Chapter

Sparse Linear Antenna Arrays: A Review

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Abstract

Linear sparse antenna arrays have been widely studied in array processing literature. They belong to the general class of non-uniform linear arrays (NULAs). Sparse arrays need fewer sensor elements than uniform linear arrays (ULAs) to realize a given aperture. Alternately, for a given number of sensors, sparse arrays provide larger apertures and higher degrees of freedom than full arrays (ability to detect more source signals through direction-of-arrival (DOA) estimation). Another advantage of sparse arrays is that they are less affected by mutual coupling compared to ULAs. Different types of linear sparse arrays have been studied in the past. While minimum redundancy arrays (MRAs) and minimum hole arrays (MHAs) existed for more than five decades, other sparse arrays such as nested arrays, co-prime arrays and super-nested arrays have been introduced in the past decade. Subsequent to the introduction of co-prime and nested arrays in the past decade, many modifications, improvements and alternate sensor array configurations have been presented in the literature in the past five years (2015–2020). The use of sparse arrays in future communication systems is promising as they operate with little or no degradation in performance compared to ULAs. In this chapter, various linear sparse arrays have been compared with respect to parameters such as the aperture provided for a given number of sensors, ability to provide large hole-free co-arrays, higher degrees of freedom (DOFs), sharp angular resolutions and susceptibility to mutual coupling. The chapter concludes with a few recommendations and possible future research directions.

Keywords: Antenna Arrays, Array Signal Processing, Co-array MUSIC, Co-prime arrays, Degrees of Freedom (DOFs), Difference co-array (DCA), Direction of Arrival (DOA), Fractal arrays, Fragility of sparse arrays, Minimum redundancy arrays (MRAs), Nested arrays and Sparse linear arrays

1. Introduction

Array processing research has flourished and raked-in much attention in the past five to six decades. It has been an evergreen topic that has fancied many researchers due to the sheer variety of its applications. Array Processing is a scientific field of study which involves the processing of information-bearing signals received by an array of sensors operating in an environment of interest [1], for example, on the ground, above ground or under water. An array has two or more sensors which are arranged in a specific geometrical layout. An array has better directional properties than an individual sensor.

Sensor arrays find application in diverse fields such as radar (radio detection and ranging), space exploration, sonar (sound navigation and ranging), seismology, chemical sensing, medical imaging, wireless communications, navigation, source localization etc. Depending on the application, the sensors could be antennas, microphones, hydrophones, geophones, ultrasonic probes etc. For example, hydrophone arrays are used in sonar; acoustic arrays are used for audio source localization; piezoelectric sensors are used in medical ultrasound; geophone arrays are used in seismology etc. More specifically, antenna arrays are used for electromagnetic applications such as radar, radio astronomy, remote sensing, wireless communications, positioning and navigation [2]. Exhaustive reading for the topic of array processing can be found in [3–5]. A mention of the multidisciplinary nature of this topic is given in the introductory chapter of [4] and final chapters of [3, 5]. A thorough coverage of phased array theory and array processing applications in the modern era is provided in [6]. Arrays can have different geometries – such as linear, planar, circular, hexagonal, spherical etc. An array spanning across more dimensions can extract more details about the scene of interest.

In many of the above applications, it is of primary interest to sense the directions from which the source signals impinge the array. The signal of interest may be an electromagnetic wave, a sound wave, an underwater acoustic wave, a gas diffused into the surroundings, the location of a tumor etc. The use of multiple sensors bestows the array with a sense of direction. Individual sensors cannot sense direction.

A uniform linear array (array in which the antenna elements are arranged along a straight line) with an inter-element spacing of d is given in **Figure 1**.

Direction of Arrival (DOA) estimation involves determining the directions/angles at which electromagnetic sources are located with reference to the receiver array or the directions from which electromagnetic signals strike the array [7]. DOA estimation methods have been classified into three broad approaches, namely, classical methods, subspace methods and maximum-likelihood (ML) methods [8]. A new class of DOA estimation algorithms based on compressed sensing (CS) and sparse recovery have received much attention in the recent past [9–12]. Dealing with coherent arrivals is a main problem in DOA estimation of practical sources. Two signals are said to be coherent if one is a scaled and shifted version of the other. Multipath interference and intentional jamming are the main causes for sources being coherent to each other [13]. When coherent sources arrive at the array, the array covariance matrix becomes non-diagonal, singular and rank-deficient. That is, its rank would be less than the number of incoming signals [14]. Hence, when subspace methods are used for DOA estimation, an additional step of spatial smoothing would be needed to restore the rank of the covariance matrix. Also, a major drawback of subspace methods is that they need prior information about the number of source angles to be detected, which is often impossible in practical

