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Asymptotic formulas for the left truncated moments of sums with consistently varying distributed increments*

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Abstract. In this paper, we consider the sum $S_n^\xi = \xi_1 + \cdots + \xi_n$ of possibly dependent and nonidentically distributed real-valued random variables $\xi_1, \dots, \xi_n$ with consistently varying distributions. By assuming that collection $\{\xi_1, \dots, \xi_n\}$ follows the dependence structure, similar to the asymptotic independence, we obtain the asymptotic relations for $\mathbf{E}((S_n^\xi)^\alpha \mathbf{1}_{\{S_n^\xi > x\}})$ and $\mathbf{E}((S_n^\xi - x)^+)^\alpha$, where α is an arbitrary nonnegative real number. The obtained results have applications in various fields of applied probability, including risk theory and random walks.

Keywords: sum of random variables, asymptotic independence, tail moment, truncated moment, heavy tail, consistently varying distribution.

1 Introduction

Let $n \in \mathbb{N} := \{1, 2, \dots\}$ and let $\{\xi_1, \dots, \xi_n\}$ be a collection of possibly dependent real-valued random variables (r.v.s) with heavy-tailed distributions. Denote

$$S_n^\xi := \xi_1 + \cdots + \xi_n. \tag{1}$$

Throughout the paper, we assume that random summands have consistently varying distributions. This is a subclass of heavy-tailed distributions. We recall some definitions. We say that a distribution function (d.f.) is supported on $\mathbb{R}$ if its tail $\overline{F} = 1 - F$ satisfies $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$.

- *A d.f. F supported on $\mathbb{R}$ is said to be heavy-tailed, written as $F \in \mathcal{H}$, if for every $h > 0$, it holds that*

$$\int_{-\infty}^{\infty} \mathrm{e}^{hx}\,\mathrm{d}F(x) = \infty.$$

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- *A d.f. F on $\mathbb{R}$ is said to be dominatedly varying, written as $F \in \mathcal{D}$, if for any fixed $y \in (0,1)$, it holds that*
$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$
- *A d.f. F on $\mathbb{R}$ is said to be consistently varying, written as $F \in \mathcal{C}$, if*
$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$
- *A d.f. F on $\mathbb{R}$ is said to be regularly varying with index $\gamma \geqslant 0$, written as $F \in \mathcal{R}_\gamma$, if for any $y > 0$, it holds that*
$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\gamma}.$$

It is well known (see, for instance, [5]) that
$$\mathcal{R} := \bigcup_{\gamma\geqslant 0} \mathcal{R}_\gamma \subset \mathcal{C} \subset \mathcal{D} \subset \mathcal{H}.$$

The following two indices are important to the determination whether d.f. F belongs to the aforementioned heavy-tailed distribution classes. The first index is the so-called *upper Matuszewska index* (see, e.g., [2, Sect. 2], [9, 23]), defined as
$$J_F^+ = \inf_{y>1} \left\{ -\frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\}.$$
Another index, so-called *L-index*, is defined as
$$L_F = \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$
This index was used by [16, 19, 33], among others.

The definitions of the aforementioned heavy-tailed distribution classes imply that
$$\begin{aligned}
F \in \mathcal{D} \quad &\Longleftrightarrow \quad J_F^+ < \infty \quad \Longleftrightarrow \quad L_F > 0,\\
F \in \mathcal{C} \quad &\Longleftrightarrow \quad L_F = 1,\\
F \in \mathcal{R}_\gamma \quad &\Longrightarrow \quad L_F = 1, \quad J_F^+ = \gamma.
\end{aligned}$$

The classes $\mathcal{R}$ and $\mathcal{D}$ have been extensively used in real analysis and various areas of probability (see, e.g., [2, 12, 25, 27]). The class $\mathcal{C}$ of consistently varying distributions was introduced as a generalization of the class $\mathcal{R}$ in [8], and was named there as a class of distributions with "intermediate regular variation". The concept of consistent variation has been used in various papers in the context of applied probability, such as queueing systems, graph theory and ruin theory (see, e.g., [1, 3–7, 9, 13, 17, 22, 32]).

