Chapter

A Study of Nonlinear Boundary Value Problem

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Abstract

In this chapter, firstly we apply the iterative method to establish the existence of the positive solution for a type of nonlinear singular higher-order fractional differential equation with fractional multi-point boundary conditions. Explicit iterative sequences are given to approximate the solutions and the error estimations are also given. Secondly, we cover the multi-valued case of our problem. We investigate it for nonconvex compact valued multifunctions via a fixed point theorem for multivalued maps due to Covitz and Nadler. Two illustrative examples are presented at the end to illustrate the validity of our results.

Keywords: Positive solution, Uniqueness, Iterative sequence, Green's function, Fractional differential equation and inclusion, Existence, Nonlocal boundary value problem, Fixed point theorem

1. Introduction

In this chapter, we are interested in the existence of solutions for the nonlinear fractional boundary value problem (BVP)

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, & t \in (0,1), \\ u^{(i)}(0) = 0, i \in \{0,1,2,\dots,n-2\}, \;\; D_{0^+}^{\beta}u(1) = \sum\limits_{j=1}^{p} a_j D_{0^+}^{\beta} u\left(\eta_j\right). \end{cases} \tag{1}$$

We also cover the multi-valued case of problem

$$\begin{cases} -D_{0^+}^{\alpha}u(t) \in F(t,u(t)), & t \in (0,1), \\ u^{(i)}(0) = 0, i \in \{0,1,2,\dots,n-2\}, \;\; D_{0^+}^{\beta}u(1) = \sum\limits_{j=1}^{p} a_j D_{0^+}^{\beta} u\left(\eta_j\right), \end{cases} \tag{2}$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivative of order

$$\alpha \in (n-1,n], \beta \in [1,n-2] \;\; for \; n \in \mathbb{N}^* \;\; and \; n \geq 3,$$

where D_{0+}^{α}, D_{0+}^{β} are the stantard Riemann-Liouville fractional derivative of order $\alpha \in (n-1,n], \beta \in [1,n-2] for \; n \geq 3$, the function $f \in C((0,1) \times \mathbb{R}, \mathbb{R})$, the multifunction $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ are allowed to be singular at $t = 0$ and/or $t = 1$ and $a_j \in \mathbb{R}^+$, $j = 1, 2, \dots, p$, $0 < \eta_1 < \eta_2 < \dots < \eta_p < 1$, $for p \in \mathbb{N}^*$.

The first definition of fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus, was mentioned already in 1695 by Leibniz [1] and L'Hospital [2]. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For more details we refer the reader to [1–6] and the references cited therein.

Boundary value problems for nonlinear differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. A point of central importance in the study of nonlinear boundary value problems is to understand how the properties of nonlinearity in a problem influence the nature of the solutions to the boundary value problems. The multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the sensors located, at intermediate points, see [7, 8] and the references therein. We quote also that realistic problems arising from economics, optimal control, stochastic analysis can be modeled as differential inclusion. The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim [9]. Also, recently, several qualitative results for fractional differential inclusion were obtained in [10–13] and the references therein.

The techniques of nonlinear analysis, as the main method to deal with the problems of nonlinear differential equations (DEs), nonlinear fractional differential equations (FDEs), nonlinear partial differential equations (PDEs), nonlinear fractional partial differential equations (FPDEs), nonlinear stochastic fractional partial differential equations (SFPDEs), plays an essential role in the research of this field, such as establishing the existence, uniqueness and multiplicity of solutions (or positive solutions) and mild solutions for nonlinear of different kinds of FPDEs, FPDEs, SFPDEs, inclusion differential equations and inclusion fractional differential equations with various boundary conditions, by using different techniques (approaches). For more details, see [14–37] and the references therein. For example, iterative method is an important tool for solving linear and nonlinear Boundary Value Problems. It has been used in the research areas of mathematics and several branches of science and other fields. However, Many authors showed the existence of positive solutions for a class of boundary value problem at resonance case. Some recent devolopment for resonant case can be found in [38, 39]. Let us cited few papers. In [40], the authors studied the boundary value problems of the fractional order differential equation:

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \;\; D^{\beta}_{0+}u(1) = aD^{\beta}_{0+}u(\eta), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \eta < 1$, $0 < a, \beta < 1$, $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$ and D^{α}_{0+}, D^{β}_{0+} are the stantard Riemann-Liouville fractional derivative of order α. They obtained the multiple positive solutions by the Leray-Schauder nonlinear alternative and the fixed point theorem on cones.

In 2020 Li et al. [41] consider the existence of a positive solution for the following BVP of nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} \left({}^C D_{0+}^q u\right)(t) + f(t, u(t)) = 0, \quad t \in [0,1], \\ u''(0) = 0, \\ \alpha u(0) - \beta u'(0) = \int_0^1 h_1(s) u(s) ds, \\ \gamma u(1) + \delta\left({}^C D_{0+}^{\sigma} u\right)(1) = \int_0^1 h_2(s) ds, \end{cases}$$

where $2 < q \le 3$, $0 < \sigma \le 1$, $\alpha, \gamma, \delta \ge 0$, and $\beta > 0$ satisfying $0 < \rho(\alpha + \beta)\gamma + \frac{\alpha\delta}{\Gamma(2-\sigma)} < \beta\left[\gamma + \frac{\delta\Gamma(q)}{\Gamma(q-\sigma)}\right]$, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ and $h_i (i = 1, 2) : [0,1] \to [0, +\infty)$ are continuous. To obtain the existence results, the authors used the well-known GuoKrasnoselskiis fixed point theorem.

