

DOI: <https://doi.org/10.24297/jam.v21i.9136>

Visualization In Mathematics Teaching

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Abstract

In recent years, there has been an increased use of information and communication technologies and mathematical software in mathematics teaching. Numerous studies of the effectiveness of mathematical learning have shown the justification and usefulness of the implementation of new teaching aids. They also showed that learning with educational software has a great impact on students' achievement in the overall acquisition of mathematical knowledge during the school year as well as in the final exam at the end of primary education. Teaching realized by using computers and software packages is interesting for students, increases their interest and active participation. It is indisputable that the use of computers and mathematical software has great benefits that have been proven and presented in their works by many researchers of effective learning. It is also indisputable that one of the main tasks of teaching mathematics is to develop constructive thinking of students. Visualization and representation of mathematical laws are of great importance in the realization of mathematics teaching. They should be applied everywhere and whenever possible.

Keywords: educational software, mathematics teaching, visualization and representation, constructive thinking.

1. Introduction

Modernization of teaching is also helped by monitoring the development of technology, ie the introduction of new teaching aids (computers, calculators, graphing calculators ...) in order to bring mathematics closer to students, improve understanding, discovery and adoption of mathematical concepts, phenomena and laws. When choosing individual programs (software), the teacher must consider whether this particular software can help teach mathematics, increase knowledge, develop skills, improve understanding of mathematical ideas, solve problems, create teaching materials, etc.

A significant number of researchers in mathematics education have pointed to another extremely important aspect of this issue. Namely, some difficulties in mastering the content related to the functions are related to the limited experience of students in working with individual teams. Mathematics teachers traditionally focus their instructions on algebraic representations and analytical elaboration of content, while graphic (geometric) and intuitive representations are in some way neglected (Hitt, 2002: 2-3). Marginalization of visual representations and visual approaches to learning, leads to imbalance between representations and unsynchronized development of different forms of mathematical thinking. Pupils do not acquire complete but fragmented knowledge, and are not able to interpret different representations, nor to flexibly and competently transform one representation into another (Duval, 1999, 2006) [1]. Favoring an algebraic approach results in students relying primarily on problem-solving methods that involve working with algebraic representations,

thereby reducing the sense and need to use graphical representations and visual approaches to problem solving (Knuth, 2000) [2].

Modern technology, mathematical representations and visualization are closely related research issues in mathematics education. Computer technology and mathematical software increase the ability to use and analyze multiple related representations (Yerushalmy & Shternberg, 2001) and can help students gain a better understanding of the use of representations and build a thorough understanding of mathematical concepts and ideas (Kaput, 1992) [3]. The potential of computer technologies to connect mathematical concepts with visual representations in a way that encourages mathematical reasoning and conceptual understanding is very important for teaching/learning mathematics (Lopez Jr., 2001) [4]. Visual-dynamic and interactive functionality of computer technologies opens numerous creative possibilities for the realization of content in the field of functions.

Cognitive-visual approach is a modern didactic-methodical concept that promotes the visualization of the learning process, the importance of connecting visual and symbolic representations and the use of computer technology in creating a visual learning environment. Purposeful, planned and wide use of the cognitive function of obviousness in order to develop and optimally use the resources of visual thinking of students, is the basic premise of the cognitive-visual approach to teaching / learning mathematics (Dalinger, 2006a). The obviousness of visual representations is very important for the learning process, where the decisive factor in its efficiency is the way and degree of engagement of students in activities that are more than just watching. In that sense, the cognitive function of obviousness comes to full expression when the visualization of teaching contents is didactically-methodically designed so that the role of visual representations from an auxiliary, illustrative means is transposed into a leading, productive cognitive means (Dalinger and Knyazeva, 2004). The application of computers and appropriate educational software provides teachers with a wide range of opportunities for the realization of cognitive-visual approach in the study of functions, their properties and graphics in a visual, dynamic and interactive environment. The visual nature of human perception and cognitive processes, the potential of visual thinking, and the accelerating development of technology are the main reasons for the wide, purposeful and planned application of graphic representations as a cognitive-visual tool in modern education.

This paper presents the basic settings of the cognitive-visual approach to teaching / learning mathematical analysis, and gives a comparative overview of the traditional and cognitive-visual concepts in the processing of functions. Mathematical problems with graphic contents and/or requirements are presented and examples of tasks are given by which connections are established between algebraic and graphical representations of a function and derivatives of a function that are not represented in traditional teaching contents. Given that there is a wide range of tasks that can be used in teaching, the classification was performed according to different criteria that reflect their methodological features and based on that, their typology is given.

