Nonlinear Analysis: Modelling and Control, Vol. 26, No. 5, 928–946
https://doi.org/10.15388/namc.2021.26.24176

Vilnius University Press

Existence of positive S-asymptotically periodic solutions of the fractional evolution equations in ordered Banach spaces*

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Received: October 11, 2020 / **Revised:** January 15, 2021 / **Published online:** September 1, 2021

Abstract. In this paper, we discuss the asymptotically periodic problem for the abstract fractional evolution equation under order conditions and growth conditions. Without assuming the existence of upper and lower solutions, some new results on the existence of the positive S-asymptotically ω-periodic mild solutions are obtained by using monotone iterative method and fixed point theorem. It is worth noting that Lipschitz condition is no longer needed, which makes our results more widely applicable.

Keywords: fractional evolution equation, S-asymptotically periodic solution, positive C_0-semigroup, positive mild solution, monotone iterative method.

1 Introduction

Let $(E, \|\cdot\|)$ be an ordered Banach space, whose positive cone $K := \{x \in E\colon x \geqslant \theta\}$ is a normal cone with normal constant N, θ is the zero element of E. In this paper, we discuss the positive S-asymptotically ω-periodic mild solutions for the following fractional evolution equation:

$$\begin{aligned} &{}^cD_t^q u(t) + Au(t) = F\bigl(t, u(t)\bigr), \quad t \geqslant 0, \\ &u(0) = u_0, \end{aligned} \tag{1}$$

*This work was supported by NNSF of China (11871302), China Postdoctoral Science Foundation (No. 2020M682140), NSF of Shanxi, China (201901D211399), and Scientific and Technologial Innovation Programs of Higher Education Institutions in Shanxi Province (No. 2020L0243).

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where ${}^cD_t^q$ is a Caputo fractional derivative of the order $q \in (0,1)$ with the lower limits zero, $A : D(A) \subset E \to E$ is a closed linear (not necessarily bounded) operator, and $-A$ generates a C_0-semigroup $T(t)$ $(t \geqslant 0)$ in E, $F : \mathbb{R}^+ \times E \to E$ is a given continuous function, which will be specified later.

In the past decades, in view of the wide practical background and application prospects of fractional calculus in physics, chemistry, engineering, biology, financial sciences, and other applied disciplines, numerous scholars pay more attention to fractional differential equations and have found that in many practical applications, fractional differential equations can more truthfully describe the process and phenomena of things' motion development than integer differential equations (see [1, 30, 40] and references therein). Since fractional evolution equations are abstract models from many practical applications, the study for fractional evolution equations has attracted more and more attention of mathematicians (see [6, 12, 17, 37, 38, 41] and references therein).

Recently, the periodicity problems or asymptotic periodicity problems have extensive physical background and realistic mathematical model, hence, it has been considerably developed and many properties of its solutions have been studied (see [3, 11, 13, 14, 18, 20, 21, 23–25, 31, 36, 39] and references therein). On the other hand, because fractional derivative has genetic or memory properties, the solutions of periodic boundary value problems for fractional differential equations cannot be extended periodically to time t in $\mathbb{R}^+$. Specially, the nonexistence of nontrivial periodic solutions of fractional evolution equations had been shown in [32]. In 2008, Henríquez et al. [15] formally introduced the concept of S-asymptotically ω-periodic function, which is a more general approximate period function. Since then, the S-asymptotically periodic functions have been widely studied in fractional evolution equations, and the existence and uniqueness of S-asymptotically ω-periodic solutions have been well studied (see [3, 9, 10, 19, 29, 32, 33, 35]). It is not difficult to find that in most of the above work, the Lipschitz-type conditions for nonlinear functions are necessary. In fact, for equations arising in complicated reaction–diffusion processes, the nonlinear function represents the source of material or population, which depends on time in diversified manners in many contexts.

It is well known that in many practice models, such as heat conduction equations, neutron transport equations, reaction diffusion equations, etc., only positive solutions are significant. But as far as we know, only a few scholars are concerned about the existence of positive solutions for fractional evolution equations on infinite interval (see [7, 8, 35]). In [7, 8], by means of the monotone iterative method Chen presented the existence and uniqueness of the positive mild solutions for the abstract fractional evolution equations under certain initial conditions. In [35], Shu studied a class of semilinear neutral fractional evolution equations with delay and obtained the existence and uniqueness of the positive S-asymptotically ω-periodic mild solutions by using contraction mapping principle in positive cone.

Inspired by the above literature, we will use a completely different method to improve and extend the results mentioned above, which will make up the research in this area blank. In Section 3, we investigate the positive S-asymptotically ω-periodic mild solutions for problem (1) under order conditions and growth conditions. Without assuming the existence of upper and lower solutions, some new results on the existence of the positive S-asymptotically ω-periodic mild solution are obtained by using monotone iterative method and fixed point theorem. It is worth noting that we no longer require nonlinear functions to satisfy Lipschitz condition, which makes our results more widely applicable. Thus, our conclusions are new in some respects. In Section 2, some notions, definitions, and preliminary facts are introduced, and at last, an example of time-fractional partial differential equation is given to illustrate the application of our results.

2 Preliminaries

In this paper, we always assume that $(E, \|\cdot\|)$ is an ordered Banach space, whose positive cone $K = \{u \in E\colon u \geqslant \theta\}$ is a normal cone with normal constant N, θ is the zero element of E.

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and nondecreasing function such that $h(t) \geqslant 1$ for all $t \in \mathbb{R}^+$ and $\lim_{t\to\infty} h(t) = \infty$. Thus, we can define a Banach space

$$C_h(E) = \left\{u \in C\big(\mathbb{R}^+, E\big)\colon \lim_{t\to\infty} \frac{\|u(t)\|}{h(t)} = 0\right\}$$

with the norm $\|u\|_h = \sup_{t\geqslant 0} \|u(t)\|/h(t)$. For the Banach space $C_h(E)$, we have the following result.

Lemma 1. (*See* [16].) *A set $B \subset C_h(E)$ is relatively compact in $C_h(E)$ if and only if*

(i) *B is equicontinuous*;
(ii) $\lim_{t\to\infty} \|u(t)\|/h(t) = 0$ *uniformly for $u \in B$*;
(iii) *The set $B(t) = \{u(t)\colon u \in B\}$ is relatively compact in E for every $t \geqslant 0$.*

Next, we introduce a standard definition of S-asymptotically ω-periodic function. Let $C_b(\mathbb{R}^+, E)$ denote the Banach space of bounded and continuous functions from $\mathbb{R}^+$ to E with the norm $\|u\|_C = \sup_{t\in\mathbb{R}^+} \|u(t)\|$.

Definition 1. (See [15].) A function $u \in C_b(\mathbb{R}^+, E)$ is called S-asymptotically ω-periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty} \|u(t+\omega) - u(t)\| = 0$. Thus, ω is called an asymptotic period of u.

Let $\mathrm{SAP}_\omega(E)$ be the subspace of $C_b(\mathbb{R}^+, E)$ consisting of all the E-valued S-asymptotically ω-periodic functions equipped with norm $\|\cdot\|_C$. Then $\mathrm{SAP}_\omega(E)$ is a Banach space [15].

Define a positive cone $K_h \subset C_h(E)$ by

$$K_h = \big\{u \in C_h(E)\colon u(t) \geqslant \theta \ \forall t \geqslant 0\big\}.$$

Thus, $C_h(E)$ is an ordered Banach space, whose partial order relation "$\leqslant$" is induced by the cone K_h.