In 2017, Rezapour et al. [42] investigated a Caputo fractional inclusion with integral boundary condition for the following problem

$$\begin{cases} {}^c D^{\alpha} u(t) \in F\left(t, u(t), {}^c D^{\beta} u(t), u'(t)\right), \\ u(0) + u'(0) + {}^c D^{\beta} u(0) = {}_0^{\eta} u(s) ds, \\ u(1) + u'(1) + {}^c D^{\beta} u(1) = {}_0^{\nu} u(s) ds, \end{cases}$$

where $1 < \alpha \le 2$, $\eta, \nu, \beta \in (0,1)$, $F : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction and ${}^c D^{\alpha}$ denotes the Caputo fractional derivative of order α.

In 2018, Bouteraa and Benaicha [10] studied the existence of solutions for the Caputo fractional differential inclusion

$$ {}^c D^{\alpha} u(t) \in F(t, u(t), u'(t)), \quad t \in J = [0,1], $$

subject to three-point boundary conditions

$$\begin{cases} \beta u(0) + \gamma u(1) = u(\eta), \\ \quad u(0) = {}_0^{\eta} u(s) ds, \\ \quad \beta {}^c D^p u(0) + \gamma {}^c D^p u(1) = {}^c D^p u(\eta), \end{cases}$$

where $2 < \alpha \le 3$, $1 < p \le 2$, $0 < \eta < 1$, $\beta, \gamma \in \mathbb{R}^+$, $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction and ${}^c D^{\alpha}$ denotes the Caputo fractional derivative of order α.

In 2019, Ahmad et al. [43] investigated the existence of solutions for the boundary value problem of coupled Caputo (Liouville-Caputo) type fractional differential inclusions:

$$\begin{cases} {}^C D^{\alpha} x(t) \in F(t, x(t), y(t)), \quad t \in [0, T], \ 1 < \alpha \le 2, \\ {}^C D^{\beta} y(t) \in F(t, x(t), y(t)), \quad t \in [0, T], \ 1 < \beta \le 2, \end{cases}$$

subject to the coupled boundary conditions:

$$\begin{aligned} x(0) = \nu_1 y(T), \ \ x'(0) = \nu_2 y'(T), \\ y(0) = \mu_1 x(T), \ \ y'(0) = \mu_2 x'(T), \end{aligned}$$

where ${}^{C}D^{\alpha}$, ${}^{C}D^{\beta}$ denote the Caputo fractional derivatives of order α and β respectively, $F, G : [0, T] \times \mathbb{R} \times \mathbb{R}$ are given multivalued maps, $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\nu_i, \mu_i, i = 1, 2$ are real constants with $\nu_i\mu_i \neq 1, i = 1, 2$.

Inspired and motivated by the works mentioned above, we focus on the uniqueness of positive solutions for the nonlocal boundary value problem (1) with the iterative method and properties of $f(t, u)$, explicit iterative sequences are given to approximate the solutions and the error estimations are also given. We also cover the multi-valued case of problem (2) when the right-hand side is nonconvex compact valued multi-functions via a fixed point theorem for multivalued maps due to Covitz and Nadler.

The chapter is organized as follows. In Section 2, we present some notations and lemmas that will be used to prove our main results of problem (1) and we discuss the uniqueness of problem (1). Finally, we give an example to illustrate our result. In Section 3, we introduce some definitions and preliminary results about essential properties of multifunction that will be used in the remainder of the chapter and we present existence results for the problem (2) when the right-hand side is a non-convex compact multifunction. We shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler [44] to prove the uniqueness of solution of problem (2). Finally, we give an example to ascertain the main result.

2. Existence and uniqueness results for problem (2)

2.1 Preliminaries

In this section, we recall some definitions and facts which will be used in the later analysis. These details can be found in the recent literature; see [2, 4, 6, 45–47] and the references therein.

Let $AC^{i}([0, 1], \mathbb{R})$ denote the space of $i - times$ differentiable functions $u : [0, 1] \to \mathbb{R}$ whose $i - th$ derivative $u^{(i)}$ is absolutely continuous and $[\alpha]$ donotes the integer part of number α.

Definition 2.1. Let $\alpha > 0,\ \ n - 1 < \alpha < n,\ \ n = [\alpha] + 1$ and $u \in AC^{n}([0, \infty), \mathbb{R})$.

The Caputo derivative of fractional order α for the function $u : [0, +\infty) \to \mathbb{R}$ is defined by

$$ {}^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n - \alpha)} \int_{0}^{t} (t - s)^{n-\alpha-1} u^{(n)}(s) ds. $$

The Riemann-Liouville fractional derivative order α for the function $u : [0, +\infty) \to \mathbb{R}$ is defined by

$$ D^{\alpha}_{0^{+}}u(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_{0}^{t} (t - s)^{n-\alpha-1} u(s) ds, \quad t > 0, $$

provided that the right hand side is pointwise defined in $(0, \infty)$ and the function $\Gamma : (0, \infty) \to \mathbb{R}$, defined by

$$ \Gamma(u) = \int_{0}^{\infty} t^{u-1} e^{-t} dt, $$

is called Euler's gamma function.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t > 0,$$

provided that the right hand side is pointwise defined in $(0, \infty)$.

We recall in the following lemma some properties involving Riemann-Liouville fractional integral and Riemann-Liouville fractional derivative or Caputo fractional derivative which are need in Lemma 2.4.