We explain some notations which will be used throughout the paper. For two positive functions f, g, we write:

$$\begin{aligned}
&f(x) \underset{x\to\infty}{\lesssim} g(x) && \text{if } \limsup_{x\to\infty} \frac{f(x)}{g(x)} \leqslant 1;\\
&f(x) = O\bigl(g(x)\bigr) && \text{if } \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty;\\
&f(x) \asymp g(x) && \text{if } f(x) = O\bigl(g(x)\bigr) \text{ and}\\
& && \quad g(x) = O\bigl(f(x)\bigr);\\
&f(x) \underset{x\to\infty}{\sim} g(x) && \text{if } \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.
\end{aligned}$$

In this paper, we suppose that the random variables $\xi_1, \dots, \xi_n$ are pairwise quasi-asymptotically independent. This dependence structure was introduced in [7] and considered in [14,20,21,30,31] and other papers. In the definition below and elsewhere, we use the standard notations: $x^+ := \max\{0, x\}$, $x^- := \max\{0, -x\}$.

Definition 1. Real-valued random variables $\xi_1, \dots, \xi_n$ with distributions supported on $\mathbb{R}$ are called *pairwise quasi-asymptotically independent* (pQAI) if for all pairs of indices $k, l \in \{1, 2, \dots, n\}$, $k \neq l$, it holds that

$$\lim_{x\to\infty} \frac{\mathbf{P}(\xi_k^+ > x,\, \xi_l^+ > x)}{\mathbf{P}(\xi_k^+ > x) + \mathbf{P}(\xi_l^+ > x)} = \lim_{x\to\infty} \frac{\mathbf{P}(\xi_k^+ > x,\, \xi_l^- > x)}{\mathbf{P}(\xi_k^+ > x) + \mathbf{P}(\xi_l^+ > x)} = 0.$$

The following statement is Theorem 3.1 in [7]. The statement provides the asymptotic results for tail probability of sums of pQAI r.v.s having distributions from class $\mathcal{C}$.

Theorem 1. *Let $\{\xi_1, \dots, \xi_n\}$ be a collection of real-valued pQAI r.v.s such that $F_{\xi_k} \in \mathcal{C}$ for $k \in \{1, \dots, n\}$. Then*

$$\mathbf{P}\bigl(S_n^\xi > x\bigr) \underset{x\to\infty}{\sim} \sum_{k=1}^n \overline{F}_{\xi_k}(x).$$

The following assertion with slightly narrower dependence structure and r.v.s from a wider class $\mathcal{D}$ is derived in Theorem 2.1 of [18].

Theorem 2. *Let $\{\xi_1, \dots, \xi_n\}$ be a collection of real-valued r.v.s such that*

$$\begin{aligned}
\lim_{x\to\infty} \sup_{u\geqslant x} \mathbf{P}\bigl(\xi_k^+ > x \bigm| \xi_l^+ > u\bigr) &= \lim_{x\to\infty} \sup_{u\geqslant x} \mathbf{P}\bigl(\xi_k^- > x \bigm| \xi_l^+ > u\bigr)\\
&= \lim_{x\to\infty} \sup_{u\geqslant x} \mathbf{P}\bigl(\xi_k^+ > x \bigm| \xi_l^- > u\bigr)\\
&= 0
\end{aligned}$$

for all pairs of indices $k, l \in \{1, 2, \dots, n\}$. In addition, suppose that $F_{\xi_1} \in \mathcal{D}$, $\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x)$, $\overline{F}_{\xi_k^-}(x) = O(\overline{F}_{\xi_1}(x))$ for $k \in \{1, \dots, n\}$, and $\mathbf{E}\,|\xi_1|^m < \infty$ for some

$m \in \mathbb{N}_0 := \{0, 1, \dots, \}$. *Then*

$$\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbf{E}\big(\xi_k^m \mathbf{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbf{E}\big(\big(S_n^\xi\big)^m \mathbf{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbf{E}\big(\xi_k^m \mathbf{1}_{\{\xi_k > x\}}\big).$$

In this paper, we obtain asymptotic relationships for

$$\mathbf{E}\big(\big(S_n^\xi\big)^\alpha \mathbf{1}_{\{S_n^\xi > x\}}\big) \tag{2}$$

and

$$\mathbf{E}\big(\big(S_n^\xi - x\big)^+\big)^\alpha \tag{3}$$

for arbitrary power $\alpha \in [0, \infty)$ and for r.v.s $\xi_1, \dots, \xi_n$ following wider, pQAI, dependence structure. Asymptotic behavior of the left truncated moments of random sums was considered in various fields of applied probability, including risk theory and random walks [10,11,24]. In addition, quantity in (3) is closely related with the Haezendonck–Goovaerts risk measure (see, for instance, [15, 18, 28] and [29]). To get the precise asymptotic equivalence relationship, we consider r.v.s with d.f.s from class $\mathcal{C}$. The main results on the asymptotics of (2) and (3) are presented in Theorems 3 and 4 below.

