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On the boundary value problems of piecewise differential equations with left-right fractional derivatives and delay

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Abstract. In this paper, we study the multi-point boundary value problems for a new kind of piecewise differential equations with left and right fractional derivatives and delay. In this system, the state variables satisfy the different equations in different time intervals, and they interact with each other through positive and negative delay. Some new results on the existence, no-existence and multiplicity for the positive solutions of the boundary value problems are obtained by using Guo–Krasnoselskii's fixed point theorem and Leggett–Williams fixed point theorem. The results for existence highlight the influence of perturbation parameters. Finally, an example is given out to illustrate our main results.

Keywords: boundary value problem, piecewise differential equation, left and right fractional derivative, delay, disturbance parameter, fixed point theorem.

1 Introduction

In recent decades, fractional calculus has been widely used in various fields of science and technology, and the theoretical research of fractional differential equations has also received extensive attention, see [5–8, 10, 12–15, 17, 18, 22, 23, 25, 27, 30–34, 36] and the references therein. And the differential equation with left and right fractional derivatives have been studied extensively due to the wide application [2–4, 16, 24, 29]. In [2], the following nonlocal boundary value problems of integro-differential equations involving mixed left and right fractional derivatives and left and right fractional integrals are studied

$$\begin{aligned}&{}^{c}D_{1-}^{\alpha}\,{}^{RL}D_{0+}^{\beta}y(t)+\lambda I_{1-}^{p}I_{0+}^{q}h\bigl(t,y(t)\bigr)=f\bigl(t,y(t)\bigr),\quad t\in J:=[0,1],\\&y(0)=y(\xi)=0,\quad y(1)=\delta y(\mu),\quad 0<\xi<\mu<1,\end{aligned}$$

where $1<\alpha\leqslant 2$, $0<\beta\leqslant 1$ and $p,q>0$, $f,h:[0,1]\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, and $\delta,\lambda,\mu\in\mathbb{R}$ are constants. At the same time, the differential equations

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with delay have many successful applications in the fields of communication engineering, population control and so on; see [1, 9, 20, 21, 26, 28, 35].

Motivated by above works, we discuss the multi-point boundary value problems for piecewise differential equations with left and right fractional derivatives and delay

$$\begin{aligned} &{}_t^cD_{\xi^-}^{\alpha} u(t) + f\big(t, u(t), u(t+\tau_1)\big) = 0, \quad t \in [0, \xi], \\ &{}_{\xi^+}^{c}D_t^{\beta} u(t) + g\big(t, u(t), u(t-\tau_2)\big) = 0, \quad t \in (\xi, 1], \\ &u'(\xi^-) = \rho_2 u'(0) + a = -\big(\rho_1 u'(\xi^+) + b\big), \\ &u(0) = \gamma_1 u(\xi^-), \qquad u(1) = \gamma_2 u(\xi^+), \end{aligned} \tag{1}$$

where ${}_t^cD_{\xi^-}^{\alpha}$ is the right Caputo fractional derivative, ${}_{\xi^+}^{c}D_t^{\beta}$ is the left Caputo fractional derivative, $1<\alpha, \beta \leqslant 2$. $\xi \in (0,1)$, $u(\xi^-) = \lim_{\varepsilon\to 0^-} u(\xi+\varepsilon)$, $u(\xi^+) = \lim_{\varepsilon\to 0^+} u(\xi+\varepsilon)$. $\gamma_i, \rho_i, \tau_i \in \mathbb{R}$ and $0 < \gamma_i < 1$, $\rho_1 > 0$, $0 \leqslant \rho_2 < 1$, $a, b \geqslant 0$, $0 \leqslant \tau_1 \leqslant 1-\xi$, $0 \leqslant \tau_2 \leqslant \xi$. $f \in C([0,\xi] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $g \in C([\xi,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$.

In boundary value problem (1), the state variable $u = u(t)$ satisfies the different equations in different time intervals, and they interact with each other through positive delay τ_1 and negative delay $-\tau_2$. The parameters a and b in the boundary conditions represent the error in certain measurement. Some new results on the existence, no-existence and multiplicity for the positive solutions of the boundary value problems are obtained by using Guo–Krasnoselskii's fixed point theorem and Leggett–Williams fixed point theorem. The results for existence highlight the influence of perturbation parameters. Finally, an example is given out to illustrate our main results.

2 Preliminaries

For convenience of reading, in this section, we give out some definitions about the fractional calculus and some lemmas.

Definition 1. (See [17].) Let $\alpha > 0$, $a < b \in \mathbb{R}$, and the left and right Riemann–Liouville fractional integral of $u : [a,b] \to \mathbb{R}$ are defined as

$$\begin{aligned} {}_{a^+}I_t^{\alpha} u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)\,\mathrm{d}s, \\ {}_tI_{b^-}^{\alpha} u(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s)\,\mathrm{d}s, \end{aligned}$$

respectively, for $t \in [a,b]$.

Definition 2. (See [17]). Let $\alpha > 0$, $a < b \in \mathbb{R}$, and the left Caputo fractional derivative and right Caputo fractional derivative of function $u : [a,b] \to \mathbb{R}$ are defined as

$${}_{a^+}^{c}D_t^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

$$ {}_t^cD_{b^-}^{\alpha}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{u^{(n)}(s)}{(s-t)^{\alpha-n+1}}\,\mathrm{d}s, $$

respectively, provided the right-sided integral converges, where $t \in [a,b]$, $n-1<\alpha<n$, $n \in \mathbb{N}$.

Lemma 1. (*See* [17]). *If* $\alpha > 0$, *then*

$$ {}_{a^+}I_t^{\alpha}\bigl({}_{a^+}^{c}D_t^{\alpha}u(t)\bigr) = u(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}, $$
$$ {}_tI_{b^-}^{\alpha}\bigl({}_t^cD_{b^-}^{\alpha}u(t)\bigr) = u(t) + d_0 + d_1(b-t) + d_2(b-t)^2 + \cdots + d_{n-1}(b-t)^{n-1}, $$

where c_i, $d_i \in \mathbb{R}$, $i = 0,1,\dots,n-1$, $n \in \mathbb{N}$.

Lemma 2. (*See* [11].) *Let* E *be a Banach space and* $P \subset E$ *is a cone. Assume that* Ω_1, Ω_2 *are bounded open subsets of* E *with* $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, *and let* $T : P\cap(\overline{\Omega}_2\setminus\Omega_1) \to P$ *be a completely continuous operator such that either*

(i) $\|Tx\| \leqslant \|x\|$, $x \in P\cap\partial\Omega_1$, *and* $\|Tx\| \geqslant \|x\|$, $x \in P\cap\partial\Omega_2$, *or*
(ii) $\|Tx\| \geqslant \|x\|$, $x \in P\cap\partial\Omega_1$, *and* $\|Tx\| \leqslant \|x\|$, $x \in P\cap\partial\Omega_2$.

Then the operator T *has at least one fixed point on* $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Lemma 3. (*See* [19].) *Assume* P *is a cone in Banach space,* ω *is a nonnegative continuous concave functional on* P, *the constants* $0<d<q<c\leqslant r$. *Denote* $\overline{P}_r = \{u \in P\colon \|u\| \leqslant r\}$ *and* $P(\omega,q,c) = \{u \in P\colon q \leqslant \omega(u)$ *and* $\|u\| \leqslant c\}$. *Let* $T : \overline{P}_r \to \overline{P}_r$ *be a completely continuous operator such that* $\omega(u) \leqslant \|u\|$ *for* $x \in \overline{P}_r$ *such that*

(i) $\{u \in P(\omega,q,c)P\colon \omega(u) > q\} \neq \emptyset$ *and* $\omega(Tu) > q$ *for* $u \in P(\omega,q,c)$;
(ii) $\|Tu\| < d$ *for* $u \in \overline{P}_d$;
(iii) $\omega(Tu) > q$ *for any* $u \in P(\omega,q,r)$ *and* $\|Tu\| > c$.