Let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ ($t \geqslant 0$) in E. For a general C_0-semigroup $T(t)$ ($t \geqslant 0$), there exist $M \geqslant 1$ and $\nu \in \mathbb{R}$ such that (see [28])

$$\big\|T(t)\big\| \leqslant M\mathrm{e}^{\nu t}, \quad t \geqslant 0.$$

Specially, C_0-semigroup $T(t)(t \geqslant 0)$ is called to be uniformly bounded if

$$\|T(t)\| \leqslant M, \quad t \geqslant 0.$$

The growth exponent of $T(t)$ $(t \geqslant 0)$ is defined by

$$\nu_0 = \inf\bigl\{\nu \in \mathbb{R} \bigm| \exists\, M \geqslant 1\colon \ \|T(t)\| \leqslant M\mathrm{e}^{\nu t} \ \forall t \geqslant 0\bigr\}.$$

If $\nu_0 < 0$, then $T(t)$ $(t \geqslant 0)$ is said to be exponentially stable. Clearly, the exponentially stable C_0-semigroup $T(t)$ $(t \geqslant 0)$ is uniformly bounded. If C_0-semigroup $T(t)$ is continuous in the uniform operator topology for every $t > 0$ in E, it is well known that ν_0 can also be determined by $\sigma(A)$ (the resolvent set of A)

$$\nu_0 = -\inf\{\operatorname{Re}\lambda\colon \ \lambda \in \sigma(A)\},$$

where $-A$ is the infinitesimal generator of C_0-semigroup $T(t)$ $(t \geqslant 0)$. We know that $T(t)$ $(t \geqslant 0)$ is continuous in the uniform operator topology for $t > 0$ if $T(t)$ $(t \geqslant 0)$ is a compact semigroup. For more details about positive C_0-semigroups and compact semigroups, we can refer to [5, 22, 27, 34].

For the definition of Caputo fractional derivation, we can refer to many references (see [6, 37] and so on), so we will not repeat it here. Next, we define operator families $U(t)$ $(t \geqslant 0)$ and $V(t)$ $(t \geqslant 0)$ in E as following:

$$U(t) = \int_0^\infty \xi_q(s)T\bigl(t^q s\bigr)\,\mathrm{d}s, \qquad V(t) = q\int_0^\infty s\xi_q(s)T\bigl(t^q s\bigr)\,\mathrm{d}s, \tag{2}$$

where

$$\xi_q(s) = \frac{1}{\pi q}\sum_{n=1}^{\infty}(-s)^{n-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \qquad s \in (0,\infty)$$

is a probability density function satisfying

$$\begin{aligned} &\xi_q(s) \geqslant 0, \quad s \in (0,\infty),\\ &\int_0^\infty \xi_q(s)\,\mathrm{d}s = 1, \qquad \int_0^\infty s\xi_q(s)\,\mathrm{d}s = \frac{1}{\Gamma(1+q)}. \end{aligned} \tag{3}$$

Lemma 2. *The operator families $U(t)$ $(t \geqslant 0)$ and $V(t)$ $(t \geqslant 0)$ defined by* (2) *have the following properties*:

(i) *$U(t)$ $(t \geqslant 0)$ and $V(t)$ $(t \geqslant 0)$ are strongly continuous operators, i.e., for any $x \in E$ and $0 \leqslant t_1 \leqslant t_2$,*

$$\|U(t_2)x - U(t_1)x\| \to 0, \quad \|V(t_2)x - V(t_1)x\| \to 0 \quad \text{as } t_2 - t_1 \to 0. \tag{4}$$

(ii) *If $T(t)$ ($t \geqslant 0$) is uniformly bounded , then $U(t)$ and $V(t)$ are linear bounded operators for any fixed $t \in \mathbb{R}^+$, i.e.,*

$$\|U(t)x\| \leqslant M\|x\|, \quad \|V(t)x\| \leqslant \frac{M}{\Gamma(q)}\|x\| \quad \forall\, x \in E. \tag{5}$$

(iii) *If $T(t)$ ($t \geqslant 0$) is compact, then $U(t)$ and $V(t)$ are compact operators for every $t > 0$.*

(iv) *If $T(t)$ ($t \geqslant 0$) is equicontinuous, then $U(t)$ and $V(t)$ are uniformly continuous for $t > 0$.*

(v) *If $T(t)$ ($t \geqslant 0$) is positive, then $U(t)$ and $V(t)$ are positive operators.*

(vi) *If $T(t)$ ($t \geqslant 0$) is exponentially stable with the growth exponent $\nu_0 < 0$, then*

$$\|U(t)\| \leqslant ME_q(\nu_0 t^q), \qquad \|V(t)\| \leqslant ME_{q,q}(\nu_0 t^q) \tag{6}$$

for every $t \geqslant 0$, where $E_q(\cdot)$ and $E_{q,q}(\cdot)$ are the Mittag-Leffler functions.

Remark. The proof of statements (i)–(v) can be found in [6, 12, 37, 41], while the last one was proved in [4].

Definition 2. A function $u : [0, \infty) \to E$ is said to be a mild solution of problem (1) if $u \in C([0, \infty), E)$ and satisfies

$$u(t) = U(t)u(0) + \int_0^t (t-s)^{q-1} V(t-s) F(s, u(s))\, \mathrm{d}s$$

for all $t \geqslant 0$. Moreover, if $u(t) \geqslant \theta$ for all $t \geqslant 0$, then it is said to be a positive mild solution of problem (1).

In the proof, we also need the following lemma.

Lemma 3. (*See* [26].) *Let D be a convex, bounded and closed subset of a Banach space E. If $\mathcal{Q} : D \to D$ is a condensing map, then $\mathcal{Q}$ has a fixed poind in D.*

3 Main results

Theorem 1. *Let E be an ordered Banach space, whose positive cone K is normal cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate an exponentially stable, positive, and compact semigroup $T(t)$ ($t \geqslant 0$) in E, whose growth exponent denotes $\nu_0 < 0$, $u_0 \geqslant \theta$. Assume that $F : \mathbb{R}^+ \times E \to E$ is a continuous function and the following conditions hold*:

(H1) *There exist nonnegative constants $a \in (0, |\nu_0|/M)$ and $b \geqslant 0$ such that*

$$\|F(t, h(t)x)\| \leqslant a\|x\| + b, \quad t \geqslant 0,\ x \in E;$$

(H2) *F is nondecreasing with respect to the second variable, i.e., for $x_2 \geqslant x_1 \geqslant \theta$,*

$$F(t, x_2) \geqslant F(t, x_1) \geqslant \theta, \quad t \geqslant 0;$$

(H3) *There exists $\omega > 0$ such that*

$$\lim_{t\to\infty} \big\|F(t+\omega, x) - F(t, x)\big\| = 0, \quad t \geqslant 0,\ x \in E.$$

Then there exists a minimal positive S-asymptotically ω-periodic mild solution u^* of problem (1).

Proof. Define an operator $\mathcal{Q}$ on $C_h(E)$ by

$$\mathcal{Q}u(t) = U(t)u(0) + \int_0^t (t-s)^{q-1} V(t-s) F\big(s, u(s)\big)\,\mathrm{d}s, \quad t \geqslant 0. \tag{7}$$

It is easy to find $\mathcal{Q} : C_h(E) \to C_h(E)$ is well defined. In fact, for every $u \in C_h(E)$ and $t \geqslant 0$, we have $\|u(t)\| = h(t)\|u(t)\|/h(t) \leqslant h(t)\|u\|_h$. Hence, from (H1), (3), and (5) it follows that for $t \geqslant 0$,

$$\begin{aligned}
\frac{\|\mathcal{Q}u(t)\|}{h(t)} &\leqslant \frac{1}{h(t)}\left(\big\|U(t)u(0)\big\| + \left\|\int_0^t (t-s)^{q-1} V(t-s) F\big(s, u(s)\big)\,\mathrm{d}s\right\|\right) \\
&\leqslant \frac{1}{h(t)}\left(M\|u_0\| + \int_0^t (t-s)^{q-1} \big\|V(t-s)\big\| \big\|F\big(s, u(s)\big)\big\|\,\mathrm{d}s\right) \\
&\leqslant \frac{M}{h(t)}\left(\|u_0\| + \int_0^t \int_0^\infty q\sigma\xi_q(\sigma)(t-s)^{q-1} \mathrm{e}^{\nu_0 (t-s)^q \sigma} \big(a\|u\|_h + b\big)\,\mathrm{d}\sigma\,\mathrm{d}s\right) \\
&\leqslant \frac{M}{h(t)}\left(\|u_0\| + \big(a\|u\|_h + b\big) \int_0^\infty \xi_q(\sigma)\,\mathrm{d}\sigma \int_0^\infty \mathrm{e}^{\nu_0 s}\,\mathrm{d}s\right) \\
&= \frac{M}{h(t)}\left(\|u_0\| + \frac{b}{|\nu_0|} + \frac{a\|u\|_h}{|\nu_0|}\right),
\end{aligned} \tag{8}$$

which implies that $\mathcal{Q} : C_h(E) \to C_h(E)$ is well defined.

