.2.64}$$

By choosing s_j to be the coefficients of Chebyshev polynomials in (4.2.64), we recover the moments of X_k. Thus, (4.2.64) is the moment generating function of X_k. For a given k, by fixing s_j to be the Chebyshev coefficients in T_k,

$$\mathbb{E}_N^{(H)}\left[e^{\xi X_k}\right]=\sum_{n\geq 0}\frac{\xi^n}{n!}\mathbb{E}_N^{(H)}[X_k^n]=\frac{Z(\mathbf{s},\xi)}{Z(0,\xi)}. \tag{4.2.65}$$

By matching the terms in the L.H.S. and R.H.S. of (4.2.65) by powers in ξ, we recover the moments of X_k. The correlators of Tr $\mathcal{M}^j$ are connected to the problem of enumerating ribbon graphs as discussed above. The trace correlators count ribbon graphs that are connected and also multiplicatively count ribbon graphs that are disconnected. When we have a generating series that counts disconnected objects multiplicatively, taking the logarithm counts only

connected objects [138]. Hence, the cumulant generating function is given by

$$\begin{aligned}\log \mathbb{E}_N^{(H)}\left[e^{\xi X_k}\right] &= \log \frac{Z(\mathbf{s},\xi)}{Z(0,\xi)} = \sum_{n\geq 1}\frac{\xi^n}{n!}\kappa_n \\ &= s_0 N\xi + \sum_{n\geq 1}\xi^n \sum_{n_1\cdots+n_k=n}\frac{s_1^{n_1}}{n_1!}\cdots\frac{s_k^{n_k}}{n_k!}\mathbb{E}_N^{(H)}\left[\prod_{j=1}^{k}(\mathrm{Tr}\ \mathcal{M}^j)^{n_j}\right]_c .\end{aligned} \tag{4.2.66}$$

For $\mu = (1^{n_1}\dots k^{n_k}) \equiv (\mu_1,\dots,\mu_l)$, the connected correlators in (4.2.66) are given by (4.2.58).

We are now ready to estimate the cumulants of X_k. We treat k even and odd cases separately.

(1) *k odd*: In this case, the parameters $s_{2j} = 0$ for $0 \leq j \leq \frac{k-1}{2}$, and

$$s_{2j+1} = (-1)^{\frac{k-1}{2}-j}k\frac{(\frac{k-1}{2}+j)!}{(\frac{k-1}{2}-j)!(2j+1)!}2^{2j}, \quad 0 \leq j \leq \frac{k-1}{2}. \tag{4.2.67}$$

When k is odd, all the odd moments are zero. Hence all the odd cumulants are also zero. By inserting (4.2.58) in (4.2.66), the even cumulants are given by

$$\kappa_{2n} = \frac{2n!}{N^{2n-2}}\sum_g \sum_{n_1+n_3+\cdots+n_k=2n}\frac{1}{2^{\sum_j jn_j}}\frac{a_g}{N^{2g}}\frac{s_1^{n_1}}{n_1!}\frac{s_3^{n_3}}{n_3!}\cdots\frac{s_k^{n_k}}{n_k!}. \tag{4.2.68}$$

(2) *k even:* In this case, the parameters $s_{2j+1} = 0$ for $0 \leq j \leq \frac{k}{2}-1$, and

$$\begin{aligned} s_0 &= (-1)^{\frac{k}{2}} - c_k, \\ s_{2j} &= (-1)^{\frac{k}{2}-j}k\frac{(\frac{k}{2}+j-1)!}{(\frac{k}{2}-j)!(2j)!}2^{2j-1}, \quad 1 \leq j \leq \frac{k}{2}. \end{aligned} \tag{4.2.69}$$

The first cumulant is zero by the definition of X_k. So the first term in (4.2.66) is cancelled by $n = 1$ contribution coming from the second term. Hence,

$$\log \mathbb{E}_N^{(H)}\left[e^{\xi X_k}\right] = \sum_{n\geq 2}\xi^n \sum_{n_2+\cdots+n_k=n}\frac{s_2^{n_2}}{n_2!}\frac{s_4^{n_4}}{n_4!}\cdots\frac{s_k^{n_k}}{n_k!}\mathbb{E}_N^{(H)}\left[\prod_{j=1}^{k/2}(\mathrm{Tr}\ \mathcal{M}^{2j})^{n_{2j}}\right]_c . \tag{4.2.70}$$

By inserting (4.2.58) in (4.2.66), the cumulants are given by

$$\kappa_n = \frac{n!}{N^{n-2}}\sum_g \sum_{n_2+\cdots+n_k=n}\frac{1}{2^{\sum_j jn_j}}\frac{a_g}{N^{2g}}\frac{s_2^{n_2}}{n_2!}\frac{s_4^{n_4}}{n_4!}\cdots\frac{s_k^{n_k}}{n_k!}, \quad n \geq 2. \tag{4.2.71}$$

Third and higher order cumulants of a Gaussian random variable are identically equal to zero. Since X_k converges to $\mathcal{N}(0, k/4)$ as $N \to \infty$, the cumulants of X_k, $\kappa_n \to 0$ as $N \to \infty$ for all $n \geq 3$. For a fixed n, we see from (4.2.68) and (4.2.71) that κ_n decay as N^{-n+2}.

Example: The simplest non-trivial example is to calculate the cumulants of X_2. By mapping the problem to counting ribbon graphs (see App. A),

$$\mathbb{E}_N^{(H)}[(\mathrm{Tr}\,\mathcal{M}^2)^n]_c = \frac{1}{(4N)^n}2^{n-1}(n-1)!N^2 = (n-1)!\frac{1}{2^{n+1}}\frac{1}{N^{n-2}}. \tag{4.2.72}$$

For X_2, $s_0 = -\frac{1}{2}$, $s_2 = 2$, and $s_j = 0$ for $j \neq 0, 2$. Hence

$$\kappa_n = s_2^n \, \mathbb{E}_N^{(H)}[\mathrm{Tr}(\mathcal{M}^2)^n]_c = \frac{1}{2} \frac{(n-1)!}{N^{n-2}}. \tag{4.2.73}$$

4.2.3 Laguerre and Jacobi ensembles

The correlators and the connected correlators of traces of the LUE and the JUE also have combinatorial interpretations. For the LUE, the joint moments of traces can be expressed in terms of double monotone Hurwitz numbers [57] and Hodge integrals [73, 74, 127]. In a similar spirit, JUE correlators are related to triple monotone Hurwitz numbers [126].

For the LUE, consider the rescaled matrix $\mathcal{M} = M/(2N)$. The random variables

$$Y_k(\mathcal{M}) = \mathrm{Tr}\, T_k(\mathcal{M} - 1) - \mathbb{E}_N^{(L)}\big[\, \mathrm{Tr}\, T_k(\mathcal{M} - 1)\big], \quad k \in \mathbb{N}, \tag{4.2.74}$$

converge in distribution to $\mathcal{N}(0, k/4)$ as $N \to \infty$. More generally,

$$(Y_1, \ldots, Y_{2m}) \overset{d}{\Rightarrow} \big(\tfrac{1}{2} r_1, \ldots, \tfrac{\sqrt{2m}}{2} r_{2m}\big). \tag{4.2.75}$$

Similarly, for a JUE matrix M of size N, the random variables

$$Z_k = \mathrm{Tr}\, T_k(2M - 1) - \mathbb{E}_N^{(J)}\big[\, \mathrm{Tr}\, T_k(2M - 1)\big], \quad k \in \mathbb{N}, \tag{4.2.76}$$

have the Gaussian distribution in the limit $N \to \infty$,

$$Z_k \overset{d}{\Rightarrow} \frac{\sqrt{k}}{2} r_k. \tag{4.2.77}$$

Furthermore, variables Z_k also satify the multivariate CLT,

$$(Z_1, \ldots, Z_{2m}) \overset{d}{\Rightarrow} \big(\tfrac{1}{2} r_1, \ldots, \tfrac{\sqrt{2m}}{2} r_{2m}\big). \tag{4.2.78}$$

For the Jacobi ensemble, the support of eigenvalues is compact. Therefore, no further rescaling is required unlike the Gaussian and Laguerre ensembles. For special values of γ_1 and γ_2 in the JUE, namely $\gamma_1, \gamma_2 = \pm 1/2$, the j.p.d.f. of eigenvalues represents that of compact groups $O(2N)_\pm$, $O(2N+1)_\pm$, $Sp(2N)$. In [156], Johansson studied the rate of convergence in the CLT for these groups. Error estimates in Thm. 4.1.5 can also be calculated for the Laguerre and Jacobi ensembles using the machinery discussed in Sec. 4.2.

4.3 Mixed moments

Since X_k converges to independent Gaussian normals, the correlators of X_k also converge to random Gaussian variables as $N \to \infty$. For instance,

$$\mathbb{E}_N^{(H)}[X_i X_j] = \frac{\sqrt{ij}}{4} \mathbb{E}[r_i r_j] + O(N^{-1}). \tag{4.3.1}$$

For $\mu = (1^{a_1} \dots k^{a_k})$, let

$$X_\mu = \prod_{j=1}^{k} X_j^{a_j}. \tag{4.3.2}$$

Then,

$$\mathbb{E}_N^{(H)}[X_\mu] = \mathbb{E}_N^{(H)}\left[\prod_{j=1}^{k} X_j^{a_j}\right] = \mathbb{E}\left[\prod_{j=1}^{k} \frac{j^{\frac{a_j}{2}}}{2^{a_j}} r_j^{a_j}\right] + O(N^{-1}). \tag{4.3.3}$$

In this work, we do not pursue the correlations of X_j in detail, but we give explicit expressions in the following tables. Results similar to (4.3.3) also hold for the variables Y_k and Z_k.

X_μ	$\mathbb{E}_N^{(H)}[X_\mu]$
X_2	0
X_1^2	$\frac{1}{4}$
X_4	0
X_3X_1	0
X_2^2	$\frac{1}{2}$
$X_2X_1^2$	$\frac{1}{4N}$
X_1^4	$\frac{3}{16}$
X_6	0
X_5X_1	$\frac{5}{4N^2}$
X_4X_2	$\frac{1}{N^2}$
$X_4X_1^2$	$\frac{1}{2N}$
X_3^2	$\frac{3}{4} + \frac{3}{4N^2}$
$X_3X_2X_1$	$\frac{3}{4N}$
$X_3X_1^3$	$\frac{3}{8N^2}$
X_2^3	$\frac{1}{N}$
$X_2^2X_1^2$	$\frac{1}{8} + \frac{1}{2N^2}$
$X_2X_1^4$	$\frac{3}{8N}$
X_1^6	$\frac{15}{64}$

Table 4.4: Mixed moments of Chebyshev polynomials of traces of the GUE. Since $\mathbb{E}_N^{(H)}[X_\mu(M)] = 0$ for odd $|\mu|$, we listed correlations corresponding to partitions of even integers.

Y_μ	$\mathbb{E}_N^{(L)}[Y_\mu]$
Y_1	0
Y_2	0
Y_1^2	$\frac{1}{4}+\frac{\gamma}{4N}$
Y_3	0
Y_2Y_1	$\frac{\gamma}{2N}+\frac{\gamma^2}{2N^2}$
Y_1^3	$\frac{1}{4N}+\frac{\gamma}{4N^2}$
Y_4	0
Y_3Y_1	$\frac{3\gamma}{4N}+3(2\gamma^2+1)\frac{1}{4N^2}+3\gamma(\gamma^2+1)\frac{1}{4N^3}$
Y_2^2	$\frac{1}{2}+\frac{\gamma}{N}+(3\gamma^2+1)\frac{1}{2N^2}+\gamma(2\gamma^2+1)\frac{1}{2N^3}$
$Y_2Y_1^2$	$\frac{1}{2N}+\frac{5\gamma}{4N^2}+\frac{3\gamma^2}{4N^3}$
Y_1^4	$\frac{3}{16}+\frac{3\gamma}{8N}+3(\gamma^2+2)\frac{1}{16N^2}+\frac{3\gamma}{8N^3}$
Y_5	0
Y_4Y_1	$\frac{\gamma}{N}+4(\gamma^2+1)\frac{1}{N^2}+\gamma(4\gamma^2+9)\frac{1}{N^3}+\gamma^2(\gamma^2+5)\frac{1}{N^4}$
Y_3Y_2	$\frac{3\gamma}{2N}+3(3\gamma^2+2)\frac{1}{N^2}+3\gamma(3\gamma^2+5)\frac{1}{2N^3}+3\gamma^2(\gamma^2+3)\frac{1}{2N^4}$
$Y_3Y_1^2$	$\frac{3}{4N}+\frac{15\gamma}{4N^2}+3(3\gamma^2+1)\frac{1}{N^3}+3\gamma(\gamma^2+1)\frac{1}{N^4}$
$Y_2^2X_1$	$\frac{1}{N}+\frac{4\gamma}{N^2}+(5\gamma^2+1)\frac{1}{N^3}+\gamma(2\gamma^2+1)\frac{1}{N^4}$
$Y_2Y_1^3$	$\frac{3\gamma}{8N}+3(\gamma^2+2)\frac{1}{4N^2}+3\gamma(\gamma^2+8)\frac{1}{8N^3}+\frac{3\gamma^2}{2N^4}$
Y_1^5	$\frac{5}{8N}+\frac{5\gamma}{4N^2}+(5\gamma^2+6)\frac{1}{8N^3}+\frac{3\gamma}{4N^4}$
Y_6	0
Y_5Y_1	$\frac{5\gamma}{4N}+5(6\gamma^2+11)\frac{1}{4N^2}+5\gamma(11\gamma^2+41)\frac{1}{4N^3}$ $+5(7\gamma^4+45\gamma^2+8)\frac{1}{4N^4}+5\gamma(\gamma^4+12\gamma^2+8)\frac{1}{4N^5}$
Y_4Y_2	$\frac{2\gamma}{2N}+3(3\gamma^2+4)\frac{1}{N^2}+2\gamma(8\gamma^2+23)\frac{1}{N^3}+(11\gamma^4+54\gamma^2+8)\frac{1}{N^4}$ $+2\gamma(\gamma^4+10\gamma^2+4)\frac{1}{N^5}$
$Y_4Y_1^2$	$\frac{1}{N}+\frac{17\gamma}{2N^2}+(18\gamma^2+13)\frac{1}{N^3}+\gamma(26\gamma^2+51)\frac{1}{2N^4}+5\gamma^2(\gamma^2+5)\frac{1}{2N^5}$

(To be continued)

Y_μ	$\mathbb{E}_N^{(L)}[Y_\mu]$
Y_3^2	$\frac{3}{4}+\frac{9\gamma}{4N}+3(6\gamma^2+7)\frac{1}{2N^2}+3\gamma(11\gamma^2+29)\frac{1}{2N^3}$ $+9(8\gamma^4+23\gamma^2+4)\frac{1}{4N^4}+3\gamma(3\gamma^4+25\gamma^2+12)\frac{1}{4N^5}$
$Y_3Y_2Y_1$	$\frac{3}{2N}+\frac{39\gamma}{4N^2}+3(27\gamma^2+14)\frac{1}{4N^3}+3\gamma(21\gamma^2+29)\frac{1}{4N^4}$ $+15\gamma^2(\gamma^2+3)\frac{1}{4N^5}$
$Y_3Y_1^3$	$\frac{9\gamma}{16N}+3(9\gamma^2+25)\frac{1}{16N^2}+3\gamma(9\gamma^2+80)\frac{1}{16N^3}$ $+3(3\gamma^4+75\gamma^2+20)\frac{1}{16N^4}+15\gamma(\gamma^2+1)\frac{1}{4N^5}$
Y_2^3	$\frac{2}{N}+\frac{11\gamma}{N^2}+2(11\gamma^2+4)\frac{1}{N^3}+18\gamma(\gamma^2+1)\frac{1}{N^4}+5\gamma^2(\gamma^2+2)\frac{1}{N^5}$
$Y_2^2Y_1^2$	$\frac{1}{8}+\frac{3\gamma}{8N}+(9\gamma^2+37)\frac{1}{8N^2}+\gamma(13\gamma^2+122)\frac{1}{8N^3}$ $+(6\gamma^4+125\gamma^2+20)\frac{1}{8N^4}+5\gamma(2\gamma^2+1)\frac{1}{2N^5}$
$Y_2Y_1^4$	$\frac{3}{4N}+\frac{25\gamma}{8N^2}+(8\gamma^2+9)\frac{1}{2N^3}+\gamma(13\gamma^2+66)\frac{1}{8N^4}+\frac{15\gamma^2}{4N^5}$
Y_1^6	$\frac{15}{64}+\frac{45}{64N}+5(9\gamma^2+26)\frac{1}{64N^2}+5\gamma(3\gamma^2+52)\frac{1}{64N^3}$ $+5(13\gamma^2+12)\frac{1}{32N^4}+\frac{15\gamma}{8N^5}$

Table 4.5: Mixed moments of Chebyshev polynomials of traces of the LUE.

Z_μ	$\mathbb{E}_N^{(J)}[Z_\mu]$
Z_1	0
Z_2	0
Z_1^2	$\frac{\Gamma(2N+\gamma_1+\gamma_2)}{\Gamma(2N+\gamma_1+\gamma_2+2)}\frac{\Gamma(2N+\gamma_1+\gamma_2-1)}{\Gamma(2N+\gamma_1+\gamma_2+1)}$ $\times 4N(N+\gamma_1)(N+\gamma_2)(N+\gamma_1+\gamma_2)$
Z_3	0
Z_2Z_1	$\frac{\Gamma(2N+\gamma_1+\gamma_2)}{\Gamma(2N+\gamma_1+\gamma_2+3)}\frac{\Gamma(2N+\gamma_1+\gamma_2-2)}{\Gamma(2N+\gamma_1+\gamma_2+1)}$ $\times 16(\gamma_1^2-\gamma_2^2)N(N+\gamma_1)(N+\gamma_2)(N+\gamma_1+\gamma_2)$

(To be continued)

Z_μ	$\mathbb{E}_N^{(J)}[Z_\mu]$
Z_1^3	$-\frac{1}{(2N+\gamma_1+\gamma_2)}\frac{\Gamma(2N+\gamma_1+\gamma_2)}{\Gamma(2N+\gamma_1+\gamma_2+3)}\frac{\Gamma(2N+\gamma_1+\gamma_2-2)}{\Gamma(2N+\gamma_1+\gamma_2+1)}$ $\times 16(\gamma_1^2-\gamma_2^2)N(N+\gamma_1)(N+\gamma_2)(N+\gamma_1+\gamma_2)$

Table 4.6: Mixed moments of Chebyshev polynomials of traces of the JUE.

Chapter 5

Asymptotics

This chapter is a part of the paper *On the moments of characteristic polynomials* [164], which is a joint work with J. P. Keating and F. Mezzadri. The present author entirely carried the project with the advisement from J. P. Keating and F. Mezzadri.

5.1 Introduction

Characteristic polynomials of random matrices have received considerable attention in the recent years. As discussed in Ch. 1, one of the main motivations is due to the connection to number theory to study the Riemann zeta function and other families of $L-$functions. [145, 171, 172]. The non-trivial zeros of the Riemann zeta function $\zeta(s)$ and the eigenvalues of random unitary matrices, both on the scale of their mean spacing, have the same limiting distribution. Not just the unitary matrices, but a wide class of Hermitian random matrices share this remarkable resemblance to the $\zeta-$function.

Consider a rescaled GUE matrix $\mathcal{M} = M/\sqrt{N}$. We choose the rescaling parameter to be $\sqrt{N}$ instead of $\sqrt{4N}$ to make the results consistent with the literature. With this scaling, the asymptotic spectral density is

$$\varrho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbb{1}_{x\in[-2,2]}, \tag{5.1.1}$$

In this chapter, we study the large N limits of correlations $\mathbb{E}_N[\prod_j \det(t_j - \mathcal{M})]$, and moments $\mathbb{E}_N[\det(t-\mathcal{M})^p]$ of characteristic polynomials of $N-$dimensional unitary invariant Hermitian random matrices.