Then T *has at least three fixed points* $u_1,u_2,u_3 \in \overline{P}_r$ *such that* $\|u_1\| < d$, $\omega(u_2) > q$ *and* $\|u_3\| > d$ *with* $\omega(u_3) < q$.

Lemma 4. *Let* $h \in C([0,\xi],\mathbb{R}^+)$, $y \in C([\xi,1],\mathbb{R}^+)$, *then the boundary value problem*

$$ \begin{aligned} &{}_t^cD_{\xi^-}^{\alpha}u(t) + h(t) = 0, \quad t \in [0,\xi],\\ &{}_{\xi^+}^{c}D_t^{\beta}u(t) + y(t) = 0, \quad t \in (\xi,1],\\ &u'(\xi^-) = \rho_2u'(0) + a = -\bigl(\rho_1u'(\xi^+) + b\bigr),\\ &u(0) = \gamma_1u(\xi^-), \qquad u(1) = \gamma_2u(\xi^+) \end{aligned} \tag{2} $$

has a unique solution given by

$$ u(t) = \begin{cases} \int_0^{\xi} G_1(t,s)h(s)\,\mathrm{d}s + \frac{1}{1-\rho_2}(\frac{\xi\gamma_1}{1-\gamma_1}+t)(\frac{\rho_2}{\Gamma(\alpha-1)}\int_0^{\xi} s^{\alpha-2}h(s)\,\mathrm{d}s + a),\\ \quad t \in [0,\xi],\\ \int_{\xi}^1 G_2(t,s)y(s)\,\mathrm{d}s - \frac{1}{\rho_1(1-\rho_2)}(\frac{\gamma_2\xi-1}{1-\gamma_2}+t)(\frac{\rho_2}{\Gamma(\alpha-1)}\int_0^{\xi} s^{\alpha-2}h(s)\,\mathrm{d}s + a\\ \quad +(1-\rho_2)b\bigr), \quad t \in (\xi,1], \end{cases} \tag{3} $$

where

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{1}{1-\gamma_1}s^{\alpha-1}, & 0\leqslant s<t\leqslant\xi,\\ -(s-t)^{\alpha-1}+\frac{1}{1-\gamma_1}s^{\alpha-1}, & 0\leqslant t\leqslant s\leqslant\xi;\end{cases} \tag{4}$$

$$G_2(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}\frac{1}{1-\gamma_2}(1-s)^{\beta-1}-(t-s)^{\beta-1}, & \xi\leqslant s<t\leqslant 1,\\ \frac{1}{1-\gamma_2}(1-s)^{\beta-1}, & \xi\leqslant t\leqslant s\leqslant 1.\end{cases} \tag{5}$$

Proof. From Lemma 1 the general solution of the linear differential equation ${}_t^cD_{\xi-}^{\alpha}u(t)+h(t)=0$ is given by

$$\begin{aligned}u(t)&=-{}_tI_{\xi-}^{\alpha}h(t)-c_0-c_1(\xi-t)\\&=-\frac{1}{\Gamma(\alpha)}\int_t^{\xi}(s-t)^{\alpha-1}h(s)\mathrm{d}s-c_0-c_1(\xi-t),\quad t\in[0,\xi],\end{aligned} \tag{6}$$

and $u'(t)=(1/\Gamma(\alpha-1))\int_t^{\xi}(s-t)^{\alpha-2}h(s)\,\mathrm{d}s+c_1$.

The general solution of the linear differential equation ${}_{\xi+}^{c}D_t^{\beta}u(t)+y(t)=0$ is given by

$$\begin{aligned}u(t)&=-{}_{\xi+}I_t^{\beta}y(t)-c_2-c_3t\\&=-\frac{1}{\Gamma(\beta)}\int_{\xi}^{t}(t-s)^{\beta-1}y(s)\,\mathrm{d}s-c_2-c_3t,\quad t\in(\xi,1],\end{aligned} \tag{7}$$

and $u'(t)=-(1/\Gamma(\beta-1))\int_{\xi}^{t}(t-s)^{\beta-2}y(s)\,\mathrm{d}s-c_3$.

By the boundary value conditions $u'(\xi^-)=\rho_2u'(0)+a=-(\rho_1u'(\xi^+)+b)$ we can easily get that

$$\begin{aligned}c_1&=\frac{1}{1-\rho_2}\left(\frac{\rho_2}{\Gamma(\alpha-1)}\int_0^{\xi}s^{\alpha-2}h(s)\,\mathrm{d}s+a\right),\\c_3&=\frac{1}{\rho_1(1-\rho_2)}\left(\frac{\rho_2}{\Gamma(\alpha-1)}\int_0^{\xi}s^{\alpha-2}h(s)\,\mathrm{d}s+a+(1-\rho_2)b\right).\end{aligned} \tag{8}$$

By (6)–(8) and the boundary conditions $u(0)=\gamma_1u(\xi^-)$, $u(1)=\gamma_2u(\xi^+)$ we can also get that

$$c_0=\frac{1}{\gamma_1-1}\left(\int_0^{\xi}\frac{1}{\Gamma(\alpha)}s^{\alpha-1}h(s)\,\mathrm{d}s+\frac{\xi}{1-\rho_2}\left(\int_0^{\xi}\frac{\rho_2}{\Gamma(\alpha-1)}\mathrm{s}^{\alpha-2}\mathrm{h(s)\,ds+a}\right)\right),$$

$$c_2 = \frac{1}{\gamma_2 - 1}\left(\frac{1}{\Gamma(\beta)}\int\limits_{\xi}^{1}(1-s)^{\beta-1}y(s)\,\mathrm{ds} + \frac{1-\gamma_2\xi}{\rho_1(1-\rho_2)}\left(\frac{\rho_2}{\Gamma(\alpha-1)}\int\limits_{0}^{\xi}s^{\alpha-2}h(s)\,\mathrm{ds} + \mathrm{a} + (1-\rho_2)\mathrm{b}\right)\right).$$

Thus, by substituting c_0 and c_1 into (6) we can get that for $t \in [0, \xi]$,

$$u(t) = \int\limits_{0}^{\xi} G_1(t,s)h(s)\,\mathrm{d}s + \frac{1}{1-\rho_2}\left(\frac{\xi\gamma_1}{1-\gamma_1} + t\right)\left(\frac{\rho_2}{\Gamma(\alpha-1)}\int\limits_{0}^{\xi}s^{\alpha-2}h(s)\,\mathrm{d}s + a\right),$$

and by substituting c_2 and c_3 into (7) we can get that for $t \in (\xi, 1]$,

$$u(t) = \int\limits_{\xi}^{1} G_2(t,s)y(s)\,\mathrm{d}s - \frac{1}{\rho_1(1-\rho_2)}\left(\frac{\gamma_2\xi - 1}{1-\gamma_2} + t\right)\left(\frac{\rho_2}{\Gamma(\alpha-1)}\int\limits_{0}^{\xi}s^{\alpha-2}h(s)\,\mathrm{d}s + a + (1-\rho_2)b\right).$$

Hence, $u(t)$ satisfies equation (3) if it is the solution of the boundary value problem (2) and vice versa. □

Lemma 5. *Suppose* $G_i(t,s)$ $(i = 1, 2)$ *are defined by* (4), (5), *then* $G_i(t,s)$ *has the following properties, respectively*:

(i) $G_1(t,s)$ *is continuous and* $0 \leqslant \gamma_1 G_1(\xi,s) \leqslant G_1(t,s) \leqslant G_1(\xi,s)$, $\partial G_1(t,s)/\partial t \geqslant 0$ *on* $(t,s) \in [0,\xi]\times[0,\xi]$;

(ii) $G_2(t,s)$ *is continuous and* $0 \leqslant \gamma_2 G_2(\xi,s) \leqslant G_2(t,s) \leqslant G_2(\xi,s)$, $\partial G_2(t,s)/\partial t \leqslant 0$ *on* $(t,s) \in [\xi,1]\times[\xi,1]$.