The rest of the paper is organized as follows. In Section 2, we provide formulations of the main results. In Section 3, we present the proofs of the asymptotic formulas for the left truncated moments of S_n^ξ. The last Section 4 deals with the examples illustrating the obtained results.

2 Main results

The first assertion generalizes results of Theorem 1 which can be derived from theorem below by supposing $\alpha = 0$. In addition, for class $\mathcal{C}$, theorem below gives an analogous result to Theorem 2 for r.v.s $\xi_1, \dots, \xi_n$ following a wider dependence structure and for a real-valued nonnegative moment order α.

Theorem 3. *Let $\{\xi_1, \dots, \xi_n\}$ be a collection of real-valued pQAI r.v.s such that $F_{\xi_k} \in \mathcal{C}$ and $\mathbf{E}(\xi_k^+)^\alpha < \infty$ for all $k \in \{1, \dots, n\}$ and for some $\alpha \geqslant 0$. Then*

$$\mathbf{E}\big(\big(S_n^\xi\big)^\alpha \mathbf{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \mathbf{E}\big(\xi_k^\alpha \mathbf{1}_{\{\xi_k > x\}}\big). \tag{4}$$

The second theorem shows that the asymptotic behaviour of the left truncated moments of sums depends on consistently varying distributed increments but does not depend on asymptotically lighter increments.

Theorem 4. *Let $\{\xi_1, \dots, \xi_n\}$ be a collection of real-valued r.v.s such that, for each $k \in \{1, \dots, n\}$, it holds that $F_{\xi_k} \in \mathcal{C}$ or $\mathbf{P}(|\xi_k| > x) = o(\overline{F}_{\xi_1}(x))$. Suppose that $F_{\xi_1} \in \mathcal{C}$ and $\mathbf{E}(\xi_k^+)^\alpha < \infty$ for all $k \in \{1, \dots, n\}$ and some $\alpha \geqslant 0$. Let $\mathcal{I} \subseteq \{1, \dots, n\}$ be a subset of indices k such that $F_{\xi_k} \in \mathcal{C}$. If the subcollection $\{\xi_k,\ k \in \mathcal{I}\}$ consists of pQAI r.v.s, then,* *for each* $\beta \in [0, \alpha]$*,*

$$\mathbf{E}\big(\big(S_n^{\xi}\big)^{\beta}\mathbf{1}_{\{S_n^{\xi}>x\}}\big) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}} \mathbf{E}\big(\xi_k^{\beta}\mathbf{1}_{\{\xi_k>x\}}\big), \tag{5}$$

and, for $\beta \in (0, \alpha]$*, it holds that*

$$\mathbf{E}\big(\big(S_n^{\xi}-x\big)^{+}\big)^{\beta} \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}} \mathbf{E}\big(\big(\xi_k-x\big)^{+}\big)^{\beta}. \tag{6}$$

We notice that the basic index in the formulation of Theorem 4, which is equal to one, can be replaced by any index $l \in \{1, \dots, n\}$. In addition, it should be noted that dependence of r.v.s ξ_k, $k \in \mathcal{I}^c$, as well as mutual dependence between the sets $\{\xi_k,\, k \in \mathcal{I}\}$ and $\{\xi_k,\, k \in \mathcal{I}^c\}$, can be arbitrary.

3 Proofs of main results

We present two auxiliary lemmas before providing proofs of the main results.

Lemma 1. *Let* ξ *be a real-valued r.v. such that* $\mathbf{E}(\xi^{+})^p < \infty$ *for some* $p > 0$*. Then, for any* $x \geqslant 0$*, we have*

$$\mathbf{E}\big(\xi^p\mathbf{1}_{\{\xi>x\}}\big) = x^p\mathbf{P}(\xi > x) + p\int_x^{\infty} u^{p-1}\mathbf{P}(\xi > u)\,\mathrm{d}u \tag{7}$$

and

$$\mathbf{E}\big((\xi-x)^{+}\big)^p = p\int_x^{\infty} (u-x)^{p-1}\mathbf{P}(\xi > u)\,\mathrm{d}u. \tag{8}$$

Proof. Both equalities of the lemma follow directly from the following well-known formula

$$\mathbf{E}\eta^p = p\int_0^{\infty} u^{p-1}\mathbf{P}(\eta > u)\,\mathrm{d}u, \tag{9}$$

provided that $p > 0$ and η is a nonnegative r.v. (see, for instance, [26, p. 208, Cor. 2]).