For the GUE, Brezin and Hikami [40] showed that in the Dyson limit, $N \to \infty$, $t_i - t_j \to 0$ and $N(t_i - t_j)$ is finite, the moments of characteristic polynomials are equal to

$$\mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^{2p}\right] = e^{-Np}e^{Np\frac{t^2}{2}}(2\pi N\varrho_{sc}(t))^{p^2}\prod_{j=0}^{p-1}\frac{j!}{(p+j)!}. \tag{5.1.2}$$

This should be compared to the $2p^{th}$ moment of the zeta function which is conjectured to

be [55, 171]

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2p}\,dT \sim a_p(\log T)^{p^2}\prod_{j=0}^{p-1}\frac{j!}{(p+j)!}, \quad \text{as } T\to\infty. \tag{5.1.3}$$

Here a_p, given in (1.6.16), is a constant related to the Dirichlet coefficient. As already discussed in Sec. 1.6, random matrices and $\zeta-$function have several features in common: characteristic polynomial replaces the Riemann $\zeta-$function, one-point density $2\pi N\varrho_{sc}(t)$ replaces $\log T$, and $\prod_{j=0}^{p-1} j!/(p+j)!$ is a universal constant that holds its place.

In this chapter, we investigate the conditions under which the semi-circle law in (5.1.2) is recovered. As mentioned in Ch. 3, the correlations of characteristic polynomials have a determinantal structure involving classical orthogonal polynomials. For the GUE, these correlations are given by

$$\mathbb{E}_N^{(H)}\big[\prod_{j=1}^{2p}\det(t_j-\mathcal{M})\big] = \frac{1}{\Delta(t_1,\ldots,t_{2p})}\det[N^{-\frac{N-j}{2}-p}H_{N+2p-j}(\sqrt{N}t_k)]_{1\le j,k\le 2p}. \tag{5.1.4}$$

As a consequence, the moments also take a determinantal form comprising derivatives of Hermite polynomials. But the Hermite polynomials depend on the parity of the degree n via

$$H_n(-x) = (-1)^n H_n(x). \tag{5.1.5}$$

Therefore, the moments of characteristic polynomials also depend on the parity of the degree, which in turn depends on the parity of the dimension of the matrix N. As a result, we expect that the asymptotic behaviour of the moments of characteristic polynomials should be different for even and odd dimensional matrices. This parity dependence of Hermite polynomials is captured by our approach and is reflected in the large N limit. This is in contrast to other Hermitian ensembles such as the LUE and the JUE. For the LUE and the JUE, both even and odd dimensional matrices have the same contribution to the moments.

In [40], the authors used orthogonal polynomial techniques to arrive at (5.1.2). In fact, all the studies on the asymptotics of the moments of characteristic polynomials rely on the orthogonal polynomial and saddle point techniques [19, 40, 41], the Riemann-Hilbert method [224], Hankel determinants with Fisher-Hartwig symbols [95, 121, 175], and supersymmetric tools [12,107,118,226]. In the present chapter, we express moments in terms of the multivariate orthogonal polynomials and take a combinatorial approach to compute the asymptotics of moments using the properties of these polynomials. By doing so, we discover that the even and odd dimensional GUE matrices give different contributions in the large N limit, and that only a formal average between these two contributions gives the semi-circle law. In Sec. 5.4.2.1, this phenomenon is discussed in detail for the second moment of the characteristic polynomial.

In addition to connections with number theory, characteristic polynomials have found numerous applications in quantum chaos [12], mesoscopic systems [106], quantum chromodynamics [61], and in a variety of combinatorial problems [71, 223]. The asymptotic study of negative moments and ratios of characteristic polynomials is another active area of research, see for example Sec. 1.7 in Ch. 1 and also [7, 19, 31, 36, 39, 96, 108, 110, 120, 224]. More recently, the statistics of the maximum of the characteristic polynomial are being extensively studied,

motivated by the relations to logarithmically correlated Gaussian processes. For example, see [109, 111, 113, 115] and references therein. We expect that the techniques developed here will have applications to those calculations as well.

This chapter is structured as follows. After introducing the required tools in Sec. 5.2, we recall the moments of characteristic polynomials of the GUE, LUE and JUE in Sec. 5.3. In Sec. 5.4, we compute the asymptotics of moments of the GUE and illustrate how to recover the semi-circle law in the limit as the matrix size goes to infinity. In the last section Sec. 5.5, as an application of the results discussed, we compute the correlators of secular coefficients, which are the coefficients of a characteristic polynomial when expanded as a function of the spectral variable.

5.2 Background

The joint probability density function for the rescaled GUE, LUE and JUE is

$$\rho^{(H)}(x_1,\dots,x_N) = \frac{1}{\mathcal{Z}_N^{(H)}}\Delta^2(x_1,\dots,x_N)\prod_{j=1}^{N} e^{-\frac{Nx_j^2}{2}}, \tag{5.2.1}$$

$$\rho^{(L)}(x_1,\dots,x_N) = \frac{1}{\mathcal{Z}_N^{(L)}}\Delta^2(x_1,\dots,x_N)\prod_{j=1}^{N} x_j^{\gamma} e^{-2Nx_j}, \tag{5.2.2}$$

$$\rho^{(J)}(x_1,\dots,x_N) = \frac{1}{\mathcal{Z}_N^{(J)}}\Delta^2(x_1,\dots,x_N)\prod_{j=1}^{N} x_j^{\gamma_1}(1-x_j)^{\gamma_2}, \tag{5.2.3}$$

with

$$\mathcal{Z}_N^{(H)} = \frac{(2\pi)^{\frac{N}{2}}}{N^{\frac{N^2}{2}}}\prod_{j=1}^{N} j!, \tag{5.2.4}$$

$$\mathcal{Z}_N^{(L)} = \frac{N!}{(2N)^{N(N+\gamma)}} G_0(N,\gamma)G_0(N,0), \tag{5.2.5}$$

$$\mathcal{Z}_N^{(J)} = N!\prod_{j=0}^{N-1}\frac{j!\,\Gamma(j+\gamma_1+1)\Gamma(j+\gamma_2+1)\Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+2)\Gamma(2j+\gamma_1+\gamma_2+1)}. \tag{5.2.6}$$

Note that $\mathcal{Z}_N^{(J)}$ is same as $Z_N^{(J)}$ in (3.5.51) as no rescaling is required for the Jacobi ensemble. Here $G_\lambda(N,\gamma)$ is given in (3.2.6). The multivariate polynomials

$$\mathcal{H}_\mu(x_1,\dots,x_n) = \frac{\det[N^{-\frac{1}{2}(\mu_j+n-j)}H_{\mu_j+n-j}(\sqrt{N}x_k)]}{\Delta(x_1,\dots,x_n)}, \tag{5.2.7}$$

$$\mathcal{L}_\mu^{(\gamma)}(x_1,\dots,x_n) = \frac{\det[(\frac{-1}{2N})^{\mu_j+n-j}(\mu_j+n-j)!L_{\mu_j+n-j}^{(\gamma)}(2Nx)]}{\Delta(x_1,\dots,x_n)}, \tag{5.2.8}$$

$$\mathcal{J}_\mu^{(\gamma_1,\gamma_2)}(x_1,\dots,x_n) = \frac{\det[(-1)^{\mu_j+n-j}(\mu_j+n-j)!\frac{\Gamma(\mu_j+n-j+\gamma_1+\gamma_2+1)}{\Gamma(2(\mu_j+n-j)+\gamma_1+\gamma_2+1)}J_{\mu_j+n-j}^{(\gamma_1,\gamma_2)}(x)]}{\Delta(x_1,\dots,x_n)}, \tag{5.2.9}$$

are orthogonal with respect to the densities in (5.2.1), (5.2.2) and (5.2.3). The orthogonality relations read to be

$$\frac{1}{\mathcal{Z}_n^{(H)}}\int_{(-\infty,\infty)^n}\mathcal{H}_\mu(\mathbf{x})\mathcal{H}_\nu(\mathbf{x})\Delta^2(\mathbf{x})\prod_{j=1}^n e^{-\frac{Nx_j^2}{2}}\,dx_j=\frac{1}{N^{|\mu|}}C_\mu(n)\delta_{\mu\nu}, \tag{5.2.10}$$

$$\frac{1}{\mathcal{Z}_n^{(L)}}\int_{[0,\infty)^n}\mathcal{L}_\mu(\mathbf{x})\mathcal{L}_\nu(\mathbf{x})\Delta^2(\mathbf{x})\prod_{j=1}^n x_j^\gamma e^{-2Nx_j}\,dx_j=\frac{1}{(2N)^{2|\mu|}}\frac{G_\mu(n,\gamma)}{G_0(n,\gamma)}C_\mu(n)\delta_{\mu\nu}, \tag{5.2.11}$$

$$\begin{aligned}&\frac{1}{\mathcal{Z}_n^{(J)}}\int_{[0,1]^n}\mathcal{J}_\mu^{(\gamma_1,\gamma_2)}(\mathbf{x})\mathcal{J}_\nu^{(\gamma_1,\gamma_2)}(\mathbf{x})\Delta^2(\mathbf{x})\prod_{j=1}^n x_j^{\gamma_1}(1-x_j)^{\gamma_2}\,dx_j\\ =&\prod_{j=1}^N\frac{\Gamma(2n-2j+\gamma_1+\gamma_2+1)\Gamma(2n-2j+\gamma_1+\gamma_2+2)}{\Gamma(2\lambda_j+2n-2j+\gamma_1+\gamma_2+1)\Gamma(2\lambda_j+2n-2j+\gamma_1+\gamma_2+2)}\\ &\times\frac{G_\mu(n,\gamma_1+\gamma_2)G_\mu(n,\gamma_1)G_\mu(n,\gamma_2)}{G_0(n,\gamma_1+\gamma_2)G_0(n,\gamma_1)G_0(n,\gamma_2)}C_\mu(n)\delta_{\lambda\mu},\end{aligned} \tag{5.2.12}$$

where C_μ is given in (3.2.6).

The polynomials $\mathcal{H}_\mu$, $\mathcal{L}_\mu^{(\gamma)}$ and $\mathcal{J}_\mu^{(\gamma_1,\gamma_2)}$ are chosen such that the leading coefficient of these polynomials in the Schur basis is 1. More precisely,

$$\Phi_\lambda(x_1,\dots,x_n)=\sum_{\mu\subseteq\lambda}\Upsilon_{\lambda\mu}S_\mu(x_1,\dots,x_n), \tag{5.2.13}$$

where Φ_λ is one of the $\mathcal{H}_\lambda$, $\mathcal{L}_\lambda^{(\gamma)}$, $\mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}$, and

$$\begin{aligned}\Upsilon_{\lambda\mu}^{(H)}&=\frac{1}{N^{\frac{|\lambda|-|\mu|}{2}}}\kappa_{\lambda\mu}^{(H)}\\ &=\left(\frac{-1}{2N}\right)^{\frac{|\lambda|-|\mu|}{2}}\frac{C_\lambda(n)}{C_\mu(n)}D_{\lambda\mu}^{(H)},\end{aligned} \tag{5.2.14}$$

$$\begin{aligned}\Upsilon_{\lambda\mu}^{(L)}&=\frac{(-1)^{|\lambda|+\frac{1}{2}N(N-1)}}{(2N)^{|\lambda|-|\mu|}}G_\lambda(N,0)\kappa_{\lambda\mu}^{(L)}\\ &=\left(\frac{-1}{2N}\right)^{|\lambda|-|\mu|}\frac{G_\lambda(n,\gamma)G_\lambda(n,0)}{G_\mu(n,\gamma)G_\mu(n,0)}D_{\lambda\mu}^{(L)},\end{aligned} \tag{5.2.15}$$

$$\begin{aligned}\Upsilon_{\lambda\mu}^{(J)}&=(-1)^{|\lambda|+\frac{1}{2}n(n-1)}\frac{G_\lambda(n,\gamma_1+\gamma_2)G_\lambda(n,\gamma_1)}{\prod_{j=1}^n\Gamma(2\lambda_j+2n-2j+\gamma_1+\gamma_2+1)}\kappa_{\lambda\mu}^{(J)}\\ &=(-1)^{|\lambda|+|\mu|}\left(\prod_{j=1}^n\frac{1}{\Gamma(2\lambda_j+2n-2j+\gamma_1+\gamma_2+1)}\right)\frac{G_\lambda(n,\gamma_1)G_\lambda(n,0)}{G_\mu(n,\gamma_1)G_\mu(n,0)}\tilde{\mathcal{D}}_{\lambda\mu}^{(J)}.\end{aligned} \tag{5.2.16}$$

Here $\kappa_{\lambda\mu}^{(H)}$, $\kappa_{\lambda\mu}^{(L)}$ and $\kappa_{\lambda\mu}^{(J)}$ are given in (3.5.16), (3.5.44) and (3.5.56), respectively; and $D_{\lambda\nu}^{(H)}$, $D_{\lambda\nu}^{(L)}$ and $D_{\lambda\nu}^{(J)}$ are given in (3.5.17), (3.5.45) and (3.5.61), respectively. Equivalently, Schur polynomials can be expanded as

$$S_\lambda(x_1,\dots,x_n)=\sum_{\nu\subseteq\lambda}\Psi_{\lambda\nu}\Phi_\nu(x_1,\dots,x_n), \tag{5.2.17}$$

where

$$\Psi_{\lambda\nu}^{(H)} = \frac{1}{N^{\frac{|\lambda|-|\nu|}{2}}}\psi_{\lambda\nu}^{(H)}$$
$$= \left(\frac{1}{2N}\right)^{\frac{|\lambda|-|\nu|}{2}} \frac{C_\lambda(n)}{C_\nu(n)} D_{\lambda\nu}^{(H)}, \tag{5.2.18}$$
$$\Psi_{\lambda\nu}^{(L)} = \frac{(-1)^{|\nu|+\frac{1}{2}n(n-1)}}{(2N)^{|\lambda|-|\nu|}} \frac{1}{G_\nu(n,0)}\psi_{\lambda\nu}^{(L)}$$
$$= \frac{1}{(2N)^{|\lambda|-|\nu|}} \frac{G_\lambda(n,\gamma)G_\lambda(n,0)}{G_\nu(n,\gamma)G_\nu(n,0)} D_{\lambda\mu}^{(L)}, \tag{5.2.19}$$
$$\Psi_{\lambda\nu}^{(J)} = (-1)^{|\nu|+\frac{1}{2}n(n-1)} \frac{\prod_{j=1}^{n}\Gamma(2\nu_j+2n-2j+\gamma_1+\gamma_2+1)}{G_\nu(n,\gamma_1+\gamma_2)G_\nu(n,0)}\psi_{\lambda\mu}^{(J)}$$
$$= \frac{G_\lambda(n,\gamma_1)G_\lambda(n,0)}{G_\nu(n,\gamma_1)G_\nu(n,0)} \left(\prod_{j=1}^{n}\Gamma(2\nu_j+2n-2j+\gamma_1+\gamma_2+2)\right) \mathcal{D}_{\lambda\nu}^{(J)}, \tag{5.2.20}$$

with $\mathcal{D}_{\lambda\nu}^{(J)}$ same as given in (3.5.57), and $\psi_{\lambda\nu}^{(H)}$, $\psi_{\lambda\nu}^{(L)}$, $\psi_{\lambda\nu}^{(J)}$ are given in (3.5.16), (3.5.44), (3.5.56), respectively.

5.3 Moments of characteristic polynomials

The identity in (2.1.164) is the main tool to study the moments of characteristic polynomials.

Proposition 5.3.1. *We have,*

$$(a) \quad \mathbb{E}_N^{(H)}\Big[\prod_{j=1}^{p}\det(t_j-\mathcal{M})\Big] = \mathcal{H}_{(N^p)}(t_1,\ldots,t_p) \tag{5.3.1a}$$

$$(b) \quad \mathbb{E}_N^{(L)}\Big[\prod_{j=1}^{p}\det(t_j-\mathcal{M})\Big] = \mathcal{L}_{(N^p)}^{(\gamma)}(t_1,\ldots,t_p) \tag{5.3.1b}$$

$$(c) \quad \mathbb{E}_N^{(J)}\Big[\prod_{j=1}^{p}\det(t_j-M)\Big] = \mathcal{J}_{(N^p)}^{(\gamma_1,\gamma_2)}(t_1,\ldots,t_p) \tag{5.3.1c}$$

Note that the polynomials used in (5.2.7), (5.2.8) and (5.2.9) are monic, which is why there are no pre factors in (5.3.1) as compared to Thm. 3.2.2 in Ch. 3. Moments can be readily computed from the above formulae by taking the limit $t_j \to t$ for $j = 1,\ldots,p$. This leads to a determinantal formula for the moments involving the derivatives of orthogonal polynomials:

$$\frac{(-1)^{\frac{1}{2}p(p-1)}}{\prod_{j=0}^{p-1} j!} \begin{vmatrix} \varphi_N(t) & \varphi_{N+1}(t) & \ldots & \varphi_{N+p-1}(t) \\ \varphi_N'(t) & \varphi_{N+1}'(t) & \ldots & \varphi_{N+p-1}'(t) \\ \vdots & \vdots & & \vdots \\ \varphi_N^{(p-1)}(t) & \varphi_{N+1}^{(p-1)}(t) & \ldots & \varphi_{N+p-1}^{(p-1)}(t) \end{vmatrix}. \tag{5.3.2}$$

Here $\varphi_n(t)$ are the rescaled Hermite, Laguerre and Jacobi polynomials for the GUE, LUE and JUE, respectively. The polynomials $\varphi_n(t)$ are rescaled to account for rescaling the matrix and

to make $\varphi_n(t)$ monic. By expressing the multivariate polynomials in the Schur basis and using

$$C_\lambda(n) = \prod_{j=1}^{l(\lambda)} \frac{(\lambda_j + n - j)!}{(n-j)!} = |\lambda|! \frac{S_\lambda(1^n)}{\dim V_\lambda}, \tag{5.3.3}$$

we have the following proposition.

Proposition 5.3.2. *Let $\lambda = (N^p)$. The moments of the characteristic polynomial are given by*

$$\mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^p\right] = C_\lambda(p) \sum_{\nu\subseteq\lambda} \left(\frac{-1}{2N}\right)^{\frac{|\lambda|-|\nu|}{2}} \frac{\dim V_\nu}{|\nu|!} D_{\lambda\nu}^{(H)} t^{|\nu|} \tag{5.3.4}$$

$$\mathbb{E}_N^{(L)}\left[\det(t-\mathcal{M})^p\right] = \left(\frac{-1}{2N}\right)^{Np} \frac{G_\lambda(p,\gamma)G_\lambda(p,0)}{G_0(p,0)} \sum_{\nu\subseteq\lambda} \frac{(-2N)^{|\nu|}}{G_\nu(p,\gamma)} \frac{\dim V_\nu}{|\nu|!} D_{\lambda\nu}^{(L)} t^{|\nu|} \tag{5.3.5}$$

$$\begin{aligned}\mathbb{E}_N^{(J)}\left[\det(t-M)^p\right] = &\left(\prod_{j=N}^{N+p-1} \frac{1}{\Gamma(2j+\gamma_1+\gamma_2+1)}\right)(-1)^{Np} \frac{G_\lambda(p,\gamma_1)G_\lambda(p,0)}{G_0(p,0)} \\ &\times \sum_{\nu\subseteq\lambda} \frac{(-1)^{|\nu|}}{|\nu|!\, G_\nu(p,\gamma_1)} \dim V_\nu \tilde{\mathcal{D}}_{\lambda\nu}^{(J)} t^{|\nu|}\end{aligned} \tag{5.3.6}$$

These expansions are exact and hold for any finite N. In the next section we are concerned with the large N asymptotics of the moments of characteristic polynomials. In the large N limit, only even moments are interesting since the odd moments result in oscillatory behaviour.