Proof. (i) Obviously, $G_1(t,s)$ is continuous on $(t,s) \in [0,\xi]\times[0,\xi]$.

For $0 \leqslant s < t \leqslant \xi$,

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)(1-\gamma_1)}s^{\alpha-1} \geqslant 0, \qquad \frac{\partial G_1(t,s)}{\partial t} = 0,$$

and for $0 \leqslant t \leqslant s \leqslant \xi$,

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1-\gamma_1}s^{\alpha-1} - (s-t)^{\alpha-1}\right),$$
$$\frac{\partial G_1(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha-1)}(s-t)^{\alpha-2} \geqslant 0.$$

Hence, $G_1(t,s)$ is monotone increasing for any $t \in [0,\xi]$, and

$$0 \leqslant G_1(0,s) \leqslant G_1(t,s) \leqslant G_1(s,s) = G_1(\xi,s).$$

Because $G_1(0,s) = (\gamma_1/((1-\gamma_1)\Gamma(\alpha)))s^{\alpha-1} = \gamma_1 G_1(\xi,s)$, then

$$0 \leqslant \gamma_1 G_1(\xi,s) \leqslant G_1(t,s) \leqslant G_1(\xi,s), \qquad \frac{\partial G_1(t,s)}{\partial t} \geqslant 0, \quad (t,s) \in [0,\xi]\times[0,\xi].$$

(ii) Similar to the proof of (i), we can prove that (ii) holds. □

Let $J = [0,1]$, $J_0 = J \setminus \{\xi\}$, $E = PC(J,\mathbb{R}) = \{u : J \to \mathbb{R} : u$ is continuous in J_0. $u(\xi^+)$ and $u(\xi^-)$ exist, and $u(\xi^-) = u(\xi)\}$. Obviously, E is a Banach space with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$.

Denote $\|u\|_{[0,\xi]} = \sup_{t\in[0,\xi]} |u(t)|$, $\|u\|_{(\xi,1]} = \sup t \in (\xi,1]|u(t)|$, then $\|u\| = \max\{\|u\|_{[0,\xi]}, \|u\|_{(\xi,1]}\}$. Set

$$P = \Bigl\{u \in E\colon u(t) \geqslant 0,\ t\in[0,1],\ \inf_{t\in[0,\xi]} u(t) \geqslant \gamma_1\|u\|_{[0,\xi]},\ \inf_{t\in(\xi,1]} u(t) \geqslant \gamma_2\|u\|_{(\xi,1]}\Bigr\},$$

then $P \subset E$ is a cone. Define a operator $T : P \to E$ by

$$Tu(t) = \begin{cases} \int_0^\xi G_1(t,s) f(s,u(s),u(s+\tau_1))\,\mathrm{d}s + \frac{1}{1-\rho_2}(\frac{\xi\gamma_1}{1-\gamma_1} + t) \\ \qquad \times(\frac{\rho_2}{\Gamma(\alpha-1)}\int_0^\xi s^{\alpha-2} f(s,u(s),u(s+\tau_1))\,\mathrm{d}s + a), \quad t\in[0,\xi], \\ \int_\xi^1 G_2(t,s) g(s,u(s),u(s-\tau_2))\,\mathrm{d}s - \frac{1}{\rho_1(1-\rho_2)}(\frac{\gamma_2\xi-1}{1-\gamma_2} + t) \\ \qquad \times(\frac{\rho_2}{\Gamma(\alpha-1)}\int_0^\xi s^{\alpha-2} f(s,u(s),u(s+\tau_1)) ds + a + (1-\rho_2)b), \\ \quad t\in(\xi,1]. \end{cases} \tag{9}$$

Obviously, $u = u(t)$ is a positive solution of (1) if and only if u is a fixed point of the operator T in P.

Lemma 6. *The operator $T : P \to P$ is completely continuous.*

Proof. By Lemma 5 we can easily obtain that $T : P \to P$.

Let $\{u_n\} \subset P$, $u \in P$, and $\|u_n - u\| \to 0$ as $n \to \infty$. There exists a constant $\overline{M}_0 > 0$ such that $\|u_n\| \leqslant \overline{M}_0$ and $\|u\| \leqslant \overline{M}_0$.

By the continuity of $f(t,u,v)$, $g(t,u,v)$ we have

$$\lim_{n\to\infty} \bigl(f\bigl(t,u_n(t),u_n(t+\tau_1)\bigr) - f\bigl(t,u(t),u(t+\tau_1)\bigr)\bigr) = 0,$$
$$\lim_{n\to\infty} \bigl(g\bigl(t,u_n(t),u_n(t-\tau_2)\bigr) - g\bigl(t,u(t),u(t-\tau_2)\bigr)\bigr) = 0,$$

and there is a constant $\overline{M}_1 > 0$, which makes $\sup_{(t,u,v)\in A} |f(t,u,v)| \leqslant \overline{M}_1$ and $\sup_{(t,u,v)\in B} |g(t,u,v)| \leqslant \overline{M}_1$, where $A = [0,\xi]\times[-\overline{M}_0,\overline{M}_0]\times[-\overline{M}_0,\overline{M}_0]$, $B = [\xi,1]\times[-\overline{M}_0,\overline{M}_0]\times[-\overline{M}_0,\overline{M}_0]$.

It follows from Lemma 5 and the Lebesgue dominated convergence theorem that $\lim_{n\to\infty} \|Tu_n - Tu\|_{[0,\xi]} = 0$ and $\lim_{n\to\infty} \|Tu_n - Tu\|_{(\xi,1]} = 0$.

Thus, $\lim_{n\to\infty} \|Tu_n - Tu\| = 0$, which implies that the operator T is a continuous operator.

Let $\Omega \subset P$ be bounded. By the continuity of f, g, we can get that there is a constant $\overline{M}_2 > 0$ such that $|f(t,u,v)| \leqslant \overline{M}_2$ for any $t \in [0,\xi]$, $u, v \in \Omega$, and $|g(t,u,v)| \leqslant \overline{M}_2$ for all $t \in (\xi,1]$, $u, v \in \Omega$.

By Lemma 5 we can show that $(Tu)'(t) \geqslant 0$ for $t \in [0,\xi]$ and $(Tu)'(t) \leqslant 0$ for $t \in (\xi,1]$. Hence,

$$\begin{aligned}
\|Tu\|_{[0,\xi]} &= Tu(\xi) \\
&\leqslant \frac{1}{1-\gamma_1}\left(\frac{\overline{M}_2\xi^\alpha}{\Gamma(\alpha+1)} + \frac{\xi}{1-\rho_2}\left(\frac{\rho_2\xi^{\alpha-1}\overline{M}_2}{\Gamma(\alpha)} + a\right)\right), \\
\|Tu\|_{(\xi,1]} &= Tu(\xi^+) \\
&\leqslant \frac{1}{1-\gamma_2}\left(\frac{\overline{M}_2(1-\xi)^\beta}{\Gamma(\beta+1)} + \frac{1-\xi}{\rho_1(1-\rho_2)}\left(\frac{\overline{M}_2\rho_2\xi^{\alpha-1}}{\Gamma(\alpha)} + a + (1-\rho_2)b\right)\right).
\end{aligned}$$

Consequently, $T(\Omega)$ is uniformly bounded.

Since $G_1(t,s)$ is continuous, it is uniformly continuous on $(t,s) \in [0,\xi] \times [0,\xi]$. Hence, for any $\varepsilon > 0$, there exists a constant

$$0 < \delta_1 < \frac{\varepsilon(1-\rho_2)\Gamma(\alpha)}{2(\overline{M}_2\rho_2\xi^{\alpha-1} + a\Gamma(\alpha) + 1)}$$

such that $|G_1(t_1,s) - G_1(t_2,s)| < \varepsilon/(2\overline{M}_2)$ for all t_1, t_2, $s \in [0,\xi]$ and $|t_1 - t_2| < \delta_1$.

