AN EXTENSION OF THE LANDWEBER REGULARIZATION FOR A BACKWARD TIME FRACTIONAL WAVE PROBLEM

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Abstract. In this paper, we investigate numerical methods for a backward problem of the time-fractional wave equation in bounded domains. We propose two fractional filter regularization methods, which can be regarded as an extension of the classical Landweber iteration for the time-fractional wave backward problem. The idea is first to transform the ill-posed backward problem into a weighted normal operator equation, then construct the regularization methods for the operator equation by introducing suitable fractional filters. Both a priori and a posteriori regularization parameter choice rules are investigated, together with an estimate for the smallest regularization parameter according to a discrepancy principle. Furthermore, an error analysis is carried out to derive the convergence rates of the regularized solutions generated by the proposed methods. The theoretical estimate shows that the proposed fractional regularizations efficiently overcome the well-known over-smoothing drawback caused by the classical regularizations. Some numerical examples are provided to confirm the theoretical results. In particular, our numerical tests demonstrate that the fractional regularization is actually more efficient than the classical methods for problems having low regularity.

1. **Introduction.** The fractional partial differential equations are attracting increasing attention as a modelling tool for the phenomenon related to memory and spatial heterogeneity. They have applications in a broad range of fields; see, e.g., [24, 3, 6, 22, 1, 2, 25, 9, 23] and the references therein. As the basic and core of most fractional partial differential equations, the time-fractional diffusion-wave equation

$$\partial_t^\alpha w(x,t) = \Delta w(x,t) \tag{1}$$

is of importance for the fact that it reflexes the main feature and difficulty of general fractional equations of its kind. In the above equation, ∂_t^α is the fractional derivative of order $\alpha \in (0, 2)$ [25]. This equation itself has also been used to model diffusive/propagation processes associated with the so-called anomalous diffusion/propagation, which can be found in control theory, biology, electrochemical processes, porous media, viscoelastic materials, polymer, finance, and etc.

There has been extensive numerical investigation for the "forward" problem associated to these equations; see [20, 30, 10, 19, 29, 12] to list a few, covering from finite difference methods to spectral methods. However, there is also a need to solve the "backward" problem (also called inverse problem) in practice [11, 27].

For the time-fractional diffusion equation, i.e., eq.(1) with $\alpha \in (0, 1)$, the corresponding backward problems have been studied intensely in the literature. In [21], by using eigenfunction expansion of elliptic operator in space, Liu and Yamamoto proposed a quasi-reversibility method for the inverse problem of different type. Wang and Liu [33] investigated a backward problem for a time-fractional diffusion process in inhomogeneous media. Based on the eigenfunction expansion with respect to the spatial variable, they proposed a regularization technique with the number of truncation terms as the regularization parameter. Wang et al. [32] transformed their backward problem into a Fredholm integral equation of the first kind, then solved the transformed problem by Tikhonov regularization method. Wei and Wang [34] proposed a modified regularization method for an inverse quasi-boundary value problem. Wang and Wei [31] proposed an iterative method inspired by the work [7, 8, 5] to solve a backward problem, and gave convergence estimates under two kinds of regularization parameter choice rule. In [15], Han et al. obtained a regularization solution by the fractional Landweber iterative regularization method for identifying the initial condition of the time-fractional diffusion equation.

As compared to the case $\alpha \in (0, 1)$, investigation for $\alpha \in (1, 2)$, i.e., fractional wave equation, is relatively sparse. In [35], Wei and Zhang considered two backward problems for (1) with $\alpha \in (1, 2)$. Based on the series expression of the solution for the forward problem, the backward problem for determining the initial data was converted into solving the Fredholm integral equation of the first kind. For the lack of the uniqueness for backward problems, the authors turned to consider the best-approximate solution. Then they used the Tikhonov regularization method to deal with the integral equation. The convergence rate of the regularized solution to the best-approximate solution were presented under certain regularization parameter choice rule. More recently, Yang et al. [37] considered the inverse problem of identifying the initial value problem of a space-time fractional wave equation. Then classical Landweber iterative regularization method was used to solve this problem. However, it is known that the classical regularization methods such as Landweber or Tikhonov method suffer from the over-smoothing approximate solutions. Moreover, the classical Landweber/Tikhonov method requires too large/small regularization parameter, this leads to increasing numerical instabilities and cost.

In this paper we aim at proposing and analyzing an improved algorithm to compute the best-approximate solution of a typical backward problem associated to the time fractional wave equation. Inspired by some recent work [36, 35, 37], our proposed algorithm makes use of the modified Landweber iteration method with a suitably chosen fractional operator. This idea was originally proposed by Klann et al. [18] in a general framework based on the filter regularization technique for solving a linear inverse problem. It was then studied and applied in some extent

by many authors with success, see [16, 13, 4, 36, 15] for examples. Our goal is to graft this approach to the time-fractional inverse problem considered in this paper. Precisely, we will present two fractional Landweber regularization methods for the time-fractional wave backward problem, which are based on the reformulation of the problem using a weighted normal operator equation. Our analysis and numerical tests carried out in the paper show that, compared to the classical Landweber method, the improved algorithm not only requires a smaller regularization parameter but also reduces the effect of over-smoothing of the approximate solutions.

The rest of this paper is organized as follows. In section 2, we first present the direct and associated backward problems, describe basic properties of the problems, including the well-posedness of the reformulated equation. In Section 3, we construct our fractional regularization methods — explicit and implicit, and analyze the influence of the fractional parameter on the smoothness of the regularized solutions. The error analysis is carried out in Section 4 to derive error estimates of the regularized solutions, together with investigation of the regularization parameter choice rules and estimation of the smallest regularization parameter based on an a posteriori principle. Some numerical examples are provided in Section 5 to validate the proposed methods. In Section 6 we give a few concluding remarks. Some preparatory materials are presented in the final appendix section.

2. Inverse problem and well-posedness.

2.1. Statement of the problem.
Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 1, 2, 3$) with sufficient smooth boundary $\partial\Omega$. We consider the following time-fractional wave problem with homogeneous Dirichlet boundary condition: Given the initial conditions $f, h \in L^2(\Omega)$, find w such that:

$$\begin{cases} \partial_t^\alpha w(x,t) = -(Lw)(x,t) + G(x,t), & x \in \Omega, \quad t \in (0,T], \\ w(x,t) = 0, & x \in \partial\Omega, \quad t \in [0,T], \\ w(x,0) = f(x), & x \in \Omega, \\ w_t(x,0) = h(x), & x \in \Omega, \end{cases} \tag{2}$$

where $\alpha \in (1,2)$, $G(x,t) \in L^2(0,T; L^2(\Omega))$, ∂_t^α is defined as

$$\partial_t^\alpha w(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{w_{ss}}{(t-s)^{\alpha-1}} \, \mathrm{d}s \tag{3}$$

with $\Gamma(\cdot)$ being the Gamma function. In (2), L is symmetric uniformly elliptic defined in $\mathcal{D}(L) = H^2(\Omega) \cap H_0^1(\Omega)$, which can be $-\Delta$ or the following more general operator:

$$Lw(x) = -\sum_{i=1}^d \frac{\partial}{\partial x_i} \left(\sum_{i=1}^d a_{ij}(x) \frac{\partial}{\partial x_j} w(x) \right) + b(x)w(x)$$

with the coefficients $\{a_{ij}\}$ and b satisfying

$$a_{ij} = a_{ji}, \quad a_{ij} \in C^1(\bar{\Omega}), \quad 1 \le i, j \le d,$$

$$a_0 \sum_{i=1}^d \xi_i^2 \le \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j, \quad \forall x \in \bar{\Omega}, \quad \forall \xi \in \mathbb{R}^d, \quad a_0 > 0,$$

$$b(x) \ge 0, \quad b \in C(\bar{\Omega}), \quad \forall x \in \bar{\Omega}.$$

We are interested to the following **backward problem**: given the noisy final state $w^\delta(x,T)$, find the initial condition $f(x)$, such that the solution $w(x,t)$ of (2) satisfies

$$\|w(x,T) - w^\delta(x,T)\| \leq \delta, \tag{4}$$

where $\delta > 0$ is a noise level. Hereafter we use $\|\cdot\|$ to denote the standard $L^2(\Omega)$ norm.

It is readily seen that the problem (2) is equivalent to the following two problems:

$$\begin{cases} \partial_t^\alpha v(x,t) = -(Lv)(x,t) + G(x,t), & x \in \Omega, \quad t \in (0,T], \\ v(x,t) = 0, & x \in \partial\Omega, \quad t \in [0,T], \\ v(x,0) = 0, & x \in \Omega, \\ v_t(x,0) = h(x), & x \in \Omega, \end{cases} \tag{5}$$

and

$$\begin{cases} \partial_t^\alpha u(x,t) = -(Lu)(x,t), & x \in \Omega, \quad t \in (0,T], \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in [0,T], \\ u(x,0) = f(x), & x \in \Omega, \\ u_t(x,0) = 0, & x \in \Omega \end{cases} \tag{6}$$

with $w(x,t) = u(x,t) + v(x,t)$. Since (5) is a direct and well-posed problem, we are led to consider the backward problem associated to (6). That is

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Given $g^\delta(x) := u^\delta(x,T) = w^\delta(x,T) - v(x,T)$, determine $f(x) \in L^2(\Omega)$ such that the solution $u(x,t)$ to (6) satisfies

$$\|g^\delta(x) - g(x)\| \leq \delta, \tag{7}$$

where $g(x) := u(x,T)$.