5.4 Asymptotics

In the rest of the chapter, we consider the asymptotics of the moments of characteristic polynomials for the GUE. By exploiting the integral representation of classical orthogonal polynomials, Brezin and Hikami [40] showed that in the Dyson limit, $N \to \infty$, $t_i - t_j \to 0$ and $N(t_i - t_j)$ is finite, the moments of characteristic polynomials behave as

$$\mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^{2p}\right] = e^{-Np} e^{Np\frac{t^2}{2}} (2\pi N \varrho_{sc}(t))^{p^2} \prod_{j=0}^{p-1} \frac{j!}{(p+j)!}, \quad \text{as } N \to \infty, \tag{5.4.1}$$

where the asymptotic eigenvalue density is

$$\varrho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}. \tag{5.4.2}$$

Using (5.3.4), we show in Sec. 5.4.1 that

$$\mathbb{E}_N^{(H)}\left[\det(\mathcal{M})^{2p}\right] = e^{-Np}(2N)^{p^2} \prod_{j=0}^{p-1} \frac{j!}{(p+j)!}, \tag{5.4.3}$$

which coincides with (5.4.1) for $t = 0$. For $t \neq 0$, we discover that the asymptotic behaviour is different for even and odd dimensional GUE matrices and these contributions combine in a

special way to produce the semi-circle law. These cases are discussed in Sec. 5.4.1 and Sec. 5.4.2 in more detail.

5.4.1 At the center of the semi-circle

Let $\lambda = (N^{2p})$. For any finite N,

$$\mathbb{E}_N^{(H)}\left[\det \mathcal{M}^{2p}\right] = \left(-\frac{1}{2N}\right)^{Np} C_\lambda(2p) D_{\lambda 0}^{(H)}. \tag{5.4.4}$$

Proposition 5.4.1. *We have*

$$\begin{aligned} D_{\lambda 0}^{(H)} &= \prod_{j=0}^{p-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m,\ m \in \mathbb{N}, \\ D_{\lambda 0}^{(H)} &= (-1)^p \frac{m!}{(m+p)!} \prod_{j=0}^{p-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m+1, \quad m \in \mathbb{N}. \end{aligned} \tag{5.4.5}$$

Proof. If $N = 2m$, then

$$\begin{aligned} D_{\lambda 0}^{(H)} &= \det\left[\mathbb{1}_{k-j=0 \bmod 2}\left((m+\tfrac{k-j}{2})!\right)^{-1}\right]_{1\le j,k\le p} \\ &= \prod_{j=0}^{p-1} \frac{1}{(m+j)!^2} \begin{vmatrix} 1 & 0 & m & 0 & \dots & \frac{m!}{(m-p+1)!} & 0 \\ 0 & 1 & 0 & m & \dots & 0 & \frac{m!}{(m-p+1)!} \\ 1 & 0 & m+1 & 0 & \dots & \frac{(m+1)!}{(m-p+2)!} & 0 \\ 0 & 1 & 0 & m+1 & \dots & 0 & \frac{(m+1)!}{(m-p+2)!} \\ & & & & \vdots & & \\ 1 & 0 & m+p-1 & 0 & \dots & \frac{(m+p-1)!}{m!} & 0 \\ 0 & 1 & 0 & m+p-1 & \dots & 0 & \frac{(m+p-1)!}{m!} \end{vmatrix}. \end{aligned} \tag{5.4.6}$$

Do the row operations $R_{2j} = R_{2j} - R_{2j-2}$, $R_{2j-1} = R_{2j-1} - R_{2j-3}$ with j running from $p, p-1, \dots, 2$ in that order. Using the Pascal's rule for binomial coefficients,

$$\binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}, \tag{5.4.7}$$

we obtain

$$D_{\lambda 0}^{(H)} = (p-1)!^2 \prod_{j=0}^{p-1} \frac{1}{(m+j)!^2} \begin{vmatrix} 1 & 0 & m & 0 & \dots & \frac{m!}{(m-p+1)!} & 0 \\ 0 & 1 & 0 & m & \dots & 0 & \frac{m!}{(m-p+1)!} \\ 0 & 0 & 1 & 0 & \dots & \frac{m!}{(m-p+2)!} & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & \frac{m!}{(m-p+2)!} \\ & & & & \vdots & & \\ 0 & 0 & 1 & 0 & \dots & \frac{(m+p-2)!}{m!} & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & \frac{(m+p-2)!}{m!} \end{vmatrix}. \tag{5.4.8}$$

Next perform $R_{2j} = R_{2j} - R_{2j-2}$, $R_{2j-1} = R_{2j-1} - R_{2j-3}$ with j running from $p, p-1, \ldots, 3$ in that order. Repeat this process $p-2$ more times to reach an upper triangular matrix, with 1's on the diagonal, whose determinant is given in (5.4.5). Similarly, $D_{\lambda 0}^{(H)}$ can be calculated for N odd. ■

Define

$$\begin{aligned} D_e(N) &= \prod_{j=0}^{p-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m, \\ D_o(N) &= (-1)^p \frac{m!}{(m+p)!} \prod_{j=0}^{p-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m+1. \end{aligned} \tag{5.4.9}$$

Using this notation, (5.4.4) reads to be

$$\mathbb{E}_N^{(H)} \left[(\det \mathcal{M})^{2p} \right] = \left(-\frac{1}{2N} \right)^{Np} \times \begin{cases} C_\lambda(2p) D_e(N), & N \text{ even}, \\ C_\lambda(2p) D_o(N), & N \text{ odd}, \end{cases} \tag{5.4.10}$$

The functions $C_\lambda(2p)D_e(N)$ and $C_\lambda(2p)D_o(N)$, $\lambda = (N^{2p})$, are the ratios of the factorials:

$$\begin{aligned} &C_{(N^{2p})}(2p) D_e(N) = \prod_{j=0}^{p-1} \frac{(2m+j)!(2m+p+j)!}{(m+j)!^2} \frac{j!}{(p+j)!}, \quad N = 2m, \\ &C_{(N^{2p})}(2p) D_o(N) \\ &= (-1)^p \frac{m!}{(m+p)!} \prod_{j=0}^{p-1} \frac{(2m+1+j)!(2m+1+p+j)!}{(m+j)!^2} \frac{j!}{(p+j)!}, \quad N = 2m+1. \end{aligned} \tag{5.4.11}$$

Denote

$$\gamma_p = \prod_{j=0}^{p-1} \frac{j!}{(p+j)!}. \tag{5.4.12}$$

It is interesting to note the presence of the universal constant γ_p in the moments for any finite N. To compute the large N limit, we require the asymptotic expansion of (5.4.11). In App. B, we compute the first few terms in this expansion. As $N \to \infty$,

$$\begin{aligned} C_{(N^{2p})}(2p) D_e(N) &\sim e^{-Np} (2N)^{Np+p^2} \gamma_p \left[1 + \frac{p}{6N}(4p^2+1) + O(N^{-2}) \right], \quad N \text{ even}, \\ C_{(N^{2p})}(2p) D_o(N) &\sim (-1)^p e^{-Np} (2N)^{Np+p^2} \gamma_p \left[1 + \frac{p}{3N}(2p^2-1) + O(N^{-2}) \right], \; N \text{ odd}. \end{aligned} \tag{5.4.13}$$

Plugging (5.4.13) in (5.4.10), the leading term in the moments is

$$e^{-Np} (2N)^{p^2} \gamma_p, \tag{5.4.14}$$

for both N even and N odd, which coincides with (5.4.1) for $t = 0$. On the other hand, the sub-leading behaviour depends on the parity of N.

5.4.2 Outside the centre of the semi-circle

For $t_j = t$, by using (5.3.3),

$$\mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^{2p}\right] = C_\lambda(2p)\sum_{\nu\subseteq\lambda}\left(-\frac{1}{2N}\right)^{\frac{|\lambda|-|\nu|}{2}}\frac{\dim V_\nu}{|\nu|!}D_{\lambda\nu}^{(H)}t^{|\nu|}. \tag{5.4.15}$$

To compute the asymptotics near the center of the semi-circle, $t \neq 0$, we need to evaluate $D_{\lambda\nu}^{(H)}$ for a non-empty partition ν. In Table. 5.1, we give the values of $D_{\lambda\nu}^{(H)}$ when ν runs over the first few partitions.

$D_{\lambda\nu}^{(H)}$	$N=2m$	$N=2m+1$
$D_{\lambda 0}^{(H)}$	D_e	D_o
$D_{\lambda(2)}^{(H)}$	mpD_e	mpD_o
$D_{\lambda(1^2)}^{(H)}$	$-mpD_e$	$-(m+1)pD_o$
$D_{\lambda(4)}^{(H)}$	$\frac{1}{2}m(m-1)p(p+1)D_e$	$\frac{1}{2}m(m-1)p(p+1)D_o$
$D_{\lambda(3,1)}^{(H)}$	$-\frac{1}{2}m(m-1)p(p+1)D_e$	$-\frac{1}{2}m(m+1)p(p+1)D_o$
$D_{\lambda(2^2)}^{(H)}$	$m^2p^2D_e$	$m(m+1)p^2D_o$
$D_{\lambda(1^2 2)}^{(H)}$	$-\frac{1}{2}m(m+1)p(p-1)D_e$	$-\frac{1}{2}m(m+1)p(p-1)D_o$
$D_{\lambda(1^4)}^{(H)}$	$\frac{1}{2}m(m+1)p(p-1)D_e$	$\frac{1}{2}(m+2)(m+1)p(p-1)D_o$
$D_{\lambda(6)}^{(H)}$	$\frac{1}{6}\frac{m!}{(m-3)!}\frac{(p+2)!}{(p-1)!}D_e$	$\frac{1}{6}\frac{m!}{(m-3)!}\frac{(p+2)!}{(p-1)!}D_o$
$D_{\lambda(5,1)}^{(H)}$	$-\frac{1}{6}\frac{m!}{(m-3)!}\frac{(p+2)!}{(p-1)!}D_e$	$-\frac{1}{6}\frac{(m+1)!}{(m-2)!}\frac{(p+2)!}{(p-1)!}D_o$
$D_{\lambda(4,2)}^{(H)}$	$\frac{1}{2}m^2(m-1)p^2(p+1)D_e$	$\frac{1}{2}\frac{(m+1)!}{(m-2)!}p^2(p+1)D_o$
$D_{\lambda(1^2 4)}^{(H)}$	$-\frac{1}{3}\frac{(m+1)!}{(m-2)!}\frac{(p+1)!}{(p-2)!}D_e$	$-\frac{1}{3}\frac{(m+1)!}{(m-2)!}\frac{(p+1)!}{(p-2)!}D_o$
$D_{\lambda(3^2)}^{(H)}$	$-\frac{1}{2}m^2(m-1)p^2(p+1)D_e$	$-\frac{1}{2}m^2(m+1)p^2(p+1)D_o$
$D_{\lambda(3,2,1)}^{(H)}$	0	0
$D_{\lambda(1^3 3)}^{(H)}$	$\frac{1}{3}\frac{(m+1)!}{(m-2)!}\frac{(p+1)!}{(p-2)!}D_e$	$\frac{1}{3}\frac{(m+2)!}{(m-1)!}\frac{(p+1)!}{(p-2)!}D_o$
$D_{\lambda(2^3)}^{(H)}$	$\frac{1}{2}m^2(m+1)p^2(p-1)D_e$	$\frac{1}{2}m(m+1)^2p^2(p-1)D_o$

(To be continued)

$D^{(H)}_{\lambda\nu}$	$N=2m$	$N=2m+1$
$D^{(H)}_{\lambda(1^2 2^2)}$	$-\frac{1}{2}m^2(m+1)p^2(p-1)D_e$	$-\frac{1}{2}\frac{(m+2)!}{(m-1)!}p^2(p-1)D_o$
$D^{(H)}_{\lambda(1^4 2)}$	$\frac{1}{6}\frac{(m+2)!}{(m-1)!}\frac{p!}{(p-3)!}D_e$	$\frac{1}{6}\frac{(m+2)!}{(m-1)!}\frac{p!}{(p-3)!}D_o$
$D^{(H)}_{\lambda(1^6)}$	$-\frac{1}{6}\frac{(m+2)!}{(m-1)!}\frac{p!}{(p-3)!}D_e$	$-\frac{1}{6}\frac{(m+3)!}{m!}\frac{p!}{(p-3)!}D_o$
$D^{(H)}_{\lambda(8)}$	$\frac{1}{24}\frac{m!}{(m-4)!}\frac{(p+3)!}{(p-1)!}D_e$	$\frac{1}{24}\frac{m!}{(m-4)!}\frac{(p+3)!}{(p-1)!}D_o$
$D^{(H)}_{\lambda(7,1)}$	$-\frac{1}{24}\frac{m!}{(m-4)!}\frac{(p+3)!}{(p-1)!}D_e$	$-\frac{1}{24}\frac{(m+1)!}{(m-3)!}\frac{(p+3)!}{(p-1)!}D_o$
$D^{(H)}_{\lambda(6,2)}$	$\frac{1}{6}m\frac{m!}{(m-3)!}p\frac{(p+2)!}{(p-1)!}D_e$	$\frac{1}{6}\frac{(m+1)!}{(m-3)!}p\frac{(p+2)!}{(p-1)!}D_o$
$D^{(H)}_{\lambda(1^2 6)}$	$-\frac{1}{8}\frac{(m+1)!}{(m-3)!}\frac{(p+2)!}{(p-2)!}D_e$	$-\frac{1}{8}\frac{(m+1)!}{(m-3)!}\frac{(p+2)!}{(p-2)!}D_o$
$D^{(H)}_{\lambda(5,3)}$	$-\frac{1}{6}m\frac{m!}{(m-3)!}p\frac{(p+2)!}{(p-1)!}D_e$	$-\frac{1}{6}m\frac{(m+1)!}{(m-2)!}p\frac{(p+2)!}{(p-1)!}D_o$
$D^{(H)}_{\lambda(5,2,1)}$	0	0
$D^{(H)}_{\lambda(1^3 5)}$	$\frac{1}{8}\frac{(m+1)!}{(m-3)!}\frac{(p+2)!}{(p-2)!}D_e$	$\frac{1}{8}\frac{(m+2)!}{(m-2)!}\frac{(p+2)!}{(p-2)!}D_o$
$D^{(H)}_{\lambda(4^2)}$	$\frac{1}{4}m^2(m-1)^2p^2(p+1)^2D_e$	$\frac{1}{4}m\frac{(m+1)!}{(m-2)!}p^2(p+1)^2D_o$
$D^{(H)}_{\lambda(4,3,1)}$	$-\frac{1}{12}m\frac{(m+1)!}{(m-2)!}p\frac{(p+1)!}{(p-2)!}D_e$	$-\frac{1}{12}m\frac{(m+1)!}{(m-2)!}p\frac{(p+1)!}{(p-2)!}D_o$
$D^{(H)}_{\lambda(2^2 4)}$	$\frac{1}{3}m\frac{(m+1)!}{(m-2)!}p\frac{(p+1)!}{(p-2)!}D_e$	$\frac{1}{3}(m+1)\frac{(m+1)!}{(m-2)!}p\frac{(p+1)!}{(p-2)!}D_o$
$D^{(H)}_{\lambda(1^2 24)}$	$-\frac{1}{4}m\frac{(m+1)!}{(m-2)!}p\frac{(p+1)!}{(p-2)!}D_e$	$-\frac{1}{4}\frac{(m+2)!}{(m-2)!}p\frac{(p+1)!}{(p-2)!}D_o$
$D^{(H)}_{\lambda(1^4 4)}$	$\frac{1}{8}\frac{(m+2)!}{(m-2)!}\frac{(p+1)!}{(p-3)!}D_e$	$\frac{1}{8}\frac{(m+2)!}{(m-2)!}\frac{(p+1)!}{(p-3)!}D_o$
$D^{(H)}_{\lambda(23^2)}$	$-\frac{1}{4}m\frac{(m+1)!}{(m-2)!}p\frac{(p+1)!}{(p-2)!}D_e$	$-\frac{1}{4}m^2(m+1)^2p\frac{(p+1)!}{(p-2)!}D_o$
$D^{(H)}_{\lambda(1^2 3^2)}$	$\frac{1}{3}m\frac{(m+1)!}{(m-2)!}p\frac{(p+1)!}{(p-2)!}D_e$	$\frac{1}{3}m\frac{(m+2)!}{(m-1)!}p\frac{(p+1)!}{(p-2)!}D_o$
$D^{(H)}_{\lambda(3,2,2,1)}$	$-\frac{1}{12}m\frac{(m+1)!}{(m-2)!}p\frac{(p+1)!}{(p-2)!}D_e$	$-\frac{1}{12}(m+1)\frac{(m+2)!}{(m-1)!}p\frac{(p+1)!}{(p-2)!}D_o$
$D^{(H)}_{\lambda(1^3 23)}$	0	0
$D^{(H)}_{\lambda(1^5 3)}$	$-\frac{1}{8}\frac{(m+2)!}{(m-2)!}\frac{(p+1)!}{(p-3)!}D_e$	$-\frac{1}{8}\frac{(m+3)!}{(m-1)!}\frac{(p+1)!}{(p-3)!}D_o$
$D^{(H)}_{\lambda(2^4)}$	$\frac{1}{4}m^2(m+1)^2p^2(p-1)^2D_e$	$\frac{1}{4}(m+1)\frac{(m+2)!}{(m-1)!}p^2(p-1)^2D_o$

(To be continued)

$D^{(H)}_{\lambda\nu}$	$N=2m$	$N=2m+1$
$D^{(H)}_{\lambda(1^2 2^3)}$	$-\frac{1}{6}m\frac{(m+2)!}{(m-1)!}p\frac{p!}{(p-3)!}D_e$	$-\frac{1}{6}(m+1)\frac{(m+2)!}{(m-1)!}p\frac{p!}{(p-3)!}D_o$
$D^{(H)}_{\lambda(1^4 2^2)}$	$\frac{1}{6}m\frac{(m+2)!}{(m-1)!}p\frac{p!}{(p-3)!}D_e$	$\frac{1}{6}\frac{(m+3)!}{(m-1)!}p\frac{p!}{(p-3)!}2D_o$
$D^{(H)}_{\lambda(1^6 2)}$	$-\frac{1}{24}\frac{(m+3)!}{(m-1)!}\frac{p!}{(p-4)!}D_e$	$-\frac{1}{24}\frac{(m+3)!}{(m-1)!}\frac{p!}{(p-4)!}D_o$
$D^{(H)}_{\lambda(1^8)}$	$\frac{1}{24}\frac{(m+3)!}{(m-1)!}\frac{p!}{(p-4)!}D_e$	$\frac{1}{24}\frac{(m+4)!}{m!}\frac{p!}{(p-4)!}D_o$

Table 5.1: The values of determinant $D^{(H)}_{\lambda\nu}$ for $\lambda=(N^{2p})$. The determinants D_e and D_o are given in (5.4.9).