Thus, for $u \in \Omega$, t_1, $t_2 \in [0,\xi]$, $|t_1 - t_2| < \delta_1$, we have

$$\begin{aligned}
\left|Tu(t_2) - Tu(t_1)\right| &\leqslant \overline{M}_2 \int_0^\xi \left|G_1(t_1,s) - G_1(t_2,s)\right| \mathrm{d}s \\
&\quad + \frac{1}{1-\rho_2}\left(\frac{\overline{M}_2\rho_2\xi^{\alpha-1}}{\Gamma(\alpha)} + a\right)|t_1 - t_2| < \varepsilon.
\end{aligned}$$

Similarly, due to that $G_2(t,s)$ is continuous on $[\xi,1] \times [\xi,1]$, for above mentioned $\varepsilon > 0$, there exists a constant $\delta_2 > 0$ such that for t_3, t_4, $s \in (\xi,1]$, $|t_3 - t_4| < \delta_2$, we have $|Tu(t_3) - Tu(t_4)| < \varepsilon$.

Hence, $T(\Omega)$ is equicontinuous on $[0,\xi]$, $(\xi,1]$, respectively.

By Arzela–Ascoli theorem we know that operator T is a relative compactness operator, and because operator T is a continuous operator, it is a completely continuous operator. □

3 Existence of the positive solutions

Denote

$$\begin{aligned}
f_\varphi &= \liminf_{u+v\to\varphi} \inf_{t\in[\xi-\tau_0,\xi]} \frac{f(t,u,v)}{u+v}, \qquad g_\varphi = \liminf_{u+v\to\varphi} \inf_{t\in(\xi,\xi+\tau_0]} \frac{g(t,u,v)}{u+v}, \\
f^\varphi &= \limsup_{u+v\to\varphi} \sup_{t\in[0,\xi]} \frac{f(t,u,v)}{u+v}, \qquad g^\varphi = \limsup_{u+v\to\varphi} \sup_{t\in(\xi,1]} \frac{g(t,u,v)}{u+v},
\end{aligned}$$

$$M_1 = \frac{(1-\gamma_1)(1-\rho_2)\Gamma(\alpha+1)}{\xi^\alpha(2+2(\alpha-1)\rho_2+\Gamma(\alpha+1))},$$
$$M_2 = \frac{\Gamma(\beta+1)(1-\gamma_2)}{2(1-\xi)^\beta}\left(1-\frac{(1-\xi)(1-\gamma_1)}{\rho_1\xi(1-\gamma_2)}\right),$$
$$M_3 = \min\left\{\frac{M_1}{2}, \frac{M_1\rho_1\xi(1-\gamma_2)}{2(1-\xi)(1-\gamma_1)}\right\}, \qquad M_4 = \frac{\Gamma(\beta+1)(1-\gamma_2)}{6(1-\xi)^\beta},$$
$$N_1 = \frac{(1-\gamma_1)\Gamma(\alpha+1)}{\gamma_1\gamma(\xi^\alpha-(\xi-\tau_0)^\alpha)}, \qquad N_2 = \frac{(1-\gamma_2)\Gamma(\beta+1)}{\gamma_2\gamma((1-\xi)^\beta-(1-\xi-\tau_0)^\beta)},$$
$$N = \max\{\gamma N_1, \gamma N_2\},$$

where $\varphi = 0^+$ or $+\infty$ and $\gamma = \min\{\gamma_1,\gamma_2\}$, $0 < \tau_0 \leqslant \min\{\tau_1,\tau_2\}$.

Theorem 1. *Assume that $M_2 > 0$ and the following conditions hold*:

(H1) $f^0 < M_1$ *and* $g^0 < M_2$;
(H2) $f_\infty > N_1$ *or* $g_\infty > N_2$.

Then there exist constants a_0, $b_0 \geqslant 0$ such that boundary value problem (1) *has at least one positive solution for the parameters a and b with $0 \leqslant a \leqslant a_0$, $0 \leqslant b \leqslant b_0$.*

Proof. Because $f^0 < M_1$, there exists a constant $r_1 > 0$ such that $f(t,u,v) < M_1(u+v)$ for any $t \in [0,\xi]$, $u+v \in (0,r_1)$. Similarly, by $g^0 < M_2$ there is a constant $r_2 > 0$ such that $g(t,u,v) < M_2(u+v)$ for any $t \in (\xi,1]$, $u+v \in (0,r_2)$.

Let
$$r = \min\left\{\frac{r_1}{2}, \frac{r_2}{2}\right\}, \qquad \Omega_1 = \{u \in P\colon \|u\| \leqslant r\},$$
and $a_0 = \xi^{\alpha-1}M_1 r$, $b_0 = 2\xi^{\alpha-1}M_1 r/\Gamma(\alpha+1)$. For any $u \in P \cap \partial\Omega_1$, we have $\|u\| = r$.

When $0 \leqslant a \leqslant a_0$, $0 \leqslant b \leqslant b_0$, for any $u \in P \cap \partial\Omega_1$, we have $0 \leqslant u(s)+u(s+\tau_1) \leqslant 2r \leqslant r_1$ for $s \in [0,\xi]$, and

$$\begin{aligned}
\|Tu\|_{[0,\xi]} &= Tu(\xi) \\
&< M_1\int_0^\xi G_1(\xi,s)\big(u(s)+u(s+\tau_1)\big)\,\mathrm{d}s \\
&\quad + \frac{\xi}{(1-\rho_2)(1-\gamma_1)}\left(\frac{M_1\rho_2}{\Gamma(\alpha-1)}\int_0^\xi s^{\alpha-2}\big(u(s)+u(s+\tau_1)\big)\,\mathrm{d}s + a_0\right) \\
&< \frac{\xi^\alpha(2+2(\alpha-1)\rho_2)+\Gamma(\alpha+1)}{(1-\gamma_1)(1-\rho_2)\Gamma(\alpha+1)}M_1 r = r = \|u\|.
\end{aligned}$$

Similarly,

$$\|Tu\|_{(\xi,1]} = Tu(\xi^+) < \left(\frac{2(1-\xi)^\beta M_2}{\Gamma(\beta+1)(1-\gamma_2)} + \frac{(1-\xi)(1-\gamma_1)}{\rho_1\xi(1-\gamma_2)}\right)r.$$

In view of

$$\frac{2(1-\xi)^{\beta}M_2}{\Gamma(\beta+1)(1-\gamma_2)}+\frac{(1-\xi)(1-\gamma_1)}{\rho_1\xi(1-\gamma_2)}=1,$$

we have $\|Tu\|_{(\xi,1]}<r=\|u\|$.

Then for any $u\in P\cap\partial\Omega_1$, we get $\|Tu\|\leqslant\|u\|$.

If $f_\infty>N_1$, there exists a constant $R_1>0$ such that $f(t,u,v)>N_1(u+v)$ for any $t\in[\xi-\tau_0,\,\xi]$, $u+v\in[R_1,+\infty)$.