Lemma 2.1. (([45], Prop.4.3), [46]) *Let $\alpha, \beta \geq 0$ and $u \in L^1(0,1)$. Then the next formulas hold.*

i. $\left(D^{\beta} I^{\alpha} u\right)(t) = I^{\alpha-\beta} u(t)$,

ii. $(D^{\alpha} I^{\alpha} u)(t) = u(t)$,

iii. $I^{\alpha}_{0^+} I^{\beta}_{0^+} u(t) = I^{\alpha+\beta}_{0^+} u(t)$.

iv. *If $\beta > \alpha > 0$, then $D^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta) t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$. where D^{α} and D^{β} represents Riemann-Liouville's or Caputo's fractional derivative of order α and β respectively.*

Lemma 2.2 [47]. *Let $\alpha > 0$ and $y \in L^1(0,1)$. Then, the general solution of the fractional differential equation $D^{\alpha}_{0^+} u(t) + y(t) = 0, \ \ 0 < t < 1$ is given by*

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad 0 < t < 1,$$

where $c_0, c_1, \ldots, c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Based on the previous Lemma 2.2, we will define the integral solution of our problem (1).

Lemma 2.3. *Let $\sum_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1} \in [0,1), \alpha \in (n-1, n], \beta \in [1, n-2], n \geq 3$ and $y(\cdot) \in C[0,1]$. Then the solution of the fractional boundary value problem*

$$\begin{cases} D^{\alpha}_{0+} u(t) + y(t) = 0, \\ u^{(i)}(0) = 0, \ \ i \in \{0, 1, 2, \ldots, n-2\}, \\ D^{\beta}_{0+} u(1) = \sum_{j=1}^{p} a_j D^{\beta}_{0+} u\left(\eta_j\right), \end{cases} \tag{3}$$

is given by

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{4}$$

where

$$G(t,s) = g(t,s) + \frac{t^{\alpha-1}}{d}\sum_{j=1}^{p} a_j h\left(\eta_j, s\right), \tag{5}$$

$$g(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases} \tag{6}$$

$$h(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases} \tag{7}$$

where $d = 1 - \sum_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1}$.

Proof. By using Lemma 2.2, the solution of the equation $D_{0+}^{\alpha}u(t) + y(t) = 0$ is

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_1, c_2 \dots, c_n$ are arbitrary real constants.

From the boundary condition in (1), one can $c_2 = c_3 \dots = c_{n-2} = c_{n-1} = c_n = 0$. Hence

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1 t^{\alpha-1}.$$

By the last above equation and Lemma 2.1(i), we get

$$D_{0+}^{\beta}u(t) = \frac{1}{\Gamma(\alpha-\beta)}\left[c_1\Gamma(\alpha)t^{\alpha-\beta-1} - \int_0^t (t-s)^{\alpha-\beta-1}y(s)ds\right],$$

this and by $D_{0+}^{\beta}u(1) = \sum_{j=1}^{p} a_j D_{0+}^{\beta}u\left(\eta_j\right)$, we have

$$c_1 = \frac{1}{d\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-\beta-1}y(s)ds - \sum_{j=1}^{p} a_j \int_0^{\eta_j}\left(\eta_j - s\right)^{\alpha-\beta-1}y(s)ds\right].$$

Then, the unique solution of the problem (1) is given by

$$u(t)=\frac{t^{\alpha-1}}{d\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-\beta-1}y(s)ds-\sum_{j=1}^{p}a_j\int_0^{\eta_j}\left(\eta_j-s\right)^{\alpha-\beta-1}y(s)ds\right]-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds,$$

$$=\frac{1}{\Gamma(\alpha)}\left[\int_0^t\left[t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}\right]y(s)ds+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds\right.$$

$$\left.+\frac{1-d}{d}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds-\frac{t^{\alpha-1}}{d}\sum_{j=1}^{p}a_j\int_0^{\eta_j}\left(\eta_j-s\right)^{\alpha-\beta-1}y(s)ds\right]$$

$$=\int_0^1 g(t,s)y(s)ds+\frac{t^{\alpha-1}}{d}\sum_{j=1}^{p}a_j\left[\int_{\eta_j}^1\eta_j^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}y(s)ds\right.$$

$$\left.+\int_0^{\eta_j}\left[\eta_j^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}-\left(\eta_j-s\right)^{\alpha-\beta-1}\right]y(s)ds\right]$$

$$=\int_0^1 g(t,s)y(s)ds+\frac{t^{\alpha-1}}{d}\sum_{j=1}^{p}a_j{}_0^1h\left(\eta_j,s\right)y(s)ds$$

$$=\int_0^1 G(t,s)y(s)ds.$$

The proof is completed. □

Lemma 2.4. *Let* $\sum_{j=1}^{p}a_j\eta_j^{\alpha-\beta-1}\in[0,1),\alpha\in(n-1,n],\beta\in[1,n-2],n\geq 3$. *Then, the functions* $g(t,s)$ *and* $h(t,s)$ *defined by* (6) *and* (7) *have the following properties:*

i. *The functions* $g(t,s)$ *and* $h(t,s)$ *are continuous on* $[0,1]\times[0,1]$ *and for all* $t,s\in(0,1)$

$$g(t,s)>0,\quad h(t,s)>0.$$

ii. $g(t,s)\leq\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ *for all* $t,s\in[0,1]$.

iii. $g(t,s)\geq t^{\alpha-1}g(1,s)$ *for all* $t,s\in[0,1]$, *where*

$$g(1,s)=\frac{1}{\Gamma(\alpha)}\left[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}\right].$$

From the above properties, we deduce the following properties:

iv. *The function* $G(t,s)\geq 0$ *is continuous on* $[0,1]\times[0,1]$ *and* $G(t,s)>0$ *for all* $t,s\in(0,1)$.

v. $\max_{t\in[0,1]}G(t,s)=G(1,s)$, *for all* $s\in[0,1]$, *where*

$$G(1,s)=g(1,s)+\frac{1}{d}\sum_{j=1}^{p}a_jh\left(\eta_j,s\right)\leq\frac{(1-s)^{\alpha-\beta-1}}{d\Gamma(\alpha)}.$$

Proof. It is easy to chek that (i), (v), (vi) holds. So we prove that (ii) is true. Note that (6) and $0\leq(1-s)^{\alpha-\beta-1}\leq1$. It follows that $g(t,s)\leq\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for all $t,s\in[0,1]$. It remains to prove (iii). We divide the proof into two cases and by (1), we have.