Namely, by supposing $\eta = \xi\,\mathbf{1}_{\{\xi>x\}}$, from (9) we obtain

$$\begin{aligned}\mathbf{E}\big(\xi^p\,\mathbf{1}_{\{\xi>x\}}\big) &= p\int_0^{\infty} u^{p-1}\mathbf{P}\big(\xi\,\mathbf{1}_{\{\xi>x\}} > u\big)\,\mathrm{d}u\\ &= p\,\mathbf{P}(\xi > x)\int_0^{x} u^{p-1}\,\mathrm{d}u + p\int_x^{\infty} u^{p-1}\mathbf{P}(\xi > u)\,\mathrm{d}u,\end{aligned}$$

and equality (7) follows.

Similarly, by supposing $\eta = (\xi - x)^+$, from (9) equality (8) holds because

$$\begin{aligned}
&\mathbf{E}\bigl((\xi - x)^+\bigr)^p \\
&\quad = p\int\limits_0^\infty u^{p-1}\mathbf{P}\bigl((\xi - x)^+ > u\bigr)\,\mathrm{d}u \\
&\quad = p\int\limits_0^\infty u^{p-1}\bigl(\mathbf{P}\bigl((\xi - x)^+ > u,\ \xi > x\bigr) + \mathbf{P}\bigl((\xi - x)^+ > u,\ \xi \leqslant x\bigr)\bigr)\,\mathrm{d}u \\
&\quad = p\int\limits_0^\infty u^{p-1}\mathbf{P}(\xi > x + u)\,\mathrm{d}u. \qquad \square
\end{aligned}$$

Lemma 2. *Let ξ and η be two arbitrarily dependent r.v.s. If $F_\xi \in \mathcal{C}$ and $\mathbf{P}(|\eta| > x) = o(\overline{F}_\xi(x))$, then*

$$\mathbf{P}\bigl(\xi + \eta > x\bigr) \underset{x\to\infty}{\sim} \overline{F}_\xi(x). \tag{10}$$

Proof. Proof of the lemma is presented in [34] (see part (i) of Lemma 3.3). $\square$

Proof of Theorem 3. In the case $\alpha = 0$, the assertion of Theorem 3 follows from Theorem 1 immediately. Hence, further, we can suppose that α is positive. By Lemma 1, for all $x \geqslant 0$, we have

$$\begin{aligned}
&\frac{\mathbf{E}((S_n^\xi)^\alpha \mathbf{1}_{\{S_n^\xi > x\}})}{\sum_{k=1}^n \mathbf{E}(\xi_k^\alpha \mathbf{1}_{\{\xi_k > x\}})} \\
&\quad = \frac{x^\alpha \mathbf{P}(S_n^\xi > x) + \alpha\int_x^\infty u^{\alpha-1}\mathbf{P}(S_n^\xi > u)\,\mathrm{d}u}{\sum_{k=1}^n x^\alpha \mathbf{P}(\xi_k > x) + \alpha\int_x^\infty u^{\alpha-1}\sum_{k=1}^n \mathbf{P}(\xi_k > u)\,\mathrm{d}u} \\
&\quad \leqslant \max\left\{\frac{\mathbf{P}(S_n^\xi > x)}{\sum_{k=1}^n \mathbf{P}(\xi_k > x)}, \frac{\int_x^\infty u^{\alpha-1}\frac{\mathbf{P}(S_n^\xi > u)}{\sum_{k=1}^n \mathbf{P}(\xi_k > u)}\sum_{k=1}^n \mathbf{P}(\xi_k > u)\,\mathrm{d}u}{\int_x^\infty u^{\alpha-1}\sum_{k=1}^n \mathbf{P}(\xi_k > u)\,\mathrm{d}u}\right\} \\
&\quad \leqslant \max\left\{\frac{\mathbf{P}(S_n^\xi > x)}{\sum_{k=1}^n \overline{F}_{\xi_k}(x)}, \sup_{u\geqslant x}\frac{\mathbf{P}(S_n^\xi > u)}{\sum_{k=1}^n \overline{F}_{\xi_k}(u)}\right\}
\end{aligned}$$

due to right inequality in min-max inequality

$$\min\left\{\frac{a_1}{b_1}, \dots, \frac{a_r}{b_r}\right\} \leqslant \frac{a_1 + \dots + a_r}{b_1 + \dots + b_r} \leqslant \max\left\{\frac{a_1}{b_1}, \dots, \frac{a_r}{b_r}\right\}, \tag{11}$$

provided that $a_i \geqslant 0$ and $b_i > 0$ for $i \in \{1, \dots, r\}$.