It is well known that this inverse problem can be reformulated into an integral equation by using spectral decomposition in the spatial variable. Let

$$L\varphi_n = \lambda_n \varphi_n, \quad \varphi_n\big|_{\partial\Omega} = 0,$$

where λ_n are the eigenvalues of L, and $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ are the corresponding orthonormal eigenfunctions. The eigenvalues λ_n satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \quad \lim_{n\to\infty} \lambda_n = +\infty. \tag{8}$$

It was proved in [28] that if $f \in L^2(\Omega)$, then there is a unique solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (6), and

$$u(x,t) = \sum_{n=1}^\infty f_n E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x), \tag{9}$$

where $f_n = (f, \varphi_n)$, $E_{\alpha,1}$ is the Mittage-Leffler function; see also Definition 7.1 in the appendix.

Therefore the initial condition f can be determined by taking $t = T$ in (9)

$$\sum_{n=1}^\infty (f, \varphi_n) E_{\alpha,1}(-\lambda_n T^\alpha)\varphi_n(x) = u(x,T) =: g(x), \quad \forall x \in \Omega,$$

which is nothing than the first kind integral equation:

$$(Kf)(x) = g(x), \quad \forall x \in \Omega, \tag{10}$$

where the integral operator K is defined by

$$(Kf)(x) = \int_\Omega k(x,\xi)f(\xi)\mathrm{d}\xi, \quad \forall x \in \Omega \tag{11}$$

with the kernel function $k(x, \xi)$ being

$$k(x, \xi) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n T^\alpha) \varphi_n(x) \varphi_n(\xi). \tag{12}$$

Obviously K is a self-adjoint operator. Furthermore it follows from (9) that if $f \in L^2(\Omega)$, then $g \in H^2(\Omega)$. As a consequence the operator K is compact from $L^2(\Omega)$ to $L^2(\Omega)$ since $H^2(\Omega)$ is compactly imbedded into $L^2(\Omega)$. Therefore, according to a classical theory; see, e.g., [17], the integral equation (10) is ill-posed.

2.2. **Existence, uniqueness, and stability.** We begin with studying the singular values, denoted by σ_n, of the integral operator K.

First, by the orthogonality of $\{\varphi_n\}_{n=1}^{\infty}$, it is easy to see that

$$\sigma_n = |E_{\alpha,1}(-\lambda_n T^\alpha)|, \quad n = 1, 2, \cdots. \tag{13}$$

On the other side we deduce from (8) and Lemma 7.2 in the appendix that, for large enough n, it holds

$$E_{\alpha,1}(-\lambda_n T^\alpha) \leq \frac{1}{2\Gamma(1-\alpha)\lambda_n T^\alpha} < 0,$$

and $\sigma_n \to 0$ as $n \to \infty$. Then we define

$$\phi_n(x) = \begin{cases} \varphi_n(x), & E_{\alpha,1}(-\lambda_n T^\alpha) \geq 0, \\ -\varphi_n(x), & E_{\alpha,1}(-\lambda_n T^\alpha) < 0, \end{cases} \quad n = 1, 2, \cdots.$$

It is seen that $\{\phi_n\}_{n=1}^{\infty}$ are orthonormal in $L^2(\Omega)$ and

$$K\varphi_n(\xi) = E_{\alpha,1}(-\lambda_n T^\alpha)\varphi_n(x) = \sigma_n \phi_n(x),$$
$$K^* \phi_n(x) = E_{\alpha,1}(-\lambda_n T^\alpha)\phi_n(\xi) = \sigma_n \varphi_n(\xi),$$

where K^* is the adjoint of K. Thus $\{(\sigma_n; \varphi_n, \phi_n); \ \sigma_n > 0\}$ is the singular system of K.

Then we define the index set

$$I = \left\{ n \in \mathbb{N}^+; \ \sigma_n = 0 \right\}.$$

It was proved in [35] that the set I is finite (possibly empty). Hence the integral kernel defined in (12) can be rewritten as

$$k(x, \xi) = \sum_{n=1, n \notin I}^{\infty} E_{\alpha,1}(-\lambda_n T^\alpha) \varphi_n(x) \varphi_n(\xi).$$

Observe that the kernel space and the range space of operator K are respectively

$$N(K) = \text{span}\{\varphi_n; \ n \in I\},$$

$$R(K) = \left\{ g \in L^2(\Omega); \ (g, \phi_n) = 0 \text{ for } n \in I, \text{ and } \sum_{n=1, n \notin I}^{\infty} \frac{1}{\sigma_n^2} |(g, \phi_n)|^2 < +\infty \right\}.$$

Clearly the integral equation (10) has a solution if and only if $g \in R(K)$. For $g \notin R(K)$, we turn to look for a *least-squares solution* to (10): find $\hat{f} \in L^2(\Omega)$ such that

$$\|K\hat{f} - g\| = \inf_{f \in L^2(\Omega)} \|Kf - g\|.$$

An obvious fact is, if $g \in R(K)$, then every solution of (10) is a least-squares solution.

For a given $g \in L^2(\Omega)$, we denote by S_g the set of all least-squares solutions of (10), i.e.,

$$S_g := \left\{ \hat{f} \in L^2(\Omega); \ \|K\hat{f} - g\| = \inf_{f \in L^2(\Omega)} \|Kf - g\| \right\}.$$

Let K^+ denote the Moore-Penrose pseudo-inverse of K and $D(K^+)$ stands for the domain of K^+. Then it is known from [11] that $D(K^+) = R(K) + R(K)^\perp$, and $S_g \neq \emptyset$ if and only if $g \in D(K^+)$. Moreover, $\hat{f} \in S_g$ if and only if $\hat{f}$ is a solution of the normal equation

$$K^*K\hat{f} = K^*g. \tag{14}$$

Note that when $I \neq \emptyset$, K is not injective. In this case a unique least-squares solution from S_g can be identified by imposing certain additional restrictions. For example the unique *best-approximate solution* $f^+ \in N(K)^\perp$ is determined such that

$$\|f^+\| = \inf_{\hat{f} \in S_g} \|\hat{f}\|. \tag{15}$$

The following theorem gives the existence and uniqueness result of the solutions for the integral equation (10), which can be directly proved by applying the classical Picard criterion (see [11], Theorem 2.8).

Theorem 2.1. *Given* $g \in D(K^+)$. *If* $I = \emptyset$, *there exists a unique least squares solution in* $L^2(\Omega)$ *to* (10), *which is given by*

$$f^+(x) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n}(g, \phi_n)\varphi_n(x).$$

If $I \neq \emptyset$, *there exist infinitely many least squares solutions to* (10). *However only one best-approximate solution is allowed in* $L^2(\Omega)$, *given by*

$$f^+(x) = \sum_{n=1, n \notin I}^{\infty} \frac{1}{\sigma_n}(g, \phi_n)\varphi_n(x). \tag{16}$$

Define the space

$$\mathcal{D}(L^\gamma) = \left\{ \psi \in L^2(\Omega); \ \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \varphi_n)|^2 < \infty \right\}, \quad \gamma \geq 0, \tag{17}$$

equipped with the norm

$$\|\psi\|_{\mathcal{D}(L^\gamma)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \varphi_n)|^2 \right)^{\frac{1}{2}}.$$

Then $\mathcal{D}(L^\gamma)$ is a Hilbert space. In particular, we have $\mathcal{D}(L^0) = L^2(\Omega)$, $\mathcal{D}(L^\gamma) \subset H^{2\gamma}(\Omega)$, and $\mathcal{D}(L^{\frac{1}{2}}) = H_0^1(\Omega)$.

Finally the following conditional stability result is very useful in the error analysis of the numerical methods to be proposed in the next section.

Theorem 2.2. *(See [35]) For any* $p > 0, E > 0$, *it holds*

$$\|f\| \leq cE^{\frac{2}{p+2}} \|Kf\|^{\frac{p}{p+2}}, \tag{18}$$

if $f(x) \in \mathcal{D}(L^{\frac{p}{2}}) \cap N(K)^\perp$ *and* $\|f\|_{\mathcal{D}(L^{\frac{p}{2}})} \leq E$, *where* c *is a positive constant depending on* α, T *and* p.

3. **Fractional Landweber regularization.** The classical Landweber iterative methods for approximating the best-approximate solution of (10) consists in first transforming the original problem (10) into the normal equation (14), then approximating the solutions by solving an equivalent fixed point equation.

Here we propose a generalized Landweber method for finding the best-approximate solution of (10). The main idea of the generalization is: instead of solving standard normal equation (14), we propose to solve a weighted normal equation as follows:

$$(K^*K)^{\frac{\nu-1}{2}} K^*Kf = (K^*K)^{\frac{\nu-1}{2}} K^*g, \tag{19}$$

where $0 \leq \nu \leq 1$, $(K^*K)^{\frac{\nu-1}{2}}$ is defined with the aid of the Moore-Penrose pseudo-inverse of K^*K. Then the explicit Landweber iteration for solving (19) reads:

• **Explicit fractional Landweber regularization (ExFLR)**

$$
\begin{aligned}
\text{Set} \quad & f^0: \quad \text{initial guess,} \\
\text{Compute} \quad & f^m: \quad = f^{m-1} + a\,(K^*K)^{\frac{\nu-1}{2}} K^*(g - Kf^{m-1}) \\
& \qquad = (I - a\,(K^*K)^{\frac{\nu+1}{2}})f^{m-1} + a\,(K^*K)^{\frac{\nu-1}{2}} K^*g, \quad m = 1, 2, \cdots.
\end{aligned}
$$

In the above iteration, the iteration number m plays the role of regularization parameter, and a is an acceleration parameter satisfying

$$0 < a < \frac{1}{\|K\|^{\nu+1}}. \tag{20}$$

Next we are going to analyze the convergence of the sequence $\{f^m\}$ and the error with respect to the noise level δ. Without loss of generality, we assume $f^0 = 0$. Observe that

$$f^m = a \sum_{j=0}^{m-1} (I - a\,(K^*K)^{\frac{\nu+1}{2}})^j\,(K^*K)^{\frac{\nu-1}{2}} K^*g =: R_m g, \quad m = 1, 2, \cdots.$$

The sequence corresponding to the noisy data g^δ is denoted by f_δ^m, i.e.,

$$f_\delta^m = R_m g^\delta, \quad m = 1, 2, \cdots.$$

By singular value decomposition for compact self-adjoint operator (see, e.g., [14]), we can also derive an alternative expression for f_δ^m:

$$f_\delta^m = \sum_{n=1, n\notin I}^{\infty} \frac{1 - \left(1 - a\sigma_n^{\nu+1}\right)^m}{\sigma_n}(g^\delta, \phi_n)\varphi_n, \quad m = 1, 2, \cdots. \tag{21}$$

Remark 1. (a) The method (21) can be regarded as a filter regularization method; see, e.g., [11, 17, 36].