Among researchers in mathematics education, there is a growing interest in multiple representations and mathematical visualization. The importance of representations and visualizations and their close connection in the learning process is emphasized in the works of numerous authors, especially in the report of the working group "Representations and Mathematics Visualization", from the 20th PME-NA conference (Hitt, 2002) [5]. The authors agree that the teaching of mathematics, at all levels, is traditionally focused on the use of algebraic representations and that geometric (graphic) and intuitive representations are neglected. Marginalization of visual representations and visual approaches to learning, leads to imbalance between representations and unsynchronized development of different forms of mathematical thinking. The difficulties of some students stem precisely from the limited experience in working with visual representations and their connection with symbolic representations. This is one of the main problems that traditional mathematics teaching faces. The results of empirical research indicate that teaching approaches based on visualization and multiple representations are more effective than traditional ones.

2. VISUALIZATION IS THE ABILITY, PROCESS AND PRODUCT OF CREATIVITY

The meaning of the word "visualization" itself means to make something visible. It is the ability to present and imagine something in pictures. Based on non-visual data, an image is created in our brain. This means that data from something that is abstract or not immediately visible is transformed into visible. From this we can conclude

that a diagram, sketch, drawing, etc. emerges from the visualization. which is not always easy to visualize. The result of the visualization must be accurate and legible. The most important criterion of visualization is that from visualization we must acquire some new knowledge that we connect with previous knowledge in order for visualization to make sense. Visualization is the ability, process and product of creativity, interpretation, consideration of images, sketches, drawings, diagrams that are in our mind, on paper or displayed on a computer. Visualization leads to the development of new ideas using old knowledge. We can say that visualization is a skill of mental programming by which we achieve our goals such as solving a mathematical problem. Having the ability to visualize means having the power in your mind to imagine an image that helps us come up with a solution to a problem. Visualization has a wide range of uses for example better learning, in the fight against stress, healing, quitting smoking.

The skill of visualization is the intertwining of virtues such as creativity, imagination, determination, perseverance, consistency, diligence. Not every visualization is effective so it is necessary to practice and practice visualization in order for it to become a skill. When Einstein solved some problems, he always tried to look at the core of the problem in as many ways as possible, including pictures and diagrams. He visualized solutions and believed that numbers and words as such did not play an important role in the process of his thinking and creation. It is the visualizations that led him to his goal. The importance of visualization was confirmed by Nikola Tesla, who said: "I can thank visualization for everything I have created. The events of my life and my discoveries before my eyes are real, visible like any phenomenon and object. In my youth I was afraid of it not knowing what it really was, but later I received that power as a gift and a fortune. I nurtured it and jealously guarded it. I made corrections to most of the inventions by visualization, and then, thus completed, I made them. I also solve complicated mathematical equations with it, without writing numbers"[6].

2.1 The importance of visualization in mathematics teaching

Visualization in mathematics has a long tradition and the list of famous mathematicians who have used and advocated the use of visualization in mathematics is long. One notable example is certainly the blind mathematician Euler whose limitation had no effect on his creative power and power of visualization. During the years of his blindness he wrote more than 355 works because of his visual power and of course, his phenomenal memory. In learning and understanding mathematics, visualization can be a powerful tool for researching mathematical problems and can give clearer meaning to mathematical concepts and the connections between them. Visualization reduces complexity when dealing with a large amount of information. The following story could convey the charm of visualization in mathematics teaching much better than many analyzes. The main role in this story is played by Professor Norbert, but surely every man who has had contact with mathematics will be able to identify the story with more than one professor or friend he knows. Norbert gave a lecture at a well-known university in front of a large audience and was engrossed in solving complicated evidence on the board. The board was almost a full formula and he was certainly heading for a solution. He stopped abruptly as he got lost among the multitude of formulas. To observers, it seemed unbelievable how confusing the professor could be. It was a few minutes before the professor seemed to know what he was doing. He walked to the corner of the board that remained unprinted and began to draw a picture. He didn't say a word. He finally shrugged in relief, erased the drawing he had made, and returned to where he had stopped proving, and the proof was easily completed [7].