Let

$$R=\frac{R_1}{\gamma},\qquad \Omega_2=\{u\in P\colon\ \|u\|\leqslant R\}.$$

For $u\in P\cap\partial\Omega_2$, we have $\|u\|=R$, and

$$\inf_{t\in[\xi-\tau_0,\xi]}u(t)\geqslant\inf_{t\in[0,\xi]}u(t)\geqslant\gamma_1\|u\|_{[0,\xi]},\qquad \inf_{t\in(\xi,\,\xi+\tau_0]}u(t)\geqslant\inf_{t\in(\xi,1]}u(t)\geqslant\gamma_2\|u\|_{(\xi,1]}.$$

Because $\|u\|=\max\{\|u\|_{[0,\xi]},\,\|u\|_{(\xi,1]}\}$ for $t\in(\xi-\tau_0,\,\xi]\subset[0,\xi]$, then $t+\tau_1\in(\xi+\tau_1-\tau_0,\,\xi+\tau_1]\subset(\xi,1]$, and

$$\begin{aligned}u(t)+u(t+\tau_1)&\geqslant\inf_{t\in(\xi-\tau_0,\xi]}u(t)+\inf_{t\in(\xi+\tau_1-\tau_0,\,\xi+\tau_1]}u(t)\\&\geqslant\inf_{t\in[0,\xi]}u(t)+\inf_{t\in(\xi,1]}u(t)\geqslant\gamma_1\|u\|_{[0,\xi]}+\gamma_2\|u\|_{(\xi,1]}\\&\geqslant\gamma\big(\|u\|_{[0,\xi]}+\|u\|_{(\xi,1]}\big)\geqslant\gamma\|u\|.\end{aligned}$$

So that

$$u(t)+u(t+\tau_1)\geqslant\gamma\|u\|=\gamma R=R_1,\quad t\in(\xi-\tau_0,\,\xi].\tag{10}$$

By Lemma 5 and (9) we can easily get that

$$\begin{aligned}Tu(0)&=\gamma_1Tu(\xi)\geqslant\gamma_1\int_0^{\xi}G_1(\xi,s)f\big(s,u(s),u(s+\tau_1)\big)\,\mathrm{d}s\\&>N_1\gamma_1\int_{\xi-\tau_0}^{\xi}G_1(\xi,s)\big(u(s)+u(s+\tau_1)\big)\,\mathrm{d}s\geqslant N_1\gamma_1\gamma R\int_{\xi-\tau_0}^{\xi}G_1(\xi,s)\,\mathrm{d}s\\&=\frac{\gamma_1\gamma(\xi^{\alpha}-(\xi-\tau_0)^{\alpha})}{(1-\gamma_1)\Gamma(\alpha+1)}N_1R=R=\|u\|.\end{aligned}$$

Then for $u\in P\cap\partial\Omega_2$, we have $\|Tu\|\geqslant\|u\|$.

According to Lemma 2, T has at least one fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Similarly, if $g_\infty>N_2$, there is a constant $R_2>0$, which makes $g(t,u,v)>N_2(u+v)$ for any $t\in(\xi,\,\xi+\tau_0]$, $u+v\in[R_2,+\infty)$.

Let

$$R_0=\frac{R_2}{\gamma},\qquad \Omega_3=\big\{u\in P\colon\ \|u\|\leqslant R_0\big\}.$$

We have $\|u\| = R_0$ for any $u \in P \cap \partial\Omega_3$ and for $t \in (\xi, \xi + \tau_0] \subset (\xi, 1]$, then $t - \tau_2 \in (\xi - \tau_2, \xi - (\tau_2 - \tau_0)] \subset [0, \xi]$

$$u(t) + u(t - \tau_2) \geqslant \inf_{t \in (\xi,1]} u(t) + \inf_{t \in [0,\xi]} u(t) \geqslant \gamma_1 \|u\|_{[0,\xi]} + \gamma_2 \|u\|_{(\xi,1]} \\ \geqslant \gamma(\|u\|_{[0,\xi]} + \|u\|_{(\xi,1]}) \geqslant \gamma\|u\|.$$

Thus

$$u(t) + u(t - \tau_2) \geqslant \gamma\|u\| = \gamma R_0 = R_2, \quad t \in (\xi, \xi + \tau_0], \tag{11}$$

and because

$$\begin{aligned} Tu(1) &= \gamma_2 Tu(\xi^+) \\ &\geqslant \gamma_2 \int_\xi^1 G_2(\xi, s) g\big(s, u(s), u(s - \tau_2)\big)\, \mathrm{d}s > N_2 \gamma_2 \gamma R_0 \int_\xi^{\xi+\tau_0} G_2(\xi, s)\, \mathrm{d}s \\ &= \frac{\gamma_2 \gamma((1 - \xi)^\beta - (1 - \xi - \tau_0)^\beta)}{(1 - \gamma_2)\Gamma(\beta + 1)} N_2 R_0 = R_0 = \|u\|, \end{aligned}$$

then for any $u \in P \cap \partial\Omega_3$, we have $\|Tu\| \geqslant \|u\|$.

According to Lemma 2, T has at least one fixed point in $P \cap (\overline{\Omega}_3 \setminus \Omega_1)$, which implies that boundary value problem (1) has at least one positive solution. □

In particular, the following result holds by Theorem 1.

Corollary 1. *Assume that the following conditions hold*:

(H1′) $f^0 = g^0 = 0$;
(H2′) $f_\infty = +\infty$ *or* $g_\infty = +\infty$.

Then there exist constants $a_0, b_0 \geqslant 0$ *such that boundary value problem* (1) *has at least one positive solution for* $0 \leqslant a \leqslant a_0$, $0 \leqslant b \leqslant b_0$.

Theorem 2. *Assume that the following conditions* (H3) *and* (H4) *hold*:

(H3) $f^\infty < M_3$ *and* $g^\infty < M_4$;
(H4) $f_0 > N_1$ *or* $g_0 > N_2$.

Then there exist constants $a_0, b_0 \geqslant 0$ *such that boundary value problem* (1) *has at least one positive solution for* $0 \leqslant a \leqslant a_0$, $0 \leqslant b \leqslant b_0$.

Proof. Due to $f^\infty < M_3$, then there exists a constant $\lambda_1 > 0$ such that $f(t, u, v) < M_3(u + v)$ for any $t \in [0, \xi]$, $u + v \in [\lambda_1, +\infty)$.

Let

$$D = \big\{(t, u, v)\colon\ t \in [0, \xi],\ u \geqslant 0,\ v \geqslant 0\big\} := D_1 \cup D_2,$$

where $D_1 = \{(t, u, v) \in D\colon\ u + v \leqslant \lambda_1\}$, $D_2 = \{(t, u, v) \in D\colon\ u + v > \lambda_1\}$.

Since f is bounded on D_1, then there is $L_1 > 0$, which makes $|f(t,u,v)| \leqslant L_1$ for any $(t,u,v) \in D_1$. Hence, for all $(t,u,v) \in D$, we have

$$\bigl|f(t,u,v)\bigr| < L_1 + M_3(u+v).$$

Similarly, because $g^\infty < M_4$, then there is a constant $\lambda_2 > 0$ such that $g(t,u,v) < M_4(u+v)$ for any $t \in (\xi,1]$, $u+v \in [\lambda_2,+\infty)$.

Let $\widetilde{D} = \{(t,u,v)\colon\ u+v \leqslant \lambda_2,\ t \in [\xi,1],\ u \geqslant 0,\ v \geqslant 0\}$, then there is $L_2 > 0$ such tat $|g(t,u,v)| \leqslant L_2$ for any $(t,u,v) \in \widetilde{D}$. Then for any $t \in [\xi,1]$ and $u,v \in [0,+\infty]$, we have $|g(t,u,v)| < L_2 + M_4(u+v)$. Denote

$$\begin{aligned}
\lambda = \max\biggl\{&\lambda_1,\ \lambda_2,\ \frac{2\xi^\alpha(1+(\alpha-1)\rho_2)L_1}{\Gamma(\alpha+1)(1-\gamma_1)(1-\rho_2)},\\
&6\biggl(\frac{(1-\xi)\rho_2\xi^{\alpha-1}}{\rho_1(1-\rho_2)(1-\gamma_2)\Gamma(\alpha)}L_1 + \frac{(1-\xi)^\beta}{\Gamma(\beta+1)(1-\gamma_2)}L_2\biggr)\biggr\},\\
\Omega_4 = \bigl\{&u \in P\colon\ \|u\| \leqslant \lambda\bigr\}, \qquad a_0 = \frac{\xi^{\alpha-1}M_3\lambda}{2}, \qquad b_0 = \frac{\xi^{\alpha-1}M_3\lambda}{\Gamma(\alpha+1)}.
\end{aligned}$$