(b) In the case $\nu = 1$, (21) is indeed the standard filter regularization method based on the classical Landweber iteration; see also [11] for details. This is the reason we also term the proposed method as fractional Landweber regularization. It is worth to emphasize that the new method can be implemented at essentially no additional computational cost in comparison to the classical method ($\nu = 1$).

(c) A similar method has been used in [36] for minimizing a weighted least squares functional model for ill-posed operator equations. The difference here is, although no essential, (21) is directly derived from the weighted normal equation (19).

• **Implicit fractional Landweber regularization (ImFLR)**

We also propose the implicit Landweber iteration as follows:

Set $\qquad f^0 : \ = 0,$

Compute $\quad f^m : \ = f^{m-1} + a\,(K^*K)^{\frac{\nu-1}{2}}\,K^*(g - Kf^m), \quad m = 1, 2, \cdots,$

where $a > 0$, which is less restrictive as compared to (20) for the explicit formula. A simple reformulation gives

$$(I + a\,(K^*K)^{\frac{\nu+1}{2}})f^m = f^{m-1} + a\,(K^*K)^{\frac{\nu-1}{2}}\,K^*g, \quad m = 1, 2, \cdots.$$

Further observation yields

$$f^m = \bar{R}_m g, \quad m = 1, 2, \cdots,$$

where the operator $\bar{R}_m$ is defined by

$$\bar{R}_m = \sum_{j=1}^{m} a(I + a\,(K^*K)^{\frac{\nu+1}{2}})^{-j}\,(K^*K)^{\frac{\nu-1}{2}}\,K^*.$$

Let

$$f_\delta^m = \bar{R}_m g^\delta, \quad m = 1, 2, \cdots.$$

Again, using singular value decomposition, we obtain

$$f_\delta^m = \sum_{n=1, n \notin I}^{\infty} \frac{1 - \left(1 + a\sigma_n^{\nu+1}\right)^{-m}}{\sigma_n}(g^\delta, \phi_n)\varphi_n, \quad m = 1, 2, \cdots. \tag{22}$$

Notice that the implicit regularization removes the upper bound restriction on the acceleration parameter a, thus is more flexible than the explicit algorithm.

Next we show the regularization effect of the proposed fractional Landweber methods.

Theorem 3.1. *Let f_δ^m be the sequence generated by the explicit fractional Landweber regularization (21) with $0 < a < \frac{1}{\|K\|^{\nu+1}}$ or implicit version (22) with $a > 0$. Then for any $m = 1, 2, \cdots$, f_δ^m belongs to $\mathcal{D}(L^\nu)$ for $0 \le \nu \le 1$ and $g^\delta \in L^2(\Omega)$.*

Proof of Theorem 3.1. We only give the proof for the explicit fractional Landweber regularization. The implicit version can be proved in a similar way. First it follows from (17) and (21), for any $\gamma \ge 0$,

$$\|f_\delta^m\|_{\mathcal{D}(L^\gamma)}^2 = \sum_{n=1}^{\infty} \lambda_n^{2\gamma}|(f_\delta^m, \varphi_n)|^2 = \sum_{n=1, n \notin I}^{\infty} \lambda_n^{2\gamma}\frac{[1 - (1 - a\sigma_n^{\nu+1})^m]^2}{\sigma_n^2}|(g^\delta, \phi_n)|^2. \tag{23}$$

To study the convergence of this series, we use the following known result [35]:

$$|E_{\alpha,1}(-\lambda_n T^\alpha)| = O(\lambda_n^{-1}), \quad n \notin I. \tag{24}$$

Thus, by definition (13), $\sigma_n = |E_{\alpha,1}(-\lambda_n T^\alpha)| \to 0$ as $n \to \infty$. As a consequence, we have for $\nu \ge 0$ and $m \ge 1$, $\dfrac{1 - (1 - a\sigma_n^{\nu+1})^m}{\sigma_n} \sim ma\sigma_n^\nu$ as $n \to \infty$. Combining this with (24) gives

$$\frac{[1 - (1 - a\sigma_n^{\nu+1})^m]^2}{\sigma_n^2} = O(\sigma_n^{2\nu}) = O(\lambda_n^{-2\nu}) \text{ as } n \to \infty.$$

Therefore the series (23) converges if and only if the series $\sum\limits_{n=1, n \notin I}^{\infty} \lambda_n^{2(\gamma-\nu)}|(g^\delta, \phi_n)|^2$ converges. The latter is true for any $g^\delta \in L^2(\Omega)$ and $0 \le \gamma \le \nu$. This ends the proof. $\qquad \square$

Remark 2. It has been well known that the classical regularization methods such as Landweber and Tikhonov methods suffer from the over-smoothing problem. We see from the above theorem that for the noisy final solution data $g^\delta \in L^2(\Omega)$, the approximations f_δ^m to the initial condition $f \in L^2(\Omega)$ by the fractional Landweber iterations belong to the space $\mathcal{D}(L^\nu) \subset H^{2\nu}$. This means that the fractional Landweber iteration has also regularization effect on the original solutions. However we would like to point out that compared to the classical Landweber method ($\nu = 1$) for which $f_\delta^m \in H^2$, the new method reduces the smoothness of the approximative solutions, particularly for smaller ν, thus efficiently overcomes the over-smoothing drawback caused by the classical regularization methods.

4. **Error estimation under two parameter choice rules.** In this section, we carry out the convergence analysis for the proposed fractional Landweber regularization methods under both a priori and a posteriori regularization parameter choice rules. Hereafter we use c or C, with or without subscripts, bars, to mean generic positive constants, which may not be the same at different occurrences.

4.1. **Under a priori parameter choice rule.**

Theorem 4.1. *Let f^+ be the best-approximate solution defined in (15), f_δ^m is the solution of the fractional Landweber regularization (21) or (22). Assume the noisy data satisfies (7) and the solution satisfies the a priori condition $\|f^+\|_{\mathcal{D}(L^{\frac{p}{2}})} \le E$ for some $p > 0$. Then we have*

$$\|f_\delta^m - f^+\| \le cE^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}} \tag{25}$$

under the choices for the a priori regularization parameter m and acceleration factor a:

$$m = \left\lceil \left(\frac{E}{\delta}\right)^{\frac{2(\nu+1)}{p+2}} \right\rceil + 1, \quad 0 < a \le \frac{1}{\|K\|^{\nu+1}} \tag{26}$$

for the explicit fractional Landweber regularization (21); or

$$m = \left\lceil \left(\frac{E}{\delta}\right)^{\frac{2(\nu+1)}{p+2}} + \frac{p}{2(\nu+1)} \right\rceil + 1, \quad a > 0 \tag{27}$$

for the implicit fractional Landweber regularization (22). Where c is a positive constant, which may depend on a, ν, and p. $\lceil r \rceil$ stands for the largest integer less than or equal to r.

Proof of Theorem 4.1. Using the triangle inequality

$$\|f_\delta^m - f^+\| \le \|f_\delta^m - f^m\| + \|f^m - f^+\|, \tag{28}$$

we are led to estimate the two terms on the right hand side.

(i) We start with the explicit fractional Landweber regularization (21). First, for the first term, we deduce from (21) and (7)

$$\|f_\delta^m - f^m\|^2 = \sum_{n=1, n \notin I}^{\infty} \left(\frac{1 - (1 - a\sigma_n^{\nu+1})^m}{\sigma_n}\right)^2 \left((g^\delta, \phi_n) - (g, \phi_n)\right)^2 \le \left(\sup_{n \notin I} A_n\right)^2 \delta^2,$$

where

$$A_n := \frac{1 - (1 - a\sigma_n^{\nu+1})^m}{\sigma_n}, \quad n \notin I.$$

By the Bernoulli inequality, we have

$$A_n \leq \frac{\sqrt[\nu+1]{1 - \left(1 - a\sigma_n^{\nu+1}\right)^m}}{\sigma_n} \leq \frac{\sqrt[\nu+1]{1 - \left(1 - ma\sigma_n^{\nu+1}\right)}}{\sigma_n} = \sqrt[\nu+1]{ma}.$$

Therefore

$$\|f_\delta^m - f^m\| \leq \sqrt[\nu+1]{ma}\delta \leq \sqrt[\nu+1]{a\left(\left(\frac{E}{\delta}\right)^{\frac{2(\nu+1)}{p+2}} + 1\right)}\delta \leq \sqrt[\nu+1]{2a}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \qquad (29)$$