It is easy to show that $\mathcal{Q} : C_h(E) \to C_h(E)$ is continuous. Let $\{u_n\} \subset C_h(E)$ and $u_n \to u$ in $C_h(E)$ as $n \to \infty$, that is, for arbitrary $\varepsilon > 0$, there exists sufficiently large n such that $\|u_n - u\|_h < \varepsilon$ for. For above ε, by the continuity of F it is easy to see

$$\big\|F\big(t, u_n(t)\big) - F\big(t, u(t)\big)\big\| \leqslant \frac{|\nu_0|\varepsilon}{M}, \quad t \geqslant 0, \tag{9}$$

and the dominated convergence theorem ensures

$$\begin{aligned}
&\frac{\|\mathcal{Q}u_n(t) - \mathcal{Q}u(t)\|}{h(t)} \\
&\quad \leqslant \frac{1}{h(t)} \int\limits_0^t (t-s)^{q-1} \|V(t-s)\| \|F(s, u_n(s)) - F(s, u(s))\| \,\mathrm{d}s \\
&\quad \leqslant \frac{|\nu_0|\varepsilon}{Mh(t)} \int\limits_0^t (t-s)^{q-1} \|V(t-s)\| \,\mathrm{d}s \leqslant \varepsilon.
\end{aligned}$$

Hence, we conclude that $\mathcal{Q}$ is continuous from $C_h(E)$ to $C_h(E)$. Specially, by (8) one can find that for every $u \in C_h(E)$,

$$\|Qu\|_h \leqslant M\|u_0\| + \frac{Mb}{|\nu_0|} + \frac{Ma}{|\nu_0|}\|u\|_h = \gamma + \beta\|u\|_h, \tag{10}$$

where $\gamma = M\|u_0\| + Mb/|\nu_0|$ and $\beta = Ma/|\nu_0| < 1$. Therefore, from Definition 2 it follows that the fixed points of $\mathcal{Q}$ are mild solutions to problem (1).

Next, we will prove that $\mathcal{Q}(\mathrm{SAP}_\omega(E)) \subset \mathrm{SAP}_\omega(E)$. Choose $u \in \mathrm{SAP}_\omega(E)$, then for any $\varepsilon > 0$, there exists a constant $t_{\varepsilon,1} > 0$ such that $\|u(t+\omega) - u(t)\| \leqslant \varepsilon$ for all $t \geqslant t_{\varepsilon,1}$. Thus, by continuity of F we have

$$\|F(t, u(t+\omega)) - F(t, u(t))\| \leqslant \frac{|\nu_0|}{M}\varepsilon \quad \forall t \geqslant t_{\varepsilon,1}. \tag{11}$$

On the other hand, by (H3) there is a sufficiently large constant $t_{\varepsilon,2}$ such that for $t \geqslant t_{\varepsilon,2}$,

$$\|F(t+\omega, u(t+\omega)) - F(t, u(t+\omega))\| \leqslant \frac{|\nu_0|}{M}\varepsilon. \tag{12}$$

Hence, for $t > t_\varepsilon := \max\{t_{\varepsilon,1}, t_{\varepsilon,2}\}$, from (7) it follows that

$$\begin{aligned}
&\mathcal{Q}u(t+\omega) - \mathcal{Q}u(t) \\
&\quad = U(t+\omega)u(0) + \int\limits_0^{t+\omega} (t+\omega-s)^{q-1} V(t+\omega-s) F(s, u(s)) \,\mathrm{d}s \\
&\qquad - U(t)u(0) - \int\limits_0^t (t-s)^{q-1} V(t-s) F(s, u(s)) \,\mathrm{d}s \\
&\quad = U(t+\omega)u(0) - U(t)u(0) \\
&\qquad + \int\limits_0^{\omega} (t+\omega-s)^{q-1} V(t+\omega-s) F(s, u(s)) \,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t (t-s)^{q-1} V(t-s)\big(F(s,\, u(s+\omega)) - F\big(s, u(s)\big)\big)\,\mathrm{d}s \\
&+ \int_0^t (t-s)^{q-1} V(t-s)\big(F(s+\omega,\, u(s+\omega)) - F\big(s,\, u(s+\omega)\big)\,\mathrm{d}s \\
&:= I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{aligned}
$$

According to (6), let

$$
M_0 = M \max\Big\{ \sup_{t\geqslant 0} E_q\big(\nu_0 t^q\big)(1+t)^q,\, \sup_{t\geqslant 0} E_{q,q}\big(\nu_0 t^q\big)(1+t)^{2q} \Big\},
$$

then

$$
\big\|U(t)\big\| \leqslant \frac{M_0}{(1+t)^q}, \quad \big\|V(t)\big\| \leqslant \frac{M_0}{(1+t)^{2q}}, \quad t \in \mathbb{R}^+. \tag{13}
$$

Thus, by (13) and (H1) one can find that

$$
\begin{aligned}
\big\|I_1(t)\big\| &\leqslant \big\|U(t+\omega)u(0)\big\| + \big\|U(t)u(0)\big\| \leqslant \big(\big\|U(t+\omega)\big\| + \big\|U(t)\big\|\big)\|u_0\| \\
&\leqslant \frac{2M_0\|u_0\|}{(1+t)^q}
\end{aligned}
$$

and

$$
\begin{aligned}
\big\|I_2(t)\big\| &\leqslant \int_0^\omega (t+\omega-s)^{q-1}\big\|V(t+\omega-s)\big\|\big\|F\big(s,u(s)\big)\big\|\,\mathrm{d}s \\
&\leqslant \int_0^\omega (t+\omega-s)^{q-1}\frac{M_0(a\|u(s)\|+b)}{(1+t+\omega-s)^{2q}}\,\mathrm{d}s \\
&\leqslant M_0\big(a\|u\|_C+b\big)\frac{((t+\omega)^q - t^q)}{q(1+t)^{2q}} \leqslant M_0\big(a\|u\|_C+b\big)\frac{\omega^q}{q(1+t)^{2q}}.
\end{aligned}
$$

Hence, we deduce that $\|I_1\|, \|I_2\|$ tend to 0 as $t \to \infty$. By (13), (11), and (H1) one can obtain that

$$
\begin{aligned}
\big\|I_3(t)\big\| &\leqslant \int_0^{t_\varepsilon} (t-s)^{q-1}\big\|V(t-s)\big\|\big\|F\big(s,\, u(s+\omega)\big) - F\big(s, u(s)\big)\big\|\,\mathrm{d}s \\
&\quad + \int_{t_\varepsilon}^t (t-s)^{q-1}\big\|V(t-s)\big\|\big\|F\big(s,\, u(s+\omega)\big) - F\big(s,\, u(s+\omega)\big)\big\|\,\mathrm{d}s \\
&\leqslant 2M_0\big(a\|u\|_C+b\big)\int_0^{t_\varepsilon}\frac{(t-s)^{q-1}}{(1+t-s)^{2q}}\,\mathrm{d}s + \int_{t_\varepsilon}^t (t-s)^{q-1}\big\|V(t-s)\big\|\,\mathrm{d}s\frac{|\nu_0|\varepsilon}{M}
\end{aligned}
$$