For any $u \in P \cap \partial\Omega_4$, which implies $\|u\| = \lambda$. Since

$$M_3 = \min\biggl\{\frac{M_1}{2}, \frac{M_1\rho_1\xi(1-\gamma_2)}{2(1-\xi)(1-\gamma_1)}\biggr\}.$$

Then for $0 \leqslant a \leqslant a_0$ and $0 \leqslant b \leqslant b_0$, we have

$$\begin{aligned}
\|Tu\|_{[0,\xi]} &= Tu(\xi)\\
&\leqslant \biggl(\frac{\xi^\alpha(2+2(\alpha-1)\rho_2+\Gamma(\alpha+1))}{(1-\gamma_1)(1-\rho_2)\Gamma(\alpha+1)}\biggr)M_3\lambda + \frac{\xi^\alpha\bigl(1+(\alpha-1)\rho_2\bigr)L_1}{\Gamma(\alpha+1)(1-\gamma_1)(1-\rho_2)}\\
&\leqslant \biggl(\frac{\xi^\alpha(2+2(\alpha-1)\rho_2+\Gamma(\alpha+1))}{(1-\gamma_1)(1-\rho_2)\Gamma(\alpha+1)}\biggr)\frac{M_1}{2}\lambda + \frac{\xi^\alpha(1+(\alpha-1)\rho_2)L_1}{\Gamma(\alpha+1)(1-\gamma_1)(1-\rho_2)}\\
&\leqslant \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda = \|u\|
\end{aligned}$$

and

$$\begin{aligned}
\|Tu\|_{(\xi,1]} &= Tu(\xi^+)\\
&< \frac{2(1-\xi)^\beta}{\Gamma(\beta+1)(1-\gamma_2)}M_4\lambda + \frac{(1-\xi)\xi^{\alpha-1}(2+2(\alpha-1)\rho_2+\Gamma(\alpha+1))}{\rho_1(1-\gamma_2)(1-\rho_2)\Gamma(\alpha+1)}M_3\lambda\\
&\quad + \frac{(1-\xi)\rho_2\xi^{\alpha-1}}{\rho_1(1-\rho_2)(1-\gamma_2)\Gamma(\alpha)}L_1 + \frac{(1-\xi)^\beta}{\Gamma(\beta+1)(1-\gamma_2)}L_2\\
&\leqslant \frac{\lambda}{3} + \frac{(1-\xi)(1-\gamma_1)}{\rho_1\xi(1-\gamma_2)M_1}\cdot\frac{M_1\rho_1\xi(1-\gamma_2)}{2(1-\xi)(1-\gamma_1)}\lambda + \frac{\lambda}{6}\\
&\leqslant \frac{\lambda}{3} + \frac{\lambda}{2} + \frac{\lambda}{6} = \lambda = \|u\|.
\end{aligned}$$

Then for any $u \in P \cap \partial\Omega_4$, we have $\|Tu\| \leqslant \|u\|$.

If $f_0 > N_1$, then there exists a constant $\lambda_1 > \mu_1 > 0$ such that $f(t,u,v) > N_1(u+v)$ for any $t \in [\xi - \tau_0,\, \xi]$, $u + v \in (0, \mu_1]$.

Let

$$\Omega_5 = \{u \in P\colon\ \|u\| \leqslant \mu\}, \quad 0 < \mu \leqslant \frac{\mu_1}{2}.$$

Hence, for any $u \in \partial\Omega_5$, we have $\|u\| = \mu$. It is similar to (10), for $t \in (\xi - \tau_0,\, \xi]$,

$$\gamma\mu = \gamma\|u\| \leqslant u(t) + u(t+\tau_1) \leqslant 2\|u\| \leqslant \mu_1.$$

Then for any $u \in P \cap \partial\Omega_5$,

$$\begin{aligned} Tu(0) &= \gamma_1 Tu(\xi) \geqslant N_1\gamma_1\gamma\mu \int\limits_{\xi-\tau_0}^{\xi} G_1(\xi,s)\,\mathrm{d}s \\ &= \frac{\gamma_1\gamma(\xi^\alpha - (\xi-\tau_0)^\alpha)}{(1-\gamma_1)\Gamma(\alpha+1)} N_1\mu = \mu = \|u\|. \end{aligned}$$

Thus, for any $u \in P \cap \partial\Omega_5$, there is $\|Tu\| \geqslant \|u\|$. According to Lemma 2, T has at least one fixed point in $P \cap (\overline{\Omega}_4 \setminus \Omega_5)$.

Similarly, if $g_0 > N_2$, there is a constant $\lambda_2 > \mu_2 > 0$ that makes $g(t,u,v) > N_2(u+v)$ for any $t \in (\xi,\, \xi+\tau_0]$, $u+v \in (0,\mu_2]$.

Let

$$\Omega_6 = \{u \in P\colon\ \|u\| \leqslant \overline{\mu}\}, \quad 0 < \overline{\mu} \leqslant \frac{\mu_2}{2}.$$

Then for any $u \in P \cap \partial\Omega_6$, we have $\|u\| = \overline{\mu}$. Similar to (11), for $t \in (\xi,\, \xi+\tau_0]$, we have

$$\gamma\overline{\mu} = \gamma\|u\| \leqslant u(t) + u(t-\tau_2) \leqslant 2\|u\| \leqslant \mu_2,$$

and for any $u \in P \cap \partial\Omega_6$,

$$\begin{aligned} Tu(1) &= \gamma_2 Tu(\xi^+) \geqslant \gamma_2 N_2 \gamma\overline{\mu} \int\limits_{\xi}^{\xi+\tau_0} G_2(\xi,s)\,\mathrm{d}s \\ &= \frac{\gamma_2\gamma((1-\xi)^\beta - (1-\xi-\tau_0)^\beta)}{(1-\gamma_2)\Gamma(\beta+1)} N_2\overline{\mu} = \overline{\mu} = \|u\|. \end{aligned}$$

Thus, for any $u \in P \cap \partial\Omega_6$, we have $\|Tu\| \geqslant \|u\|$.

According to Lemma 2, T has at least one fixed point in $P \cap (\overline{\Omega}_4 \setminus \Omega_6)$, which implies that boundary value problem (1) has at least one positive solution. □

In particular, the following result holds by Theorem 2.

Corollary 2. *Assume that the following conditions hold*:

(H3′) $f^\infty = g^\infty = 0$*;*
(H4′) $f_0 = +\infty$ *or* $g_0 = +\infty$.

Then there exist constants $a_0, b_0 \geqslant 0$ *such that boundary value problem* (1) *has at least one positive solution for* $0 \leqslant a \leqslant a_0$*,* $0 \leqslant b \leqslant b_0$.

Theorem 3. *Assume $f_\infty > N_1$ holds. Then there exists a large enough positive constant $a_1 > 0$ such that boundary value problem* (1) *has no positive solution for $a > a_1$.*

Proof. If $f_\infty > N_1$, there exists a constant $R > 0$ such that for any $t \in [\xi - \tau_0, \xi]$, $u + v \in [\gamma R, +\infty)$, we have $f(t, u, v) > N_1(u + v)$. Assume that for any large enough $a > 0$, boundary value problem (1) has a positive solution $u = u(t)$.

Let
$$a_1 > \frac{(1-\rho_2)(1-\gamma_1)R}{\xi\gamma_1}, \qquad a > a_1.$$
In fact, since $Tu = u$, we have
$$u(0) \geqslant \frac{\xi\gamma_1 a}{(1-\rho_2)(1-\gamma_1)} > R.$$
Hence, $\|u\| > R$.