Then, for the second term in (28), we derive from (16)

$$f^m - f^+ = \sum_{n=1, n \notin I}^{\infty} \left(\frac{1 - \left(1 - a\sigma_n^{\nu+1}\right)^m}{\sigma_n} - \frac{1}{\sigma_n}\right)(g, \phi_n)\varphi_n$$

$$= \sum_{n=1, n \notin I}^{\infty} -\left(1 - a\sigma_n^{\nu+1}\right)^m \frac{1}{\sigma_n}(g, \phi_n)\varphi_n$$

$$= \sum_{n=1, n \notin I}^{\infty} -\left(1 - a\sigma_n^{\nu+1}\right)^m (f^+, \varphi_n)\varphi_n.$$

Under the a priori bound condition, we have

$$\|f^m - f^+\|^2 = \sum_{n=1, n \notin I}^{\infty} |(f^+, \varphi_n)|^2 \lambda_n^p \left(1 - a\sigma_n^{\nu+1}\right)^{2m} \lambda_n^{-p} \leq \left(\sup_{n \notin I} B_n\right)^2 E^2, \qquad (30)$$

where $B_n := \left(1 - a\sigma_n^{\nu+1}\right)^m \lambda_n^{-\frac{p}{2}}$, $n \notin I$. It follows from (24) and Lemma 7.3 in the appendix, for $n \notin I$,

$$B_n \leq c \left(1 - a\sigma_n^{\nu+1}\right)^m \sigma_n^{\frac{p}{2}}$$

$$\leq c\left(\frac{p}{a(2m(\nu+1)+p)}\right)^{\frac{p}{2(\nu+1)}} \left(\frac{2m(\nu+1)}{2m+p}\right)^m$$

$$\leq c\left(\frac{2m(\nu+1)}{2m(\nu+1)+p}\right)^{m+\frac{p}{2(\nu+1)}} \left(2m(\nu+1)\right)^{-\frac{p}{2(\nu+1)}}$$

$$\leq cm^{-\frac{p}{2(\nu+1)}},$$

where c may depend on a, ν, and p. Inserting this estimate into (30), then using (26) gives

$$\|f^m - f^+\| \leq cm^{-\frac{p}{2(\nu+1)}}E \leq cE^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \qquad (31)$$

We then obtain (25) by bringing (29) and (31) into (28).

(ii) We now turn to the case of the implicit Landweber regularization (21). Similar to the explicit formula, we use (22) and (7) to get

$$\|f_\delta^m - f^m\|^2 = \sum_{n=1, n \notin I}^{\infty} \left(\frac{1 - (1 - a\sigma_n^{\nu+1})^{-m}}{\sigma_n}\right)^2 \left((g^\delta, \phi_n) - (g, \phi_n)\right)^2 \leq \left(\sup_{n \notin I} \bar{A}_n\right)^2 \delta^2,$$

where

$$\bar{A}_n := \frac{1 - (1 - a\sigma_n^{\nu+1})^{-m}}{\sigma_n}, \quad n \notin I.$$

Using the Bernoulli inequality

$$\left(1 - \frac{a\sigma_n^{\nu+1}}{1 + a\sigma_n^{\nu+1}}\right)^m \geq 1 - \frac{ma\sigma_n^{\nu+1}}{1 + a\sigma_n^{\nu+1}}, \quad m \in \mathbb{N}$$

leads to

$$\bar{A}_n \leq \frac{1}{\sigma_n} \sqrt[\nu+1]{1 - (1 + a\sigma_n^{\nu+1})^{-m}}$$

$$= \frac{1}{\sigma_n} \sqrt[\nu+1]{1 - \left(1 - \frac{a\sigma_n^{\nu+1}}{1 + a\sigma_n^{\nu+1}}\right)^m}$$

$$\leq \frac{1}{\sigma_n} \sqrt[\nu+1]{\frac{ma\sigma_n^{\nu+1}}{1 + a\sigma_n^{\nu+1}}}$$

$$\leq \sqrt[\nu+1]{ma}.$$

Consequently,

$$\|f_\delta^m - f^m\| \leq \sqrt{ma}\delta \leq \sqrt[\nu+1]{\frac{p + 2(\nu+1)}{2(\nu+1)}} a E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{32}$$

On the other side, we have from (16)

$$f^m - f^+ = \sum_{n=1, n \notin I}^{\infty} \left(\frac{1 - \left(1 + a\sigma_n^{\nu+1}\right)^{-m}}{\sigma_n} - \frac{1}{\sigma_n}\right)(g, \phi_n)\varphi_n$$

$$= \sum_{n=1, n \notin I}^{\infty} -\left(1 + a\sigma_n^{\nu+1}\right)^{-m} \frac{1}{\sigma_n}(g, \phi_n)\varphi_n$$

$$= \sum_{n=1, n \notin I}^{\infty} -\left(1 + a\sigma_n^{\nu+1}\right)^{-m} \frac{1}{\sigma_n}(f^+, \varphi_n)\varphi_n.$$

Under the a priori bound condition, we obtain

$$\|f^m - f^+\|^2 = \sum_{n=1, n \notin I}^{\infty} |(f^+, \varphi_n)|^2 \lambda_n^p \left(1 + a\sigma_n^{\nu+1}\right)^{-2m} \lambda_n^{-p} \leq \left(\sup_{n \notin I} \bar{B}_n\right)^2 E^2,$$

where

$$\bar{B}_n := \left(1 + a\sigma_n^{\nu+1}\right)^{-m} \lambda_n^{-\frac{p}{2}}, \quad n \notin I.$$

To bound $\bar{B}_n$, we use (24), Lemma 7.4, and (27) to get

$$\bar{B}_n \leq c \left(1 + a\sigma_n^{\nu+1}\right)^{-m} \sigma_n^{\frac{p}{2}}$$

$$\leq c\left(\frac{p}{a(2m(\nu+1) - p)}\right)^{\frac{p}{2(\nu+1)}} \left(\frac{2m(\nu+1) - p}{2m(\nu+1)}\right)^m$$

$$\leq c\left(\frac{2m(\nu+1) - p}{2m(\nu+1)}\right)^{m - \frac{p}{2(\nu+1)}} (2m(\nu+1))^{-\frac{p}{2(\nu+1)}}$$

$$\leq cm^{-\frac{p}{2(\nu+1)}}, \quad n \notin I.$$

Thus

$$\|f^m - f^+\| \leq cm^{-\frac{p}{2(\nu+1)}} E \leq cE^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{33}$$

Finally we conclude by combining (32), (33), and the triangle inequality.

$\square$

Remark 3. The formula for the regularization parameter for the implicit fractional Landweber regularization, i.e., (27), was chosen so that $m > \frac{p}{2(\nu+1)}$, which was used in establishing the error estimate in Theorem 4.1. However it should be emphasized that this is only a technical choice because usually the noise level δ is small, thus $\left(\frac{E}{\delta}\right)^{\frac{2(\nu+1)}{p+2}}$ is much larger than $\frac{p}{2(\nu+1)}$. In practice it suffices to choose $m = O\left(\left(\frac{E}{\delta}\right)^{\frac{2(\nu+1)}{p+2}}\right)$ as for the explicit method.

4.2. **Under a posteriori parameter choice rule.** As a priori bound condition is unknown in practice, there is a need to consider an a posterior parameter choice rule to determine necessary regularization parameter m.

Here we consider the so-called Morozov discrepancy principle [11]. The idea is to estimate the smallest iteration number $m = m(\delta)$ such that

$$\|Kf_\delta^m - \Pi g^\delta\| \leq \tau\delta, \tag{34}$$

where $\tau > 1$ is a fixed number, Π is the L^2-orthogonal projector from $L^2(\Omega)$ to $\overline{R(K)}$. In order for the discrepancy principle (34) to be meaningful, it is common to assume $\|\Pi g^\delta\| > \tau\delta$, otherwise (34) admits a trivial solution $f_\delta^m = 0$.

The following lemma shows that the Morozov discrepancy principle (34) for the fractional Landweber regularization is meaningful and realizable.

Lemma 4.2. *Let f_δ^m be generated by the fractional Landweber regularization (21) with $0 < a < \frac{1}{\|K\|^{\nu+1}}$ or (22) with $a > 0$. Define $\rho(m) = \|Kf_\delta^m - \Pi g^\delta\|, m > 0$. Then the following results hold*

(1) $\rho(m)$ is continuous with respect to the integer variable m;

(2) $\lim\limits_{m \to +\infty} \rho(m) = 0$;

(3) $\lim\limits_{m \to 0} \rho(m) = \|\Pi g^\delta\|$;

(4) $\rho(m)$ is a strictly decreasing function with respect to m.

Proof of Lemma 4.2. The proof is straightforward by using the following expressions:

$$\rho(m) = \sqrt{\sum_{n=1, n \notin I}^{\infty} \left(1 - a\sigma_n^{\nu+1}\right)^{2m} |(g^\delta, \phi_n)|^2}, \quad 0 < a < \frac{1}{\|K\|^{\nu+1}}$$

for the explicit formula (21); or

$$\rho(m) = \sqrt{\sum_{n=1, n \notin I}^{\infty} \left(1 + a\sigma_n^{\nu+1}\right)^{-2m} |(g^\delta, \phi_n)|^2}, \quad a > 0$$

for the implicit formula (22). □

The next lemma provides an upper-bound for the regularization parameter m following the discrepancy principle (34).