Case1. When $0\leq s\leq t\leq1$, we have

$$g(t,s)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}\left[(1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-1}\right]\geq t^{\alpha-1}g(1,s).$$

Case2. When $0\leq t\leq s\leq1$, we have

$$g(t,s)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1}\geq t^{\alpha-1}g(1,s).$$

Hence $g(t,s)\geq t^{\alpha-1}g(1,s)$ for all $t,s\in[0,1]$. □

2.2 Existence results

First, for the uniqueness results of problem (1), we need the following assumptions.

(A_1) $f(t,u_1)\leq f(t,u_2)$ for any $0<t<1$, $0\leq u_1\leq u_2$.

(A_2) For any $r\in(0,1)$, there exists a constant $q\in(0,1)$ such that

$$f(t,ru)\geq r^qf(t,u),\quad (t,u)\in(0,1)\times[0,\infty). \tag{8}$$

(A_3) $0<\int_0^1 f(s,s^{\alpha-1})ds<\infty$.

We shall consider the Banach space $E=C[0,1]$ equipped with the norm $\|u\|=\max_{0\leq t\leq1}|u(t)|$ and let

$$D=\left\{u\in C^+[0,1]:\exists M_u\geq m_u\geq0,\backslash m_ut^{\alpha-1}\leq u(t)\leq M_ut^{\alpha-1},fort\in[0,1]\right\}, \tag{9}$$

where

$$C^+[0,1]=\{u\in E:u(t)\geq0,t\in[0,1]\}.$$

In view of Lemma 2.3, we define an operator T as

$$(Tu)(t)=\int_0^1G(t,s)y(s)ds, \tag{10}$$

where $G(t,s)$ is given by (5).

By (A_1) it is easy to see that the operator $T:D\to C^+[0,1]$ is increasing. Observe that the BVP (1) has a solution if and only if the operator T has a fixed point.

Obviously, from (A_1) we obtain

$$f(t,ru)\leq r^qf(t,u),\quad\forall r>1,q\in(0,1),\quad (t,u)\in(0,1)\times[0,\infty).$$

In what follows, we first prove $T:D\to D$. In fact, for any $u\in D$, there exist a positive constants $0<m_u<1<M_u$ such that

$$m_u s^{\alpha-1} \le u(s) \le M_u s^{\alpha-1}, \quad s \in [0,1].$$

Then, from (A_1), $f(t,u)$ non-decreasing respect to u and (A_2), we can imply that for $s \in (0,1)$, $q \in (0,1)$

$$(m_u)^q f\left(s, s^{\alpha-1}\right) \le f(s,u(s)) \le (M_u)^q f\left(s, s^{\alpha-1}\right), \quad s \in (0,1). \tag{11}$$

From (11) and Lemma 2.4, we obtain

$$\begin{aligned} Tu(t) &= \int_0^1 g(t,s) f(s,u(s))ds + \frac{t^{\alpha-1}}{d} \sum_{j=1}^{p} a_j \int_0^1 h\left(\eta_j, s\right) f(s,u(s))ds, \\ &\le t^{\alpha-1} \left[\frac{1}{\Gamma(\alpha)} \int_0^1 f(s,u(s))ds + \frac{1}{d} \sum_{j=1}^{p} a_j \int_0^1 h\left(\eta_j, s\right) f(s,u(s))ds \right], \\ &\le t^{\alpha-1} \left[\frac{(M_u)^q}{\Gamma(\alpha)} \int_0^1 f\left(s, s^{\alpha-1}\right)ds + \frac{(M_u)^q}{d} \sum_{j=1}^{p} a_j \int_0^1 h\left(\eta_j, s\right) f\left(s, s^{\alpha-1}\right)ds \right], \quad t \in [0,1], \end{aligned} \tag{12}$$

and

$$\begin{aligned} Tu(t) &= \int_0^1 g(t,s) f(s,u(s))ds + \frac{t^{\alpha-1}}{d} \sum_{j=1}^{p} a_j \int_0^1 h\left(\eta_j, s\right) f(s,u(s))ds, \\ &\ge t^{\alpha-1} \left[\frac{1}{\Gamma(\alpha)} \int_0^1 g(1,s) f(s,u(s))ds + \frac{1}{d} \sum_{j=1}^{p} a_j \int_0^1 h\left(\eta_j, s\right) f(s,u(s))ds \right], \\ &\ge t^{\alpha-1} \left[\frac{(m_u)^q}{\Gamma(\alpha)} \int_0^1 g(1,s) f\left(s, s^{\alpha-1}\right)ds + \frac{(m_u)^q}{d} \sum_{j=1}^{p} a_j \int_0^1 h\left(\eta_j, s\right) f\left(s, s^{\alpha-1}\right)ds \right], \quad t \in [0,1]. \end{aligned} \tag{13}$$

Eqs. (12) and (13) and assumption (A_3) imply that $T: D \to D$.

Now, we are in the position to give the first main result of this chapter.