$$\begin{aligned}
&\leqslant 2M_0(a\|u\|_C+b)\int_0^{t_\varepsilon}(t-s)^{-q-1}\,\mathrm{d}s+\int_0^t(t-s)^{q-1}\big\|V(t-s)\big\|\,\mathrm{d}s\frac{|\nu_0|\varepsilon}{M}\\
&\leqslant 2M_0\big(a\|u\|_C+b\big)\frac{(t-t_\varepsilon)^{-q}-t^{-q}}{q}\\
&\quad+q\int_0^t\int_0^\infty\sigma\xi_q(\sigma)(t-s)^{q-1}\mathrm{e}^{-|\nu_0|(t-s)^q\sigma}\,\mathrm{d}\sigma\,\mathrm{d}s\,|\nu_0|\varepsilon\\
&\leqslant 2M_0\big(a\|u\|_C+b\big)\frac{(t-t_\varepsilon)^{-q}-t^{-q}}{q}+\int_0^\infty\xi_q(\sigma)\,\mathrm{d}\sigma\int_0^\infty\mathrm{e}^{-|\nu_0|s}\,\mathrm{d}s\,|\nu_0|\varepsilon\\
&\leqslant 2M_0\big(a\|u\|_C+b\big)\frac{(t-t_\varepsilon)^{-1}-t^{-q}}{q}+\varepsilon,
\end{aligned}$$

which implies that $\|I_3(t)\|$ tends to 0 as $t\to\infty$. Similarly, by (13), (12), and (H1) we can get that $\|I_4(t)\|$ tends to 0 as $t\to\infty$.

Thus, from the above results we can deduce that

$$\lim_{t\to\infty}\big\|\mathcal{Q}u(t+\omega)-\mathcal{Q}u(t)\big\|=0,$$

namely, $\mathcal{Q}u\in\mathrm{SAP}_\omega(E)$, which implies that $\mathcal{Q}(\mathrm{SAP}_\omega(E))\subset\mathrm{SAP}_\omega(E)$.

Now, we prove the existence of positive solutions by monotone iterative technique. For any $u,v\in K_h$ with $u\leqslant v$, by (H2), (7), the positivity of $U(t)$, $V(t)$, and $u_0\geqslant\theta$, one can find that for all $t\in[0,\infty)$,

$$\theta\leqslant\mathcal{Q}u(t)\leqslant\mathcal{Q}v(t).$$

Thus $\mathcal{Q}$ is monotone increasing.

Let $v_0(t)\equiv\theta$. Clearly, $v_0\in K_h\cap\mathrm{SAP}_\omega(E)$. Now, define a sequence $\{v_i\}$ by

$$v_i=\mathcal{Q}v_{i-1},\quad i=1,2,\dots. \tag{14}$$

From the definition and properties of $\mathcal{Q}$, (14), (10) one can find $\{v_i\}\subset K_h\cap\mathrm{SAP}_\omega(E)$ and

$$\begin{gathered}
v_0\leqslant v_1\leqslant\cdots\leqslant v_i\leqslant\cdots,\\
\|v_i\|_h\leqslant\gamma+\beta\|v_{i-1}\|_h.
\end{gathered}\tag{15}$$

Since $\|v_0\|_h\equiv 0$, from (15) it follows that

$$\|v_i\|_h\leqslant\gamma+\beta\gamma+\cdots+\beta^{i-1}\gamma=\gamma\frac{1-\beta^i}{1-\beta}\leqslant\frac{\gamma}{1-\beta}, \tag{16}$$

thus, the sequence $\{v_i\}$ is uniformly bounded. Next, we prove that the sequence $\{v_i\}$ is uniformly convergent.

Firstly, $\{v_i(t)\}$ is relatively compact on E for $t \in [0,\infty)$. Let $\mathcal{V} = \{v_i\}$ and $\mathcal{V}_0 = \mathcal{V} \cup \{v_0\}$. Obviously, $\mathcal{V}(t) = (\mathcal{Q}\mathcal{V}_0)(t)$ for $t \in [0,\infty)$. For arbitrary $r_0 \in [0,\infty)$, one can obtain that $\{v_i(t)\}$ is relatively compact on E for $t \in [0, r_0]$. In fact, for all $\varepsilon \in (0,t)$ and for all $\delta > 0$, we define a set $\mathcal{Q}_{\varepsilon,\delta}\mathcal{V}_0(t)$ by

$$\mathcal{Q}_{\varepsilon,\delta}\mathcal{V}_0(t) := \bigl\{\mathcal{Q}_{\varepsilon,\delta}v_i(t)\colon\ v_i \in \mathcal{V}_0,\ t \in [0, r_0]\bigr\},$$

where

$$\begin{aligned}
&\mathcal{Q}_{\varepsilon,\delta}v_i(t)\\
&\quad= U(t)v_{i-1}(0) + q\int_0^{t-\varepsilon}\int_\delta^\infty \tau(t-s)^{q-1}\xi_q(\tau)T\bigl((t-s)^q\tau\bigr)F\bigl(s, v_{i-1}(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s\\
&\quad= U(t)v_{i-1}(0)\\
&\qquad+ qT\bigl(\varepsilon^q\delta\bigr)\int_0^{t-\varepsilon}\int_\delta^\infty \tau(t-s)^{q-1}\xi_q(\tau)T\bigl((t-s)^q\tau - \varepsilon^q\delta\bigr)F\bigl(s, v_{i-1}(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}$$

By the compactness of $U(t)$ and $T(\varepsilon^q\delta)$ the set $\mathcal{Q}_{\varepsilon,\delta}\mathcal{V}_0(t)$ is relatively compact in E. Moreover, for every $v_i \in \mathcal{V}_0$ and $t \in [0, r_0]$, one has

$$\begin{aligned}
&\|\mathcal{Q}v_i(t) - \mathcal{Q}_{\varepsilon,\delta}v_i(t)\|\\
&\quad\leqslant \left\|q\int_0^t\int_0^\delta \tau(t-s)^{q-1}\xi_q(\tau)T\bigl((t-s)^q\tau\bigr)F\bigl(s, v_{i-1}(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s\right\|\\
&\qquad+ \left\|q\int_{t-\varepsilon}^t\int_\delta^\infty \tau(t-s)^{q-1}\xi_q(\tau)T\bigl((t-s)^q\tau\bigr)F\bigl(s, v_{i-1}(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s\right\|\\
&\quad\leqslant qM\biggl(\frac{a\gamma}{1-\beta} + b\biggr)\int_0^t (t-s)^{q-1}\,\mathrm{d}s\int_0^\delta \tau\xi_q(\tau)\,\mathrm{d}\tau\\
&\qquad+ qM\biggl(\frac{a\gamma}{1-\beta} + b\biggr)\int_{t-\varepsilon}^t (t-s)^{q-1}\,\mathrm{d}s\int_\delta^\infty \tau\xi_q(\tau)\,\mathrm{d}\tau\\
&\quad\leqslant M\biggl(\frac{a\gamma}{1-\beta} + b\biggr)\biggl(r_0^q\int_0^\delta \tau\xi_q(\tau)\,\mathrm{d}\tau + \frac{\varepsilon^q}{\Gamma(1+q)}\biggr)\\
&\quad\to 0 \quad \text{as } \varepsilon \to 0,\ \delta \to 0.
\end{aligned}$$

Hence, the set $(\mathcal{Q}\mathcal{V}_0)(t)$ is relatively compact, which implies that $\{v_i(t)\}$ is relatively compact on E for $t \in [0, r_0]$. Therefore, by the definition of $\mathcal{Q}$ and the arbitrariness of r_0 we can obtain that $\{v_i(t)\}$ is relatively compact on E for $t \in [0,\infty)$.