By Theorem 1 we get

$$\limsup_{x\to\infty}\frac{\mathbf{E}((S_n^\xi)^\alpha \mathbf{1}_{\{S_n^\xi > x\}})}{\sum_{k=1}^n \mathbf{E}(\xi_k^\alpha \mathbf{1}_{\{\xi_k > x\}})} \leqslant \limsup_{x\to\infty}\sup_{u\geqslant x}\frac{\mathbf{P}(S_n^\xi > u)}{\sum_{k=1}^n \overline{F}_{\xi_k}(u)} = 1. \tag{12}$$

Similarly, using the left inequality in (11), we obtain

$$\liminf_{x\to\infty} \frac{\mathbf{E}\big((S_n^\xi)^\alpha \mathbf{1}_{\{S_n^\xi > x\}}\big)}{\sum_{k=1}^n \mathbf{E}(\xi_k^\alpha \mathbf{1}_{\{\xi_k > x\}})} \geqslant \liminf_{x\to\infty} \inf_{u\geqslant x} \frac{\mathbf{P}(S_n^\xi > u)}{\sum_{k=1}^n \overline{F}_{\xi_k}(u)} = 1. \tag{13}$$

The derived estimates (12) and (13) complete the proof of Theorem 3. □

Proof of Theorem 4. If $\mathcal{I} = \{1, \dots, n\}$, then relation (5) follows immediately from Theorem 3. Hence, let us suppose that $\mathcal{I}^c \neq \emptyset$ and denote

$$S_n^{(1)} = \sum_{k\in\mathcal{I}} \xi_k, \qquad S_n^{(2)} = \sum_{k\in\mathcal{I}^c} \xi_k.$$

Summands in $S_n^{(1)}$ are pQAI r.v.s with consistently varying d.f.s. Hence, Theorem 1 implies that

$$\mathbf{P}\big(S_n^{(1)} > x\big) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x). \tag{14}$$

This asymptotic relation and inequality (11) imply that d.f. $F_{S_n^{(1)}}(x) = \mathbf{P}(S_n^{(1)} \leqslant x)$ belongs to the class $\mathcal{C}$ due to the following estimate

$$\limsup_{x\to\infty} \frac{\mathbf{P}(S_n^{(1)} > yx)}{\mathbf{P}(S_n^{(1)} > x)} = \limsup_{x\to\infty} \frac{\sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(yx)}{\sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x)} \leqslant \max_{k\in\mathcal{I}} \left\{ \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(yx)}{\overline{F}_{\xi_k}(x)} \right\},$$

provided that $y \in (0, 1)$.

In addition, each r.v. ξ_k with index $k \in \mathcal{I}^c$ satisfies condition $\mathbf{P}(|\xi_k| > x) = o(\overline{F}_{\xi_1}(x))$ according to requirements of the theorem. The fact that $F_{\xi_1} \in \mathcal{C} \subset \mathcal{D}$ and asymptotic equality (14) imply that

$$\mathbf{P}\big(\big|S_n^{(2)}\big| > x\big) = o\big(\mathbf{P}\big(S_n^{(1)} > x\big)\big) \tag{15}$$

because

$$\begin{aligned} \frac{\mathbf{P}(|S_n^{(2)}| > x)}{\mathbf{P}(S_n^{(1)} > x)} &\leqslant \frac{\mathbf{P}(\bigcup_{k\in\mathcal{I}^c} \{|\xi_k| > \frac{x}{r}\})}{\sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x)} \frac{\sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x)}{\mathbf{P}(S_n^{(1)} > x)} \\ &\leqslant \frac{\sum_{k\in\mathcal{I}^c} \mathbf{P}(|\xi_k| > \frac{x}{r})}{\overline{F}_{\xi_1}(\frac{x}{r})} \frac{\overline{F}_{\xi_1}(\frac{x}{r})}{\overline{F}_{\xi_1}(x)} \frac{\sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x)}{\mathbf{P}(S_n^{(1)} > x)}, \end{aligned}$$

where $r = |\mathcal{I}^c| \leqslant n - 1$.