Theorem 1.1 Suppose $(A_1) - (A_3)$ hold. Then problem (1) has a unique, nondecreasing solution $u^* \in D$, moreover, constructing successively the sequence of functions

$$h_n(t) = \int_0^1 G(t,s) f(s, h_{n-1}(s))ds, \quad t \in [0,1], \quad n = 1, 2, \dots, \tag{14}$$

for any initial function $h_0(t) \in D$, then $\{h_n(t)\}$ must converge to $u^*(t)$ uniformly on $[0,1]$ and the rate of convergence is

$$\max_{t \in [0,1]} |h_n(t) - u^*(t)| = O\left(1 - \theta^{q^n}\right), \tag{15}$$

where $0 < \theta < 1$, which depends on the initial function $h_0(t)$.

Proof. For any $h_0 \in D$, we let

$$l_{h_0} = \sup\{l > 0 : lh_0(t) \leq (Th_0)(t), t \in [0,1]\}, \tag{16}$$

$$L_{h_0} = \inf\{L > 0 : Lh_0(t) \geq (Th_0)(t), t \in [0,1]\}, \tag{17}$$

$$m = \min\left\{1, (l_{h_0})^{\frac{1}{1-q}}\right\}, \quad M = \max\left\{1, (L_{h_0})^{\frac{1}{1-q}}\right\}, \tag{18}$$

and

$$u_0(t) = mh_0(t), \quad v_0(t) = Mh_0(t), \tag{19}$$

$$u_n(t) = Tu_{n-1}(t), \quad v_n(t) = Tv_{n-1}(t), \quad n = 0, 1, \dots, . \tag{20}$$

Since the operator T is increasing, $(A_1), (A_2)$ and (16)–(20) imply that there exist iterative sequences $\{u_n\}, \{v_n\}$ satisfying

$$u_0(t) \leq u_1(t) \leq \dots \leq u_n(t) \leq \dots \leq v_n(t) \leq \dots \leq v_1(t) \leq v_0(t), \ \ t \in [0,1]. \tag{21}$$

In fact, from (19) and (20), we have

$$u_0(t) \leq v_0(t), \tag{22}$$

$$\begin{aligned} u_1(t) = Tu_0(t) &= \int_0^1 G_1(t,s)f(s,mh_0(s))ds + \frac{t^{\alpha-1}}{d}\sum_{i=1}^{n} a_j \int_0^1 G_2\left(\eta_j,s\right)f(s,mh_0(s))ds, \\ &\geq m^q\left[\int_0^1 G_1(t,s)f(s,h_0(s))ds + \frac{t^{\alpha-1}}{d}\sum_{i=1}^{n} a_j \int_0^1 G_2\left(\eta_j,s\right)f(s,h_0(s))ds\right], \\ &\geq m^q Th_0(t) \geq mh_0(t) = u_0(t), \end{aligned} \tag{23}$$

and

$$\begin{aligned} v_1(t) = Tv_0(t) &= \int_0^1 G_1(t,s)f(s,Mh_0(s))ds + \frac{t^{\alpha-1}}{d}\sum_{i=1}^{n} a_j \int_0^1 G_2\left(\eta_j,s\right)f(s,Mh_0(s))ds, \\ &\leq M^q\left[\int_0^1 G_1(t,s)f(s,h_0(s))ds + \frac{t^{\alpha-1}}{d}\sum_{i=1}^{n} a_j \int_0^1 G_2\left(\eta_j,s\right)f(s,h_0(s))ds\right] \\ &\leq M^q Th_0(t) \leq Mh_0(t) = v_0(t). \end{aligned} \tag{24}$$

Then, by (22)–(24) and induction, the iterative sequences $\{u_n\}, \{v_n\}$ satisfy

$$u_0(t) \leq u_1(t) \leq \dots \leq u_n(t) \leq \dots \leq v_n(t) \leq \dots \leq v_1(t) \leq v_0(t), \quad \forall t \in [0,1].$$

Note that $u_0(t) = \frac{m}{M}v_0(t)$, from (A_1), (10), (19) and (20), it can obtained by induction that

$$u_n(t) \geq \theta^{q^n} v_n(t), \ \ t \in [0,1], \ \ n = 0, 1, 2, \dots, \tag{25}$$

where $\theta = \frac{m}{M}$.

From (21) and (25) we know that

$$0 \le u_{n+p}(t) - u_n(t) \le v_n(t) - u_n(t) \le \left(1 - \theta^{q^n}\right) M h_0(t), \quad \forall n, p \in \mathbb{N}, \tag{26}$$

and since $\left(1 - \theta^{q^n}\right) M h_0(t) \to 0, \ \ as \ \ n \to \infty$,this yields that there exists $u^* \in D$ such that

$$u_n(t) \to u^*(t), \quad (uniformly\, on\, [0,1]).$$

Moreover, from (26) and

$$\begin{aligned} 0 \le v_n(t) - u^*(t) &= v_n(t) - u_n(t) + u_n(t) - u^*(t), \\ &\le \left(1 - \theta^{q^n}\right) M h_0(t) \to 0, \quad as \ \ n \to \infty, \end{aligned}$$

we have

$$v_n(t) \to u^*(t), \quad (uniformly\, on\, [0,1]),$$

so,

$$u_n(t) \to u^*(t), \ \ v_n(t) \to u^*(t), \quad (uniformly\, on\, [0,1]). \tag{27}$$

Therefore

$$u_n(t) \le u^*(t) \le v_n(t), \quad t \in [0,1], n = 0, 1, 2, \dots, , \tag{28}$$

From (A_1), (19) and (20), we have

$$u_{n+1}(t) = Tu_n(t) \le Tu^*(t) \le Tv_n(t) = v_{n+1}(t), n = 0, 1, 2, \dots, .$$