Secondly, $\{v_i\} \subset K_h \cap \mathrm{SAP}_\omega(E)$ is equicontinuous in $[0, \infty)$. In general, let $0 \leqslant t_1 < t_2$. For any $u \in \{v_i\}$, by the definition of Q one can see

$$\begin{aligned}
&\big\|\mathcal{Q}u(t_2) - \mathcal{Q}u(t_1)\big\| \\
&\quad = \left\| U(t_2)u(0) + \int_0^{t_2} (t_2 - s)^{q-1} V(t_2 - s) F\big(s, u(s)\big)\,\mathrm{d}s \right. \\
&\qquad \left. - U(t_1)u(0) - \int_0^{t_1} (t_1 - s)^{q-1} V(t_1 - s) F\big(s, u(s)\big)\,\mathrm{d}s \right\| \\
&\quad \leqslant \big\|U(t_2)u(0) - U(t_1)u(0)\big\| \\
&\qquad + \int_0^{t_1} \big((t_2 - s)^{q-1} - (t_1 - s)^{q-1}\big) \big\|V(t_2 - s)\big\| \big\|F\big(s, u(s)\big)\big\|\,\mathrm{d}s \\
&\qquad + \int_0^{t_1} (t_1 - s)^{q-1} \big\|V(t_2 - s) - V(t_1 - s)\big\| \big\|F\big(s, u(s)\big)\big\|\,\mathrm{d}s \\
&\qquad + \int_{t_1}^{t_2} (t_2 - s)^{q-1} \big\|V(t_2 - s)\big\| \big\|F\big(s, u(s)\big)\big\|\,\mathrm{d}s \\
&\quad := J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Next, we check if $\|J_i\|$ $(i = 1, 2, 3, 4)$ tend to 0 as $t_2 - t_1 \to 0$ independently of $u \in \{v_i\}$. From (4) it follows that $J_1 \to 0$ as $t_2 - t_1 \to 0$. By (H1), (13), and (16) we can obtain

$$\begin{aligned}
J_2 &= \int_0^{t_1} \big((t_2 - s)^{q-1} - (t_1 - s)^{q-1}\big) \big\|V(t_2 - s)\big\| \big\|F\big(s, u(s)\big)\big\|\,\mathrm{d}s \\
&\leqslant M_0 \int_0^{t_1} \frac{(t_1 - s)^{q-1} - (t_2 - s)^{q-1}}{(1 + t_2 - s)^{2q}} \big(a\|u\|_h + b\big)\,\mathrm{d}s \\
&\leqslant \frac{M_0}{q} \left(\frac{a\gamma}{1 - \beta} + b\right) \frac{t_1^q - t_2^q + (t_2 - t_1)^q}{(1 + t_2 - t_1)^{2q}} \\
&\leqslant \frac{2M_0}{q} \left(\frac{a\gamma}{1 - \beta} + b\right) (t_2 - t_1)^q \to 0 \quad \text{as } t_2 - t_1 \to 0,
\end{aligned}$$

$$\begin{aligned}
J_4 &\leqslant M_0 \left(\frac{a\gamma}{1 - \beta} + b\right) \int_{t_1}^{t_2} (t_2 - s)^{q-1}\,\mathrm{d}s \\
&= \frac{M_0}{q} \left(\frac{a\gamma}{1 - \beta} + b\right) (t_2 - t_1)^q \to 0 \quad \text{as } t_2 - t_1 \to 0.
\end{aligned}$$

If $t_1 = 0$ and $t_2 > 0$, then it easy to see that $J_3 = 0$. For $t_1 > 0$ and $\epsilon > 0$ small enough, by (H1), (13), (16), and Lemma 2(iv) we get that

$$\begin{aligned} J_3 &= \int_0^{t_1} (t_1 - s)^{q-1} \big\| V(t_2 - s) - V(t_1 - s) \big\| \big\| F\big(s, u(s)\big) \big\| \, \mathrm{d}s \\ &\leqslant \left(\frac{a\gamma}{1-\beta} + b \right) \int_0^{t_1-\epsilon} (t_1 - s)^{q-1} \big\| V(t_2 - s) - V(t_1 - s) \big\| \, \mathrm{d}s \\ &\quad + \left(\frac{a\gamma}{1-\beta} + b \right) \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{q-1} \big\| V(t_2 - s) - V(t_1 - s) \big\| \, \mathrm{d}s \\ &\leqslant \left(\frac{a\gamma}{1-\beta} + b \right) \sup_{s \in [0, t_1 - \epsilon]} \big\| V(t_2 - s) - V(t_1 - s) \big\| \int_0^{t_1-\epsilon} (t_1 - s)^{q-1} \, \mathrm{d}s \\ &\quad + 2M_0 \left(\frac{a\gamma}{1-\beta} + b \right) \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{q-1} \, \mathrm{d}s \\ &\leqslant \left(\frac{a\gamma}{1-\beta} + b \right) \left(\sup_{s \in [0, t_1 - \epsilon]} \big\| V(t_2 - s) - V(t_1 - s) \big\| \frac{t_1^q - \epsilon^q}{q} + \frac{2M_0 \epsilon^q}{q} \right) \\ &\to 0 \quad \text{as } \epsilon \to 0,\ t_2 - t_1 \to 0. \end{aligned}$$

As a result, $\|\mathcal{Q}u(t_2) - \mathcal{Q}u(t_1)\|$ tends to 0 as $t_2 - t_1 \to 0$ independently of $u \in \{v_i\}$, thus $\{v_i\}$ is equicontinuous.

Thirdly, for every $u \in \{v_i\}$, by (8) and (16) one can find

$$\frac{\|\mathcal{Q}u(t)\|}{h(t)} \leqslant \frac{M}{h(t)} \left(\|u_0\| + \frac{b}{|\nu_0|} + \frac{a\gamma}{|\nu_0|(1-\beta)} \right), \quad t \geqslant 0,$$

which implies that $\|\mathcal{Q}u(t)\|/h(t) \to 0$ as $t \to \infty$ uniformly for $u \in \{v_i\}$. Therefore, Lemma 1 allows us to deduce that $\{v_i\}$ is relatively compact in $K_h \cap \mathrm{SAP}_\omega(E)$, thus there is convergent subsequence in $\{v_i\}$. From the monotonicity of sequence and the normality of cone we can obtain that $\{v_i\}$ itself is uniformly convergent, which means that there is $u^* \in K_h \cap \mathrm{SAP}_\omega(E)$ such that $\lim_{i \to \infty} v_i = u^*$.

Taking the limit in (14), one can see $u^* = \mathcal{Q}u^*$. Thus $u^* \in K_h \cap \mathrm{SAP}_\omega(E)$ is fixed point of $\mathcal{Q}$, which is a positive S-asymptotically ω-periodic mild solution of problem (1). It is easy to prove that u^* is the minimal positive mild solution. To this end, let $\widetilde{u} \in K_h \cap \mathrm{SAP}_\omega(E)$ be a positive S-asymptotically ω-periodic mild solution of problem (1), namely, $\widetilde{u}(t) = \mathcal{Q}\widetilde{u}(t)$ for every $t \in [0, \infty)$. Obviously, $\widetilde{u}(t) \geqslant v_0$, and from the monotonicity of Q it follows that

$$\widetilde{u}(t) = (\mathcal{Q}\widetilde{u})(t) \geqslant (\mathcal{Q}v_0)(t) = v_1(t), \tag{17}$$

namely, $\widetilde{u} \geqslant v_1$. In general, $\widetilde{u} \geqslant v_i, \quad i = 1, 2, \dots$. Taking the limit in (17) as $i \to \infty$, we have $\widetilde{u} \geqslant u^*$, which means that u^* is the minimal positive S-asymptotically ω-periodic mild solution of problem (1). □

Next, we always assume that the positive cone K is regeneration cone. By the characteristic of positive semigroups (see [22]), for sufficiently large $\lambda_0 > -\inf\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\}$, we have that $\lambda_0 I + A$ has positive bounded inverse operator $(\lambda_0 I + A)^{-1}$. Since $\sigma(A) \neq \emptyset$, the spectral radius

$$r\big((\lambda_0 I + A)^{-1}\big) = \frac{1}{\operatorname{dist}(-\lambda_0, \sigma(A))} > 0.$$

By the famous Krein–Rutmann theorem A has the smallest eigenvalue $\lambda_1 > 0$, which has a positive eigenfunction e_1, and

$$\lambda_1 = \inf\big\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\big\},$$

which implies that $\nu_0 = -\lambda_1$. Hence, by Theorem 1 we have the following results.