Lemma 4.3. *Let f^+ be the best-approximate solution defined in (15). Under the a priori condition $\|f^+\|_{\mathcal{D}(L^{\frac{p}{2}})} \leq E$ and the noise assumption (7). If m is the smallest iteration number such that (34) holds, then we have the following upper bound estimator for m:*

$$m \leq c\left(\frac{E}{\delta(\tau-1)}\right)^{\frac{2(\nu+1)}{p+2}}. \tag{35}$$

Proof of Lemma 4.3. (i) For the explicit formula (21), we have

$$f^m = R_m g = \sum_{n=1, n \notin I}^{\infty} \frac{1 - \left(1 - a\sigma_n^{\nu+1}\right)^m}{\sigma_n} (g, \phi_n)\varphi_n$$

from which we deduce

$$\|KR_m g - \Pi g\|^2 = \sum_{n=1, n \notin I}^{\infty} \left(1 - a\sigma_n^{\nu+1}\right)^{2m} |(g, \phi_n)|^2.$$

Since the acceleration factor a is chosen such that $|1 - a\sigma_n^{\nu+1}| < 1$, we get

$$\|KR_m g - \Pi g\| \leq \|g\|, \quad \forall m \geq 0.$$

This means $\|KR_m - \Pi\| \leq 1$, $\forall m \geq 0$. Suppose now m is the smallest number such that (34) holds, then $\|KR_{m-1}g^\delta - \Pi g^\delta\| > \tau\delta$, and hence

$$\begin{aligned}
\|KR_{m-1}g - \Pi g\| &\geq \|KR_{m-1}g^\delta - \Pi g^\delta\| - \|(KR_{m-1} - \Pi)(g - g^\delta)\| \\
&\geq \tau\delta - \|KR_{m-1} - \Pi\|\delta \\
&\geq (\tau - 1)\delta.
\end{aligned} \tag{36}$$

Using (24) and the a priori assumption on f^+, we get

$$\begin{aligned}
\|KR_{m-1}g - \Pi g\|^2 &= \sum_{n=1, n \notin I}^{\infty} \left(1 - a\sigma_n^{\nu+1}\right)^{2m-2} |(g, \phi_n)|^2 \\
&= \sum_{n=1, n \notin I}^{\infty} \left(1 - a\sigma_n^{\nu+1}\right)^{2m-2} \sigma_n^2 |(f^+, \varphi_n)|^2 \\
&= \sum_{n=1, n \notin I}^{\infty} \left(1 - a\sigma_n^{\nu+1}\right)^{2m-2} \sigma_n^2 \lambda_n^{-p} \lambda_n^p |(f^+, \varphi_n)|^2 \\
&\leq \sup_{n \notin I} \widetilde{A}_n^2 E^2, \tag{37}
\end{aligned}$$

where $\widetilde{A}_n := \left(1 - a\sigma_n^{\nu+1}\right)^{m-1} \sigma_n \lambda_n^{-\frac{p}{2}}$, $n \notin I$. From (24) and Lemma 7.5, we know

$$\begin{aligned}
\widetilde{A}_n &\leq c\left(1 - a\sigma_n^{\nu+1}\right)^{m-1} \sigma_n^{\frac{p}{2}+1} \\
&\leq c\left(\frac{p+2}{a(2(m-1)(\nu+1)+p+2)}\right)^{\frac{p+2}{2(\nu+1)}} \left(\frac{2(m-1)(\nu+1)}{2(m-1)(\nu+1)+p+2}\right)^{m-1} \\
&= c\left(\frac{2a(\nu+1)}{p+2}\right)^{-\frac{p+2}{2(\nu+1)}} \left(\frac{2(m-1)(\nu+1)}{2(m-1)(\nu+1)+p+2}\right)^{\frac{2(m-1)(\nu+1)+p+2}{2(\nu+1)}} (m-1)^{-\frac{p+2}{2(\nu+1)}} \\
&\leq c(m-1)^{-\frac{p+2}{2(\nu+1)}}, \quad n \notin I. \tag{38}
\end{aligned}$$

Combining (36), (37), and (38), we obtain (35).
(ii) In the case of the implicit formula (22), we have

$$f_\delta^m = \bar{R}_m g = \sum_{n=1, n \notin I}^{\infty} \frac{1 - \left(1 + a\sigma_n^{\nu+1}\right)^{-m}}{\sigma_n} (g^\delta, \phi_n)\varphi_n.$$

By following the same lines as in the explicit case, we can obtain

$$\|K\bar{R}_{m-1}g - \Pi g\| \geq (\tau - 1)\delta. \tag{39}$$

Then we deduce from (24) and the a priori condition

$$
\begin{aligned}
\|KR_{m-1}g - \Pi g\|^2 &= \sum_{n=1, n \notin I}^{\infty} \left(1 + a\sigma_n^{\nu+1}\right)^{-(2m-2)} |(g, \phi_n)|^2 \\
&= \sum_{n=1, n \notin I}^{\infty} \left(1 + a\sigma_n^{\nu+1}\right)^{-(2m-2)} \sigma_n^2 |(f^+, \varphi_n)|^2 \\
&= \sum_{n=1, n \notin I}^{\infty} \left(1 + a\sigma_n^{\nu+1}\right)^{-(2m-2)} \sigma_n^2 \lambda_n^{-p} \lambda_n^p |(f^+, \varphi_n)|^2 \\
&\leq \sup_{n \notin I} \widetilde{B}_n^2 E^2,
\end{aligned}
\tag{40}
$$

where $\widetilde{B}_n := (1 + a\sigma_n^{\nu+1})^{-(m-1)} \sigma_n \lambda_n^{-\frac{p}{2}}$, $n \notin I$. Let's suppose $m > \frac{p+2+2(\nu+1)}{2(\nu+1)}$, otherwise we obtain a trivial upper bound which is independent of E and δ. Then it follows from (24) and Lemma 7.6

$$
\begin{aligned}
\widetilde{B}_n &\leq c\left(1 + a\sigma_n^{\nu+1}\right)^{-(m-1)} \sigma_n^{\frac{p}{2}+1} \\
&\leq c\left(\frac{p+2}{a(2(m-1)(\nu+1) - p - 2)}\right)^{\frac{p+2}{2(\nu+1)}} \left(\frac{2(m-1)(\nu+1) - p - 2}{2(m-1)(\nu+1)}\right)^{m-1} \\
&= c\left(\frac{2a(\nu+1)}{p+2}\right)^{-\frac{p+2}{2(\nu+1)}} \left(\frac{2(m-1)(\nu+1) - p - 2}{2(m-1)(\nu+1)}\right)^{\frac{2(m-1)(\nu+1)-p-2}{2(\nu+1)}} (m-1)^{-\frac{p+2}{2(\nu+1)}} \\
&\leq c(m-1)^{-\frac{p+2}{2(\nu+1)}}, \quad n \notin I.
\end{aligned}
\tag{41}
$$

Finally we conclude by combining (39), (40), and (41). □

Theorem 4.4. *Let f^+ be the best-approximate solution defined in (15), f_δ^m is the solution of the fractional Landweber regularization (21) with $0 < a < \frac{1}{\|K\|^{\nu+1}}$ or (22) with $a > 0$. Assume the noisy data satisfies (7) and the solution satisfies the a priori condition $\|f^+\|_{\mathcal{D}(L^{\frac{p}{2}})} \leq E$ for some $p > 0$. If the regularization parameter m is chosen according to the discrepancy principle (34), then we have*

$$
\|f_\delta^m - f^+\| \leq c\left(\left(\frac{E}{\tau-1}\right)^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}} + E^{\frac{2}{p+2}} (\delta(\tau+1))^{\frac{p}{p+2}}\right),
\tag{42}
$$

where c is a positive constant, which may depend on a, ν, and p.

Proof of Theorem 4.4. In virtue of the triangle inequality

$$
\|f_\delta^m - f^+\| \leq \|f_\delta^m - f^m\| + \|f^m - f^+\|,
\tag{43}
$$

we are brought to estimating the two terms on the right hand sides.
(i) Explicit case (21). First, similar to the proof of (29), we can obtain

$$
\|f_\delta^m - f^m\| \leq \sqrt[\nu+1]{ma}\delta.
$$

Using the upper bound for m provided in Lemma 4.3, we have

$$
\|f_\delta^m - f^m\| \leq c\left(\frac{E}{\tau-1}\right)^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.
\tag{44}
$$

For the second term in (43), we derive from the definitions of Kf^m and Kf^+

$$K\left(f^m - f^+\right) = \sum_{n=1,n\notin I}^{\infty} -(1 - a\sigma_n^{\nu+1})^m (g, \phi_n)\phi_n$$

$$= \sum_{n=1,n\notin I}^{\infty} -(1 - a\sigma_n^{\nu+1})^m (g - g^\delta, \phi_n)\phi_n + \sum_{n=1,n\notin I}^{\infty} -(1 - a\sigma_n^{\nu+1})^m (g^\delta, \phi_n)\phi_n$$

$$= \sum_{n=1,n\notin I}^{\infty} -(1 - a\sigma_n^{\nu+1})^m (g - g^\delta, \phi_n)\phi_n + Kf_\delta^m - \Pi g^\delta.$$

Using the assumption on a and the discrepancy principle (34), we obtain

$$\|K\left(f^m - f^+\right)\| \le \delta + \tau\delta = (\tau+1)\delta.$$

Furthermore, it follows from the a priori bound condition $\|f^+\|_{\mathcal{D}(L^{\frac{p}{2}})} \le E$

$$\|f^m - f^+\|_{\mathcal{D}(L^{\frac{p}{2}})} = \sum_{n=1,n\notin I}^{\infty} \left(\left(1 - a\sigma_n^{\nu+1}\right)^{2m} |(f^+, \varphi_n)|^2 \lambda_n^p\right)^{\frac{1}{2}} \le E.$$

Then applying Theorem 2.2 gives

$$\|f^m - f^+\| \le cE^{\frac{2}{p+2}}(\delta(\tau+1))^{\frac{p}{p+2}}. \tag{45}$$

Finally the estimate (42) is obtained by bringing (44) and (45) into (43).