This together with (27) and uniqueness of limit imply that u^* satisfy $u^* = Tu^*$, that is $u^* \in D$ is a solution of BVP (1) and (2).
From (19)–(21) and (A_1), we obtain

$$u_n(t) \le h_n(t) \le v_n(t), n = 0, 1, 2, \dots, . \tag{29}$$

It follows from (26)–(29) that

$$\begin{aligned} |h_n(t) - u^*(t)| &\le |h_n(t) - u_n(t)| + |u_n(t) - u^*(t)|, \\ &\le |h_n(t) - u_n(t)| + |u^*(t) - u_n(t)|, \\ &\le 2|v_n(t) - u_n(t)|, \\ &\le 2M\left(1 - \theta^{q^n}\right)|h_0(t)|. \end{aligned}$$

Therefore

$$\max_{t \in [0,1]} |h_n(t) - u^*(t)| \le 2M\left(1 - \theta^{q^n}\right) \max_{t \in [0,1]} |h_0(t)|.$$

Hence, (15) holds. Since $h_0(t)$ is arbitrary in D we know that $u^*(t)$ is the unique solution of the boundary value problem (1) in D. □

We construct an example to illustrate the applicability of the result presented.
Example 2.1. *Consider the following boundary value problem*

$$\begin{cases} D_{0^+}^{\frac{5}{2}}u(t) + \dfrac{(u)^{\frac{2}{3}-\frac{1}{6}\cos(t)}}{\sqrt{t}} = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0,\ u'(1) = \dfrac{\sqrt{2}}{2}u'\left(\dfrac{1}{2}\right), \end{cases} \tag{30}$$

where $\alpha = \frac{5}{2}, \beta = 1, a_1 = \frac{\sqrt{2}}{2}, \eta_1 = \frac{1}{2}$ *and* $f(t,u) = \frac{(u)^{\frac{2}{3}-\frac{1}{6}\cos(t)}}{\sqrt{t}}$ *is increasing function with respect to u for all* $t \in (0,1)$, *so, assumption* (A_1) *satisfied.*
By simple calculation we have $d = 1 - \frac{\sqrt{2}}{2}\left(\sqrt{\frac{1}{2}}\right) = \frac{1}{2}$.
For any $r \in (0,1)$, *there exists* $q = \frac{1}{2} \in (0,1)$ *such that*

$$f(t,ru) = \frac{(ru)^{\frac{2}{3}-\frac{1}{6}\cos(t)}}{\sqrt{t}} \geq r^{\frac{1}{2}}\frac{(u)^{\frac{2}{3}-\frac{1}{6}\cos(t)}}{\sqrt{t}} = r^{\frac{1}{2}}f(t,u),$$

thus, $f(t,u)$ *satisfies* (A_2) *and is singular at* $t = 0$.
On the other hand,

$$\int_0^1 f\left(t, t^{2,5-1}\right)dt \leq \int_0^1 t^{\frac{1}{4}}dt = \frac{4}{5} < \infty,$$

so, assumption (A_3) *is satisfied.*
Hence, all the assumptions of Theorem 1.1 *are satisfied. Which implies that the boundary value* (30) *has an unique, nondecreasing solution* $u^* \in D$.

3. Existence result for inclusion problem (2)

We provide another result about the existence of solutions for the problem (2) by using the assumption of nonconvex compact values for multifunction. Our strategy to deal with this problem is based on the Covitz-Nadler theorem for the contraction multivalued maps [44] for lower semi-continuous maps with decomposable values.

First, we will present notations, definitions and preliminary facts from multivalued analysis which are used throughout this chapter. For more details on the multivalued maps, see the book of Aubin and Cellina [48], Demling [49], Gorniewicz [50] and Hu and Papageorgiou [51], see also [44, 48, 49, 52–54].

Here $(C[0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|u\| = \sup\{|u(t)| : for\,all\,t \in [0,1]\}$, $L^1([0,1],\mathbb{R})$, the Banach space of measurable functions $u : [0,1] \to \mathbb{R}$ which are Lebesgue integrable, normed by $\|u\|_{L^1} = {}_0^1|u(t)|dt$.

Let (X,d) be a metric space induced from the normed space $(X,\|\cdot\|)$. We denote

$$\begin{aligned} &P_0(X) = \{A \in P(X) : A \neq \phi\}, \\ &P_b(X) = \{A \in P_0(X) : A\, is\ bounded\}, \\ &P_{cl}(X) = \{A \in P_0(X) : A\, is\ closed\}, \\ &P_{cp}(X) = \{A \in P_0(X) : A\, is\ compact\}, \\ &P_{b,cl}(X) = \{A \in P_0(X) : A\, is\ closed\ \ and\ \ bounded\}, \end{aligned}$$

where $P(X)$ is the family of all subsets of X.

Definition 3.1. *A multivalued map* $G : X \to P(X)$.

1. $G(u)$ is convex (closed) valued if $G(u)$ is convex (closed) for all $u \in X$,

2. is bounded on bounded sets if $G(B) = \bigcup_{u \in B} G(u)$ is bounded in X for all $B \in P_b(X)$ i.e., $\sup_{u \in B}\{\sup\{|v|, v \in G(u)\}\} < \infty$,

3. has a fixed point if there is $u \in X$ such that $u \in G(u)$. The fixed point set of the multivalued operator G will be denote by Fix G.

Definition 3.2. A multivalued map $G : [0, 1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$ the function

$$t \mapsto d(y, G(t)) = \inf\{\|y - z\| : z \in G(t)\},$$

is measurable.

Definition 3.3. Let Y be a nonempty closed subset of a Banach space E and $G : Y \to P_{cl}(E)$ be a multivalued operator with nonempty closed values.

i. G is said to be lower semi-continuous (l.s.c) if the set $\{x \in X : G(x) \cap U \neq \phi\}$ is open for any open set U in E.

ii. G has a fixed point if there is $x \in Y$ such that $x \in G(x)$.