Mathematical concepts, ideas, methods, have great richness and connection with visualization in many different ways. The use of visual relationships in problem solving is very useful. The basics of mathematics such as distance or number operations were born from concrete and visible situations. Every expert is aware of how useful it is to connect concrete cases when studying appropriate abstract objects. The same thing happens with other abstract parts of mathematics. This way of working, with special emphasis on possible concrete representations of objects that an individual manipulates in order to have the most effective access to more abstract connections that an individual has to manage, is what we call mathematical visualization. The fact that visualization is an important aspect in mathematics and mathematics teaching is quite natural if we consider the meaning of mathematical activity and the structure of the human brain. Through mathematical activities, students try to explore different real-life situations that we then recognize as mathematical problems that we solve.

Visualization proves useful in discovering the relationship between mathematical objects and of course in the communication process that favors mathematical activity. Human perception is highly visual and therefore it is not at all surprising that visual support is implicated in mathematical tasks, not only of the geometric type, but is also implicated in other areas of mathematics where this is not so obvious. Even in mathematical activities that are abstract to our brain, mathematicians use symbols, visual diagrams, and many other mental processes that involve visualization. Connecting visualization with thought processes results in the discovery of new insights among mathematical objects in the communication process which is good for mathematical activity. Therefore, it is necessary to acquaint students with the process of visualization in learning and teaching mathematics and gradually accustom the student's brain to such thinking. It is not an easy task but a complex process that involves continuous exercise and practice over many years. Which is of course an experiential process for the teacher. Of course, the process of visualization is worth carrying out only if it is possible to bring things closer to students, make the learning process more natural, easier, more interesting, more acceptable, etc. It is true that a picture is worth 1000 words, but what matters is that the picture is understood or "decoded" in the right way and that the one who studies it understands in the right way, otherwise the picture is worth nothing [8].

However, limitations, difficulties, and even reluctance of students to use visualization in problem solving arise. While some students use visualization to modify the resulting task into a visual one. Visual task-solving processes, which are not always safe routines because they do not have a "template," have proven to be much more cognitively demanding than analytical problem-solving techniques. Questions that arise: How do students deal with a problem of a visual nature? What is the degree of visual perception of students? How much do students use visualization in problem solving? Tall points out, "The quality of using images in solving problems without enslaving them gives the mathematician an advantage, but it can cause many difficulties to the learner." Visualization should be accompanied by logical thinking in order to avoid mistakes, because sometimes the image itself can lead to wrong conclusions. When learning and teaching mathematics one should also be aware of the pitfalls that arise if we use visualization exclusively in problem solving. Visualization can be a powerful and useful tool for working with mathematical problems. One of the possibilities of how to practice and encourage visualization in students is the method of proving without words.

2.2. The concept, role and significance of mathematical visualization

Visualization is an important aspect of mathematics, mathematical thinking, understanding and reasoning. The role of visualization in the learning process and its impact on educational outcomes are the subject of many studies conducted so far. The authors point out that visual thinking is very important for the learning process, present convincing arguments about the central role of visualization in reform changes in mathematics teaching (Arcavi, 2003; Presmeg, 1992; Zimmermann & Cunningham, 1991), but also point out possible difficulties and limitations of visualization (Aspinwall, Shaw & Presmeg, 1997[9]; Baker, Cooley & Trigueros, 2000[10]; Monk, 2003; Presmeg, 2006).

Norma Presmeg (2006) in her paper gave an overview of previously conducted research related to visualization in learning and teaching mathematics. She points out that more active work on this topic began in the 1990s, more precisely, after the PME-13 (Psychology of Mathematics Education) conference, at which several papers on visual representations and visual thinking were exhibited. Presmeg notes that in the existing literature, the term "visualization" has different meanings and gives its view of this phenomenon. Visualization includes the processes of constructing and transforming both visual mental images and all sketches of a spatial nature that may be related to dealing with mathematics (*ibid.*, 218). With the development of computers and their graphics capabilities, visualization has gained more and more importance and relevance. To date, a large number of papers have been published dealing with visualization, visual thinking, visual representations, visual problem-solving methods, visual approaches to teaching / learning, etc.

The concept of visual thinking, which is widely used today, was introduced by Rudolf Arnheim (1991), defining it as thinking through visual operations. In visualization, images combine aspects of natural representation with more formal forms and thus increases cognitive understanding. Arnheim points out that visual images are not just an illustration of thought, but far more, they are the development of thought itself. The title of his work is a

very concise definition of visual thinking: "Visual thinking - The unity of image and concept" which also gives a significant implication for the teaching of mathematics.