Consequently, Lemma 2 and asymptotic relations (14), (15) imply that

$$\mathbf{P}\big(S_n^\xi > x\big) \underset{x\to\infty}{\sim} \mathbf{P}\big(S_n^{(1)} > x\big) \underset{x\to\infty}{\sim} \sum_{k\in\mathcal{I}} \overline{F}_{\xi_k}(x). \tag{16}$$

Hence, the first relation (5) of Theorem 4 holds in the case $\beta = 0$. If $\beta \in (0, \alpha]$, then using the first equality of Lemma 1 and estimates of (11), similarly as in the proof of

Theorem 3, we derive that

$$\limsup_{x\to\infty}\frac{\mathbf{E}((S_n^\xi)^\beta\mathbf{1}_{\{S_n^\xi>x\}})}{\sum_{k\in\mathcal{I}}\mathbf{E}(\xi_k^\beta\mathbf{1}_{\{\xi_k>x\}})}\leqslant\limsup_{x\to\infty}\sup_{u\geqslant x}\frac{\mathbf{P}(S_n^\xi>u)}{\sum_{k\in\mathcal{I}}\overline{F}_{\xi_k}(u)},$$
$$\liminf_{x\to\infty}\frac{\mathbf{E}((S_n^\xi)^\beta\mathbf{1}_{\{S_n^\xi>x\}})}{\sum_{k\in\mathcal{I}}\mathbf{E}(\xi_k^\beta\mathbf{1}_{\{\xi_k>x\}})}\geqslant\liminf_{x\to\infty}\inf_{u\geqslant x}\frac{\mathbf{P}(S_n^\xi>u)}{\sum_{k\in\mathcal{I}}\overline{F}_{\xi_k}(u)}.$$

Relation (5) of Theorem 4 for $\beta\in(0,\alpha]$ follows now from (16).

The second asymptotic relation (6) can be obtained in a similar way by using the second equality of Lemma 1, relation (16) and estimate (11). Theorem 4 is proved. □

4 Examples

In this section, we provide two examples illustrating our main results.

Example 1. Let r.v.s $\xi_1,\dots,\xi_n$ satisfy the assumptions of Theorem 3. Suppose that for each k, r.v. ξ_k is a copy of r.v. $\xi:=(1+\mathcal{U})2^{\mathcal{G}}$, where $\mathcal{U}$, $\mathcal{G}$ are independent, $\mathcal{U}$ is uniformly distributed on interval $[0,1]$, and $\mathcal{G}$ is geometrically distributed with parameter $q\in(0,1)$, i.e., $\mathbf{P}(\mathcal{G}=l)=(1-q)q^l$, $l\in\mathbb{N}_0$. We derive the asymptotic formulas for

$$\mathbf{E}\bigl(\bigl(S_n^\xi\bigr)^\alpha\mathbf{1}_{\{S_n^\xi>x\}}\bigr)\quad\text{and}\quad\mathbf{E}\bigl(\bigl(S_n^\xi-x\bigr)^+\bigr)^\alpha$$

in the case of $0\leqslant\alpha<\log_2(1/q)$, where $S_n^\xi=\xi_1+\dots+\xi_n$ as usual.

Due to considerations on pages 122–123 of [5], $F_\xi\in\mathcal{C}\setminus\mathcal{R}$. In addition, for $x\geqslant 1$, we have

$$\begin{aligned}\overline{F}_\xi(x)&=\sum_{l=0}^\infty\mathbf{P}\biggl(\mathcal{U}>\frac{x}{2^l}-1\biggr)\mathbf{P}(\mathcal{G}=l)\\&=\sum_{\log_2 x-1<l\leqslant\log_2 x}\biggl(2-\frac{x}{2^l}\biggr)(1-q)q^l+\sum_{l>\log_2 x}(1-q)q^l\\&=\biggl(2-\frac{x}{2^{\lfloor\log_2 x\rfloor}}\biggr)(1-q)q^{\lfloor\log_2 x\rfloor}+q^{\lfloor\log_2 x\rfloor+1}\\&=q^{\log_2 x}\bigl(\bigl(2-2^{\langle\log_2 x\rangle}\bigr)(1-q)q^{-\langle\log_2 x\rangle}+q^{1-\langle\log_2 x\rangle}\bigr)\\&=x^{\log_2 q}\bigl(q^{-\langle\log_2 x\rangle}+(1-q)q^{-\langle\log_2 x\rangle}\bigl(1-2^{\langle\log_2 x\rangle}\bigr)\bigr)\\&=x^{\log_2 q}f\bigl(\langle\log_2 x\rangle\bigr),\end{aligned}$$