Corollary 1. *Let E be an ordered Banach space, whose positive cone K is a regeneration cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate an exponentially stable, positive, and compact semigroup $T(t)$ $(t \geqslant 0)$ in E, $u_0 \geqslant \theta$. Assume that $F : \mathbb{R}^+ \times E \to E$ is a continuous function, and let conditions* (H2), (H3)*, and*

(H1′) *there exist nonnegative constants $a \in (0, \lambda_1/M)$ and $b \geqslant 0$ such that*

$$\big\|F\big(t, h(t)x\big)\big\| \leqslant a\|x\| + b, \quad t \geqslant 0,\ x \in E.$$

hold. Then problem (1) *has a minimal positive S-asymptotically ω-periodic mild solution u^*.*

Theorem 2. *Let E be an ordered Banach space, whose positive cone K is regeneration cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate an exponentially stable, positive, and compact semigroup $T(t)$ $(t \geqslant 0)$ in E. Assume that $F : \mathbb{R}^+ \times E \to E$ is a continuous function, and let conditions* (H1′), (H3)*,*

(H4) *for any $u \in C_h(E)$ with $u(t) \geqslant \sigma e_1$, there is a constant $\sigma > 0$ such that*

$$F\big(t, u(t)\big) \geqslant F(t, \sigma e_1) \geqslant \lambda_1 \sigma e_1, \quad t \geqslant 0,$$

hold and $u_0 \geqslant \sigma e_1$. Then problem (1) *has at least one positive S-asymptotically ω-periodic mild solution.*

Proof. Let $\mathcal{Q}$ be defined by (7). From the proof of Theorem 1 it follows

$$\mathcal{Q}\big(\mathrm{SAP}_\omega(E)\big) \subset \mathrm{SAP}_\omega(E).$$

Since $a \in (0, \lambda_1/M)$, we can choose $R_0 \geqslant M(\lambda_1\|u_0\| + b)/(\lambda_1 - aM)$. Denote

$$\Omega_{R_0} := \big\{u \in C_h(E)\colon \|u\|_h \leqslant R_0,\ u(t) \geqslant \sigma e_1,\ t \geqslant 0\big\}. \tag{18}$$

Then $\varOmega_{R_0} \subset C_h(E)$ is a nonempty bounded convex closed set. Hence, for any $u \in \varOmega_{R_0}$ and $t \geqslant 0$, from (H1$'$), (3), and (5) the following holds:

$$\begin{aligned}
\frac{\|\mathcal{Q}u(t)\|}{h(t)} &\leqslant \bigl\|U(t)u(0)\bigr\| + \left\|\int_0^t (t-s)^{q-1}V(t-s)F\bigl(s,u(s)\bigr)\,\mathrm{d}s\right\| \\
&\leqslant M\|u_0\| + \int_0^t (t-s)^{q-1}\bigl\|V(t-s)\bigr\|\bigl\|F\bigl(s,u(s)\bigr)\bigr\|\,\mathrm{d}s \\
&\leqslant M\|u_0\| + M\int_0^t\int_0^\infty \sigma\xi_q(\sigma)(t-s)^{q-1}\mathrm{e}^{\nu_0(t-s)^q\sigma}\bigl(a\|u\|_h+b\bigr)\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\leqslant M\|u_0\| + M\bigl(a\|u\|_h+b\bigr)\int_0^\infty \xi_q(\sigma)\,\mathrm{d}\sigma\int_0^\infty \mathrm{e}^{\nu_0 s}\,\mathrm{d}s \\
&= M\left(\|u_0\| + \frac{b}{\lambda_1} + \frac{a\|u\|_h}{\lambda_1}\right) \leqslant R_0.
\end{aligned}$$

Let $w_0 \equiv \sigma e_1$. Then $w_0(t) = \sigma e_1$ for any $t \geqslant 0$, and

$$g(t) :=^c D_t^q w_0(t) + Aw_0(t) = \lambda_1\sigma e_1 \leqslant F(t,\sigma e_1), \quad t \geqslant 0.$$

By the positivity of semigroup $T(t)$ ($t \geqslant 0$), condition (H4), and (7), for any $u \in \varOmega_{R_0}$ and $t \geqslant 0$, one can obtain that

$$\begin{aligned}
\sigma e_1 = w_0(t) &= U(t)w_0(0) + \int_0^t (t-s)^{q-1}V(t-s)g(s)\,\mathrm{d}s \\
&\leqslant U(t)\sigma e_1 + \int_0^t (t-s)^{q-1}V(t-s)F(s,\sigma e_1)\,\mathrm{d}s \\
&\leqslant U(t)u_0 + \int_0^t (t-s)^{q-1}V(t-s)F\bigl(s,u(s)\bigr)\,\mathrm{d}s = (\mathcal{Q}u)(t).
\end{aligned}$$

Thus, $\mathcal{Q}(\varOmega_{R_0}) \subset \varOmega_{R_0}$ and $(\mathcal{Q}u)(t) \geqslant \sigma e_1$ for any $u \in \varOmega_{R_0}$ and $t \geqslant 0$.

Next, we show that $\mathcal{Q}$ is completely continuous. From assumptions (H1$'$) and (H3) there is a constant M_1 such that for all $u \in \varOmega_{R_0}$,

$$\sup_{t\geqslant 0}\bigl\|F\bigl(t,u(t)\bigr)\bigr\| \leqslant M_1. \tag{19}$$

Thus, it is easy to verify that the set

$$\mathcal{Q}(\varOmega_{R_0,r_0})(t) := \bigl\{(\mathcal{Q}u)(t)\colon\ u \in \varOmega_{R_0},\ t \in [0,r_0]\bigr\}$$

is relatively compact on E for any $r_0 \in (0,\infty)$. In order to do this, we define a set

$$\mathcal{Q}_{\varepsilon,\delta}(\Omega_{R_0,r_0})(t) := \bigl\{(\mathcal{Q}_{\varepsilon,\delta}u)(t)\colon\ u \in \Omega_{R_0},\ t \in [0,r_0]\bigr\},$$

where $\varepsilon \in (0,t)$, $\delta > 0$, and

$$\begin{aligned}
&(\mathcal{Q}_{\varepsilon,\delta}u)(t)\\
&\quad= U(t)u(0) + q\int_0^{t-\varepsilon}\int_\delta^\infty \tau(t-s)^{q-1}\xi_q(\tau)T\bigl((t-s)^q\tau\bigr)F\bigl(s,u(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s\\
&\quad= U(t)u_0 + qT(\varepsilon^q\delta)\int_0^{t-\varepsilon}\int_\delta^\infty \tau(t-s)^{q-1}\xi_q(\tau)T\bigl((t-s)^q\tau-\varepsilon^q\delta\bigr)F\bigl(s,u(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}$$

By the compactness of $U(t)$ and $T(\varepsilon^q\delta)$ the set $\mathcal{Q}_{\varepsilon,\delta}(\Omega_{R_0,r_0})(t)$ is relatively compact in E. Right now, for every $u \in \Omega_{R_0}$ and $t \in [0,r_0]$, one has

$$\begin{aligned}
&\bigl\|\mathcal{Q}u(t) - \mathcal{Q}_{\varepsilon,\delta}u(t)\bigr\|\\
&\quad\leqslant \Biggl\|q\int_0^t\int_\delta^\infty \tau(t-s)^{q-1}\xi_q(\tau)T\bigl((t-s)^q\tau\bigr)F\bigl(s,u(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s\Biggr\|\\
&\qquad+ \Biggl\|q\int_{t-\varepsilon}^t\int_\delta^\infty \tau(t-s)^{q-1}\xi_q(\tau)T\bigl((t-s)^q\tau\bigr)F\bigl(s,u(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s\Biggr\|\\
&\quad\leqslant qMM_1\Biggl(\int_0^t(t-s)^{q-1}\,\mathrm{d}s\int_0^\delta \tau\xi_q(\tau)\,\mathrm{d}\tau + \int_{t-\varepsilon}^t(t-s)^{q-1}\,\mathrm{d}s\int_\delta^\infty \tau\xi_q(\tau)\,\mathrm{d}\tau\Biggr)\\
&\quad\leqslant MM_1\Biggl(r_0^q\int_0^\delta \tau\xi_q(\tau)\,\mathrm{d}\tau + \frac{\varepsilon^q}{\Gamma(1+q)}\Biggr)\\
&\quad\to 0 \quad \text{as } \varepsilon\to 0,\ \delta\to 0.
\end{aligned}$$