Therefore,

$$\begin{aligned}\mathbb{E}^{(H)}_N\left[\det(t-\mathcal{M})^{2p}\right] = &\sum_{\nu\subseteq\lambda}\left(-\frac{1}{2N}\right)^{\frac{|\lambda|-|\nu|}{2}}\frac{\dim V_\nu}{|\nu|!}t^{|\nu|}\,\mathrm{poly}_{\frac{|\nu|}{2}}(N,p)\\ &\times\begin{cases}C_\lambda(2p)D_e, & N \text{ even},\\ C_\lambda(2p)D_o, & N \text{ odd},\end{cases}\end{aligned} \tag{5.4.16}$$

where $\mathrm{poly}_j(N,p)$ represents a polynomial of degree j in variables N, p, and the explicit expressions are given in Table. 5.1 for $j\le 8$. By referring to (5.4.11), it is remarkable to see that the universal constant γ_p is a factor of the moments for any finite N and for $t\neq 0$. The first few terms in the moments of characteristic polynomials are

$$\begin{aligned}\mathbb{E}^{(H)}_N[\det(t-\mathcal{M})^{2p}] = &\left(-\frac{1}{2N}\right)^{Np}C_\lambda(2p)D_e\\ &\times\Bigg[1+\left(\frac{2^2N^2}{4!}\right)Np\,t^4+\left(\frac{2^3N^3}{6!}\right)2Np(2p-N)t^6\\ &+\left(\frac{2^4N^4}{8!}\right)Np(4N^2-17Np+16p^2+2)t^8\\ &+O(t^{10})\Bigg], \qquad N \text{ even},\\ \mathbb{E}^{(H)}_N[\det(t-\mathcal{M})^{2p}] = &\left(-\frac{1}{2N}\right)^{Np}C_\lambda(2p)D_o\Bigg[1+\left(\frac{2N}{2!}\right)pt^2+\left(\frac{2^2N^2}{4!}\right)(p^2-Np)\,t^4\\ &+\left(\frac{2^3N^3}{6!}\right)p(2N^2-3Np+p^2)t^6\\ &+\left(\frac{2^4N^4}{8!}\right)p(-4N^3+15N^2p-6Np^2-2N+p^3-4p)t^8\\ &+O(t^{10})\Bigg], \qquad N \text{ odd}.\end{aligned} \tag{5.4.17}$$

Up to a factor of $(-1)^p$, both $C_\lambda D_e$ and $C_\lambda D_o$ have the same leading term,

$$e^{-Np}(2N)^{Np+p^2}\gamma_p, \tag{5.4.18}$$

but they differ at sub-leading order as shown in (5.4.13). In App. B, we give the asymptotic expansion of $C_\lambda(N)D_e$ and $C_\lambda(N)D_o$ up to $O(N^{-6})$. Note that the coefficients of t^{2j} in (5.4.17) are polynomials in N, and both $C_\lambda D_e$ and $C_\lambda D_o$ have an expansion in $1/N$. Therefore for higher values of j, more sub-leading terms in the expansion of $C_\lambda(N)D_e$ and $C_\lambda(N)D_o$ are required to compute the right coefficients of t^j. But finding the exact asymptotic expansion of $C_\lambda D_e$ and $C_\lambda D_o$ is far from trivial as it involves a sequence of ratios of factorials, whose asymptotics is only known via recurrence relations.

In the next section, we focus on the second moment and show that we recover the semi-circle law only after averaging over even and odd matrix dimensional contributions.

5.4.2.1 Second moment

The correlations of characteristic polynomials and the correlation functions of eigenvalues of random matrices are related to each other [98, 186, 193]. This connection is briefly recalled in the following proposition.

Proposition 5.4.2. *The $k-$point correlation function of a rescaled GUE matrix given by*

$$R_{N,p}(t_1,\dots,t_p) = \frac{N!}{(N-p)!}\frac{1}{\mathcal{Z}_N^{(H)}}\int \Delta^2(t_1,\dots,t_N)e^{-\sum_{j=1}^N \frac{Nt_j^2}{2}}\,dt_{p+1}\dots dt_N \tag{5.4.19}$$

can be written in terms of the characteristic polynomial as

$$R_{N,p}(t_1,\dots,t_p) = \frac{N!}{(N-p)!}\frac{\mathcal{Z}_{N-p}^{(H)}}{\mathcal{Z}_N^{(H)}}\exp\left(-\frac{N}{2}\sum_{j=1}^p t_j^2\right)\Delta^2(t_1,\dots,t_p)\mathbb{E}_{N-p}^{(H)}\left[\prod_{j=1}^p \det(t_j-\mathcal{M})^2\right] \tag{5.4.20}$$

Proof. Consider two parameters N and n. Here n is the matrix size which we fix later. Consider the joint eigenvalue density function,

$$\frac{1}{\mathcal{Z}_n^{(H)}}\Delta^2(\mathbf{x})\exp\left(-\sum_{j=1}^n \frac{Nx_j^2}{2}\right). \tag{5.4.21}$$

According to this measure, the correlations of characteristic polynomials are given by

$$\begin{aligned}\mathbb{E}_n^{(H)}\left[\prod_{j=1}^p \det(t_j-\mathcal{M})^2\right] &= \frac{1}{\mathcal{Z}_n^{(H)}}\int e^{-\sum_{j=1}^n \frac{Nx_j^2}{2}}\Delta^2(\mathbf{x})\prod_{l=1}^p\prod_{k=1}^n (t_l-x_k)^2\,dx_1\dots dx_n \\ &= \frac{e^{\sum_{j=1}^p \frac{Nt_j^2}{2}}}{\Delta^2(\mathbf{t})}\frac{1}{\mathcal{Z}_n^{(H)}}\int e^{\sum_{l=1}^p\sum_{j=1}^n -\frac{Nx_j^2}{2}-\frac{Nt_l^2}{2}}\Delta^2(\mathbf{x})\Delta^2(\mathbf{t}) \\ &\quad\times\prod_{l=1}^p\prod_{k=1}^n (t_l-x_k)^2\,dx_1\dots dx_n.\end{aligned} \tag{5.4.22}$$

Up to a factor, the integrand in the R.H.S. is the $p-$point correlation function of a GUE matrix of size $n+p$. By using (5.4.19) and by choosing $n = N-p$, (5.4.20) is easily recovered. ■

In particular,

$$R_{N,1}(t) = \frac{N!}{(N-1)!} \frac{\mathcal{Z}_{N-1}^{(H)}}{\mathcal{Z}_N^{(H)}} \exp\left(-\frac{Nt^2}{2}\right) \mathbb{E}_{N-1}^{(H)} \left[\det(t-\mathcal{M})^2\right], \tag{5.4.23}$$

where $R_{N,1}$ is the one-point density of eigenvalues of matrices of size N. It is natural to expect the semi-circle law in the large N limit as the second moment of the characteristic polynomial is related to the density of states.

For $p = 1$, (5.4.1) can be re written as

$$\lim_{N\to\infty} \frac{1}{2N} e^N \exp\left(-\frac{Nt^2}{2}\right) \mathbb{E}_N^{(H)} \left[\det(t-\mathcal{M})^2\right] = \pi \varrho_{sc}(t), \tag{5.4.24}$$

which as an expansion in t reads to be

$$\lim_{N\to\infty} \frac{1}{2N} e^N \exp\left(-\frac{Nt^2}{2}\right) \mathbb{E}_N^{(H)} \left[\det(t-\mathcal{M})^2\right] = 1 - \frac{1}{8}t^2 - \frac{1}{128}t^4 - \frac{1}{1024}t^6 + O(t^8). \tag{5.4.25}$$

We now show that for $p = 1$, starting with (5.4.15) we arrive at (5.4.25). Inserting the asymptotics of $C_\lambda D_e$ and $C_\lambda D_o$ in (5.4.17), one obtains

$$\begin{aligned}
&e^{-\frac{Nt^2}{2}} \mathbb{E}_N^{(H)} \left[\det(t-\mathcal{M})^2\right] \\
&= 2Ne^{-N}\Bigg[1 + \left(-\frac{5}{12} - \frac{1}{2}N\right)t^2 + \left(-\frac{811}{77760} + \frac{17}{216}N + \frac{19}{72}N^2 + \frac{1}{6}N^3\right)t^4 \\
&\quad + \left(-\frac{640879}{587865600} + \frac{799}{1749600}N - \frac{3667}{291600}N^2 - \frac{323}{6480}N^3 - \frac{31}{540}N^4 - \frac{1}{45}N^5\right)t^6 \\
&\quad + O(t^8)\Bigg], \quad N \text{ even}, \\
&e^{-\frac{Nt^2}{2}} \mathbb{E}_N^{(H)} \left[\det(t-\mathcal{M})^2\right] \\
&= 2Ne^{-N}\Bigg[1 + \left(\frac{1}{6} + \frac{1}{2}N\right)t^2 + \left(-\frac{101}{19440} - \frac{17}{216}N - \frac{19}{72}N^2 - \frac{1}{6}N^3\right)t^4 \\
&\quad + \left(-\frac{15853}{18370800} - \frac{799}{1749600}N + \frac{3667}{291600}N^2 + \frac{323}{6480}N^3 + \frac{31}{540}N^4 + \frac{1}{45}N^5\right)t^6 \\
&\quad + O(t^8)\Bigg], \quad N \text{ odd}.
\end{aligned} \tag{5.4.26}$$

Treating the above expansions as a formal series in N and taking their average gives (5.4.25). In the next section, we show that the average over even and odd dimensional matrix contributions coincides with the semi-circle law up to $O(t^{10})$. Also, a general expression for the coefficient of t^{2j} in (5.4.15) is given for $p = 1$.

5.4.2.2 More on the second moment

Fix $\lambda = (N, N)$. The second moment of the characteristic polynomial is given by

$$\mathbb{E}_N^{(H)} \left[\det(t-\mathcal{M})^2\right] = \left(\frac{-1}{2N}\right)^N C_\lambda(2) \sum_{\nu\subseteq\lambda} \frac{1}{|\nu|!} (-2N)^{\frac{|\nu|}{2}} D_{\lambda\nu}^{(H)} \dim V_\nu \, t^{|\nu|}. \tag{5.4.27}$$

Let $\nu = (\nu_1, \nu_2) \subseteq \lambda$. Since $|\nu|$ is even, either both ν_1, ν_2 are even or both of them are odd. For $N = 2m$, $m \in \mathbb{N}$,

$$D^{(H)}_{\lambda\nu} = \begin{cases} \frac{1}{(m-\frac{\nu_1}{2})!(m-\frac{\nu_2}{2})!}, & \nu_1,\, \nu_2 \text{ are even,} \\ -\frac{1}{\left(m-\frac{\nu_1+1}{2}\right)!\left(m-\frac{\nu_2-1}{2}\right)!}, & \nu_1,\, \nu_2 \text{ are odd.} \end{cases} \tag{5.4.28}$$

Therefore,

$$C_\lambda(2) D^{(H)}_{\lambda\nu} = (2m)!(2m+1)! \begin{cases} \frac{1}{(m-\frac{\nu_1}{2})!}\frac{1}{(m-\frac{\nu_2}{2})!}, & \nu_1,\, \nu_2 \text{ are even,} \\ -\frac{1}{\left(m-\frac{\nu_1+1}{2}\right)!}\frac{1}{\left(m-\frac{\nu_2-1}{2}\right)!}, & \nu_1,\, \nu_2 \text{ are odd.} \end{cases} \tag{5.4.29}$$

Similarly for $N = 2m+1$, $m \in \mathbb{N}$,

$$D^{(H)}_{\lambda\nu} = \begin{cases} -\frac{1}{(m-\frac{\nu_1}{2})!\left(m-\frac{\nu_2-2}{2}\right)!}, & \nu_1,\, \nu_2 \text{ are even,} \\ \frac{1}{\left(m-\frac{\nu_1-1}{2}\right)!\left(m-\frac{\nu_2-1}{2}\right)!}, & \nu_1,\, \nu_2 \text{ are odd,} \end{cases} \tag{5.4.30}$$

and

$$C_\lambda(2) D^{(H)}_{\lambda\nu} = (2m+1)!(2m+2)! \begin{cases} -\frac{1}{(m-\frac{\nu_1}{2})!}\frac{1}{\left(m-\frac{\nu_2-2}{2}\right)!}, & \nu_1,\, \nu_2 \text{ are even,} \\ \frac{1}{\left(m-\frac{\nu_1-1}{2}\right)!}\frac{1}{\left(m-\frac{\nu_2-1}{2}\right)!}, & \nu_1,\, \nu_2 \text{ are odd.} \end{cases} \tag{5.4.31}$$

For a partition of length 2, $\nu = (\nu_1, \nu_2)$,

$$\frac{1}{|\nu|!} \dim V_\nu = \frac{\nu_1 - \nu_2 + 1}{(\nu_1+1)!\,\nu_2!}. \tag{5.4.32}$$

Inserting (5.4.29), (5.4.31), (5.4.32) in (5.4.27), and observing that ν runs over all partitions such that $0 \le |\nu| \le 2N$ gives

$$\begin{aligned} &\mathbb{E}^{(H)}_N \left[\det(t-\mathcal{M})^2\right] \\ &= \left(-\frac{1}{2N}\right)^N C_\lambda(2) D^{(H)}_{\lambda 0} \sum_{k=0}^{N} (-2N)^k t^{2k} \\ &\quad \times \Bigg[\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \left(\frac{2k+1-4j}{(2k+1-2j)!(2j)!} - \frac{2k-1-4j}{(2k-2j)!(2j+1)!} \right) \frac{(\frac{N}{2})!^2}{(\frac{N}{2}-k+j)!(\frac{N}{2}-j)!} \\ &\qquad\qquad + \frac{1}{k!(k+1)!} \frac{(\frac{N}{2})!^2}{(\frac{N}{2}-\frac{k}{2})!^2} \mathbb{1}_{k=0 \bmod 2} \Bigg]. \end{aligned} \tag{5.4.33}$$

for N even. Similarly for N odd, one gets

$$
\begin{aligned}
&\mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^2\right] \\
&= \left(-\frac{1}{2N}\right)^N C_\lambda(2) D_{\lambda 0}^{(H)} \sum_{k=0}^{N} (-2N)^k t^{2k} \\
&\times \Bigg[\sum_{j=0}^{\lfloor \frac{k-2}{2} \rfloor} \left(-\frac{2k-1-4j}{(2k-2j)!(2j+1)!} + \frac{2k-3-4j}{(2k-2j-1)!(2j+2)!} \right) \frac{\left(\frac{N-1}{2}\right)!\,\left(\frac{N+1}{2}\right)!}{\left(\left(\frac{N+1}{2}\right)-k+j\right)!\,\left(\left(\frac{N-1}{2}\right)-j\right)!} \\
&\qquad + \frac{1}{(2k)!} \frac{\left(\frac{N-1}{2}\right)!}{\left(\left(\frac{N-1}{2}\right)-k\right)!} - \frac{1}{k!(k+1)!} \frac{\left(\frac{N-1}{2}\right)!\,\left(\frac{N+1}{2}\right)!}{\left(\left(\frac{N-1}{2}\right)-\frac{k-1}{2}\right)!^2} \mathbb{1}_{k=0 \bmod 1} \Bigg].
\end{aligned}
\tag{5.4.34}
$$

Here

$$
C_\lambda(2) D_{\lambda 0}^{(H)} = \begin{cases} \frac{N!(N+1)!}{\left(\frac{N}{2}\right)!^2}, & \text{N even}, \\ -\frac{N!(N+1)!}{\left(\frac{N-1}{2}\right)!\left(\frac{N+1}{2}\right)!}, & N \text{ odd}. \end{cases}
\tag{5.4.35}
$$

The asymptotics of the ratio of the factorials are already discussed in App. B. For the sake of completion, here we again give the result for $p=1$,

$$
\begin{aligned}
C_\lambda(2) D_{\lambda 0}^{(H)} \sim e^{-N}(2N)^{N+1} \Bigg[& 1 + \frac{5}{6\,N} - \frac{11}{72\,N^2} + \frac{337}{6480\,N^3} + \frac{985}{31104\,N^4} - \frac{360013}{6531840\,N^5} \\
& - \frac{46723609}{1175731200\,N^6} + \frac{224766221}{1410877440\,N^7} + \frac{41757020981}{338610585600\,N^8} \\
& - \frac{889926952101377}{1005673439232000\,N^9} + O(N^{-10}) \Bigg], \ N \text{ even}, \\
C_\lambda(2) D_{\lambda 0}^{(H)} \sim -e^{-N}(2N)^{N+1} \Bigg[& 1 + \frac{1}{3\,N} + \frac{1}{18\,N^2} - \frac{31}{810\,N^3} - \frac{139}{9720\,N^4} + \frac{9871}{204120\,N^5} \\
& + \frac{324179}{18370800\,N^6} - \frac{8225671}{55112400\,N^7} - \frac{69685339}{1322697600\,N^8} \\
& + \frac{1674981058019}{1964205936000\,N^9} + O(N^{-10}) \Bigg], \quad N \text{ odd}.
\end{aligned}
\tag{5.4.36}
$$

Substituting the above asymptotic series in

$$
(2N)^{-1} e^{N - \frac{Nt^2}{2}} \mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^2\right]
\tag{5.4.37}
$$

and taking the average over N even and odd gives

$$
\begin{aligned}
&\lim_{N\to\infty} \frac{1}{2N} e^N \exp\left(-\frac{Nt^2}{2}\right) \mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^2\right] \\
=& 1 - \frac{1}{8}t^2 - \frac{1}{128}t^4 - \frac{1}{1024}t^6 - \frac{5}{32768}t^8 - \frac{7}{262144}t^{10} + O(t^{12}).
\end{aligned}
\tag{5.4.38}
$$

The R.H.S. in (5.4.38) coincides with $\pi\varrho_{sc}(t)$ up to $O(t^{10})$.

5.4.2.3 Higher moments

We can infer from (5.4.20) that the correlations of characteristic polynomials of matrices of size $N-p$ are related to the correlation functions of the eigenvalues of matrices of size N. The Dyson sine-kernel for the $p-$point correlation function and (5.4.1) for the moments of the characteristic polynomial are recovered in the Dyson limit: $t_i - t_j \to 0$, $N \to \infty$ and $N(t_i - t_j)$ is kept finite when $|t_j| < 2$, $j = 1, \ldots, p$.

In terms of the Schur polynomials, $\lambda = (N^{2p})$,

$$\mathbb{E}_N^{(H)}\left[\prod_{j=1}^{2p} \det(t_j - \mathcal{M})\right] = C_\lambda(2p) \sum_{\nu \subseteq \lambda} \left(-\frac{1}{2N}\right)^{\frac{|\lambda|-|\nu|}{2}} \frac{1}{C_\nu(2p)} D_{\lambda\nu}^{(H)} S_\nu(t_1, \ldots, t_{2p}). \tag{5.4.39}$$

Computing the asymptotics of the moments of the characteristic polynomials in the Dyson limit using (5.4.39) is not straight-forward. Instead, we fix $t_j = t$, $j = 1, \ldots, 2p$, and give an expansion of the moments as a function of t in the large N limit.

As $N \to \infty$, up to $O(t^2)$,

$$\begin{aligned} \mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^{2p}\right] &= (2N)^{p^2} e^{-Np} \gamma_p \left[1 + O(t^4)\right], && N \text{ even} \\ \mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^{2p}\right] &= (2N)^{p^2} e^{-Np} \gamma_p \left[1 + t^2\left(Np + \frac{p^2}{3}(2p^2-1)\right) + O(t^4)\right], && N \text{ odd.} \end{aligned} \tag{5.4.40}$$

Note that the coefficient of t^2 is identically zero for even N, where as for odd N it is a polynomial in N and p. Treating the above expansions as a formal series in N and taking their average gives

$$(2N)^{p^2} e^{-Np} \gamma_p \left(1 + \frac{Npt^2}{2}\right)\left(1 - \frac{p^2t^2}{8}\right)\left(1 + \frac{p}{12N}(8p^2-1)\right). \tag{5.4.41}$$

By comparing with (5.4.1), the terms in the first and second parenthesis of (5.4.41) are the expansions of $e^{\frac{Npt^2}{2}}$ and $(\pi\varrho_{sc}(t))^{p^2}$, respectively, up to $O(t^2)$. The last factor in (5.4.41) is sub-leading. Thus at $O(t^2)$, moments of characteristic polynomials in the Dyson limit and in the limit $t \to 0$ and $N \to \infty$ coincide.