where symbol $\lfloor a\rfloor$ denotes the integer part of a real number a, symbol $\langle a\rangle$ denotes the fractional part of a, and function f is defined by the following equality

$$f(u)=q^{-u}+(1-q)q^{-u}\bigl(1-2^u\bigr),\quad 0\leqslant u<1.$$

For the function f, we have

$$f(0)=f(1-0)=1;\qquad f(u)\geqslant 1,\quad u\in[0,1);$$

$$f(u) \leqslant f(u_{\max}) = \frac{2-q}{1-\log_2 q} q^{\log_2((1-q)(1-1/\log_2 q)/(2-q))} := \mathrm{C}_q;$$
$$u_{\max} = \log_2 \frac{(2-q)\log\frac{1}{q}}{(1-q)\log\frac{2}{q}} \in (0,1).$$

Consequently, for $x \geqslant 1$,

$$x^{-\log_2(1/q)} \leqslant \overline{F}_\xi(x) \leqslant \mathrm{C}_q\, x^{-\log_2(1/q)},$$

$$\mathbf{E}\big(\xi^\alpha \mathbf{1}_{\{\xi>x\}}\big) \geqslant \frac{\log_2\frac{1}{q}}{\log_2\frac{1}{q}-\alpha}\, x^{\alpha-\log_2(1/q)}, \quad \alpha \in \left[0, \log_2\frac{1}{q}\right),$$
$$\mathbf{E}\big(\xi^\alpha \mathbf{1}_{\{\xi>x\}}\big) \leqslant \frac{\mathrm{C}_q\log_2\frac{1}{q}}{\log_2\frac{1}{q}-\alpha}\, x^{\alpha-\log_2(1/q)}, \quad \alpha \in \left[0, \log_2\frac{1}{q}\right),$$
$$\begin{aligned}\mathbf{E}\big((\xi-x)^+\big)^\alpha &\leqslant \alpha\,\mathrm{C}_q \int_x^\infty (u-x)^{\alpha-1} u^{\log_2 q}\,\mathrm{d}u \\ &= \alpha\,\mathrm{C}_q B\left(\alpha,\, \log_2\frac{1}{q}-\alpha\right) x^{\alpha-\log_2(1/q)}, \quad \alpha \in \left(0, \log_2\frac{1}{q}\right),\end{aligned}$$
$$\mathbf{E}\big((\xi-x)^+\big)^\alpha \geqslant \alpha B\left(\alpha,\, \log_2\frac{1}{q}-\alpha\right) x^{\alpha-\log_2(1/q)}, \quad \alpha \in \left(0, \log_2\frac{1}{q}\right),$$

where B denotes the Beta function

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t, \quad a>0,\ b>0.$$

These relations and theorems 3, 4 imply that

$$\frac{n\log_2\frac{1}{q}}{\log_2\frac{1}{q}-\alpha}\, x^{\alpha-\log_2(1/q)} \underset{x\to\infty}{\lesssim} \mathbf{E}\big(\big(S_n^\xi\big)^\alpha \mathbf{1}_{\{S_n^\xi>x\}}\big) \underset{x\to\infty}{\lesssim} \frac{n\mathrm{C}_q\log_2\frac{1}{q}}{\log_2\frac{1}{q}-\alpha}\, x^{\alpha-\log_2(1/q)}$$

for $n \in \mathbb{N}$, $q \in (0,1)$ and $\alpha \in [0, \log_2(1/q))$ and

$$\mathbf{E}\big(\big(S_n^\xi - x\big)^+\big)^\alpha \underset{x\to\infty}{\lesssim} n\alpha\,\mathrm{C}_q B\left(\alpha,\, \log_2\frac{1}{q}-\alpha\right) x^{\alpha-\log_2(1/q)},$$
$$\mathbf{E}\big(\big(S_n^\xi - x\big)^+\big)^\alpha \underset{x\to\infty}{\gtrsim} n\alpha\, B\left(\alpha,\, \log_2\frac{1}{q}-\alpha\right) x^{\alpha-\log_2(1/q)}$$

for all $n \in \mathbb{N}$, $q \in (0,1)$ and $\alpha \in (0, \log_2(1/q))$.