Arcavi gave a very comprehensive definition of visualization: "Visualization is the ability, process and product of creating, interpreting and using images, diagrams, illustrations in our mind, on paper or technological means, whose purpose is to display and communicate information, think and develop previously unknown ideas and improve understanding." (Arcavi, 2003: 217).

This definition emphasizes the importance of visualization for teaching/learning mathematics in the following aspects:

- Visual representations (both external and internal) are important for presenting mathematical ideas and developing students' abilities and skills in mathematical communication.
- Visualization can give clearer meaning and significance to mathematical concepts and the connections between them and thus contribute to the development of conceptual understanding.
- Visualization can be a powerful tool for researching and solving mathematical problems.

Visual mathematical representations can be defined as a set of graphic symbols that visually encode cause-and-effect, functional, structural, and semantic properties and relationships of the mathematical concepts they represent (Sedig & Sumner, 2006) [11]. In mathematics teaching, geometry is the most natural area in which visual representations are used. However, visual representations can also be used in other areas of mathematics to encourage visual thinking. Visual representations are present in the contents of trigonometry, analytical geometry, real functions, differential and integral calculus, but they are insufficiently represented in teaching. When processing some concepts (e.g. the first derivative of a function), graphical representations are used when introducing a concept, and after that, they are often neglected and algebraic representations dominate in the work. Visualization is a central component of many processes in which the transition from the concrete to the abstract model of thought is made (Ben-Chaim, Lappan & Houang, 1989: 50) [12]. Insufficient representation of visual representations, as bearers of the meaning of concepts, can lead to premature formalization. In such situations, students rely on formulas and ready-made algorithms and perform algebraic procedures, without a clear understanding of the meaning.

The role of visualization in solving mathematical problems is very important. Visual representations present information obviously and can serve to reveal the structure of a problem and lay the foundation for its solution (Diezmann & English, 2001:77). If we compare algebraic and visual approaches to problem solving, it can be concluded that visual approaches have certain advantages:

- 1) Visualization enables reduction of complexity when working with a lot of information (Rösken & Rolka, 2006), so the task can be solved easier and faster;
- 2) Visualization can contribute to resolving cognitive conflicts between (correct) symbolic and (incorrect) intuitive solutions (Arcavi, 2003) [13];
- 3) Visual approaches can point to the conceptual basis of problems that students can easily ignore in algebraic procedures (Arcavi, 2003) [13];
- 4) The concreteness of visual representations is an important factor for creating the effect of immediacy in proving (Fischbein, 1987). Some formal proofs can be replaced by geometrically analogous, simple, and beautiful ones so that the validity of the theorem almost blinds us (Gardner, 1973) [14].

Visual representations represent a combination of concrete and abstract elements of the mathematical structure of the problem. Therefore, they can bridge the gap between the concrete and abstract side of the problem and can facilitate both the mathematization and the concretization of the problem. Representation of concrete/abstract elements determines the degree of concreteness (or abstractness) of visual representation. Specific visual representations are directly related to the real situation, are easy to understand, interesting and can enhance student motivation. However, they do not provide enough information that clearly shows the mathematical relationships and structure of the problem, i.e. the connection with the abstract side of the

problem is weak. Abstract visual representations are characterized by the use of conventional graphic symbols to represent relevant structural elements.

They help students focus on the essential characteristics of the problem, but are more difficult to understand and may require better prior knowledge. Compared to concrete, abstract representations are "more powerful" because they can transcend context in reasoning and problem-solving processes (Kaminski, Sloutsky & Heckler, 2008) [15]. If concrete and abstract visual representations are used simultaneously to visualize the problem, the advantages of both representations can be used. In this way, it is easier for students to establish relationships and connections between the concrete and abstract aspects of the problem, which ultimately leads to further improvement of the learning process and better student achievement (Moreno, Ozogul & Reisslein, 2011) [16].

By introducing visual approaches in the teaching of mathematics, students are given the opportunity to acquire complete and functional knowledge. The use of visual representations should be designed and well-balanced, so as not to lead to the other extreme. Visualization acquires its true, essential meaning only if visual representations are connected with algebraic, numerical, and verbal representations (Zimmerman & Cunningham, 1991) [17].