(ii) Implicit case (22). The desired result can be proved in a similar way as the explicit case. The proof is completed. □

Remark 4. Although the proposed fractional regularization is convergent, the convergence rate of $f_\delta^m \to f^+$ as δ tends to 0 can be arbitrarily slow without a priori assumption on the solution. That was why, to guarantee the convergence of the proposed method, we have assumed in Theorem 4.1 and Theorem 4.4 that the solution satisfies a priori condition, i.e., $f^+ \in \mathcal{D}(L^{\frac{p}{2}})$ for some $p > 0$.

5. **Numerical validation.**

5.1. **Implementation.** To provide the input data g for the reconstruction of the initial condition from the forward problem, we propose here to solve (6) by a finite difference method, which, for the sake of completeness, we give a brief description below for one-dimensional case.

Let $\Omega = (0, L)$, $\Delta x = \frac{L}{M}$, $\Delta t = \frac{T}{N}$, $x_i = i\Delta x, 0 \le i \le M$, and $t_n = n\Delta t, 0 \le n \le N$. The approximated value of function u at the grid point (x_i, t_n) is denoted by u_i^n. The time fractional derivative at $t = t_n$ is approximated by the well-known $2 - \alpha$ scheme [30, 20] as follows:

$$\partial_t^\alpha u(x_i, t_n) \approx D_t^\alpha u_i^n := \frac{(\Delta t)^{-1}}{\Gamma(2-\alpha)}\left[b_0 \delta_t u_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\delta_t u_i^{k-\frac{1}{2}}\right],$$

where

$$u_i^{n-\frac{1}{2}} = \frac{1}{2}(u_i^n + u_i^{n-1}), \quad \delta_t u_i^{n-\frac{1}{2}} = \frac{1}{\Delta t}(u_i^n - u_i^{n-1}), \quad b_l = \int_{t_l}^{t_{l+1}} \frac{\mathrm{d}t}{t^{\alpha-1}}, \quad l \ge 0.$$

The spatial differential operator L is approximated by

$$Lu(x_i, t_n) \approx F_x u_i^n := \frac{1}{\Delta x^2}\left(a_{i+\frac{1}{2}}u_{i+1}^n - (a_{i+\frac{1}{2}} + a_{i-\frac{1}{2}})u_i^n + a_{i-\frac{1}{2}}u_{i-1}^n\right) + b(x_i)u_i^n,$$

where

$$a_{i+\frac{1}{2}} = a(x_{i+\frac{1}{2}}), \quad x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}, \quad i = 1, 2, \cdots, M - 1.$$

This leads to the following scheme for (2)

$$\begin{cases} D_t^\alpha u_i^n = F_x u_i^n, & 1 \le i \le M - 1, \ 1 \le n \le N, \\ u_i^0 = f(x_i), & 0 \le i \le M, \\ u_0^n = u_M^n = 0, & 1 \le n \le N. \end{cases}$$

Both the explicit and implicit fractional regularization methods **ExFLR** (21) and **ImFLR** (22) are used to compute the regularized solutions. The number of truncation terms used in the regularization formulas is fixed to 50. The Mittag-Leffler function is evaluated using the code from [26] with accuracy 10^{-15}. The noisy data is generated by adding a random perturbation, i.e., $g^\delta = g + \varepsilon \frac{\rho}{\|\rho\|} \|g\|$, where $\varepsilon > 0$ is the relative noise level, $\rho = 2\text{rand}(\text{size}(g)) - 1$ is a vector with random entries uniformly distributed in $[-1, 1]$. This means the noise level in (7) is $\delta = \varepsilon \|g\|$. Since the a priori bound is generally unknown, we will only carry out the numerical test using the a posteriori parameter choice rule. Thus the regularization parameter m will be chosen according to the discrepancy principle (34) with $\tau = 1.1$. The accuracy of numerical solutions will be measured through the relative error:

$$e_r(f_\delta^m, \varepsilon) = \frac{\|f_\delta^m - f\|}{\|f\|}, \quad \text{with } \|f\| \approx \sqrt{\sum_{i=1}^{M} f_i^2 \Delta x}.$$

5.2. **Numerical results.** We first consider the one-dimensional problem with $L = 1$, $T = 1$, $a_{11} \equiv 1$, and $b \equiv 0$. In this case, we have $\lambda_n = n^2 \pi^2$ and $\varphi_n(x) = \sqrt{2} \sin(n\pi x)$. The discretization parameters are fixed to be $M = N = 200$. The final date g is obtained through solving the direct problem (6) with the initial conditions provided in Examples 5.1–5.3.

Example 5.1. *Smooth initial function* $f_1(x) = \sin(2\pi x)$.

Example 5.2. *Continuous piecewise smooth function*

$$f_2(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2}, \\ -2x + 2, & \frac{1}{2} \le x \le 1. \end{cases}$$

Example 5.3. *Square wave function*

$$f_3(x) = \begin{cases} 1, & \frac{1}{4} \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < \frac{3}{4}, \\ 0, & otherwise. \end{cases}$$

Note that the exact initial values in Example 5.2 and Example 5.3 are less regular that the one in Example 5.1.

In the three examples, we fix $\nu = 0.5$ and $a = 5$. Figures 1–2 show the reconstructed initial conditions with different relative noise levels ε and fractional orders $\alpha = 1.1$ and $\alpha = 1.6$ for Examples 5.1–5.3. It is first observed that the accuracy increases as the noise level decreases in all three examples. Furthermore, more regular is the exact initial condition more accurate is the reconstructed solution, as predicted by the theoretical estimate given in Theorem 4.4. Another notable observation is that the explicit regularization **ExFLR** and implicit regularization **ImFLR** produce almost same results. In Table 1, we investigate the effect of the noise level ε on the relative errors and regularization parameter m for all three

examples for $\alpha = 1.1, 1.6$. We notice that the regularization parameter m becomes larger as ε and α decrease, and smaller is ε or α better is the accuracy of the regularized solution. This table further confirms that **ExFLR** and **ImFLR** perform equally well under the same choice of the acceleration papameter a. However it is worth mentioning that the implicit regularization is more robust in the sense there is no restriction on the acceleration parameter, thus larger a is allowed. Another notable point from Table 1 is that the required regularization parameter significantly increases when the fractional order α decreases. This is probably due to the diminution of the singular values when α decreases. However we believe more investigation is needed to give a convincing explanation.
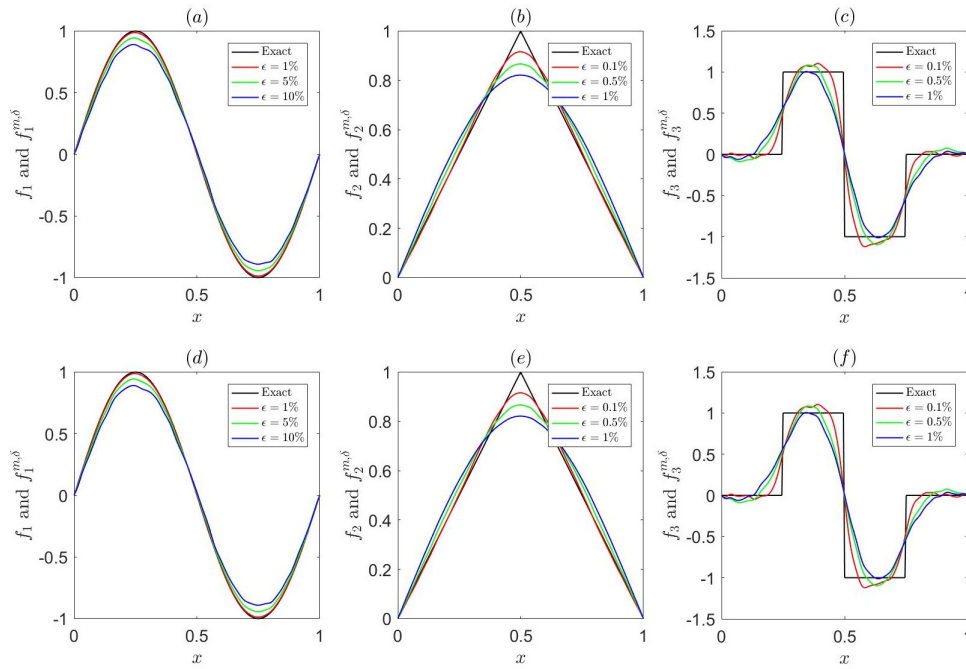

FIGURE 1. The regularized solutions with $\alpha = 1.1$ for Examples 5.1–5.3, corresponding to the three figures from left to right. (a)–(c) for **ExFLR**; (d)–(f) for **ImFLR**.