From (10) of Theorem 1 we have for any $t \in (\xi - \tau_0, \xi]$,
$$u(t) + u(t + \tau_1) \geqslant \gamma\|u\| \geqslant \gamma R.$$
Hence,
$$\begin{aligned} u(0) &= \gamma_1 u(\xi) \\ &\geqslant \gamma_1 \int\limits_{\xi-\tau_0}^{\xi} G_1(\xi, s) f\bigl(s, u(s), u(s+\tau_1)\bigr)\,\mathrm{d}s + \frac{\xi\gamma_1}{(1-\rho_2)(1-\gamma_1)} a \\ &\geqslant \frac{\gamma_1\gamma(\xi^\alpha - (\xi-\tau_0)^\alpha)}{(1-\gamma_1)\Gamma(\alpha+1)} N_1\|u\| + \frac{\xi\gamma_1}{(1-\rho_2)(1-\gamma_1)} a_1 \geqslant \|u\| + R. \end{aligned}$$

So $\|u\| \geqslant \|u\| + R$, which is a contradiction. Thus, there exists a constant $a_1 > 0$ such that the boundary value problem (1) has no positive solution for $a > a_1$. □

Theorem 4. *Assume $g_\infty > N_2$ holds. Then there exist large enough positive constants a_2, $b_1 > 0$ such that boundary value problem* (1) *has no positive solution for $a > a_2$, $b > b_1$.*

Proof. Similarly, if $g_\infty > N_2$, there exists a constant $R_0 > 0$ such that for any $t \in (\xi, \xi + \tau_0]$, $u + v \in [\gamma R_0, +\infty)$, we have $g(t, u, v) > N_2(u + v)$. Assume that for any large enough $a > 0$, $b > 0$, the boundary value problem (1) has a positive solution $u = u(t)$.

Let
$$a_2 + (1-\rho_2)b_1 > \frac{(1-\rho_2)(1-\gamma_2)\rho_1 R_0}{(1-\xi)\gamma_2}, \qquad a > a_2, \qquad b > b_1.$$
Since $Tu = u$, we have
$$u(1) \geqslant \frac{(1-\xi)\gamma_2((a_2 + (1-\rho_2)b_1)}{\rho_1(1-\rho_2)(1-\gamma_2)} > R_0.$$
Hence, $\|u\| > R_0$. Since for any $t \in (\xi, \xi + \tau_0]$,
$$u(t) + u(t - \tau_2) \geqslant \gamma\|u\| \geqslant \gamma R_0.$$

Then

$$\begin{aligned}u(1) &= \gamma_2 u(\xi^+)\\ &\geqslant \gamma_2 \int\limits_{\xi}^{\xi+\tau_0} G_2(\xi,s) g\big(s,u(s),u(s-\tau_1)\big)\,\mathrm{d}s + \frac{(1-\xi)\gamma_2}{\rho_1(1-\rho_2)(1-\gamma_2)}\big(a_2+(1-\rho_2)b_1\big)\\ &\geqslant \frac{\gamma_2\gamma((1-\xi)^\beta-(1-\xi-\tau_0)^\beta)}{(1-\gamma_2)\Gamma(\beta+1)} N_2\|u\| + \frac{(1-\xi)\gamma_2}{\rho_1(1-\rho_2)(1-\gamma_2)}\big(a_2+(1-\rho_2)b_1\big)\\ &\geqslant \|u\| + R_0.\end{aligned}$$

So $\|u\| \geqslant \|u\| + R_0$, which is a contradiction. Thus, there exist constants $a_2, b_1 > 0$ such that the boundary value problem (1) has no positive solution for $a > a_2,\ b > b_1$. □

4 Multiplicity of the positive solutions

In this section, we consider the multiplicity of solutions for boundary value problem (1) by using Lemma 3.

Let $\overline{P}_c = \{u \in P\colon \|u\| \leqslant c\}$. Define a nonnegative continuous concave functional $\omega : P \to [0,+\infty)$ by $\omega(u) = \inf_{t\in[\xi-\tau_0,\,\xi+\tau_0]} u(t)$. Obviously, $\omega(u) \leqslant \|u\|$ for any $u \in P$. Set $P(\omega,q,c) = \{u \in P\colon q \leqslant \omega(u),\ \|u\| \leqslant c\}$.

Theorem 5. *Suppose there are three constants d, q, c with $0 < d < q < c$, where $\min\{M_1c, M_2c\} \geqslant Nq > 0$, and the following hypotheses hold*:

- (H5) $f(t,u,v) < M_1 d$ *for* $(t,u,v) \in [0,\xi]\times[0,d]\times[0,d]$; $g(t,u,v) < M_2 d$ *for* $(t,u,v) \in [\xi,1]\times[0,d]\times[0,d]$;
- (H6) $f(t,u,v) > Nq$ *for* $(t,u,v) \in [\xi-\tau_0,\xi]\times[q,c]\times[0,c]$; $g(t,u,v) > Nq$ *for* $(t,u,v) \in (\xi,\xi+\tau_0]\times[q,c]\times[0,c]$;
- (H7) $f(t,u,v) \leqslant M_1 c$, $g(t,u,v) \leqslant M_2 c$ *for* $(t,u,v) \in [0,\xi]\times[0,c]\times[0,c]$; $g(t,u,v) \leqslant M_2 c$ *for* $(t,u,v) \in [\xi,1]\times[0,c]\times[0,c]$.

Then there exist constants $a_0, b_0 \geqslant 0$ such that boundary value problem (1) *has at least three positive solutions u_1, u_2, u_3 on P for $0 \leqslant a \leqslant a_0$, $0 \leqslant b \leqslant b_0$, where $\|u_1\| < d$, $\omega(u_2) > q$, $\|u_3\| > d$, $\omega(u_3) < q$.*

Proof. First of all, for any $u \in \overline{P}_c$, we have $0 \leqslant u(t) \leqslant \|u\| \leqslant c$. Let $a_0 = \xi^{\alpha-1}M_1c$ and $b_0 = 2\xi^{\alpha-1}M_1c/\Gamma(\alpha+1)$.

Define a operator $T_1 : \overline{P}_c \to P$ by

$$T_1u(t) = \begin{cases} \int_0^\xi G_1(t,s) f(s,u(s),u(s+\tau_1))\,\mathrm{d}s + \frac{1}{1-\rho_2}\big(\frac{\xi\gamma_1}{1-\gamma_1}+t\big) \\ \quad \times\big(\frac{\rho_2}{\Gamma(\alpha-1)}\int_0^\xi s^{\alpha-2} f(s,u(s),u(s+\tau_1))\,\mathrm{d}s + a\big), \quad t\in[0,\xi], \\ \int_\xi^1 G_2(t,s) g(s,u(s),u(s-\tau_2))\,\mathrm{d}s - \frac{1}{\rho_1(1-\rho_2)}\big(\frac{\gamma_2\xi-1}{1-\gamma_2}+t\big) \\ \quad \times\big(\frac{\rho_2}{\Gamma(\alpha-1)}\int_0^\xi s^{\alpha-2} f(s,u(s),u(s+\tau_1))\,\mathrm{d}s + a + (1-\rho_2)b\big), \quad t\in(\xi,1]. \end{cases}$$

Then $u = u(t)$ is a solution of (1) if and only if u is a fixed point of the operator T_1 on $\overline{P}_c$. By (H5),

$$\begin{aligned}\|T_1u\|_{[0,\xi]} &= T_1u(\xi)\\ &\leqslant \int_0^\xi G_1(\xi,s)\,\mathrm{d}s + \frac{\xi}{(1-\rho_2)(1-\gamma_1)}\left(\frac{\rho_2M_1c}{\Gamma(\alpha-1)}\int_0^\xi s^{\alpha-2}\,\mathrm{d}s + a_0\right)\\ &\leqslant M_1c\left(\frac{\xi^\alpha\big(2+2(\alpha-1)\rho_2+\Gamma(\alpha+1)\big)}{(1-\gamma_1)(1-\rho_2)\Gamma(\alpha+1)}\right) = c.\end{aligned}$$