Hence, the set $\mathcal{Q}(\Omega_{R_0,r_0})(t)$ is relatively compact on E for $t \in [0,r_0]$. Therefore, by the definition of $\mathcal{Q}$ and the arbitrariness of r_0 we can deduce that $\mathcal{Q}(\Omega_{R_0})(t)$ is relatively compact on E for $t \in [0,\infty)$. Moreover, it is easy to prove that set $\mathcal{Q}(\Omega_{R_0})$ is equicontinuous by using the method similar to Theorem 1. Also, for every $u \in \Omega_{R_0}$, by (8) and (19) one can find

$$\frac{\|\mathcal{Q}u(t)\|}{h(t)} \leqslant \frac{M}{h(t)}\biggl(\|u_0\| + \frac{b}{\lambda_1} + \frac{aR_0}{\lambda_1}\biggr), \quad t \geqslant 0,$$

which implies that $\|\mathcal{Q}u(t)\|/h(t) \to 0$ as $t\to\infty$ uniformly for $u \in \overline{\Omega}_{R_0}$. Now, we can assert that $\mathcal{Q}(\overline{\Omega}_{R_0})$ is relatively compact by Lemma 1 in C_h. Hence, $\mathcal{Q}$ is completely continuous.

From above proof

$$\mathcal{Q}: \overline{\mathrm{SAP}_\omega(E) \cap \Omega_{R_0}} \to \overline{\mathrm{SAP}_\omega(E) \cap \Omega_{R_0}}$$

is completely continuous, which implies that $\mathcal{Q}$ is a condensing mapping from $\overline{\mathrm{SAP}_\omega(E) \cap \Omega_{R_0}}$ into $\overline{\mathrm{SAP}_\omega(E) \cap \Omega_{R_0}}$. It follows from Lemma 3 that $\mathcal{Q}$ has a fixed point $u^* \in \overline{\mathrm{SAP}_\omega(E) \cap \Omega_{R_0}}$.

Finally, we show that $u^* \in \mathrm{SAP}_\omega(E)$. Let $\{u_n\} \subset \mathrm{SAP}_\omega \cap \Omega_{R_0}$ converge to u^*. Then, according to the continuity of $\mathcal{Q}$ and (18), we can find that $\{\mathcal{Q}u_n\}$ converges to $\mathcal{Q}u^* = u^*$ uniformly in $[0, \infty)$ and $u^* \geqslant \sigma e_1$, which implies that $u^* \in \mathrm{SAP}_\omega(E)$ is a positive mild solution of problem (1). This completes the proof of Theorem 2. $\square$

4 Example

We consider the following semilinear fractional parabolic equation initial boundary value problem:

$$\begin{aligned} &\frac{\partial^q}{\partial t^q} u(\xi, t) + \frac{\partial^2}{\partial \xi^2} u(\xi, t) = \frac{a \sin^2 t}{\mathrm{e}^t} u(\xi, t) + b\sqrt{\frac{2}{\pi}} \sin \xi, \quad \xi \in [0, \pi],\ t \in \mathbb{R}^+, \\ &u(0, t) = u(\pi, t), \quad t \in \mathbb{R}^+, \qquad u(\xi, 0) = u_0(\xi), \quad \xi \in [0, \pi], \end{aligned} \tag{20}$$

where $\partial^q/\partial t^q$ is the Caputo fractional partial derivative of order $q \in (0, 1)$ with the lower limits zero, $a \in (0, 1)$, $b > 0$ are constants, $u_0 : [0, \pi] \to \mathbb{R}^+$ is a continuous function.

To treat this system in the abstract form (1), we choose the space $E = L^2[0, \pi]$ equipped with the L^2-norm $\|\cdot\|$. Let $K = \{u \in E\colon u(\xi) \geqslant 0 \text{ a.e. } \xi \in [0, \pi]\}$, thus, E is an ordered Banach space, and positive cone K is a normal regeneration cone.

Define operator $A : D(A) \subset E \to E$ by

$$D(A) := \{u \in E\colon u'', u' \in E,\ u(0) = u(\pi) = 0\}, \quad Au = -\frac{\partial^2 u}{\partial \xi^2}.$$

From [2] we know that $-A$ is a self-adjoint operator in E and generates an exponentially stable analytic semigroup $T(t)$ ($t \geqslant 0$), which is contractive in E. Hence, $\|T(t)\| \leqslant M := 1$ for every $t \geqslant 0$. Moveover, A has a discrete spectrum with eigenvalues of the form n^2, $n \in \mathbb{N}$, and the associated normalized eigenfunctions are given by $e_n(\xi) = \sqrt{2/\pi} \sin(n\xi)$ for $\xi \in [0, \pi]$. On the other hand, by the maximum principle of the parabolic type it is easy to find that $T(t)$ ($t \geqslant 0$) is a positive semigroup. Since the operator A has compact resolvent in $L^2[0, \pi]$, thus, $T(t)$ ($t \geqslant 0$) is a compact semigroup (see [28]), which implies that the growth exponent of the semigroup $T(t)$ ($t \geqslant 0$) satisfies $\nu_0 = -1$.

For $\xi \in [0, \pi]$, we set $u(t)(\xi) = u(\xi, t)$ and

$$F(t, u(t))(\xi) = \frac{a \sin^2 t}{\mathrm{e}^t} u(\xi, t) + b\sqrt{\frac{2}{\pi}} \sin \xi. \tag{21}$$

It is easy to verify that $F : [0, \infty) \times E \to E$ is a continuous function. From the assumptions of problem (20) and (21) one can deduce that F satisfies the monotonicity condition (H2) and the asymptotically periodic condition (H3).

Let $h(t) = \mathrm{e}^t$, then

$$\left\|F\left(t, \mathrm{e}^t x\right)\right\| \leqslant \frac{a \sin^2 t}{\mathrm{e}^t}\left\|\mathrm{e}^t x\right\| + b\|e_1\| \leqslant a\|x\| + b. \tag{22}$$

Combining (22) and $a \in (0,1)$, one can find that condition (H1′) holds. Therefore, from Corollary 1 it follows that problem (20) has a minimal positive time S-asymptotically ω-periodic mild solution $u^* \in C([0,\infty), L^2[0,\pi]) \cap \mathrm{SAP}_\omega(L^2[0,\pi])$. On the other hand, it is clear that for $u(t,\xi) \geqslant be_1(\xi)$,

$$\frac{a \sin^2 t}{\mathrm{e}^t} u(\xi,t) + be_1(\xi) \geqslant \frac{ab \sin^2 t}{\mathrm{e}^t} e_1(\xi) + be_1(\xi) \geqslant be_1(\xi),$$

which implies that condition (H4) holds. Hence, if $u_0 \geqslant be_1$, from Theorem 2 one can obtain that for problem (20), there exists at least one positive time S-asymptotically ω-periodic mild solution $u^* \in C([0,\infty), L^2[0,\pi]) \cap \mathrm{SAP}_\omega(L^2[0,\pi])$ with $u^*(\xi,t) \geqslant be_1(\xi)$ for every $\xi \in [0,\pi]$ and $t \geqslant 0$.