Next test concerns the investigation of the effect of the fractional parameter $\nu \in [0,1]$ on the regularized solutions. For this test, we set $a = 5$, $\alpha = 1.5$, $\varepsilon = 10\%$ in Example 5.1 and $\varepsilon = 0.1\%$ in Examples 5.2 and 5.3. In Figure 3 we compare the required regularization parameter m and relative error $e_r(f_\delta^m, \varepsilon)$ for different values of the fractional parameter, computed by **ExFLR**. First of all, we observe from the top figures that the required regularization parameter m is an increasing function with respect to ν. This means that the proposed fractional regularization methods with $\nu < 1$ need smaller value of the regularization parameter m than the classical one ($\nu = 1$) to achieve the same accuracy. This is in a good agreement with the theoretical estimate given in Lemma 4.3. The impact of ν on the relative error between the regularized solutions and exact solutions is shown in the bottom figures of Figure 3. We observe from Figures 3(a), corresponding to Example 5.1, that the
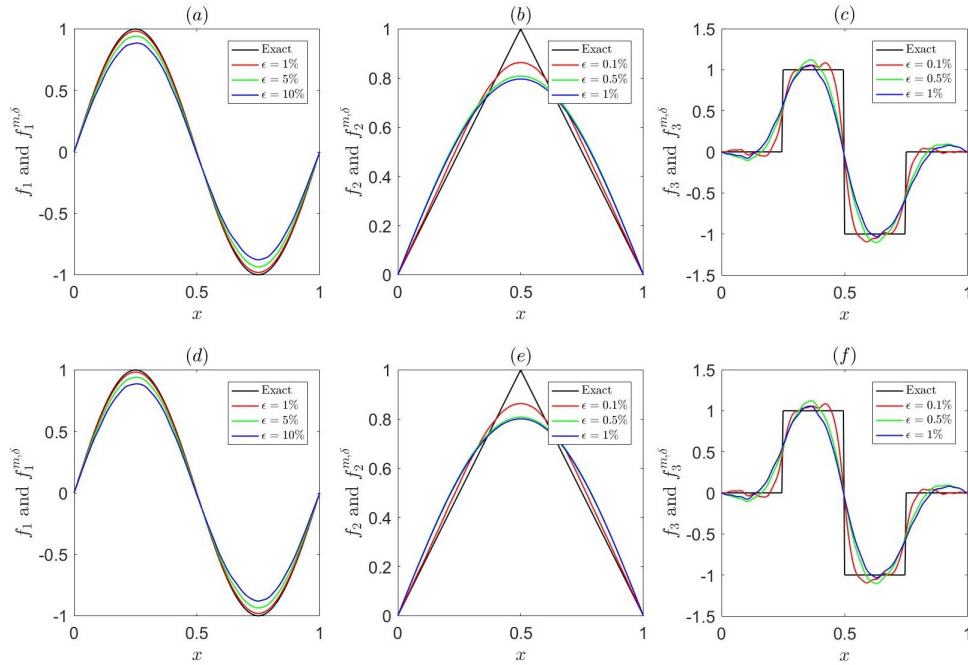
FIGURE 2. Regularized solutions with $\alpha = 1.6$ for Examples 5.1–5.3, corresponding to the three figures from left to right. (a)–(c) for **ExFLR**; (d)–(f) for **ImFLR**.

TABLE 1. Examples 5.1–5.3. Relative errors and regularization parameter versus relative noise levels.

		$\alpha = 1.1$				$\alpha = 1.6$			
		ExFLR (21)		**ImFLR** (22)		**ExFLR** (21)		**ImFLR** (22)	
f	ε	m	$e_r(f_\delta^m, \varepsilon)$	m	$e_r(f_\delta^m, \varepsilon)$	m	$e_r(f_\delta^m, \varepsilon)$	m	$e_r(f_\delta^m, \varepsilon)$
f_1	1%	7324	0.0103	7328	0.0103	147	0.0188	152	0.0187
	5%	4775	0.0506	4778	0.0506	96	0.0634	99	0.0634
	10%	3677	0.1003	3680	0.1002	73	0.1209	76	0.1188
f_2	0.1%	14913	0.0323	14915	0.0323	510	0.0647	511	0.0641
	0.5%	3846	0.0674	3846	0.0674	8	0.1183	13	0.1187
	1%	701	0.1068	706	0.1067	6	0.1211	11	0.1200
f_3	0.1%	232223	0.2947	232224	0.2947	69253	0.2728	69254	0.2728
	0.5%	32815	0.3827	32817	0.3827	7456	0.3753	7458	0.3752
	1%	20594	0.4186	20595	0.4186	4695	0.4032	4697	0.4032

error is a decreasing function with respect to ν, and the best accuracy is achieved by taking $\nu = 1$. In contrast, Figure 3(f) shows that the error is an increasing function with respect to ν, and the best solution is obtained by taking $\nu = 0$. The result for Example 5.2, presented in Figure 3(e), is between the two extreme cases, and the optimal value of ν for this example is between 0 and 1, say something close to 0.2. We want to point out that **ImFLR** has produced very similar results as **ExFLR**, which is not shown in the paper to limit the length of the paper. An interesting conclusion can be drawn from this test: the proposed fractional regularization methods are more efficient for problems having low regularity, as

demonstrated in Figure 3(e)–(f) where the initial condition to be reconstructed is respectively only of $H^1(0,1)$ and $L^2(0,1)$ regularity. One possible explanation for this observation is that changing the fractional parameter ν affects the smoothness of the computed (regularized) solution — bigger is the fractional parameter smoother is the computed solution. This explanation is evidenced by Figure 4, where the solution curves are plotted for three values of ν. It is seen that increasing ν produces smoother solution, and over-smoothing may cause a loss of accuracy of the numerical solution.
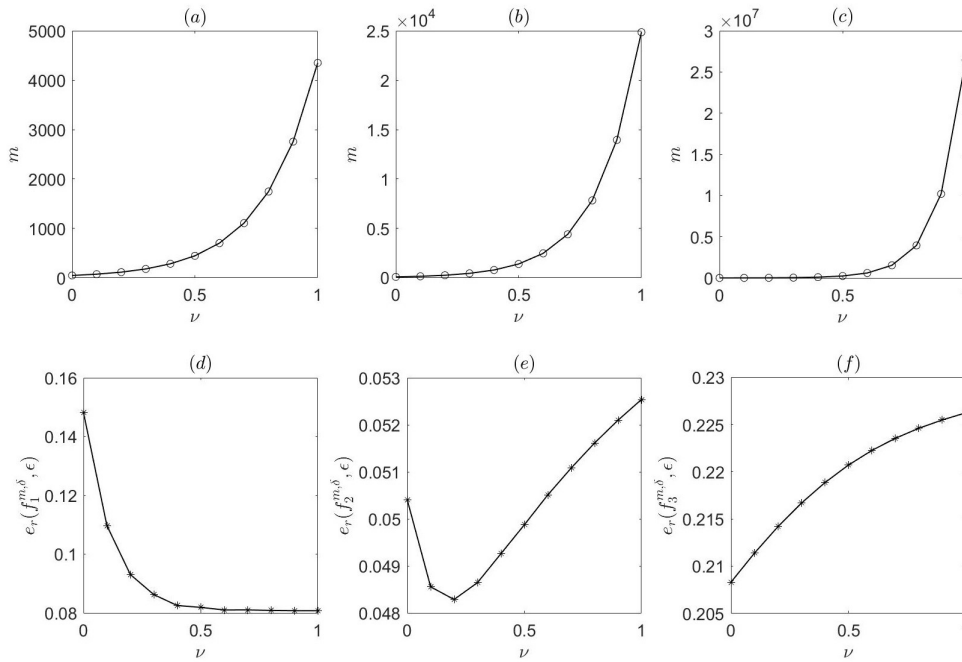

FIGURE 3. (**ExFLR**) Regularization parameter m and relative error e_r as functions of ν, vary ν in the range $\{0, 0.1, \cdots, 1\}$. (a) and (d) for Example 5.1 with $\varepsilon = 10\%$; (b) and (e) for Example 5.2 with $\varepsilon = 0.1\%$; (c) and (f) for Example 5.3 with $\varepsilon = 0.1\%$.

Finally, we perform numerical experience for a 2D example in the domain $\Omega = (0,1) \times (0,1)$ with the initial condition given in Example 5.4.

Example 5.4. *Consider the initial function*

$$f_4(x_1, x_2) = \begin{cases} 1, & (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 0.2^2, \\ 0, & otherwise. \end{cases}$$

We set $T = 1$, $\alpha = 1.3$, $a_{11}(x) = a_{22}(x) \equiv 1$, and $b(x) \equiv 0$. In this case, we have the eigenvalues $\lambda_{m,n} = (m^2 + n^2)\pi^2$ and corresponding eigenfunctions $\varphi_{m,n}(x_1, x_2) = 2\sin(m\pi x_1)\sin(n\pi x_2)$. The data of the final state is produced by solving the forward problem on the 50×50 grids in space and 100 in time domain. We choose $a = 50$ and $\nu = 0, 0.5, 1$ in the fractional regularization methods. Figure 5 plots the regularized solutions computed by the implicit regularization method, and their absolute errors with relative noise level $\varepsilon = 0.1\%$. Once again, we observe
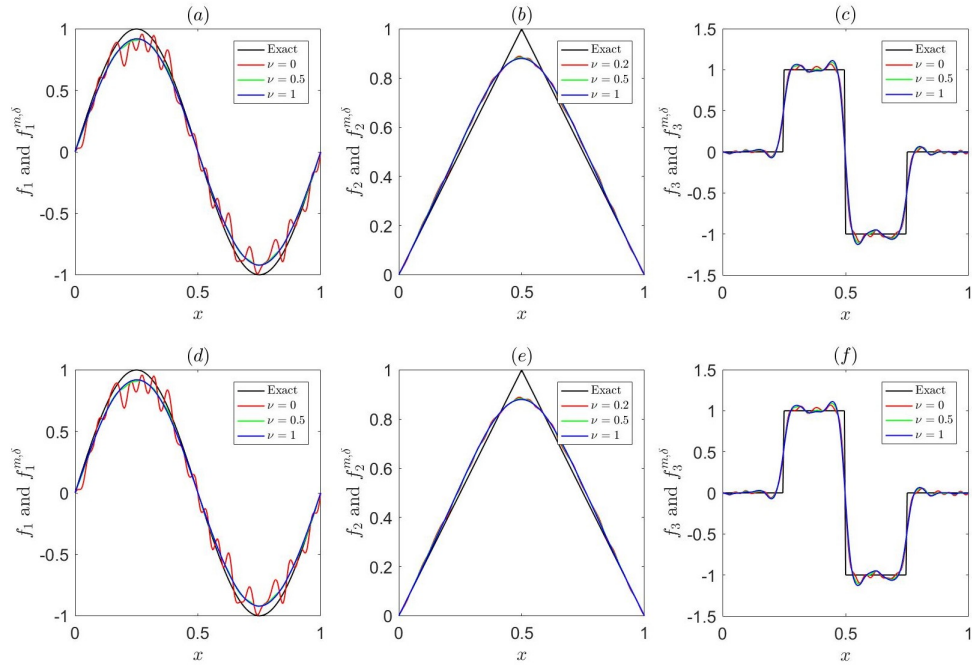
FIGURE 4. The computed (regularized) solutions for Example 5.1–5.3 (from left to right) with $\alpha = 1.6$: smoothness comparison of the computed solutions for three different values of ν. (a)–(c) for **ExFLR**; (d)–(f) for **ImFLR**.

that the optimal choice for the fractional parameter ν is 0, reflecting the fact that the initial condition to be recovered is only a $L^2(\Omega)$ function. These results confirm that the proposal fractional regularization methods are efficient and stable, and readily applicable to higher dimensional problems using eigenfunction expansion in space.