3. USE OF GEOMETRY IN TEACHING MATHEMATICS

According to Vinner (1989), the perception of the concept consists of the collection of the concept properties in mind and the whole visions identical with concepts in mind. So, when teaching a concept, the description of the concept will not be adequate, and it will be necessary to compose the concept in mind. Mirsky (1963) has been stated that there is a lot of ratios (space, plane, coordinate system, etc.) between linear algebra and geometry. Mostow and Sampson (1969) have stated that if an algebraic equation be demonstrated as geometrical, it will be possible to define the geometric constructed equations in which algebraic expressions are less used. (Wang 1989) stresses the significance of intuitional articulation and geometric visualization in the explanation of important theorem of linear algebra. Students should learn the reasons of using certain things and doing analysis and they should do visual algorithm in three-dimensional space. Wang points out that these practices will acquire to students the ability about using basic linear algebra concepts and researching system. Halmos (1987) stated that a mathematician should be born with the ability of visualization for his having a more advanced information level in mathematics. Thompson and Yaqup (1970) states that the vector space concept is actually the abstraction of certain basic geometrical constructions encountered in analytic geometry.

3.1. Affective Issues in Learning Geometry

A great deal of research has focused on the cognitive side of learning. It is commonly accepted that emotions affect learning, but emotions are more difficult to measure and study. According to Jensen (1998), three discoveries in the field of emotions have elevated the importance of studying the impact of emotions on learning: discoveries about the pathways and priorities of emotions in the brain; findings about the relationship between emotions and chemical secretions triggered by the brain; and the impact of emotions" neurological priorities and biochemicals on learning and memory.

McLeod (1988) proposed that the affective characteristics of learning be subdivided into effect, emotion and attitude [18]. Affect represents all of the feelings that relate to mathematics learning. Emotion describes the intense but short-lived sensations experienced in the moment. Attitude represents the relatively consistent, longer-term feelings about mathematics learning. Emotions create specific mind-body states, which create urges to respond or act. Key characteristics of emotions are their intensity, direction and duration. Students vary in their level of awareness of their emotional state. According to McLeod, "if problem solvers become aware of their emotional reactions, they may improve their ability to control their automatic responses to problems" (p. 137).

Students also vary in their level of control over their emotional state. Past experience and attitude are key factors in students" conscious responses to their emotional state. Students who know that struggle is part of the problem-solving process and have found success in the past after persisting, will probably calm themselves and persist again, knowing that finding the solution will be all the more satisfying given the effort to get there. Students without a track record of success will find it hard to summon the energy to try what they view as a

hopeless task. Issues such as learned helplessness, math anxiety, and causal attributions of success and failure all feed into students' attitudes about mathematics and problem solving, making the affective issues of learning complex but essential for progress. As mentioned above, Fuys et al (1988) observed several affective issues that aligned with student learning progress, including their comfort with exploration in mathematics, their degree of confidence, persistence, independence as a learner, and their tendencies for impulsive versus reflective conclusions [19].

Given that a student's affect is a key factor in learning, then the instructor's challenge is to create environments that trigger positive affective responses. Small group instruction, levels of teacher direction and scaffolding, group work and differentiated tasks are all instructional options to consider. McLeod (1988) noted that children tend to become deeply engrossed in computer work and as a result, this instructional environment can have interesting effects on affective responses. McLeod also recommended explicit instruction on affective issues. This would help students to monitor their affect and to appreciate the normal flow of feelings while working on a task [18].

3.2. The Role of Dynamic Geometry Software in Learning Geometry

Dynamic geometry software can be a means to provide students with the experiential learning that enables them to see the geometric patterns emerge. It can connect to the emotional component of learning through multisensory experience, personal investment and immediate feedback. Instead of considering perhaps a handful of static examples, students can explore an almost endless continuum of cases within moments. This allows them to recognize patterns, form conjectures and test / refine their conjectures rapidly. Learning geometry in this mode presents differences from the static examples made from compass and straightedge constructions. The emphasis on visual observations skills is much greater in a dynamic setting than with static examples (Scher, 2002).

De Villiers (1998) proposed that the advent of the computer has made experimentation in many areas of mathematics more feasible, and as a result, has changed the role of deductive proof in mathematics. In the past, "the function (or purpose) of proof is seen as only that of the verification (conviction or justification) of the correctness of mathematical statements" (p. 370) [20]. Especially in the high school geometry class, once students have personally convinced themselves of a conclusion through the use of dynamic geometry tools, the need to verify with deductive proof is not viewed as necessary. De Villiers has found that students find value in deductive proof as a means of understanding and explaining why a certain result occurs. "when proof is seen as explanation, substantial improvement in students' attitudes toward proof appears to occur (p. 388).