References

1. O. Agrawal, J. Sabatier, J. Tenreiro, *Advances in fractional calculus*, Springer, Dordrecht, 2007, `https://doi.org/10.1007/978-1-4020-6042-7`.
2. H. Amann, Periodic solutions of semilinear parabolic equations, in *Nonlinear Analysis (Collection of Papers in Honor of Erich H. Rothe)*, Academic Press, New York, 1978, pp. 1–29, `https://doi.org/10.1016/B978-0-12-165550-1.50007-0`.
3. F. Andrade, C. Cuevas, C. Silva, H. Soto, Asymptotic periodicity for hyperbolic evolution equations and applications, *Appl. Math. Comput.*, **269**:169–195, 2015, `https://doi.org/10.1016/j.amc.2015.07.046`.
4. C. Anh, T. Ke, On nonlocal problems for retarded fractional differential equations in Banach spaces, *Fixed Point Theory*, **15**:373–392, 2014.
5. J. Banasiak, L. Arlotti, *Perturbations of Positive Semigroups with Applications*, Springer, London, 2006, `https://doi.org/10.1007/1-84628-153-9`.
6. P. Chen, Y. Li, Existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions, *Z. Angew. Math. Phys.*, **65**:391–404, 2014, `https://doi.org/10.1007/s00033-013-0351-z`.
7. Y. Chen, Z. Lv, L. Zhang, Existence and uniqueness of positive mild solutions for a class of fractional evolution equations on infinite interval, *Bound Value Probl*, **2017**:120, 2017, `https://doi.org/10.1186/s13661-017-0853-2`.
8. Y. Chen, Z. Lv, L. Zhang, Existence of positive mild solutions for a class of fractional evolution equations on the half line, *Adv. Difference Equ.*, **2017**:14 pp, 2017, `https://doi.org/10.1186/s13662-017-1331-y`.
9. C. Cuevas, H.R. Henríquez, H. Soto, Asymptotically periodic solutions of fractional differential equations, *Appl. Math. Comput.*, **236**:524–545, 2014, `https://doi.org/10.1016/j.amc.2014.03.037`.

10. C. Cuevas, J. Souza, Existence of s-asymptotically ω-periodic solutions for fractional order functional integro-differential equations with infinite delay, *Nonlinear Anal., Theory Methods Appl.*, **72**:1683–1689, 2010, https://doi.org/10.1016/j.na.2009.09.007.
11. T. Diagana, *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*, Springer, New York, 2013, https://doi.org/10.1007%2F978-3-319-00849-3.
12. M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, *Chaos Solitons Fractals*, **14**:433–440, 2002, https://doi.org/10.1016/S0960-0779(01)00208-9.
13. G. Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer, Dordrecht, 2011, https://doi.org/10.1007%2F978-1-4757-4482-8.
14. Y. Guo, M. Chen, X. Shu, F. Xu, The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm, *Stochastic Anal. Appl.*, 2020, https://doi.org/10.1080/07362994.2020.1824677.
15. H. Henríquez, M. Pierri, P. Táboas, On s-asymptotically ω-periodic functions on Banach spaces and applications, *J. Math. Anal. Appl.*, **343**:1119–1130, 2008, https://doi.org/10.1016/j.jmaa.2008.02.023.
16. E. Hernández, S. Tanaka, Global solutions for abstract functional differential equations with nonlocal conditions, *Electron. J. Qual. Theory Differ. Equ.*, **50**:1–8, 2009, https://doi.org/10.14232/ejqtde.2009.1.50.
17. I. Kim, K. Kim, S. Lim, An $l_q(l_p)$-theory for the time fractional evolution equations with variable coefficients, *Adv. Math.*, **306**:123–176, 2017, https://doi.org/10.1016/j.aim.2016.08.046.
18. M. Kostić, *Almost Periodic and Almost Automorphic Type Solutions of Abstract Volterra Integro-Differential Equations*, Walter de Gruyter, Berlin, 2019, https://doi.org/10.1515/9783110641851.
19. F. Li, H. Wang, s-asymptotically ω-periodic mild solutions of neutral fractional differential equations with finite delay in Banach space, *Mediterr. J. Math.*, **14**:57, 2017, https://doi.org/10.1007/s00009-017-0855-4.
20. Q. Li, Y. Li, Positive periodic solutions for abstract evolution equations with delay, *Positivity*, **25**:379–397, 2021, https://doi.org/10.1007/s11117-020-00768-4.
21. Q. Li, M. Wei, Existence and asymptotic stability of periodic solutions for neutral evolution equations with delay, *Evol. Equ. Control Theory*, **9**:753–772, 2020, https://doi.org/10.3934/eect.2020032.
22. Y. Li, The positive solutions of abstract semilinear evolution equations and their applications, *Acta Math. Sin.*, **39**:666–672, 1996 (in Chinese).
23. Y. Li, Existence and asymptotic stability of periodic solution for evolution equations with delays, *J. Funct Anal.*, **261**:1309–1324, 2011, https://doi.org/10.1016/j.jfa.2011.05.001.
24. C. Lizama, F. Poblete, Regularity of mild solutions for a class of fractional order differential equations, *Appl. Math. Comput.*, **224**:803–816, 2013, https://doi.org/10.1016/j.amc.2013.09.009.
25. X. Ma, X. Shu, J. Mao, Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay, *Stoch. Dyn.*, **20**:2050003, 2020, https://doi.org/10.1142/S0219493720500033.

26. R.H. Martin Jr., *Nonlinear Operators and Differential Equations in Banach Spaces*, Krieger, Malabar, FL, 1986.
27. R. Nagel, *One-parameter Semigroups of Positive Operators*, Springer, Berlin, 1986, `https://doi.org/10.1007/BFb0074922`.
28. A. Pazy, *Semigroup of linear operators and applications to partial differential equations*, Springer, New York, 1993, `https://doi.org/10.1007/978-1-4612-5561-1`.
29. M. Pierri, On s-asymptotically ω-periodic functions and applications, *Nonlinear Anal., Theory Methods Appl.*, **75**:651–661, 2012, `https://doi.org/10.1016/j.na.2011.08.059`.
30. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.
31. R. Ponce, Bounded mild solutions to fractional integrodifferential equations in Banach spaces, *Semigroup Forum*, **87**:377–392, 2013, `https://doi.org/10.1007/s00233-013-9474-y`.
32. L. Ren, J. Wang, M. Fečkan, Asymptotically periodic solutions for Caputo type fractional evolution equations, *Fract. Calc. Appl. Anal.*, **21**:1294–1312, 2018, `https://doi.org/10.1515/fca-2018-0068`.
33. L. Ren, J. Wang, D. O'Regan, Asymptotically periodic behavior of solutions of fractional evolution equations of order $1 < \alpha < 2$, *Math. Slovaca*, **69**:599–610, 2019, `https://doi.org/10.1515/ms-2017-0250`.
34. W. Ruess, Summers W, Operator semigroups for functional differential equations with delay, *Trans. Am. Math. Soc.*, **341**:695–719, 1994, `https://doi.org/10.1090/S0002-9947-1994-1214785-X`.
35. X. Shu, F. Xu, Y. Shi, s-asymptotically ω-positive periodic solutions for a class of neutral fractional differential equations, *Appl. Math. Comput.*, **270**:768–776, 2015, `https://doi.org/10.1016/j.amc.2015.08.080`.
36. G. Stamov, J. Alzabut, Almost periodic solutions for abstract impulsive differential equations, *Nonlinear Anal., Theory Methods Appl.*, **72**:2457–2464, 2010, `https://doi.org/10.1016/j.na.2009.10.042`.
37. J. Wang, Y. Zhou, A class of fractional evolution equations and optimal controls, *Nonlinear Anal., Real World Appl.*, **12**:263–272, 2011, `https://doi.org/10.1016/j.nonrwa.2010.06.013`.
38. R. Wang, D. Chen, T. Xiao, Abstract fractional cauchy problems with almost sectorial operators, *J. Differ. Equations*, **252**:202–235, 2012, `https://doi.org/10.1016/j.jde.2011.08.048`.
39. T. Zhang, L. Xiong, Periodic motion for impulsive fractional functional differential equations with piecewise Caputo derivative, *Appl. Math. Lett.*, **101**:106072, 2020, `https://doi.org/10.1016/j.aml.2019.106072`.
40. Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014, `https://doi.org/10.1142/9069`.
41. Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*, Elsevier, Amstardam, 2016, `https://doi.org/10.1016/C2015-0-00813-9`.