Similarly, as $N \to \infty$, up to $O(t^4)$,

$$\begin{aligned}
\mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^{2p}\right] &= (2N)^{p^2} e^{-Np}\gamma_p \Bigg[1 + t^4 \frac{N^3 p}{6}\Big(1 + \frac{p}{6N}(4p^2+1) \\
&\quad + \frac{p^2}{72N^2}(16p^4 - 16p^2 - 11) + \frac{p}{6480N^3}(320p^8 - 1200p^6 \\
&\quad + 708p^4 + 1265p^2 - 756)\Big) + O(t^6)\Bigg], \qquad N \text{ even}, \\
\mathbb{E}_N^{(H)}\left[\det(t-\mathcal{M})^{2p}\right] &= (2N)^{p^2} e^{-Np}\gamma_p \Bigg[1 + t^2\left(Np + \frac{p^2}{3}(2p^2-1)\right) \\
&\quad + t^4 \frac{N^3 p}{6}\Big(-1 - \frac{2p}{3N}(p^2-2) - \frac{p^2}{18N^2}(4p^4 - 22p^2 + 13) \\
&\quad - \frac{p}{405N^3}(20p^8 - 210p^6 + 483p^4 - 385p^2 + 54)\Big) \\
&\quad + O(t^6)\Bigg], \qquad N \text{ odd}.
\end{aligned} \tag{5.4.42}$$

Taking average of the above series and factorising gives

$$\begin{aligned}
&(2N)^{p^2} e^{-Np}\gamma_p \left(1 + \frac{Npt^2}{2} + \frac{N^2p^2t^4}{8} + O(t^6)\right)\left(1 - \frac{p^2t^2}{8} + \frac{t^4}{128}p^2(p^2-2) + O(t^6)\right) \\
&\times \left[1 + \frac{1}{N}\left(\frac{p}{12}(8p^2-1) + \frac{pt^2}{96}(13p^2-1) + O(t^4)\right) + \frac{1}{N^2}\left(\frac{p^2}{144}(32p^4 - 56p^2 + 17) + O(t^2)\right)\right],
\end{aligned} \tag{5.4.43}$$

where the first two brackets correspond to the expansion of $e^{\frac{Npt^2}{2}}$ and $(\pi\varrho_{sc}(t))^{p^2}$, respectively, up to $O(t^4)$, and the last factor is sub-leading. Thus, asymptotics calculated by letting first $t \to 0$ and then $N \to \infty$ coincides with that of Dyson limit asymptotics up to $O(t^4)$. For higher orders in t, mismatch between the two limits start to appear[1].

5.5 Secular coefficients

Consider a matrix A of size N. Its characteristic polynomial can be expanded as

$$\det(t-A) = \prod_{j=1}^{N}(t - x_j) = \sum_{j=0}^{N}(-1)^j \mathrm{Sc}_j(A) t^{N-j}, \tag{5.5.1}$$

where Sc_j is the jth secular coefficient of the characteristic polynomial. We have

$$\mathrm{Sc}_1(A) = \mathrm{Tr}\, A, \quad \mathrm{Sc}_N(A) = \det(A). \tag{5.5.2}$$

[1]Here we discuss in detail the result at $O(t^6)$. The average of N even and N odd asymptotic series obtained by considering more terms in (5.4.42) can be factorised as shown in (5.4.43). When the Taylor expansions of $e^{\frac{Npt^2}{2}}$ and $(\pi\varrho_{sc}(t))^{p^2}$ are separated as shown in (5.4.43), we are left with the sub-leading term. In the sub-leading term obtained at the level $O(t^6)$, the coefficient of t^2/N turns out to be different from that obtained at the level $O(t^4)$. This indicates that discrepancies start to appear between the Dyson limit and the limit $N \to \infty$ and $t \to 0$. The Dyson limit indicates the presence of $e^{\frac{Npt^2}{2}}$ and $(\pi\varrho_{sc}(t))^{p^2}$ in the asymptotics, but when these are factorised as in (5.4.43), the sub-leading expansion is slightly different at each order in t starting from $O(t^6)$.

These secular coefficients are nothing but the elementary symmetric polynomials,

$$e_j(x_1, \ldots, x_N) = \sum_{1 \le k_1 < k_2 < \cdots < k_j \le N} x_{k_1} x_{k_2} \ldots x_{k_j}, \tag{5.5.3}$$

for $j \le N$ and $e_j = 0$ for $j > N$.

The correlations of secular coefficients and their connections to combinatorial objects are well studied [71, 100]. For example, the joint moments of secular coefficients of the unitary group are connected to the enumeration of magic squares: matrices with positive entries with prescribed row and column sum. In a similar way, the joint moments of secular coefficients of Hermitian ensembles, such as the GUE, are connected to matching polynomials of closed graphs. In this section, we compute these correlations and indicate their combinatorial properties.

Gaussian ensemble: Since $e_r = S_{(1^r)}$, elementary symmetric polynomials can be expanded in terms of multivariate Hermite polynomials as

$$e_r(x_1, \ldots, x_N) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \frac{1}{N^j} \psi^{(H)}_{(1^r)(1^{r-2j})} \mathcal{H}_{(1^{r-2j})}(x_1, \ldots, x_N). \tag{5.5.4}$$

Recall

$$\psi^{(H)}_{\lambda\nu} = \frac{1}{2^{\frac{|\lambda|-|\nu|}{2}}} D^{(H)}_{\lambda\nu} \prod_{k=1}^{l(\lambda)} \frac{(\lambda_k + N - k)!}{(\nu_k + N - k)!}, \tag{5.5.5}$$

where

$$D^{(H)}_{\lambda\nu} = \det \left[\mathbb{1}_{\lambda_i - \nu_k - i + k = 0 \bmod 2} \left(\left(\frac{\lambda_i - \nu_k - i + k}{2} \right)! \right)^{-1} \right]_{i,k=1,\ldots,l(\lambda)}. \tag{5.5.6}$$

For $\lambda = (1^r)$ and $\nu = (1^{r-2j})$, it is straightforward to see that $D^{(H)}_{\lambda\nu}$ simplifies as

$$\begin{aligned} D^{(H)}_{(1^r)(1^{r-2j})} &= D^{(H)}_{(1^{2j})0} \\ &= \det \left[\mathbb{1}_{1-i+k=0 \bmod 2} \frac{1}{\left(\frac{1-i+k}{2}\right)!} \right]_{i,k=1,\ldots,2j} \\ &= \frac{(-1)^j}{j!}. \end{aligned} \tag{5.5.7}$$

Therefore, (5.5.4) simplifies to

$$\psi^{(H)}_{(1^r)(1^{r-2j})} = (-1)^j \frac{(N - r + 2j)!}{2^j j! (N - r)!}. \tag{5.5.8}$$

Equivalently, we have

$$e_{2r} = \sum_{j=0}^{r} \frac{1}{N^{r-j}} \psi^{(H)}_{(1^{2r})(1^{2j})} \mathcal{H}_{(1^{2j})}, \quad e_{2r+1} = \sum_{j=0}^{r} \frac{1}{N^{r-j}} \psi^{(H)}_{(1^{2r+1})(1^{2j+1})} \mathcal{H}_{(1^{2j+1})}, \tag{5.5.9}$$

with

$$\begin{aligned}\psi^{(H)}_{(1^{2r})(1^{2j})} &= (-1)^{r-j}\frac{1}{2^{r-j}(r-j)!}\frac{(N-2j)!}{(N-2r)!},\\ \psi^{(H)}_{(1^{2r+1})(1^{2j+1})} &= (-1)^{r-j}\frac{1}{2^{r-j}(r-j)!}\frac{(N-2j-1)!}{(N-2r-1)!}.\end{aligned} \tag{5.5.10}$$

Because of the orthogonality of the $\mathcal{H}_\mu$, the first moment is

$$\mathbb{E}^{(H)}_N[\mathrm{Sc}_r] = \mathbb{E}^{(H)}_N[e_r] = \begin{cases}(-1)^{\frac{r}{2}}\frac{1}{(2N)^{\frac{r}{2}}\frac{r}{2}!}\frac{N!}{(N-r)!}, & \text{if } r \text{ is even},\\ 0, & \text{if } r \text{ is odd}.\end{cases} \tag{5.5.11}$$

These expectations are nothing but the coefficients of the rescaled Hermite polynomial of degree N. Thus, we have

$$\mathbb{E}^{(H)}_N[\det(t-\mathcal{M})] = \sum_{j=0}^{\lfloor\frac{N}{2}\rfloor}\mathbb{E}^{(H)}_N[\mathrm{Sc}_{2j}(\mathcal{M})]t^{N-2j} = N^{-\frac{N}{2}}H_N(\sqrt{N}t), \tag{5.5.12}$$

which coincides with (5.3.1a) for $p=1$. The expectation $|N^j\mathbb{E}^{(H)}_N[\mathrm{Sc}_{2j}(\mathcal{M})]|$ is equal to the number of $2j$ matchings in a complete graph [71, 100].

By using (5.5.4), the second moment of the secular coefficient can also be computed. Similar to the univariate case, multivariate Hermite polynomials $\mathcal{H}_\lambda$ corresponding to even and odd $|\lambda|$ do not mix. Therefore,

$$\mathbb{E}^{(H)}_N[\mathrm{Sc}_{2j}(\mathcal{M})\mathrm{Sc}_{2k+1}(\mathcal{M})] = 0, \tag{5.5.13}$$

and

$$\begin{aligned}\mathbb{E}^{(H)}_N[\mathrm{Sc}_{2r}(\mathcal{M})\mathrm{Sc}_{2s}(\mathcal{M})] &= \sum_{j=0}^{r}\sum_{k=0}^{s}\frac{1}{N^{r+s-j-k}}\psi^{(H)}_{(1^{2r})(1^{2j})}\psi^{(H)}_{(1^{2s})(1^{2k})}\mathbb{E}^{(H)}_N[\mathcal{H}_{(1^{2j})}\mathcal{H}_{(1^{2k})}]\\ &= \sum_{j=0}^{\min(r,s)}\frac{1}{N^{r+s}}\psi^{(H)}_{(1^{2r})(1^{2j})}\psi^{(H)}_{(1^{2s})(1^{2j})}C_{(1^{2j})}(N)\\ &= \left(-\frac{1}{2N}\right)^{r+s}\sum_{j=0}^{\min(r,s)}\frac{2^{2j}}{(r-j)!(s-j)!}\frac{N!(N-2j)!}{(N-2r)!(N-2s)!}.\end{aligned} \tag{5.5.14}$$

Similarly, one can compute that

$$\begin{aligned}&\mathbb{E}^{(H)}_N[\mathrm{Sc}_{2r+1}(\mathcal{M})\mathrm{Sc}_{2s+1}(\mathcal{M})]\\ &= \left(-\frac{1}{2N}\right)^{r+s}\sum_{j=0}^{\min(r,s)}\frac{2^{2j}}{(r-j)!(s-j)!}\frac{(N-1)!(N-2j-1)!}{(N-2r-1)!(N-2s-1)!}.\end{aligned} \tag{5.5.15}$$

Calculating higher order correlations requires evaluating integrals involving a sequence of multivariate Hermite polynomials. Busbridge [47, 48] calculated these integrals for the univariate case, but the results are still unknown for the multivariate generalisation. Instead, we take a different approach to compute correlations by first expressing the product $\prod_j(\mathrm{Sc}_j(\mathcal{M}))^{b_j}$ in

terms of the $\mathcal{H}_\mu$, and then using orthogonality for the $\mathcal{H}_\mu$.

Proposition 5.5.1. *Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$. We have*

$$\mathbb{E}_N^{(H)}\big[\prod_{j=1}^{l} \mathrm{Sc}_{\lambda_j}(\mathcal{M})\big] = \begin{cases} \sum_\mu \frac{1}{(2N)^{\frac{\mu}{2}} \frac{|\mu|}{2}!} K_{\lambda'\mu} \chi^\mu_{(2^{|\mu|/2})} C_\mu(N), & \text{if } |\lambda| \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{5.5.16}$$

Here $K_{\lambda\mu}$ are Kostka numbers and χ^μ_ν are the characters of the symmetric group.

Proof. For a partition λ, denote

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots. \tag{5.5.17}$$

Elementary symmetric polynomials e_λ can be expanded in Schur basis as

$$e_\lambda = \sum_\mu K_{\lambda'\mu} S_\mu, \tag{5.5.18}$$

where $K_{\lambda\mu}$ are Kostka numbers [182] and μ is a partition of $|\lambda|$. Using (3.5.15), we obtain

$$e_\lambda = \sum_{\mu \vdash |\lambda|} \sum_{\nu \subseteq \mu} \frac{1}{N^{\frac{|\mu|-|\nu|}{2}}} K_{\lambda'\mu} \psi^{(H)}_{\mu\nu} \mathcal{H}_\nu. \tag{5.5.19}$$

When $|\lambda|$ is odd, $\mathbb{E}_N^{(H)}[e_\lambda] = 0$ due to the orthogonality of multivariate Hermite polynomials. When $|\lambda|$ is even,

$$\begin{aligned} \mathbb{E}_N^{(H)}[e_\lambda] &= \mathbb{E}_N^{(H)}\big[\prod_{j=1}^{l} \mathrm{Sc}_{\lambda_j}(\mathcal{M})\big] \\ &= \frac{1}{N^{\frac{|\mu|-|\nu|}{2}}} \mathbb{E}_N^{(H)}\big[\sum_\mu \sum_\nu K_{\lambda'\mu} \psi^{(H)}_{\mu\nu} \mathcal{H}_\nu\big] \\ &= \frac{1}{N^{\frac{|\mu|}{2}}} \sum_{\mu \vdash |\lambda|} K_{\lambda'\mu} \psi^{(H)}_{\mu 0}. \end{aligned} \tag{5.5.20}$$

In Ch. 3, Prop. 3.5.8, we computed that

$$\psi^{(H)}_{\mu 0} = \frac{1}{2^{\frac{|\mu|}{2}} \frac{|\mu|}{2}!} \chi^\mu_{(2^{|\mu|/2})} C_\mu(N). \tag{5.5.21}$$

Putting everything together completes the proof. ■

Laguerre ensemble: All the calculations discussed for the Gaussian ensemble can be extended to the Laguerre and Jacobi cases. Polynomials e_r can be expanded as

$$e_r(x_1, \ldots, x_N) = \sum_{j=0}^{r} \frac{(-1)^{j+\frac{1}{2}N(N-1)}}{(2N)^{r-j}} \frac{1}{G_{(1^j)}(N,0)} \psi^{(L)}_{(1^r)(1^j)} \mathcal{L}^{(\gamma)}_{(1^j)}(x_1, \ldots, x_N). \tag{5.5.22}$$

By using (3.5.44), the coefficients simplify to

$$\frac{(-1)^{j+\frac{1}{2}N(N-1)}}{G_{(1^j)}(N,0)}\psi^{(L)}_{(1^r)(1^j)} = \frac{1}{(r-j)!}\frac{(N-j)!}{(N-r)!}\frac{\Gamma(N-j+\gamma+1)}{\Gamma(N-r+\gamma+1)}. \tag{5.5.23}$$

By using (5.2.11), we arrive at

$$\mathbb{E}^{(L)}_N[\mathrm{Sc}_r] = \mathbb{E}^{(L)}_N[e_r] = \frac{1}{(2N)^r}\frac{1}{r!}\frac{N!}{(N-r)!}\frac{\Gamma(N+\gamma+1)}{\Gamma(N-r+\gamma+1)}, \tag{5.5.24}$$

which are the absolute values of the coefficients of the rescaled Laguerre polynomial of degree N. Therefore, the first moment of the characteristic polynomial is

$$\mathbb{E}^{(L)}_N[\det(t-\mathcal{M})] = \sum_{j=0}^{N}(-1)^j\mathbb{E}^{(L)}_N[\mathrm{Sc}_j(\mathcal{M})]t^{N-j} = \frac{(-1)^N N!}{(2N)^N}L^{(\gamma)}_N(2Nt), \tag{5.5.25}$$

which coincides with (5.3.1b) for $p=1$. The correlations of secular coefficients can be computed similar to the Gaussian case.

Proposition 5.5.2. *For a partition $\lambda = (\lambda_1,\dots,\lambda_l)$, we have*

$$\mathbb{E}^{(L)}_N\Big[\prod_{j=1}^{l}\mathrm{Sc}_{\lambda_j}(\mathcal{M})\Big] = \sum_{\mu\vdash|\lambda|}\frac{1}{(2N)^{|\lambda|}}\frac{G_\mu(N,\gamma)G_\mu(N,0)}{G_0(N,\gamma)G_0(N,0)}\frac{\chi^{\mu}_{(1^{|\mu|})}}{|\lambda|!}K_{\lambda'\mu} \tag{5.5.26}$$

Proof. The proof is similar to the Gaussian case. By writing

$$e_\lambda = \sum_{\mu}\sum_{\nu\subseteq|\lambda|}\frac{(-1)^{|\nu|+\frac{1}{2}N(N-1)}}{(2N)^{|\mu|-|\nu|}}\frac{1}{G_\nu(N,0)}K_{\lambda'\mu}\psi^{(L)}_{\mu\nu}\mathcal{L}^{(\gamma)}_\nu, \tag{5.5.27}$$

and using (5.2.11) along with the result (3.5.47) from Ch. 3, one obtains

$$\frac{(-1)^{\frac{1}{2}N(N-1)}}{G_0(N,0)}\psi^{(L)}_{\mu 0} = \frac{G_\mu(N,\gamma)G_\mu(N,0)}{G_0(N,\gamma)G_0(N,0)}\frac{\chi^{\mu}_{(1^{|\mu|})}}{|\mu|!}. \tag{5.5.28}$$

Inserting (5.5.28) in (5.5.27) proves the proposition. ■

Jacobi ensemble. The e_r can be expanded as

$$\begin{aligned} &e_r(x_1,\dots,x_N)\\ &= \sum_{j=0}^{r}\psi^{(L)}_{(1^r)(1^j)}\frac{(-1)^{|\nu|+\frac{1}{2}N(N-1)}}{G_\nu(N,\gamma_1+\gamma_2)G_\nu(N,0)}\prod_{j=1}^{N}\frac{1}{\Gamma(2\nu_j+2N-2j+\gamma_1+\gamma_2+1)}\\ &\quad\times\mathcal{J}^{(\gamma_1,\gamma_2)}_{(1^j)}(x_1,\dots,x_N), \end{aligned} \tag{5.5.29}$$

where $\psi^{(J)}_{\lambda\nu}$ is given in (3.5.56). The expected values of e_r are related to the coefficients of the

Jacobi polynomial of degree N,

$$\mathbb{E}_N^{(J)}[\det(t-M)] = \sum_{j=0}^{N}(-1)^j \mathbb{E}_N^{(J)}[\mathrm{Sc}_j(M)]t^{N-j} = (-1)^N N! \frac{\Gamma(N+\gamma_1+\gamma_2+1)}{\Gamma(2N+\gamma_1+\gamma_2+1)} J_N^{(\gamma_1,\gamma_2)}(t). \tag{5.5.30}$$

Chapter 6

Conclusion

Diaconis, Shashahani, Bump and Gamburd very well illustrated the usefulness of symmetric function theory in compact groups. This thesis provides insights into the role of symmetric functions in unitary invariant Hermitian ensembles. By defining the generalised symmetric polynomials as a determinantal formula with orthogonal polynomials as matrix entries, we provide a concise way of computing the correlations of characteristic polynomials. One can recover the moments of characteristic polynomials from these correlations by letting all the spectral variables be the same. Our methods are different from those given in the classic papers by Brezin and Hikami [40]; Baik, Deift, Strahov [19]; Strahov and Fyodorov [224]; and Borodin and Strahov [36].

From the work of Breizin and Hikami [40], the large N limits of the moments of characteristic polynomials for a broad class of Hermitian ensembles depend on the asymptotic eigenvalue density along with a constant that also appears in number theory. This result can also be proved using supersymmetric methods or by reformulating the problem into a Riemann-Hilbert problem and using the Deift-Zhou steepest-descent method for the Riemann-Hilbert problem. For the GUE, as discussed, the asymptotic spectral density is the semi-circle law. Unlike the previous methods, we take a combinatorial approach to compute the asymptotics. Our analysis unveils that the even and odd dimensional GUE matrices have different limits for the moments of characteristic polynomials. We discover that the semi-circle law for the moments is recovered only after a formal average between the even and odd dimensional contributions. Evidence for this behaviour is provided for the second moment, but more analysis is required for higher moments.

Theorem 3.2.3 gives an explicit expression for the joint moments of traces of the GUE, LUE and JUE via the characters of the symmetric group. As emphasised, the correlations of traces of the GUE are related to the enumeration of ribbon graphs. Therefore, Thm. 3.2.3 can be used to relate the combinatorial objects, such as the size of the automorphism group of a ribbon graph of a certain genus, to the character theory of the symmetric group. The results on correlations of traces can be used to study the limiting distributions of random variables that are polynomial functions of random matrices. These polynomials are chosen to be the Chebyshev polynomials of the first kind and we obtained estimates on the bounds of the moments and cumulants of these random variables.