Olive (1998) outlined several educational implications of Geometer Sketchpad in the geometry classroom. The traditional approach of building up geometry from axioms, definitions and theorems is not appropriate when phenomena can be explored real-time. Inductive reasoning should be the focus, relying on experimentation, observation and conjecturing. He agreed with de Villiers that proof becomes more appropriately used for explanation than for verification. This approach allows students to construct mathematical relationships and meaning for themselves – which constructivists believe is the only way that learning is accomplished. Olive stated, "If used in conjunction with practical, physical experiences (such as ruler and compass constructions on paper), the computer construction tool can provide a link between the physical experiences and the mental representations" (p. 400). This statement aligns with the AIMS model of developing the interaction between concrete and representation. This also implies that dynamic geometry software aids in the development of students' spatial visualization abilities.

There are many ways in which dynamic geometry software can be implemented in a classroom. Students might use the software to create the geometric objects themselves and then explore them around a certain goal. Or, dynamic examples can be prepared in advance for the students to use, with some structure around the questions to be explored and answered. Drawings can be prepared that include measurements of lengths and angles, enabling students to conjecture on numerical formulas. Stated theorems can be captured in a drawing and explored to explain why they hold true. The possibilities depend on the teacher's expertise with the software, the access to technological resources and the time available to invest in student learning of the software. Certainly, having students create their own geometric objects requires a greater classroom time investment and

requires more independence and ownership on the part of the student. Goldenburg and Cuoco (1998) described how geometry within DGS differs from geometry on paper. In fact, except for what students do in their heads, paper-and-pencil geometry also „involves only action and does not require a description [21]. Part of what students learn in geometry is, as Poincare put it, the art of applying good reasoning to bad drawings – adding the descriptions that specify which features or relationships in the drawings are intended, which are incidental, and which are to be totally ignored as they attempt to draw inferences about the figure depicted (p. 365).

Students who can successfully make this transition have demonstrated their mastery of the third van Hiele level of abstractions: seeing the relationship between properties and figures. Geometer Sketchpad provides a great deal of flexibility on the personal computer platform and appears to be the most prominent software tool. Another currently available option is Java Sketchpad, offered by the makers of Geometer Sketchpad. According to the Geometer Sketchpad Resource Center website (n.d.), Java Sketchpad is technology for viewing and interacting with dynamic visualizations created by someone else. They can be accessed via the Internet, or in some cases, downloaded onto an individual computer. Java Sketches are simple to use. Users just click on the illustrations and drag them about. These existing models can help jumpstart the integration of dynamic geometry software into the classroom through less teacher investment in material development, virtually no learning curve for students, and robust models that should minimize technology mishaps.

4. CONCLUSION

Modern mathematical education, its harmonization with the requirements of scientific and technological development and progress, permanently requires the introduction of innovations that contribute to the modernization, rationalization and efficiency of the teaching process. In order to comprehensively understand and solve problems related to the teaching and learning of mathematics, researchers in mathematics education pay significant attention to mathematical representations, visualization and modern educational technologies, their role and importance for the learning process. Multiple representations, visualization and educational technology, recognized as necessary components of mathematics education, need to be implemented in all segments of teaching, due to their potential to promote mathematical insight and understanding and improve the learning process. These aspects are especially important for the study of functions, the fundamental concept of teaching mathematics.

One of the prominent goals of mathematics teaching is the development of general and interdisciplinary competencies that are necessary for understanding the phenomena and laws in nature and society and which will enable students to apply the acquired knowledge in solving various problems in life practice and to successfully continue mathematics education (PNPPG, 2011). Apart from mathematics, functions and their graphical representations are also used in other scientific disciplines, as a means to describe, explain, confirm or predict scientific concepts, phenomena and processes. The application of functions, especially their graphical representations as a mathematical model for solving problems from other scientific disciplines and various problems in a real environment, provides an opportunity to establish a correlation with other subjects in mathematics teaching and that students through the application of knowledge from one scientific discipline others acquire complete, lasting and functional knowledge.

In any case, the use of visualization in learning, research, and other mathematical activities is on the rise. The fact that visualization can lead us to errors should not be a good argument for against the use of visualization in mathematics because visualization in mathematics can be very efficient and effective in various processes of mathematical activity. Sometimes even formal techniques can lead us to make a mistake which is not a reason to stop dealing with such a way of dealing. Mistakes should be understood as something natural that drives us to work, and in mathematics they can sometimes lead us to new discoveries.

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