6. **Conclusion.** In this work, we consider a backward problem associated to the time-fractional wave equation in bounded domains. Two regularization methods have been proposed to solve the backward problem, which consist in first transforming the ill-posed problem into a weighted normal operator equation using eigenfunction expansion in space, then solving the resulting equation by an iterative procedure. Both a priori and a posteriori regularization parameter choice rules were investigated, and an upper-bound was estimated for the smallest regularization parameter according to a Morozov discrepancy principle. Furthermore, the convergence rates were provided for the regularized solutions generated by the proposed methods. Compared with the classical Landweber regularization method, it was shown, both theoretically and numerically, that the fractional regularization method needs smaller regularization parameter and lightens the well-known over-smoothing effect caused by the classical method. This is an interesting feature of the new method when used to treat problems having low regularity. For those problem the new method can efficiently improve the accuracy of the regularized solutions by avoiding over-smoothing.
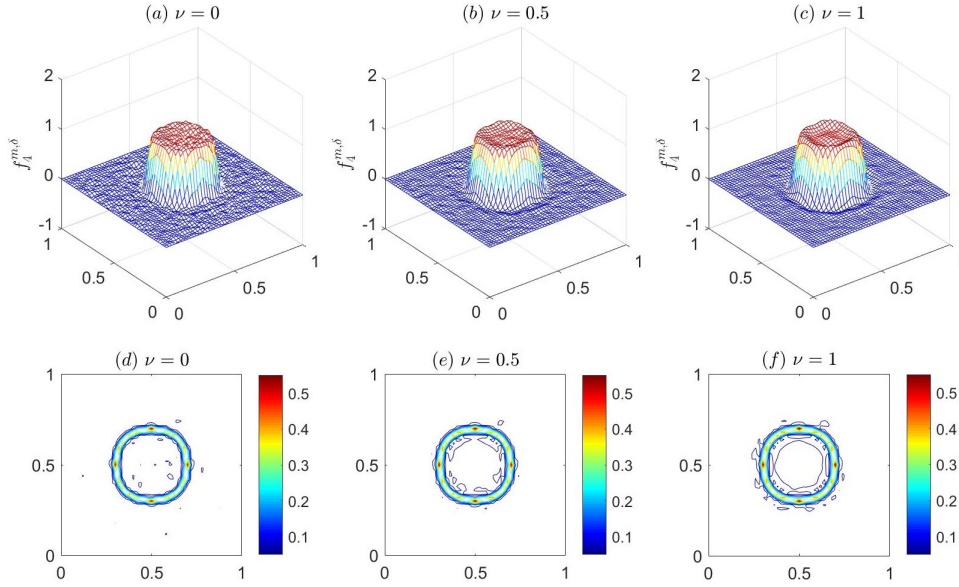
FIGURE 5. The computed initial conditions by **ImFLR** and absolute errors for Example 5.4. (a)–(c) for the regularized solutions $f_4^{m,\delta}$; (d)–(f) for the absolute errors $|f_4^{m,\delta} - f_4|$.

7. **Appendix.** We list in the appendix a number of definitions and lemmas, which have been used in the previous sections.

Definition 7.1. (See [25]) The *Mittage–Leffler* function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha, \beta > 0$.

Lemma 7.2. *(See [35, 25]) For $1 < \alpha < 2$, $\beta \in \mathbb{R}$, $\eta > 0$, it holds*

$$E_{\alpha,\beta}(-\eta) = \frac{1}{\Gamma(\beta - \alpha)\eta} + O\left(\frac{1}{\eta^2}\right), \quad \eta \to \infty. \tag{46}$$

Lemma 7.3. *For constants $a > 0, p > 0$, $0 < \nu \leq 1$, and $m \in \mathbb{N}^+$, we have*

$$s^{\frac{p}{2}}(1 - as^{\nu+1})^m \leq \left(\frac{p}{a(2m(\nu+1)+p)}\right)^{\frac{p}{2(\nu+1)}} \left(\frac{2m(\nu+1)}{2m(\nu+1)+p}\right)^m, \quad 0 < s < \frac{1}{\sqrt[\nu+1]{a}}.$$

Proof of Lemma 7.3. Define the function

$$F(s) = s^{\frac{p}{2}}(1 - as^{\nu+1})^m, \quad 0 < s < \frac{1}{\sqrt[\nu+1]{a}}.$$

It can be readily checked that

$$F'(s) = \frac{1}{2}s^{\frac{p-2}{2}}(1 - as^{\nu+1})^{m-1}\left(p - a(2m(\nu+1)+p)s^{\nu+1}\right), \quad 0 < s < \frac{1}{\sqrt[\nu+1]{a}}.$$

Let $F'(s_0) = 0$, then

$$s_0 = \sqrt[\nu+1]{\frac{p}{a(2m(\nu+1)+p)}} < \frac{1}{\sqrt[\nu+1]{a}}.$$

Observe that $F'(s) > 0$ for $s \in (0, s_0)$ and $F'(s) < 0$ for $s \in \left(s_0, \frac{1}{\sqrt[\nu+1]{a}}\right)$, thus $F(s)$ attains its maximum at $s = s_0$. That is

$$F(s) \leq F(s_0) = \left(\frac{p}{a(2m(\nu+1)+p)}\right)^{\frac{p}{2(\nu+1)}} \left(\frac{2m(\nu+1)}{2m(\nu+1)+p}\right)^m, \quad 0 < s < \frac{1}{\sqrt[\nu+1]{a}}.$$

The proof is completed.

$\square$

Lemma 7.4. *For constants $a > 0, p > 0$, $0 < \nu \leq 1$, $m > \frac{p}{2(\nu+1)}$, and $m \in \mathbb{N}^+$, we have*

$$s^{\frac{p}{2}}(1 + as^{\nu+1})^{-m} \leq \left(\frac{p}{a(2m(\nu+1)-p)}\right)^{\frac{p}{2(\nu+1)}} \left(\frac{2m(\nu+1)-p}{2m(\nu+1)}\right)^m, \quad \forall s > 0.$$

Proof of Lemma 7.4. Define the function

$$G(s) = s^{\frac{p}{2}}(1 + as^{\nu+1})^{-m}, \quad s > 0.$$

A direct calculation gives

$$G'(s) = \frac{1}{2}s^{\frac{p-2}{2}}(1 + as^{\nu+1})^{-m-1}\left(p - a(2m(\nu+1)-p)s^{\nu+1}\right), \quad s > 0.$$

Let $G'(s_0) = 0$. It is easy to get

$$s_0 = \sqrt[\nu+1]{\frac{p}{a(2m(\nu+1)-p)}} > 0.$$

A similar analysis as in the proof of Lemma 7.3 shows

$$G(s) \leq G(s_0) = \left(\frac{p}{a(2m(\nu+1)-p)}\right)^{\frac{p}{2(\nu+1)}} \left(\frac{2m(\nu+1)-p}{2m(\nu+1)}\right)^m, \quad \forall s > 0.$$

This proves the lemma.

$\square$

Similarly we can derive the following lemmas.

Lemma 7.5. *For constants $a > 0, p > 0$, $0 < \nu \leq 1$, and $m \in \mathbb{N}^+$, it holds*

$$s^{\frac{p}{2}+1}(1 - as^{\nu+1})^{m-1} \leq \left(\frac{p+2}{a(2(m-1)(\nu+1)+p+2)}\right)^{\frac{p+2}{2(\nu+1)}} \left(\frac{2(m-1)(\nu+1)}{2(m-1)(\nu+1)+p+2}\right)^{m-1}$$

for any s in $\left(0, \frac{1}{\sqrt[\nu+1]{a}}\right)$.

Lemma 7.6. *For constants $a > 0, p > 0$, $0 < \nu \leq 1$, $m > \frac{p+2+2(\nu+1)}{2(\nu+1)}$, and $m \in \mathbb{N}^+$, we have*

$$s^{\frac{p}{2}+1}(1 + as^{\nu+1})^{-(m-1)} \leq \left(\frac{p+2}{a(2(m-1)(\nu+1)-p-2)}\right)^{\frac{p+2}{2(\nu+1)}} \left(\frac{2(m-1)(\nu+1)-p-2}{2(m-1)(\nu+1)}\right)^{m-1}$$

for any $s > 0$.

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