Appendix A

Ribbon graphs and matrix integrals

Let $\mathbf{x} = (x_1, \dots, x_N)$ be an $N-$dimensional random variable. Consider the normalised Gaussian measure

$$d\mu(\mathbf{x}) = (2\pi)^{-\frac{N}{2}} \sqrt{\det A}\, e^{-\frac{1}{2}\sum_{i,j} x_i A_{ij} x_j} \prod_k dx_k, \tag{1.0.1}$$

where A is a positive definite symmetric matrix. The inverse

$$B_{ij} = (A^{-1})_{ij} \tag{1.0.2}$$

is called the propagator.

Correlations of Gaussian random variables can be computed in a combinatorial way using *Wick's theorem* [243], also known as *Isserlis' theorem*, which is stated below.

Theorem 1.0.1 (Wick's theorem)**.** *The expectation value of product of Gaussian random variables is*

$$\mathbb{E}[x_{i_1} x_{i_2} \dots x_{i_n}] = \begin{cases} 0 & \textit{if } n \textit{ is odd}, \\ B_{i_1 i_2} & \textit{if } n = 2, \\ \sum_{\textit{pairings of } (i_1,\dots,i_n)} \prod_{\textit{pairs } (k,l)} B_{i_k i_l}, & \textit{if } n \geq 2 \textit{ and even.} \end{cases} \tag{1.0.3}$$

For example,

$$\mathbb{E}[x_{i_1} x_{i_2} x_{i_3} x_{i_4}] = B_{i_1 i_2} B_{i_3 i_4} + B_{i_1 i_3} B_{i_2 i_4} + B_{i_1 i_4} B_{i_2 i_3}. \tag{1.0.4}$$

Wick's theorem becomes particularly useful when the indices i_j are repeated. The problem of computing the expectation values $\mathbb{E}[x_{i_1}^{b_1} \dots x_{i_n}^{b_n}]$ can be mapped to counting the number of ways of gluing n vertices with valencies $b_1, \dots b_n$, whose weights are determined by the propagators that correspond to their edges.

$$\mathbb{E}[x_{i_1}^{b_1} \dots x_{i_n}^{b_n}] = \sum_{\substack{\text{Graphs G with } n \text{ vertices} \\ \text{of valencies } b_j}} \prod_{(i_k, i_l) \text{ edge of G}} B_{i_k i_l}. \tag{1.0.5}$$

For example,

$$\mathbb{E}[x_{i_1}^2 x_{i_2}^2] = B_{i_1 i_1} B_{i_2 i_2} + 2 B_{i_1 i_2}^2. \tag{1.0.6}$$

Clearly many graphs in (1.0.5) are topologically identical and have the same weight because

of the symmetries among the edges and vertices. Let $\mathbf{G}$ be the group of these symmetries, $\#gluings$ be the number of gluings of obtaining a graph, and $\mathrm{Aut}(G)$ be the automorphism group of the graph. By orbit-stabiliser theorem,

$$\#\mathrm{Aut}(G) \times \#gluings = \#\mathbf{G}, \tag{1.0.7}$$

where $\#\mathbf{G}$ is the order of group relabelling. Wick's theorem can be written only in terms of non-equivalent graphs as follows:

$$\frac{1}{\#\mathbf{G}}\mathbb{E}[\prod_j x_{i_j}^{b_j}] = \sum_{\text{Non-equivalent graphs } G} \frac{1}{\#\mathrm{Aut}(G)} \prod_{(i,j) \text{ edge of } G} B_{ij}. \tag{1.0.8}$$

In the case of Gaussian matrix integrals, Wick's theorem can be applied to compute correlators of traces by studying *fat graphs* also called *ribbon graphs*.

Consider the Hermitian Gaussian matrix model with probability measure

$$d\mu_0(\mathcal{M}) = \frac{1}{\mathcal{Z}_0} e^{-2N \operatorname{Tr} \mathcal{M}^2} \prod_{j=1}^{N} d\mathcal{M}_{jj} \prod_{j<k} d\mathrm{Re}\mathcal{M}_{jk}\, d\mathrm{Im}\mathcal{M}_{jk}, \tag{1.0.9}$$

where

$$\mathcal{Z}_0 = \frac{1}{2^{N(N-1)}} \left(\frac{\pi}{N}\right)^{\frac{N^2}{2}}. \tag{1.0.10}$$

Note that here we have the rescaled GUE matrices which we denote by $\mathcal{M}$.

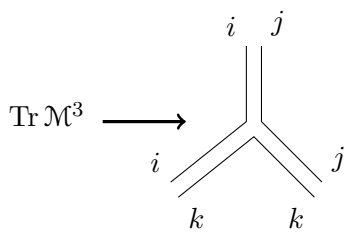
The Wick's propagator is

$$\mathbb{E}_N^{(H)}[\mathcal{M}_{ij}\mathcal{M}_{kl}] \equiv \langle \mathcal{M}_{ij}\mathcal{M}_{kl} \rangle = \frac{1}{4N}\delta_{il}\delta_{jk}. \tag{1.0.11}$$

As an example, consider

$$\mathbb{E}_N^{(H)}[(\operatorname{Tr} \mathcal{M}^3)^2] = \sum_{\substack{i,j,k,\\ l,m,n}} \mathbb{E}_N^{(H)}[\mathcal{M}_{ij}\mathcal{M}_{jk}\mathcal{M}_{ki}\mathcal{M}_{lm}\mathcal{M}_{mn}\mathcal{M}_{nl}] \tag{1.0.12}$$

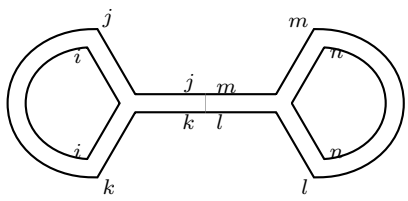
To map the problem to counting graphs, associate a vertex to each trace. The power of the matrix inside the trace gives the number of half-edges as double lines with index associated to each single line. The propagator in (1.0.11) can be used to glue these half-edges together to



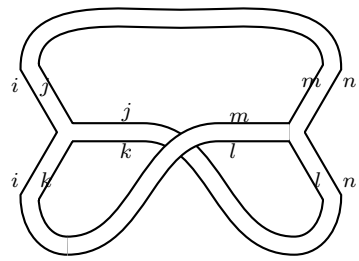
form a double line edge of the graph. Thus,

$$\begin{aligned}\mathbb{E}_N^{(H)}[(\operatorname{Tr}\mathcal{M}^3)^2] &= \sum_{\substack{i,j,k\\ l,m,n}} \langle \mathcal{M}_{ij}\mathcal{M}_{jk}\rangle\langle \mathcal{M}_{ki}\mathcal{M}_{lm}\rangle\langle \mathcal{M}_{mn}\mathcal{M}_{nl}\rangle + \langle \mathcal{M}_{ij}\mathcal{M}_{ki}\rangle\langle \mathcal{M}_{jk}\mathcal{M}_{lm}\rangle\langle \mathcal{M}_{mn}\mathcal{M}_{nl}\rangle + \dots \\ &= \frac{1}{(4N)^3}\sum_{\substack{i,j,k\\ l,m,n}} \delta_{ik}\delta_{km}\delta_{il}\delta_{ml} + \delta_{jk}\delta_{jm}\delta_{kl}\delta_{ml} + \dots \\ &= \frac{1}{4^3}\left(12 + \frac{3}{N^2}\right).\end{aligned} \tag{1.0.13}$$

There are in total $5!! = 15$ graphs in (1.0.13) with only two topologically distinct graphs shown below.



N^0



N^{-2}

If we attach to each vertex a factor of N, the N dependence of a graph is: There is a factor N per vertex, a factor N^{-1} per edge, a factor N for each single line when summed over indices. The number of single lines remaining at the end is the number of faces of the graph. So the total N dependency of a graph is

$$N^{\#\text{vertices}-\#\text{edges}+\#\text{faces}} = N^{\chi(G)}, \tag{1.0.14}$$

where $\chi(G)$ is the topological invariant of the graph called its Euler-characteristic.

This notion of counting ribbon graphs can be extended to compute correlators of the form $\mathbb{E}_N^{(H)}[\prod_j (\operatorname{Tr}\mathcal{M}^j)^{b_j}]$. When divided by $\prod_j j^{b_j} b_j!$, the order of group relabelling, matrix integrals takes a form similar to (1.0.8). This formula is due to Brezin-Itzykson-Parisi- Zuber in 1978 [42]

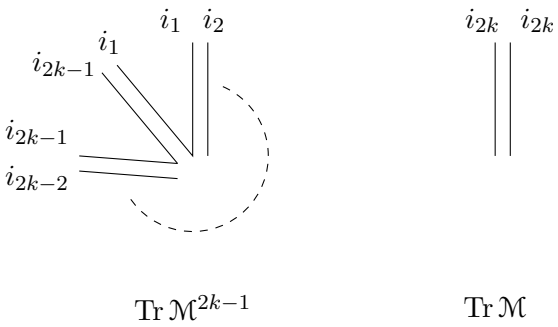
$$\mathbb{E}_N^{(H)}\left[\prod_{j=1}^{n}\frac{1}{b_j!}\left(\frac{N}{j}\operatorname{Tr}\mathcal{M}^j\right)^{b_j}\right] = \sum_{\text{Ribbon Graphs } G}\frac{1}{\#\mathrm{Aut}(G)}4^{-\#\text{edges}}N^{\chi(G)}, \tag{1.0.15}$$

where the sum is over non-topologically equivalent ribbon graphs and $\#\mathrm{Aut}(G)$ is the number of automorphisms of G. There are a total of $(\sum_j jb_j - 1)!!$ graphs (counting equivalent and non-equivalent graphs). The total number of vertices is $b = \sum_j b_j$ with j valencies for each vertex and the total number of edges is $(\sum_j jb_j)/2$.

1.0.1 Special cases

Here we consider two cases (i) $\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{2k-1}\operatorname{Tr}\mathcal{M}]$ and (ii) $\mathbb{E}_N^{(H)}[(\operatorname{Tr}\mathcal{M}^2)^n]$.

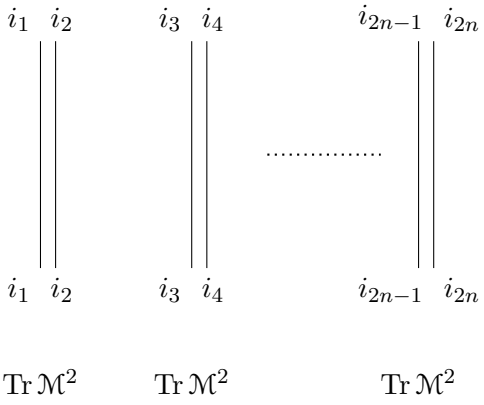
(i) $\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{2k-1}\operatorname{Tr}\mathcal{M}]$: We represent $\operatorname{Tr}\mathcal{M}^{2k-1}\operatorname{Tr}\mathcal{M}$ as two vertices with $2k-1$ and 1 valencies, respectively.



Since index i_{2k} has $2k-1$ choices, by gluing the half-edges using (1.0.11),

$$\begin{aligned}\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{2k-1}\operatorname{Tr}\mathcal{M}] &= (2k-1)\mathbb{E}_N^{(H)}[\operatorname{Tr}\mathcal{M}^{2k-2}] \\ &= N(2k-1)!!i^{-k+1}\frac{1}{k}P_{k-1}^{(1)}\left(iN,\frac{\pi}{2}\right),\end{aligned} \tag{1.0.16}$$

where $P_{k-1}^{(1)}\left(iN,\frac{\pi}{2}\right)$ is a Meixner-Pollaczek polynomial.
(ii) $\mathbb{E}_N^{(H)}[(\operatorname{Tr}\mathcal{M}^2)^n]$: Here we sketch the idea to calculate moments of $\operatorname{Tr}\mathcal{M}^2$. We represent $(\operatorname{Tr}\mathcal{M}^2)^n$ as n vertices each with two valencies as shown below. There are several ways of



gluing this set of vertices and half-edges. Trivially i_j can be glued with itself for $j=1,\dots,2n$ which gives a total contribution of $N^{2n}/(4N)^n$.

The next non-trivial contribution comes from choosing any two vertices and gluing their valencies to form an edge between them. There are $\binom{n}{2}$ ways of choosing two vertices. Let (i_p,i_{p+1}) and (i_q,i_{q+1}), $1\le p,q\le 2n$, be the indices of the valencies of these two vertices. There are two ways to pair (i_p,i_{p+1}) and (i_q,i_{q+1}). This gives a contribution of $n(n-1)N^2/(4N)^2$. The remaining $n-2$ disconnected graphs multiplicatively gives $N^{2n-4}/(4N)^{n-2}$. Hence the first two leading terms are

$$\mathbb{E}_N^{(H)}[(\operatorname{Tr}\mathcal{M}^2)^n] = \frac{1}{(4N)^n}(N^{2n}+n(n-1)N^{2n-2}+\dots) \tag{1.0.17}$$

Remaining terms in the nth moment can be likewise computed.

$$\mathbb{E}_N^{(H)}[(\operatorname{Tr}\mathcal{M}^2)^n] = \frac{1}{(4N)^n}\prod_{j=0}^{n-1}(N^2+2j). \tag{1.0.18}$$

Similar arguments can be used to show that

$$\mathbb{E}_N^{(H)}[(\operatorname{Tr}\mathcal{M}^2)^{n-k}(\operatorname{Tr}\mathcal{M})^{2k}] = (2k-1)!!\frac{1}{(4N)^n}N^k\prod_{l=k}^{n-1}(N^2+2l) \tag{1.0.19}$$

for $k \in \mathbb{N}$.

Appendix B

Asymptotics of the ratios of factorials

The asymptotics of the ratio of factorials can be computed as follows. First we look at $C_\lambda(2p)D_e$ with $\lambda = (2m, \ldots, 2m)$. Consider

$$\frac{(2m+j)!(2m+p+j)!}{(m+j)!^2} = (2m)^p \frac{(2m+j)!^2}{(m+j)!^2} \prod_{a=1}^{p} \left(1 + \frac{j+a}{2m}\right). \tag{2.0.1}$$

Now, one can see that

$$\frac{(2m+j)!}{(m+j)!} = 2^{j+1} \frac{\Gamma(2m)}{\Gamma(m)} \prod_{a=0}^{j} \frac{1+\frac{a}{2m}}{1+\frac{a}{m}}. \tag{2.0.2}$$

Using the duplication formula for the Gamma functions

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \tag{2.0.3}$$

and Stirling's series

$$\Gamma(z+h) \sim \sqrt{2\pi} e^{-z} z^{z+h-\frac{1}{2}} \prod_{j=2}^{\infty} \exp\left(\frac{(-1)^j B_j(h)}{j(j-1)z^{j-1}}\right), \quad z \to \infty, \tag{2.0.4}$$

the asymptotic expansion for the ratio of Gamma functions can be found. Here B_j is the Bernoulli polynomial of degree j. Combining all the formulae, up to first order correction,

$$C_{((2m)^{2p})}(2p)D_e \sim e^{-2mp} 2^{4mp+2p^2} m^{2mp+p^2} \left(\prod_{j=0}^{p-1} \frac{j!}{(p+j)!}\right) \left[1 + \frac{p}{12m}(4p^2+1) + O(m^{-2})\right]. \tag{2.0.5}$$

Similarly for the case $C_\lambda(2p)D_o$, we obtain

$$\frac{(2m+1+p+j)!(2m+1+j)!}{(m+j)!^2} = (2m+1)^p \frac{(2m+1+j)!^2}{(m+j)!^2} \prod_{a=1}^{p} \left(1 + \frac{j+a}{2m+1}\right). \tag{2.0.6}$$

Let $z = m + \frac{1}{2}$, then

$$\frac{(2m+1+j)!}{(m+j)!} = \frac{\Gamma(2z+j+1)}{\Gamma(z+\frac{1}{2}+j)} = 2^{j+1} z \frac{\Gamma(2z)}{\Gamma(z+\frac{1}{2})} \prod_{a=1}^{j} \frac{1+\frac{a}{2z}}{1+\frac{2a-1}{2z}}, \tag{2.0.7}$$

and

$$\frac{m!}{(m+p)!} = \frac{\Gamma(z+\frac{1}{2})}{\Gamma(z+p+\frac{1}{2})} = \frac{1}{z^p}\prod_{a=1}^{p}\frac{1}{1+\frac{2a-1}{2z}} \tag{2.0.8}$$

Combining the above formulae and using (2.0.3) and (2.0.4),

$$\begin{aligned} C_{((2m+1)^{2p})}D_o \equiv C_{((2z)^{2p})}D_o \sim & (-1)^p e^{-2zp} z^{p^2+2pz} 2^{2p^2+4pz} \left(\prod_{j=0}^{p-1}\frac{j!}{(p+j)!}\right) \\ & \times \left[1+\frac{p}{6z}(2p^2-1)+O(z^{-2})\right] \end{aligned} \tag{2.0.9}$$

Higher order corrections can also be calculated with some effort or using any commercial software like Mathematica. Writing in terms of the matrix size N, as $N \to \infty$, we have

$$\begin{aligned} C_{(N^{2p})}D_e \sim\ & e^{-Np}(2N)^{Np+p^2}\left(\prod_{j=0}^{p-1}\frac{j!}{(p+j)!}\right)\Bigg[1+\frac{p}{6N}(4p^2+1)+\frac{p^2}{72N^2}(16p^4-16p^2-11) \\ & +\frac{p}{6480N^3}(320p^8-1200p^6+708p^4+1265p^2-756) \\ & +\frac{p^2}{155520N^4}(1280p^{10}-10240p^8+25248p^6-6400p^4-56371p^2+51408) \\ & +\frac{p}{6531840N^5}\Big(7168p^{14}-98560p^{12}+499072p^{10}-982688p^8-399844p^6 \\ & \quad +4606735p^4-5598936p^2+1607040\Big) \\ & +\frac{p^2}{1175731200N^6}\Big(143360p^{16}-3010560p^{14}+25294080p^{12}-103093760p^{10} \\ & \quad +158864016p^8+298943760p^6-1697420809p^4+2663679600p^2-1390123296\Big) \\ & +O\left(\frac{1}{N^7}\right)\Bigg], \qquad N \text{ even.} \end{aligned} \tag{2.0.10}$$

$$\begin{aligned} C_{(N^{2p})}D_o \sim\ & (-1)^p e^{-Np}(2N)^{Np+p^2}\left(\prod_{j=0}^{p-1}\frac{j!}{(p+j)!}\right)\Bigg[1+\frac{p}{3N}(2p^2-1)+\frac{p^2}{18N^2}(4p^4-10p^2+7) \\ & +\frac{p}{810N^3}(40p^8-240p^6+516p^4-455p^2+108) \\ & +\frac{p^2}{9720N^4}(80p^{10}-880p^8+3828p^6-8356p^4+9509p^2-4320) \\ & +\frac{p}{204120N^5}\Big(224\,p^{14}-3920\,p^{12}+28616\,p^{10}-113428\,p^8+266818\,p^6 \\ & \quad -372127\,p^4+255528\,p^2-51840\Big) \\ & +\frac{p^2}{18370800N^6}\Big(2240\,p^{16}-57120\,p^{14}+628320\,p^{12}-3919160\,p^{10}+15363624\,p^8 \\ & \quad -39481170\,p^6+65605589\,p^4-62864640\,p^2+25046496\Big) \\ & +O\left(\frac{1}{N^7}\right)\Bigg], \qquad N \text{ odd.} \end{aligned} \tag{2.0.11}$$

Bibliography

[1] M. Adler, T. Shiota, and P. Van Moerbeke. Random matrices, vertex operators and the Virasoro algebra. *Physics Letters A*, 208(1-2):67–78, 1995.

[2] M. Adler and P. Van Moerbeke. Hermitian, symmetric and symplectic random ensembles: PDEs for the distribution of the spectrum. *Annals of Mathematics*, pages 149–189, 2001.

[3] I. Afanasiev. On the correlation functions of the characteristic polynomials of non-Hermitian random matrices with independent entries. *Journal of Statistical Physics*, 176(6):1561–1582, 2019.