The derived asymptotic formulas imply the following particular cases:

$$\frac{n}{\log_2\frac{1}{q}-1}\, x^{1-\log_2(1/q)} \underset{x\to\infty}{\lesssim} \mathbf{E}\big(\big(S_n^\xi - x\big)^+\big) \underset{x\to\infty}{\lesssim} \frac{n\mathrm{C}_q}{\log_2\frac{1}{q}-1}\, x^{1-\log_2(1/q)}$$

if $q \in (0, 1/2)$;

$$\mathbf{E}\big(\big(S_n^\xi - x\big)^+\big)^2 \underset{x\to\infty}{\gtrsim} \frac{2n}{(\log_2 \frac{1}{q} - 1)(\log_2 \frac{1}{q} - 2)}\, x^{2-\log_2(1/q)},$$
$$\mathbf{E}\big(\big(S_n^\xi - x\big)^+\big)^2 \underset{x\to\infty}{\lesssim} \frac{2n\mathrm{C}_q}{(\log_2 \frac{1}{q} - 1)(\log_2 \frac{1}{q} - 2)}\, x^{2-\log_2(1/q)}$$

if $q \in (0, 1/4)$.

Example 2. Let r.v.s $\xi_1, \xi_2, \dots, \xi_n$, $n \geqslant 2$, be pQAI. Suppose that ξ_1 is distributed according to the following tail function

$$\overline{F}_{\xi_1}(x) = \exp\bigl\{-\lfloor \log(1+x)\rfloor + \bigl(\log(1+x) - \lfloor \log(1+x)\rfloor\bigr)^{1/2}\bigr\}, \quad x \geqslant 0.$$

For other indices $k \in \{2, \dots, n\}$, let us suppose that

$$\overline{F}_{\xi_k}(x) = \mathbf{1}_{\{x<0\}} + \mathrm{e}^{-x/k}\mathbf{1}_{\{x\geqslant 0\}}.$$

Like in Example 1, we write asymptotic formulas for the left truncated moments

$$\mathbf{E}\big(\big(S_n^\xi\big)^\alpha \mathbf{1}_{\{S_n^\xi > x\}}\big) \quad \text{and} \quad \mathbf{E}\big(\big(S_n^\xi - x\big)^+\big)^\alpha$$

in the case of suitable α.

It is obvious that $\mathbf{P}(|\xi_k| > x) = o(\overline{F}_{\xi_1}(x))$ for $k \in \{2, \dots, n\}$, and, further, $F_{\xi_1} \in \mathcal{C} \setminus \mathcal{R}$ due to results of [9] (see page 87).

Therefore, Theorem 4 implies that

$$\mathbf{E}\big(\big(S_n^\xi\big)^\alpha \mathbf{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\sim} \mathbf{E}\big(\xi_1^\alpha \mathbf{1}_{\{\xi_1 > x\}}\big), \quad \alpha \in [0, 1),$$

and

$$\mathbf{E}\big(\big(S_n^\xi - x\big)^+\big)^\alpha \underset{x\to\infty}{\sim} \mathbf{E}\big(\big(\xi_1 - x\big)^+\big)^\alpha, \quad \alpha \in (0, 1)$$

Consequently,

$$\mathbf{P}\big(S_n^\xi > x\big) \underset{x\to\infty}{\sim} \exp\bigl\{-\lfloor \log(1+x)\rfloor + \bigl(\log(1+x) - \lfloor \log(1+x)\rfloor\bigr)^{1/2}\bigr\},$$
$$\mathbf{P}\big(S_n^\xi > \mathrm{e}^n - 1\big) \underset{n\to\infty}{\sim} \frac{1}{\mathrm{e}^n},$$

and, for $\alpha \in (0, 1)$,

$$\frac{1}{1-\alpha}\, x^{\alpha-1} \underset{x\to\infty}{\lesssim} \mathbf{E}\big(\big(S_n^\xi\big)^\alpha \mathbf{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\lesssim} \frac{\mathrm{e}^2}{1-\alpha}\, x^{\alpha-1},$$
$$\frac{\alpha\pi}{\sin(\alpha\pi)}\, x^{\alpha-1} \underset{x\to\infty}{\lesssim} \mathbf{E}\big(\big(S_n^\xi - x\big)^+\big)^\alpha \underset{x\to\infty}{\lesssim} \frac{\alpha\pi\mathrm{e}^2}{\sin(\alpha\pi)}\, x^{\alpha-1}.$$

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