For each $u \in (C[0, 1], \mathbb{R})$, define the set of selection of F by

$$S_{F,u} = \{v \in AC([0, 1], \mathbb{R}) : v \in F(t, u(t)), \ \textit{for almost all } \ t \in [0, 1]\}.$$

For $P(X) = 2^X$, consider the Pompeiu-Hausdorff metric (see [55]).

$H_d : 2^X \times 2^X \to [0, \infty)$ given by

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\},$$

where $d(a, B) = \inf_{b \in B} d(a, b)$ and $d(b, A) = \inf_{a \in A} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space see [8].

Definition 3.4. Let A be a subset of $[0, 1] \times \mathbb{R}$. A is $L \otimes B$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the $J \times D$, where J is Lebesgue measurable in $[0, 1]$ and D is Borel measurable in $\mathbb{R}$.

Definition 3.5. A subset A of $L^1([0, 1], \mathbb{R})$ is decomposable if all $u, v \in A$ and measurable $J \subset [0, 1] = j$, the function $u\chi_J + v\chi_{j\setminus J} \in A$, where χ_J stands for the caracteristic function of J.

Definition 3.6. Let Y be a separable metric space and $N : Y \to P(L^1([0, 1], \mathbb{R}))$ be a multivalued operator. We say N has property (BC) if N is lower semi-continuous (l.s.c) and has nonempty closed and decomposable values.

Let $F : [0, 1] \times \mathbb{R} \to P(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator

$$\Phi : C([0, 1], \mathbb{R}) \to P(L^1([0, 1], \mathbb{R})),$$

by letting

$$\Phi(u) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,u(t)) for\, a.e. t \in [0,1]\}.$$

Definition 3.7. The operator Φ is called the Niemytzki operator associated with F

. We say F is of the lower semi-continuous type (l.s.c type) if its associated Niemytzki operator Φ has (BC) property.

Definition 3.8. A multivalued operator $N : X \to P_{cl}(X)$ is called.

i. $\rho-$Lipschitz if and only if there exists $\rho > 0$ such that $H_d(N(u), N(v)) \leq \rho d(u,v)$ for each $u, v \in X$,

ii. a contraction if and only if it is $\rho-$Lipschitz with $\rho < 1$.

Lemma 3.1. ([44] Covitz-Nadler). *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \phi$, where $FixN$ is the fixed point of the operator N.*

Definition 3.9. A measurable multivalued function $F : [0,1] \to P(X)$ is said to be integrably bounded if there exists a function $g \in L^1([0,1],X)$ such that, for all $v \in F(t)$, $\|v\| \leq g(t)$ for a.e. $t \in [0,1]$.

Let us introduce the following hypotheses.

(A_4) $F : [0,1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ be a multivalued map verifying.

i. $(t,u) \mapsto F(t,u)$ is $L \otimes B$ measurable.

ii. $u \mapsto F(t,u)$ is lower semi-continuous for $a.e.\ t \in [0,1]$.

(A_5) F is integrably bounded, that is, there exists a function $m \in L^1([0,1],\mathbb{R}^+)$ such that $\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \leq m(t)$ for almost all $t \in [0,1]$.

Lemma 3.2. [56] *Let $F : [0,1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ be a multivalued map. Assume (A_4) and (A_5) hold. Then F is of the $l.s.c.$ type.*

Definition 3.10. A function $u \in AC^2([0,1],\mathbb{R})$ is called a solution to the boundary value problem (2) if u satisfies the differential inclusion in (2) $a.e.$ on $[0,1]$ and the conditions in (2).

Finally, we state and prove the second main result of this Chapter. We prove the existence of solutions for the inclusion problem (2) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler. For investigation of the problem (2) we shall provide an application of the Lemma 3.4 and the following Lemma.

Lemma 3.3. ([13]) *A multifunction $F : X \to C(X)$ is called a contraction whenever there exists $\gamma \in (0,1)$ such that $H_d(N(u), N(v)) \leq \gamma d(u,v)$ for all $u, v \in X$.*

Now, we present second main result of this section.

Theorem 1.2 Assume that the following hypothyses hold.

(H_1) $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is an integrable bounded multifunction such that the map $t \mapsto F(t,u)$ is measurable,

(H_2) $H_d(F(t,u_1), F(t,u_2)) \leq m(t)|u_1 - u_2|$ for almost all $t \in J$ and $u_1, u_2 \in \mathbb{R}$ with $m \in L^1(J,\mathbb{R})$ and $d(0, F(t,0)) \leq m(t)$ for almost all $t \in J$. Then the problem (2) has a solution provided that

$$l = \int_0^1 G(1,s) m(s) ds < 1.$$

Proof. We transform problem (2) into a fixed point problem. Consider the operator $N : C[0,1] \to P(C[0,1],\mathbb{R})$ defined by

$$N(u) = \left\{ h \in X, \exists y \in S_{F,u} \backslash h(t) = \int_0^1 G(t,s)y(s)ds, t \in J \right\}, \tag{31}$$

where $G(t,s)$ defined by (5). It is clear that fixed points of N are solution of (2).

We shall prove that N fulfills the assumptions of Covitz-Nadler contraction principle.

Note that, the multivalued map $t \mapsto F(t,u(t))$ is measurable and closed for all $u \in AC^1([0,\infty))$ (e.g., [52] Theorem III.6). Hence, it has a measurable selection and so the set $S_{F,u}$ is nonempty, so, $N(u)$ is nonempty for any $u \in C([0,\infty))$.