[4] G. Akemann. Universal correlators for multi-arc complex matrix models. *Nuclear Physics B*, 507(1-2):475–500, 1997.

[5] G. Akemann. Microscopic correlation functions for the QCD Dirac operator with chemical potential. *Physical Review Letters*, 89(7):072002, 2002.

[6] G. Akemann and Y. V. Fyodorov. Universal random matrix correlations of ratios of characteristic polynomials at the spectral edges. *Nuclear Physics B*, 664(3):457–476, 2003.

[7] G. Akemann, E. Strahov, and T. R. Würfel. Averages of products and ratios of characteristic polynomials in polynomial ensembles. *Annales Henri Poincaré*, 21(12):3973–4002, 2020.

[8] G. Akemann and G. Vernizzi. Characteristic polynomials of complex random matrix models. *Nuclear Physics B*, 660(3):532–556, 2003.

[9] A. Altland and M. R. Zirnbauer. Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures. *Physical Review B*, 55(2):1142, 1997.

[10] J. Ambjørn, L. Chekhov, C. Kristjansen, and Y. Makeenko. Matrix model calculations beyond the spherical limit. *Nuclear Physics B*, 404(1-2):127–172, 1993.

[11] J. Ambjørn, C. Kristjansen, and Y. Makeenko. Generalized Penner models to all genera. *Physical Review D*, 50(8):5193, 1994.

[12] A. Andreev and B. Simons. Correlators of spectral determinants in quantum chaos. *Physical Review Letters*, 75(12):2304, 1995.

[13] T. Assiotis, J. P. Keating, and J. Warren. On the joint moments of the characteristic polynomials of random unitary matrices. *arXiv:2005.13961*, 2020.

[14] E. Bachmat, D. Berend, L. Sapir, S. Skiena, and N. Stolyarov. Analysis of aeroplane boarding via spacetime geometry and random matrix theory. *Journal of Physics A: Mathematical and General*, 39(29):L453, 2006.

[15] Z. D. Bai and J. W. Silverstein. CLT for linear spectral statistics of large-dimensional sample covariance matrices. In *Advances In Statistics*, pages 281–333. World Scientific, 2008.

[16] J. Baik, A. Borodin, P. Deift, and T. Suidan. A model for the bus system in Cuernavaca (Mexico). *Journal of Physics A: Mathematical and General*, 39(28):8965, 2006.

[17] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *Journal of the American Mathematical Society*, 12(4):1119–1178, 1999.

[18] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the second row of a Young diagram under Plancherel measure. *Geometric & Functional Analysis GAFA*, 10(4):702–731, 2000.

[19] J. Baik, P. Deift, and E. Strahov. Products and ratios of characteristic polynomials of random Hermitian matrices. *Journal of Mathematical Physics*, 44(8):3657–3670, 2003.

[20] T. H. Baker and P. J. Forrester. Finite N fluctuation formulas for random matrices. *Journal of Statistical Physics*, 88(5-6):1371–1386, 1997.

[21] T. H. Baker and P. J. Forrester. The Calogero-Sutherland model and generalized classical polynomials. *Communications in Mathematical Physics*, 188(1):175–216, 1997.

[22] T. H. Baker and P. J. Forrester. The Calogero-Sutherland model and polynomials with prescribed symmetry. *Nuclear Physics B*, 492(3):682–716, 1997.

[23] H. U. Baranger and P. A. Mello. Mesoscopic transport through chaotic cavities: A random S-matrix theory approach. *Physical Review Letters*, 73(1):142, 1994.

[24] E. L. Basor. Distribution functions for random variables for ensembles of positive Hermitian matrices. *Communications in Mathematical Physics*, 188(2):327–350, 1997.

[25] M. Bauer and C. Itzykson. Triangulations. *Discrete Mathematics*, 156(1-3):29–81, 1996.

[26] G. Baxter. Polynomials defined by a difference system. *Journal of Mathematical Analysis and Applications*, 2(2):223–263, 1961.

[27] G. Baxter. A norm inequality for a "finite-section" Wiener-Hopf equation. *Illinois Journal of Mathematics*, 7(1):97–103, 1963.

[28] F. Bekerman, T. Leblé, and S. Serfaty. CLT for fluctuations of β-ensembles with general potential. *Electronic Journal of Probability*, 23, 2018.

[29] A. Berele. Properties of hook Schur functions with applications to p. i. algebras. *Advances in Applied Mathematics*, 41(1):52–75, 2008.

[30] S. Berezin and A. Bufetov. On the rate of convergence in the central limit theorem for linear statistics of Laguerre ensembles. *arXiv:1904.09685*, 2019.

[31] M. Berry and J. P. Keating. Clusters of near-degenerate levels dominate negative moments of spectral determinants. *Journal of Physics A: Mathematical and General*, 35(1):L1, 2001.

[32] D. Bessis, C. Itzykson, and J. B. Zuber. Quantum field theory techniques in graphical enumeration. *Advances in Applied Mathematics*, 1(2):109–157, 1980.

[33] P. Bleher and A. Its. Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model. *Annals of Mathematics*, pages 185–266, 1999.

[34] O. Bohigas, M.-J. Giannoni, and C. Schmit. Characterization of chaotic quantum spectra and universality of level fluctuation laws. *Physical Review Letters*, 52(1):1, 1984.

[35] A. Borodin, A. Okounkov, and G. Olshanski. Asymptotics of Plancherel measures for symmetric groups. *Journal of the American Mathematical Society*, 13(3):481–515, 2000.

[36] A. Borodin and E. Strahov. Averages of characteristic polynomials in random matrix theory. *Communications on Pure and Applied Mathematics*, 59(2):161–253, 2006.

[37] G. Borot and A. Guionnet. Asymptotic expansion of β matrix models in the one-cut regime. *Communications in Mathematical Physics*, 317(2):447–483, 2013.

[38] J.-P. Bouchaud and M. Potters. *Theory of financial risk and derivative pricing: from statistical physics to risk management.* Cambridge university press, 2003.

[39] J. Breuer and E. Strahov. A universality theorem for ratios of random characteristic polynomials. *Journal of Approximation Theory*, 164(6):803–814, 2012.

[40] E. Brézin and S. Hikami. Characteristic polynomials of random matrices. *Communications in Mathematical Physics*, 214(1):111–135, 2000.

[41] E. Brézin and S. Hikami. Characteristic polynomials of random matrices at edge singularities. *Physical Review E*, 62(3):3558, 2000.

[42] E. Brezin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar diagrams. *Communications in Mathematical Physics*, 59(1):35–51, 1978.

[43] E. Brézin and A. Zee. Universality of the correlations between eigenvalues of large random matrices. *Nuclear Physics B*, 402(3):613–627, 1993.

[44] D. Bump. *Lie groups*, volume 8. Springer, 2004.

[45] D. Bump and P. Diaconis. Toeplitz minors. *Journal of Combinatorial Theory, Series A*, 97(2):252–271, 2002.

[46] D. Bump and A. Gamburd. On the averages of characteristic polynomials from classical groups. *Communications in Mathematical Physics*, 265(1):227–274, 2006.

[47] I. W. Busbridge. The evaluation of certain integrals involving products of Hermite polynomials. *Journal of the London Mathematical Society*, 1(2):93–97, 1939.

[48] I. W. Busbridge. Some integrals involving Hermite polynomials. *Journal of the London Mathematical Society*, 1(2):135–141, 1948.

[49] L. Chekhov and B. Eynard. Hermitian matrix model free energy: Feynman graph technique for all genera. *Journal of High Energy Physics*, 2006(03):014, 2006.

[50] J. Choi. Some mathematical constants. *Applied Mathematics and Computation*, 187(1):122–140, 2007.

[51] B. Collins. Product of random projections, Jacobi ensembles and universality problems arising from free probability. *Probability Theory and Related Fields*, 133(3):315–344, 2005.

[52] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith. Autocorrelation of random matrix polynomials. *Communications in Mathematical Physics*, 237(3):365–395, 2003.

[53] J. B. Conrey and A. Ghosh. On mean values of the zeta-function. *Mathematika*, 31(1):159–161, 1984.

[54] J. B. Conrey and A. Ghosh. Mean values of the Riemann zeta-function III. In *Proceedings of the Amalfi Conference on Analytic Number Theory, Universita di Salerno*, 1992.

[55] J. B. Conrey and S. M. Gonek. High moments of the Riemann zeta-function. *Duke Mathematical Journal*, 107(3):577–604, 2001.

[56] O. Costin and J. L. Lebowitz. Gaussian fluctuation in random matrices. *Physical Review Letters*, 75(1):69, 1995.

[57] F. D. Cunden, A. Dahlqvist, and N. O'Connell. Integer moments of complex Wishart matrices and Hurwitz numbers. *arXiv:1809.10033*, 2018.

[58] F. D. Cunden, F. Mezzadri, N. O'Connell, and N. Simm. Moments of random matrices and hypergeometric orthogonal polynomials. *Communications in Mathematical Physics*, 369(3):1091–1145, 2019.

[59] F. D. Cunden, F. Mezzadri, N. Simm, and P. Vivo. Correlators for the Wigner–Smith time-delay matrix of chaotic cavities. *Journal of Physics A: Mathematical and Theoretical*, 49(18):18LT01, 2016.

[60] F. D. Cunden, F. Mezzadri, N. Simm, and P. Vivo. Large-N expansion for the time-delay matrix of ballistic chaotic cavities. *Journal of Mathematical Physics*, 57(11):111901, 2016.

[61] P. H. Damgaard and S. M. Nishigaki. Universal spectral correlators and massive Dirac operators. *Nuclear Physics B*, 518(1-2):495–512, 1998.

[62] P.-O. Dehaye. Averages over classical lie groups, twisted by characters. *Journal of Combinatorial Theory, Series A*, 114(7):1278–1292, 2007.

[63] P.-O. Dehaye. Joint moments of derivatives of characteristic polynomials. *Algebra & Number Theory*, 2(1):31–68, 2008.

[64] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou. Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 52(11):1335–1425, 1999.

[65] P. Desrosiers. Duality in random matrix ensembles for all β. *Nuclear Physics B*, 817(3):224–251, 2009.

[66] P. Desrosiers and D.-Z. Liu. Asymptotics for products of characteristic polynomials in classical β-ensembles. *Constructive Approximation*, 39(2):273–322, 2014.

[67] P. Desrosiers and D.-Z. Liu. Selberg integrals, super-hypergeometric functions and applications to β-ensembles of random matrices. *Random Matrices: Theory and Applications*, 4(02):1550007, 2015.

[68] P. Di Francesco. 2D quantum gravity, matrix models and graph combinatorics. In *Applications of Random Matrices in Physics*, pages 33–88. Springer, 2006.

[69] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin. 2D gravity and random matrices. *Physics Reports*, 254(1-2):1–133, 1995.

[70] P. Diaconis and S. Evans. Linear functionals of eigenvalues of random matrices. *Transactions of the American Mathematical Society*, 353(7):2615–2633, 2001.

[71] P. Diaconis and A. Gamburd. Random matrices, magic squares and matching polynomials. *The Electronic Journal of Combinatorics*, pages R2–R2, 2004.

[72] P. Diaconis and M. Shahshahani. On the eigenvalues of random matrices. *Journal of Applied Probability*, 31(A):49–62, 1994.

[73] B. Dubrovin, S.-Q. Liu, D. Yang, and Y. Zhang. Hodge–GUE correspondence and the discrete KdV equation. *Communications in Mathematical Physics*, 379(2):461–490, 2020.

[74] B. Dubrovin and D. Yang. On cubic Hodge integrals and random matrices. *arXiv:1606.03720*, 2016.

[75] B. Dubrovin and D. Yang. Generating series for GUE correlators. *Letters in Mathematical Physics*, 107(11):1971–2012, 2017.

[76] I. Dumitriu and A. Edelman. Matrix models for beta ensembles. *Journal of Mathematical Physics*, 43(11):5830–5847, 2002.

[77] I. Dumitriu, A. Edelman, and G. Shuman. MOPS: Multivariate orthogonal polynomials (symbolically). *Journal of Symbolic Computation*, 42(6):587–620, 2007.

[78] I. Dumitriu and E. Paquette. Global fluctuations for linear statistics of β-Jacobi ensembles. *Random Matrices: Theory and Applications*, 1(04):1250013, 2012.

[79] I. Dumitriu and E. Rassart. Path counting and random matrix theory. *arXiv preprint math/0307252*, 2003.

[80] F. J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *Journal of Mathematical Physics*, 3(6):1191–1198, 1962.

[81] F. J. Dyson. Statistical theory of the energy levels of complex systems. I. *Journal of Mathematical Physics*, 3(1):140–156, 1962.

[82] F. J. Dyson. The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics. *Journal of Mathematical Physics*, 3(6):1199–1215, 1962.

[83] A. Edelman and B. D. Sutton. From random matrices to stochastic operators. *Journal of Statistical Physics*, 127(6):1121–1165, 2007.

[84] S. F. Edwards and P. W. Anderson. Theory of spin glasses. *Journal of Physics F: Metal Physics*, 5(5):965, 1975.

[85] K. Efetov. Supersymmetry and theory of disordered metals. *Advances in Physics*, 32(1):53–127, 1983.

[86] K. Efetov. *Supersymmetry in disorder and chaos*. Cambridge university press, 1999.

[87] N. M. Ercolani and K.-R. McLaughlin. Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration. *International Mathematics Research Notices*, 2003(14):755–820, 2003.

[88] N. M. Ercolani and V. U. Pierce. The continuum limit of Toda lattices for random matrices with odd weights. *arXiv:1106.1143*, 2011.

[89] L. Erdős. Universality of Wigner random matrices: a survey of recent results. *Russian Mathematical Surveys*, 66(3):507, 2011.

[90] L. Erdős, S. Péché, J. A. Ramírez, B. Schlein, and H.-T. Yau. Bulk universality for Wigner matrices. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 63(7):895–925, 2010.

[91] B. Eynard. Topological expansion for the 1-Hermitian matrix model correlation functions. *Journal of High Energy Physics*, 2004(11):031, 2005.

[92] B. Eynard. Counting surfaces. *Progress in Mathematical Physics*, 70, 2016.

[93] A. R. Feier. *Methods of proof in random matrix theory.* PhD thesis, Harvard University, 2012.

[94] S. V. Fomin and N. Lulov. On the number of rim hook tableaux. *Journal of Mathematical Sciences*, 87(6):4118–4123, 1997.

[95] P. Forrester and N. Frankel. Applications and generalizations of Fisher–Hartwig asymptotics. *Journal of Mathematical Physics*, 45(5):2003–2028, 2004.

[96] P. Forrester and J. Keating. Singularity dominated strong fluctuations for some random matrix averages. *Communications in Mathematical Physics*, 250(1):119–131, 2004.

[97] P. Forrester and N. Witte. Application of the τ-function theory of Painlevé equations to random matrices: PIV, PII and the GUE. *Communications in Mathematical Physics*, 219:357–398, 05 2001.

[98] P. J. Forrester. *Log-gases and random matrices (LMS-34).* Princeton University Press, 2010.

[99] P. J. Forrester. The averaged characteristic polynomial for the Gaussian and chiral Gaussian ensembles with a source. *Journal of Physics A: Mathematical and Theoretical*, 46(34):345204, 2013.

[100] P. J. Forrester and A. Gamburd. Counting formulas associated with some random matrix averages. *Journal of Combinatorial Theory, Series A*, 113(6):934–951, 2006.

[101] P. J. Forrester, A. A. Rahman, and N. S. Witte. Large N expansions for the Laguerre and Jacobi β-ensembles from the loop equations. *Journal of Mathematical Physics*, 58(11):113303, 2017.

[102] P. J. Forrester, A. A. Rahman, and N. S. Witte. Large N expansions for the Laguerre and Jacobi β-ensembles from the loop equations. *Journal of Mathematical Physics*, 58(11):113303, 2017.

[103] C. Frenzen. Error bounds for asymptotic expansions of the ratio of two gamma functions. *SIAM Journal on Mathematical Analysis*, 18(3):890–896, 1987.

[104] K. Fukunaga. *Introduction to statistical pattern recognition.* Elsevier, 2013.

[105] W. Fulton and J. Harris. *Representation theory: a first course*, volume 129. Springer Science & Business Media, 2013.

[106] Y. Fyodorov. Mesoscopic Quantum Physics ed E Akkermans et al, 1995.

[107] Y. V. Fyodorov. Negative moments of characteristic polynomials of random matrices: Ingham–Siegel integral as an alternative to Hubbard–Stratonovich transformation. *Nuclear Physics B*, 621(3):643–674, 2002.

[108] Y. V. Fyodorov, J. Grela, and E. Strahov. On characteristic polynomials for a generalized chiral random matrix ensemble with a source. *Journal of Physics A: Mathematical and Theoretical*, 51(13):134003, 2018.

[109] Y. V. Fyodorov, G. A. Hiary, and J. P. Keating. Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function. *Physical Review Letters*, 108(17):170601, 2012.

[110] Y. V. Fyodorov and J. P. Keating. Negative moments of characteristic polynomials of random GOE matrices and singularity-dominated strong fluctuations. *Journal of Physics A: Mathematical and General*, 36(14):4035, 2003.

[111] Y. V. Fyodorov and J. P. Keating. Freezing transitions and extreme values: random matrix theory, and disordered landscapes. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2007):20120503, 2014.

[112] Y. V. Fyodorov and B. A. Khoruzhenko. Systematic analytical approach to correlation functions of resonances in quantum chaotic scattering. *Physical Review Letters*, 83(1):65, 1999.

[113] Y. V. Fyodorov and P. Le Doussal. Moments of the position of the maximum for GUE characteristic polynomials and for log-correlated Gaussian processes. *Journal of Statistical Physics*, 164(1):190–240, 2016.

[114] Y. V. Fyodorov and A. Nock. On random matrix averages involving half-integer powers of GOE characteristic polynomials. *Journal of Statistical Physics*, 159(4):731–751, 2015.

[115] Y. V. Fyodorov and N. J. Simm. On the distribution of the maximum value of the characteristic polynomial of GUE random matrices. *Nonlinearity*, 29(9):2837, 2016.

[116] Y. V. Fyodorov and H.-J. Sommers. Universality of "level curvature" distribution for large random matrices: systematic analytical approaches. *Zeitschrift für Physik B Condensed Matter*, 99(1):123–135, 1995.

[117] Y. V. Fyodorov and H.-J. Sommers. Statistics of resonance poles, phase shifts and time delays in quantum chaotic scattering: Random matrix approach for systems with broken time-reversal invariance. *Journal of Mathematical Physics*, 38(4):1918–1981, 1997.

[118] Y. V. Fyodorov and E. Strahov. Characteristic polynomials of random Hermitian matrices and Duistermaat–Heckman localisation on non-compact Kähler manifolds. *Nuclear Physics B*, 630(3):453–491, 2002.

[119] Y. V. Fyodorov and E. Strahov. On correlation functions of characteristic polynomials for chiral Gaussian unitary ensemble. *Nuclear Physics B*, 647(3):581–597, 2002.

[120] Y. V. Fyodorov and E. Strahov. An exact formula for general spectral correlation function of random Hermitian matrices. *Journal of Physics A: Mathematical and General*, 36(12):3203, 2003.

[121] T. Garoni. On the asymptotics of some large Hankel determinants generated by Fisher–Hartwig symbols defined on the real line. *Journal of Mathematical Physics*, 46(4):043516, 2005.

[122] P. Gaspard, S. Rice, H. Mikeska, and K. Nakamura. Parametric motion of energy levels: Curvature distribution. *Physical Review A*, 42(7):4015, 1990.

[123] M. Gaudin. Sur la loi limite de l'espacement des valeurs propres d'une matrice ale´ atoire. *Nuclear Physics*, 25:447–458, 1961.

[124] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov, and A. Orlov. Matrix models of two-dimensional gravity and Toda theory. *Nuclear Physics B*, 357(2-3):565–618, 1991.

[125] I. M. Gessel. Symmetric functions and P-recursiveness. *Journal of Combinatorial Theory, Series A*, 53(2):257–285, 1990.