Since $\min\{M_1c, M_2c\} \geqslant Nq > 0$, then $M_2 > 0$,

$$\frac{2(1-\xi)^\beta M_2}{\Gamma(\beta+1)(1-\gamma_2)} + \frac{(1-\xi)(1-\gamma_1)}{\rho_1\xi(1-\gamma_2)} = 1$$

and

$$\begin{aligned}\|T_1u\|_{(\xi,1]} &= T_1u(\xi^+)\\ &< c\left(M_2\int_\xi^1 G_2(\xi,s)\,\mathrm{d}s + \frac{1-\xi}{\rho_1(1-\rho_2)(1-\gamma_2)}\right.\\ &\quad \left.\times\left(\frac{M_1\rho_2}{\Gamma(\alpha-1)}\int_0^\xi s^{\alpha-2}\,\mathrm{d}s + a_0 + (1-\rho_2)b_0\right)\right)\\ &< c\left(\frac{2(1-\xi)^\beta M_2}{\Gamma(\beta+1)(1-\gamma_2)} + \frac{(1-\xi)(1-\gamma_1)}{\rho_1\xi(1-\gamma_2)}\right) = c.\end{aligned}$$

Hence, we have $\|T_1u\| \leqslant c$, which implies $T_1 : \overline{P}_c \to \overline{P}_c$.

Similarly, by (H5), we can get that $\|T_1u\| < d$ for $u \in \overline{P}_d$. Therefore, the condition (ii) in Lemma 3 is satisfied.

Select $u(t) = (q+c)/2$, $0 \leqslant t \leqslant 1$. Obviously, $u(t) = (q+c)/2 \in P(\omega, q, c)$ and $\omega(u) = \inf_{t\in[\xi-\tau_0,\xi+\tau_0]} u(t) = (q+c)/2 > q$, then $\{u \in P(\omega, q, c)\colon \omega(u) > q\} \neq \emptyset$.

For any $u \in P(\omega, q, c)$, we can get that $u(t) \geqslant q$ for $t \in [\xi-\tau_0,\ \xi+\tau_0]$ and $0 \leqslant u(t) \leqslant c$ for $t \in [0,1]$. By Lemma 5, we can easily get that T_1u is monotone increasing on $[0,\xi]$ and T_1u is monotone decreasing on $(\xi, 1]$. From (H6), we get $f(t,u,v) \geqslant Nq$, $t \in [\xi-\tau_0,\ \xi]$ and $g(t,u,v) \geqslant Nq$, $t \in (\xi,\ \xi+\tau_0]$.

Hence,

$$\begin{aligned}\inf_{t\in[\xi-\tau_0,\xi]} T_1u(t) &= T_1u(\xi-\tau_0) \geqslant T_1u(0) = \gamma_1T_1u(\xi)\\ &> \gamma_1\int_{\xi-\tau_0}^\xi G_1(\xi,s)Nq\,\mathrm{d}s\\ &\geqslant \frac{\gamma_1\gamma(\xi^\alpha-(\xi-\tau_0)^\alpha)}{(1-\gamma_1)\Gamma(\alpha+1)}N_1q = q,\end{aligned}$$

and

$$\begin{aligned}\inf_{t\in(\xi,\xi+\tau_0]} T_1u(t) &= T_1u(\xi+\tau_0) \geqslant T_1u(1) = \gamma_2 T_1u(\xi^+) \\ &> \gamma_2 \int\limits_{\xi}^{\xi+\tau_0} G_2(\xi,s)Nq\,\mathrm{d}s \\ &\geqslant \frac{\gamma_2\gamma((1-\xi)^\beta-(1-\xi-\tau_0)^\beta)}{(1-\gamma_2)\Gamma(\beta+1)}N_2q = q.\end{aligned}$$

Therefore, for $u \in P(\omega,q,c)$, we have $\omega(T_1u(t)) > q$. Hence, condition (i) in Lemma 3 holds.

Due to Lemma 3 involves paramaters d, q, c, r with $0 < d < q < c \leqslant r$. Let $c = r$, then by condition (i) in Lemma 3 it is clearly that for $u \in P(\omega,q,c)$ and $\|Tu\| > c$, we have $\omega(Tu) > q$.

Therefore, condition (iii) in Lemma 3 also satisfied. Then Lemma 3 implies that the boundary value problem (1) has at least three solutions u_1, u_2, u_3 on P_c and $\|u_1\| < d$, $\omega(u_2) > q$, $\|u_3\| > d$, $\omega(u_3) < q$. □

5 Illustration

In order to illustrate the applicability of our main results, the following boundary value problem is considered in this section.

Example 1. For the following boundary value problem

$$\begin{aligned}&{}_t^cD_{\pi/8^-}^{3/2}u(t)+u(t) \\ &\quad+\left(u(t)+u\left(t+\frac{3}{5}\right)\right)\left(\frac{1}{100}\sin t+\frac{u(t)+u(t+\frac{3}{5})}{\cos t}\right)=0, \quad t\in\left[0,\frac{\pi}{8}\right], \\ &{}_{\pi/8^+}^{\;c}D_t^{7/4}u(t) \\ &\quad+\left(u(t)+u\left(t-\frac{1}{3}\right)\right)\left(\frac{1}{4}\cos t+\left(u(t)+u\left(t-\frac{1}{3}\right)\right)^2\sin t\right)=0, \quad t\in\left(\frac{\pi}{8},1\right], \\ &u'\left(\left(\frac{\pi}{8}\right)^-\right)=-\left(2u'\left(\left(\frac{\pi}{8}\right)^+\right)+b\right)=a, \\ &u(0)=\frac{1}{2}u\left(\left(\frac{\pi}{8}\right)^-\right), \qquad u(1)=\frac{1}{2}u\left(\left(\frac{\pi}{8}\right)^+\right),\end{aligned} \tag{12}$$

we can establish the following results:

(i) If $a \in [0, 0.019191]$, $b \in [0, 0.028873]$, then boundary value problem (12) has at least one positive solution.

(ii) If $a \in (2.46995\times10^6,+\infty)$, $b \in (2.22295\times10^7,+\infty)$, then boundary value problem (12) has no positive solutions.

Proof. Boundary value problem (12) can be regarded as boundary value problem (1), where $\alpha = 3/2$, $\beta = 7/4$, $\xi = \pi/8$, $\rho_1 = 2$, $\rho_2 = 0$, $\gamma_1 = \gamma_2 = 1/2$, $\tau_1 = 3/5$, $\tau_2 = 1/3$, $f(t,u,v) = (u+v)(1/100)\sin t + (u+v)\cos t$ and $g(t,u,v) = (u+v)((1/4)\cos t + (u+v)^2 \sin t)$.

Let $\tau_0 = 1/4 < \min\{\tau_1, \tau_2\}$, we can easily obtain that $M_1 \approx 0.811256$, $M_2 \approx 0.218239 > 0$, $N_1 \approx 13.8342$ and $N_2 \approx 12.7312$, $f^0 \approx 0.00382783 < M_1$, $g^0 \approx 0.23097 < M_2$ and $g_\infty = +\infty > N_2$.

Then there exist constants $r_1 = 0.151$, $r_2 = 0.234$, $r = \min\{r_1/2, r_2/2\} = 0.0755$, $R_0 = 7.5 \times 10^6$, It is easy to get that $a_0 = (\xi^{\alpha-1} M_1 r)/2 = 0.019191$, $b_0 = (\xi^{\alpha-1} M_1 r)/\Gamma(\alpha+1) = 0.028873$, $a_2 = 2.46995 \times 10^6$ and $b_1 = 2.22295 \times 10^7$.

(i) According to Theorem 1, if $a \in [0, a_0]$ and $b \in [0, b_0]$, then boundary value problem (12) has at least one positive solution.

(ii) According to Theorem 4, if $a \in (a_2, +\infty)$ and $b \in (b_1, +\infty)$, then boundary value problem (12) has no positive solutions. □

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