First, we show that $N(u)$ is a closed subset of X for all $u \in AC^1([0,\infty),\mathbb{R})$. Let $u \in X$ and $\{u_n\}_{n\geq 1}$ be a sequence in $N(u)$ with $u_n \to u, as\ n \to \infty$ in $u \in C([0,\infty))$. For each n, choose $y_n \in S_{F,u}$ such that

$$u_n(t) = \int_0^1 G(t,s)y_n(s)ds.$$

Since F has compact values, we may pass onto a subsequence (if necessary) to obtain that y_n converges to $y \in L^1([0,1],\mathbb{R})$ in $L^1([0,1],\mathbb{R})$. In particular, $y \in S_{F,u}$ and for any $t \in [0,1]$, we have

$$u_n(t) \to u(t) = \int_0^1 G(t,s)y(s)ds,$$

i.e., $u \in N(u)$ and $N(u)$ is closed.

Next, we show that N is a contractive multifunction with constant $l < 1$. Let $u, v \in C([0,1],\mathbb{R})$ and $h_1 \in N(u)$. Then there exist $y_1 \in S_{F,u}$ such that

$$h_1(t) = \int_0^1 G(t,s)y_1(s)ds, \quad t \in J.$$

By (H_2), we have

$$H_d(F(t,u(t)),F(t,v(t))) \leq m(t)(|u(t) - v(t)|),$$

for almost all $t \in J$.

So, there exists $w \in S_{F,v}$ such that

$$|y_1(t) - w| \leq m(t)(|u(t) - v(t)|),$$

for almost all $t \in J$.

Define the multifunction $U : J \to P(\mathbb{R})$ by

$$U(t) = \{ w \in \mathbb{R} : |y_1(t) - w| \leq m(t)(|u(t) - v(t)|)\ for\ almost\ all\ t \in J \}.$$

It is easy to chek that the multifunction $V(\cdot) = U(\cdot) \cap F(\cdot, v(\cdot))$ is measurable (e.g., [52] Theorem III.4).

Thus, there exists a function $y_2(t)$ which is measurable selection for V. So, $y_2 \in S_{F,v}$ and for each $t \in J$, we have

$$\left|y_1(t) - y_2(t)\right| \leq m(t)(|u(t) - v(t)|).$$

Now, consider $h_2 \in N(u)$ which is defined by

$$h_2(t) = \int_0^1 G(t,s)y_2(s)ds, \ \ t \in J,$$

and one can obtain

$$|h_1(t) - h_2(t)| \leq \int_0^1 G(t,s)\left|y_1(s) - y_2(s)\right|ds$$

$$\leq \int_0^1 G(1,s)m(s)|u(s) - v(s)|ds.$$

Hence

$$\|h_1(t) - h_2(t)\| \leq \|p\|_\infty \|u - v\| \left[\int_0^1 G(1,s)m(s)ds\right].$$

Analogously, interchanging the roles of u and v, we obtain

$$H_d(N(u), N(v)) \leq \|u - v\| \left[\int_0^1 G(1,s)m(s)ds\right].$$

Since N is a contraction, it follows by Lemma 3.1 (by using the result of Covitz and Nadler) that N has a fixed point which is a solution to problem (2). □

We construct an example to illustrate the applicability of the result presented.

Example 3.1. *Consider the problem*

$$-D^\alpha u(t) \in F(t, u(t)), \quad t \in [0,1], \tag{32}$$

subject to the three-point boundary conditions

$$u^{(i)}(0) = 0, i \in \{0,1\}, \ \ D^\beta_{0^+} u(1) = \sum_{j=1}^{2} a_j D^\beta_{0^+} u\left(\eta_j\right), \tag{33}$$

where $\alpha = \frac{5}{2}, \beta = 1\, a_1 = \frac{1}{2}, a_2 = \frac{3}{2}, \eta_1 = \frac{1}{16}, \eta_2 = \frac{5}{16}$ *and* $F(t, u(t)) : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ *multivalued map given by*

$$u \mapsto F(t,u) = \left(0, \frac{t|u|}{2(1 + |u|)}\right), u \in \mathbb{R},$$

verifying (H_1).

Obviously,

$$\sup\{|f| : f \in F(t,u)\} \le \frac{t+1}{2},$$

we have

$$H_d(F(t,u), F(t,v)) \le \left(\frac{t+1}{2}\right)|u-v|, u, v \in \mathbb{R}, \quad t \in [0,1],$$

which shows that (H_2) *holds*
So, if $m(t) = \frac{t+1}{2}$ *for all* $t \in [0,1]$*, then*

$$H_d(F(t,u), F(t,v)) \le m(t)|u-v|.$$

It can be easily found that $d = 1 - \frac{1}{2}\left(\frac{1}{16}\right)^{\frac{5}{2}} - \frac{3}{2}\left(\frac{5}{16}\right)^{\frac{5}{2}} = 0,9176244637.$
Finally,

$$l = \int_0^1 G(1,s)m(s)ds = 0,4636273746 < 1.$$

Hence, all assumptions and conditions of Theorem 1.2 *are satisfied. So, Theorem* 1.2 *implies that the inclusion problem* (32) *and* (33) *has at least one solution.*

4. Conclusions

This chapter concerns the boundary value problem of a class of fractional differential equations involving the Riemann-Liouville fractional derivative with nonlocal boundary conditions. By using the properties of the Green's function and the monotone iteration technique, one shows the existence of positive solutions and constructs two successively iterative sequences to approximate the solutions. In the multi-valued case, an existence result is proved by using fixed point theorem for contraction multivalued maps due to Covitz and Nadler. The results of the present chapter are significantly contribute to the existing literature on the topic.

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Conflict of interest

The authors declare no conflict of interest.

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