[126] M. Gisonni, T. Grava, and G. Ruzza. Jacobi ensemble, Hurwitz numbers and Wilson polynomials. *arXiv:2011.04082*, 2020.

[127] M. Gisonni, T. Grava, and G. Ruzza. Laguerre ensemble: correlators, Hurwitz numbers and Hodge integrals. *Annales Henri Poincaré*, 21(10):3285–3339, 2020.

[128] B. Golinskiĭ and I. A. Ibragimov. On Szegö's limit theorem. *Mathematics of the USSR-Izvestiya*, 5(2):421, 1971.

[129] S. M. Gonek. On negative moments of the Riemann zeta-function. *Mathematika*, 36(1):71–88, 1989.

[130] I. P. Goulden, M. Guay-Paquet, and J. Novak. Monotone Hurwitz numbers and the HCIZ integral. *Annales Mathématiques Blaise Pascal*, 21(1):71–89, 2014.

[131] J. Gustavsson. Gaussian fluctuations of eigenvalues in the GUE. *Annales de l'IHP Probabilités et statistiques*, 41(2):151–178, 2005.

[132] F. Haake. Quantum signatures of chaos. In *Quantum Coherence in Mesoscopic Systems*, pages 583–595. Springer, 1991.

[133] F. Haake, F. Izrailev, N. Lehmann, D. Saher, and H.-J. Sommers. Statistics of complex levels of random matrices for decaying systems. *Zeitschrift für Physik B Condensed Matter*, 88(3):359–370, 1992.

[134] M. Halasz, J. Osborn, and J. Verbaarschot. Random matrix triality at nonzero chemical potential. *Physical Review D*, 56(11):7059, 1997.

[135] R. Hall. A Wirtinger type inequality and the spacing of the zeros of the Riemann zeta-function. *Journal of Number Theory*, 93(2):235–245, 2002.

[136] G. H. Hardy and J. E. Littlewood. Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. *Acta Mathematica*, 41:119–196, 1916.

[137] G. H. Hardy and S. Ramanujan. Asymptotic formulaæ in combinatory analysis. *Proceedings of the London Mathematical Society*, 2(1):75–115, 1918.

[138] J. Harer and D. Zagier. The Euler characteristic of the moduli space of curves. *Inventiones Mathematicae*, 85(3):457–485, 1986.

[139] Harish-Chandra. Differential operators on a semisimple Lie algebra. *American Journal of Mathematics*, pages 87–120, 1957.

[140] A. J. Harper. Sharp conditional bounds for moments of the Riemann zeta function. *arXiv:1305.4618*, 2013.

[141] D. Heath-Brown. Fractional moments of the Riemann zeta-function. *Journal of the London Mathematical Society*, 2(1):65–78, 1981.

[142] I. Hirschman Jr. The strong Szegö limit theorem for Toeplitz determinants. *American Journal of Mathematics*, pages 577–614, 1966.

[143] G. Hooft. A planar diagram theory for strong interactions. *Nuclear Physics B*, 72(3):461 – 473, 1974.

[144] L. Hua. Harmonic analysis of functions of many complex variables in classical domains (Peking), 1958.

[145] C. P. Hughes, J. P. Keating, and N. 'Connell. Random matrix theory and the derivative of the Riemann zeta function. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 456(2003):2611–2627, 2000.

[146] C. P. Hughes, J. P. Keating, and N. O'connell. On the characteristic polynomial of a random unitary matrix. *Communications in Mathematical Physics*, 220(2):429–451, 2001.

[147] A. Hurwitz. Über die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten. *Mathematische Annalen*, 55(1):53–66, 1901.

[148] A. E. Ingham. Mean-value theorems in the theory of the Riemann zeta-function. *Proceedings of the London Mathematical Society*, 2(1):273–300, 1928.

[149] C. Itzykson and J.-B. Zuber. The planar approximation. II. *Journal of Mathematical Physics*, 21(3):411–421, 1980.

[150] A. T. James. Normal multivariate analysis and the orthogonal group. *The Annals of Mathematical Statistics*, 25(1):40–75, 1954.

[151] A. T. James. The distribution of the latent roots of the covariance matrix. *The Annals of Mathematical Statistics*, 31(1):151–158, 1960.

[152] A. T. James. Zonal polynomials of the real positive definite symmetric matrices. *Annals of Mathematics*, pages 456–469, 1961.

[153] A. T. James. Distributions of matrix variates and latent roots derived from normal samples. *The Annals of Mathematical Statistics*, 35(2):475–501, 1964.

[154] B. Jeremie. Reviews - The Oxford handbook of random matrix theory, Gernot Akemann, Jinho Baik, Philippe Di Francesco (eds) (Oxford University Press). *The Mathematical Gazette*, pages 534–556, 2011.

[155] T. Jiang. Limit theorems for beta-Jacobi ensembles. *Bernoulli*, 19(3):1028–1046, 2013.

[156] K. Johansson. On random matrices from the compact classical groups. *Annals of Mathematics*, 145(2):519–545, 1997.

[157] K. Johansson. On fluctuations of eigenvalues of random Hermitian matrices. *Duke Mathematical Journal*, 91(1):151–204, 1998.

[158] K. Johansson. Shape fluctuations and random matrices. *Communications in Mathematical Physics*, 209(2):437–476, 2000.

[159] K. Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *Annals of Mathematics*, pages 259–296, 2001.

[160] K. Johansson and G. Lambert. Multivariate normal approximation for traces of random unitary matrices. *arXiv:2002.01879*, 2020.

[161] K. J. Johansson. Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices. *Communications in Mathematical Physics*, 215(3):683–705, 2001.

[162] I. M. Johnstone. On the distribution of the largest eigenvalue in principal components analysis. *Annals of Statistics*, pages 295–327, 2001.

[163] I. M. Johnstone. High dimensional statistical inference and random matrices. *arXiv preprint math/0611589*, 2006.

[164] B. Jonnadula, J. Keating, and F. Mezzadri. On the moments of characteristic polynomials. *arXiv:2106.11743*, 2021.

[165] B. Jonnadula, J. P. Keating, and F. Mezzadri. Symmetric function theory and unitary invariant ensembles. *arXiv:2003.02620*, 2020.

[166] M. Kac. Toeplitz matrices, translation kernels and a related problem in probability theory. *Duke Mathematical Journal*, 21(3):501–509, 1954.

[167] E. Kanzieper. Replica field theories, Painlevé transcendents, and exact correlation functions. *Physical Review Letters*, 89(25):250201, 2002.

[168] N. Katz and P. Sarnak. Zeroes of zeta functions and symmetry. *Bulletin of the American Mathematical Society*, 36(1):1–26, 1999.

[169] V. A. Kazakov, I. K. Kostov, and N. Nekrasov. D-particles, matrix integrals and KP hierarchy. *Nuclear Physics B*, 557(3):413–442, 1999.

[170] J. P. Keating, F. Mezzadri, and B. Singphu. Rate of convergence of linear functions on the unitary group. *Journal of Physics A: Mathematical and Theoretical*, 44(3):035204, 2010.

[171] J. P. Keating and N. C. Snaith. Random matrix theory and L-functions at $s = 1/2$. *Communications in Mathematical Physics*, 214(1):91–100, 2000.

[172] J. P. Keating and N. C. Snaith. Random matrix theory and $\zeta(1/2+it)$. *Communications in Mathematical Physics*, 214(1):57–89, 2000.

[173] R. Killip and I. Nenciu. Matrix models for circular ensembles. *International Mathematics Research Notices*, 2004(50):2665–2701, 2004.

[174] P. Kopel. Regularity conditions for convergence of linear statistics of GUE. *arXiv:1510.02988*, 2015.

[175] I. Krasovsky. Correlations of the characteristic polynomials in the Gaussian unitary ensemble or a singular Hankel determinant. *Duke Mathematical Journal*, 139(3):581–619, 2007.

[176] G. Lambert, M. Ledoux, and C. Webb. Quantitative normal approximation of linear statistics of β-ensembles. *The Annals of Probability*, 47(5):2619–2685, 2019.

[177] M. Larsen and A. Shalev. Characters of symmetric groups: sharp bounds and applications. *Inventiones Mathematicae*, 174(3):645, 2008.

[178] G. Livan and P. Vivo. Moments of Wishart-Laguerre and Jacobi ensembles of random matrices: application to the quantum transport problem in chaotic cavities. *arXiv:1103.2638*, 2011.

[179] T. LMFDB Collaboration. The L-functions and modular forms database. http://www.lmfdb.org, 2021. [Online; accessed 24 March 2021].

[180] F. Luo, Y. Yang, J. Zhong, H. Gao, L. Khan, D. K. Thompson, and J. Zhou. Constructing gene co-expression networks and predicting functions of unknown genes by random matrix theory. *BMC Bioinformatics*, 8(1):1–17, 2007.

[181] F. Luo, J. Zhong, Y. Yang, R. H. Scheuermann, and J. Zhou. Application of random matrix theory to biological networks. *Physics Letters A*, 357(6):420–423, 2006.

[182] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.

[183] V. A. Marčenko and L. A. Pastur. Distribution of eigenvalues for some sets of random matrices. *Mathematics of the USSR-Sbornik*, 1(4):457, 1967.

[184] J. McKay. The largest degrees of irreducible characters of the symmetric group. *Mathematics of Computation*, 30(135):624–631, 1976.

[185] M. L. Mehta. On the statistical properties of the level-spacings in nuclear spectra. *Nuclear Physics*, 18:395–419, 1960.

[186] M. L. Mehta. *Random matrices*. Elsevier, 2004.

[187] M. L. Mehta and M. Gaudin. On the density of eigenvalues of a random matrix. *Nuclear Physics*, 18:420–427, 1960.

[188] P. A. Mello, P. Pereyra, and T. H. Seligman. Information theory and statistical nuclear reactions. I. General theory and applications to few-channel problems. *Annals of Physics*, 161(2):254–275, 1985.

[189] P. A. Mello and A. D. Stone. Maximum-entropy model for quantum-mechanical interference effects in metallic conductors. *Physical Review B*, 44(8):3559, 1991.

[190] F. Mezzadri, A. K. Reynolds, and B. Winn. Moments of the eigenvalue densities and of the secular coefficients of β-ensembles. *Nonlinearity*, 30(3):1034, 2017.

[191] F. Mezzadri and N. Simm. Moments of the transmission eigenvalues, proper delay times, and random matrix theory. I. *Journal of Mathematical Physics*, 52(10):103511, 2011.

[192] F. Mezzadri and N. Simm. Moments of the transmission eigenvalues, proper delay times and random matrix theory II. *Journal of Mathematical Physics*, 53(5):053504, 2012.

[193] F. Mezzadri, N. C. Snaith, and N. Hitchin. *Recent perspectives in random matrix theory and number theory*, volume 322. Cambridge University Press, 2005.

[194] H. L. Montgomery. The pair correlation of zeros of the zeta function. In *Proceedings of Symposia in Pure Mathematics*, volume 24, pages 181–193, 1973.

[195] M. Mulase and A. Waldron. Duality of orthogonal and symplectic matrix integrals and quaternionic Feynman graphs. *Communications in Mathematical Physics*, 240(3):553–586, 2003.

[196] J. Najnudel, E. Paquette, and N. Simm. Secular coefficients and the holomorphic multiplicative chaos. *arXiv:2011.01823*, 2020.

[197] F. Nakano and K. D. Trinh. Gaussian β-ensembles at high temperature: eigenvalue fluctuations and bulk statistics. *Journal of Statistical Physics*, 173(2):295–321, 2018.

[198] M. Novaes. Statistics of quantum transport in chaotic cavities with broken time-reversal symmetry. *Physical Review B*, 78(3):035337, 2008.

[199] A. Okounkov. Random matrices and random permutations. *International Mathematics Research Notices*, 2000(20):1043–1095, 2000.

[200] J. C. Osborn. Universal results from an alternate random-matrix model for QCD with a baryon chemical potential. *Physical Review Letters*, 93(22):222001, 2004.

[201] L. Pastur. Limiting laws of linear eigenvalue statistics for Hermitian matrix models. *Journal of Mathematical Physics*, 47(10):103303, 2006.

[202] L. Pastur and M. Shcherbina. Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles. *Journal of Statistical Physics*, 86(1):109–147, 1997.

[203] V. U. Pierce. Continuum limits of Toda lattices for map enumeration. *Algebraic and Geometric Aspects of Integrable Systems and Random Matrices*, 593:317, 2013.

[204] V. Plerou, P. Gopikrishnan, B. Rosenow, L. A. N. Amaral, and H. E. Stanley. Universal and nonuniversal properties of cross correlations in financial time series. *Physical Review Letters*, 83(7):1471, 1999.

[205] M. Prähofer and H. Spohn. Universal distributions for growth processes in 1+1 dimensions and random matrices. *Physical Review Letters*, 84(21):4882, 2000.

[206] K. Ramachandra. Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series-II. *Hardy-Ramanujan Journal*, 3, 1980.

[207] J. Ramirez, B. Rider, and B. Virág. Beta ensembles, stochastic Airy spectrum, and a diffusion. *Journal of the American Mathematical Society*, 24(4):919–944, 2011.

[208] J. A. Ramírez and B. Rider. Diffusion at the random matrix hard edge. *Communications in Mathematical Physics*, 288(3):887–906, 2009.

[209] Y. Roichman. Upper bound on the characters of the symmetric groups. *Inventiones Mathematicae*, 125(3):451–485, 1996.

[210] R. L. Schilling. Reviews - The Oxford handbook of random matrix theory, Gernot Akemann, Jinho Baik, Philippe Di Francesco (eds) (Oxford University Press). *The Mathematical Gazette*, 97(539):375–376, 2013.

[211] P. Šeba. Parking and the visual perception of space. *Journal of Statistical Mechanics: Theory and Experiment*, 2009(10):L10002, 2009.

[212] A. Selberg. Berkninger om et multilet integral. *Norsk Matematisk Tidsskrift*, 26:71–78, 1944.

[213] A. N. Sergeev and A. P. Veselov. Jacobi-Trudy formula for generalised Schur polynomials. *Moscow Mathematical Journal*, 14(1):161–168, 2014.

[214] M. Shcherbina. Central limit theorem for linear eigenvalue statistics of orthogonally invariant matrix models. *Journal of Mathematical Physics, Analysis, Geometry*, 4(1):171–195, 2008.

[215] N. Simm. Central limit theorems for the real eigenvalues of large Gaussian random matrices. *Random Matrices: Theory and Applications*, 6(01):1750002, 2017.

[216] Y. Sinai and A. Soshnikov. Central limit theorem for traces of large random symmetric matrices with independent matrix elements. *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society*, 29(1):1–24, 1998.

[217] V. Sokolov and V. Zelevinsky. Dynamics and statistics of unstable quantum states. *Nuclear Physics A*, 504(3):562–588, 1989.

[218] H.-J. Sommers, Y. V. Fyodorov, and M. Titov. S-matrix poles for chaotic quantum systems as eigenvalues of complex symmetric random matrices: from isolated to overlapping resonances. *Journal of Physics A: Mathematical and General*, 32(5):L77, 1999.

[219] A. Soshnikov. The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities. *Annals of probability*, 28(3):1353–1370, 2000.

[220] K. Soundararajan. Moments of the Riemann zeta function. *Annals of Mathematics*, pages 981–993, 2009.

[221] R. P. Stanley. Enumerative Combinatorics, vol. 2. *Cambridge Studies in Advanced Mathematics*, 1999.

[222] M. A. Stephanov. Random matrix model of QCD at finite density and the nature of the quenched limit. *Physical Review Letters*, 76(24):4472, 1996.

[223] E. Strahov. Moments of characteristic polynomials enumerate two-rowed lexicographic arrays. *Electronic Journal of Combinatorics*, 10, 2003.

[224] E. Strahov and Y. V. Fyodorov. Universal results for correlations of characteristic polynomials: Riemann-Hilbert approach. *Communications in Mathematical Physics*, 241(2-3):343–382, 2003.

[225] Z. Su. On the second-order correlation of characteristic polynomials of Hermite β ensembles. *Statistics & Probability Letters*, 80(19-20):1500–1507, 2010.

[226] R. J. Szabo. Microscopic spectrum of the QCD Dirac operator in three dimensions. *Nuclear Physics B*, 598(1-2):309–347, 2001.

[227] G. Szegő. On certain Hermitian forms associated with the Fourier series of a positive function. *Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.]*, 1952(Tome Supplémentaire):228–238, 1952.

[228] G. 't Hooft. A two-dimensional model for mesons. *Nuclear Physics B*, 75(3):461 – 470, 1974.

[229] T. Tao and V. Vu. Random matrices: universality of local eigenvalue statistics. *Acta Mathematica*, 206(1):127–204, 2011.

[230] C. A. Tracy and H. Widom. Fredholm determinants, differential equations and matrix models. *Communications in Mathematical Physics*, 163(1):33–72, 1994.

[231] C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Communications in Mathematical Physics*, 159(1):151–174, 1994.

[232] J. V. Uspensky. Asymptotic formulae for numerical functions which occur in the theory of partitions. *Bulletin of the Russian Academy of Sciences*, 14(6):199–218, 1920.

[233] J. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer. Grassmann integration in stochastic quantum physics: the case of compound-nucleus scattering. *Physics Reports*, 129(6):367–438, 1985.

[234] P. Vivo and E. Vivo. Transmission eigenvalue densities and moments in chaotic cavities from random matrix theory. *Journal of Physics A: Mathematical and Theoretical*, 41(12):122004, 2008.

[235] D. Voiculescu. Limit laws for random matrices and free products. *Inventiones Mathematicae*, 104(1):201–220, 1991.

[236] D. Voiculescu. Free probability theory: random matrices and von Neumann algebras. In *Proceedings of the International Congress of Mathematicians*, pages 227–242. Springer, 1995.

[237] F. von Oppen. Exact distribution of eigenvalue curvatures of chaotic quantum systems. *Physical Review Letters*, 73(6):798, 1994.

[238] F. von Oppen. Exact distributions of eigenvalue curvatures for time-reversal-invariant chaotic systems. *Physical Review E*, 51(3):2647, 1995.

[239] K. W. Wachter. The limiting empirical measure of multiple discriminant ratios. *The Annals of Statistics*, pages 937–957, 1980.

[240] F. Wegner. The mobility edge problem: continuous symmetry and a conjecture. *Zeitschrift für Physik B Condensed Matter*, 35(3):207–210, 1979.

[241] H. Weyl. Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen. I. *Mathematische Zeitschrift*, 23(1):271–309, 1925.

[242] H. Weyl. *The classical groups: their invariants and representations*, volume 45. Princeton university press, 1946.

[243] G. C. Wick. The evaluation of the collision matrix. *Physical Review*, 80(2):268, 1950.

[244] E. P. Wigner. On the distribution of the roots of certain symmetric matrices. *Annals of Mathematics*, pages 325–327, 1958.

[245] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions I. In *The Collected Works of Eugene Paul Wigner*, pages 524–540. Springer, 1993.

[246] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions II. In *The Collected Works of Eugene Paul Wigner*, pages 541–545. Springer, 1993.

[247] S. S. Wilks. Mathematical statistics. A Wiley publication in mathematical statistics John Wiley & sons. *Inc., New York-London*, pages 0173–45805, 1962.

[248] B. Winn. Derivative moments for characteristic polynomials from the CUE. *Communications in Mathematical Physics*, 315(2):531–562, 2012.

[249] J. Wishart. The generalised product moment distribution in samples from a normal multivariate population. *Biometrika*, pages 32–52, 1928.

[250] N. S. Witte and P. J. Forrester. Moments of the Gaussian β Ensembles and the large-N expansion of the densities. *Journal of Mathematical Physics*, 55(8):083302, 2014.

[251] J. Zakrzewski and D. Delande. Parametric motion of energy levels in quantum chaotic systems. I. Curvature distributions. *Physical Review E*, 47(3):1650, 1993.

[252] M. R. Zirnbauer. Riemannian symmetric superspaces and their origin in random-matrix theory. *Journal of Mathematical Physics*, 37(10):4986–5018, 1996.