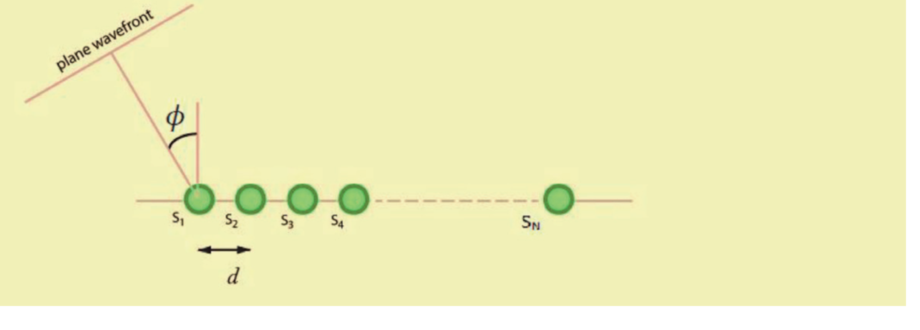

Figure 1.
A uniform linear array (ULA) with N *sensors.*

scenarios (as the number of sources is usually unknown, a priori). Akaike information criteria (AIC) test and/or minimum description length (MDL) test and their variants are generally used to estimate the number of sources beforehand. However, these methods are also susceptible to failure when the signals are coherent [15].

In a passive array signal processing system, the array of sensors just listens to the environment, as in passive sonar, radio astronomy and wireless communication. Contrarily, in an active array signal processing system, a transmitter is used to illuminate the environment and the array listens to the signal reflected by the environment and/or objects of interest, as in radar, active sonar and medical imaging [1]. Both passive and active DOA estimation are quite famous in array processing literature.

Antenna arrays offer better gain and directivity than single antennas. High directivity enables the array to confine its radiation or reception to certain directions. As the array size increases (i.e., as the number of array elements increase), so does its aperture. Arrays with wider apertures provide narrower beam-widths and finer angular resolutions than those with smaller apertures [16]. The spatial response or radiation pattern of the array indicates the directions in which the array radiates its energy or receives energy from. Due to its directional properties, an array is often regarded as a spatial filter [17].

An antenna array can serve two purposes. It can help (i) determine the directions from which source signals impinge the receiver (direction of arrival (DOA) estimation), and (ii) in focusing the radiation pattern towards certain directions based on the knowledge of desired and undesired signal directions (beamforming). Additionally, antenna arrays also offer electronic beam-steering, whereby the array's main beam can be pointed towards a desired direction just by controlling the element phases, without the need for any mechanical movement of the antenna platform.

A static array is one whose response does not change over time. The performance of such an array may be degraded severely under situations such as interfering signals, clutter returns, deliberate electronic countermeasures, non-hostile interference and multipath propagation. An adaptive array has the ability to control its response based on changing conditions of the signal environment coupled with the knowledge of desired and undesired signal directions [2].

Beamforming is the process of obtaining sharply focused beams in a given direction by applying a suitable set of complex weights (element currents and phases) to each of the sensors. Adaptive beamforming (ABF) involves the continuous adjustment of these weights to track the changes in the signal environment using adaptive algorithms that are based on certain specific optimization criteria. Generally, the optimization criteria optimize some measure of system performance such as mean square error, variance or likelihood [7].

Element spacing in an array is of high importance as it influences the occurrence of grating lobes. Grating lobes are large undesired side lobes (that are similar to the main beam in stature) that appear in the radiation pattern and radiate the energy in unwanted directions. Ideally, an inter-element spacing of half wavelength is followed in order to avoid spatial aliasing and to prevent the formation of grating lobes. In applications that require a limited field-of-view (FoV), presence of grating lobes does not hinder the array operation if the scanning region is limited to the grating-lobe-free area. An example of this is found in automotive radar systems, where the FoV is limited to $-45°$ to $+45°$. Hence, an inter-element spacing of 0.59λ is generally used. Even though grating lobes exist beyond $\pm 45°$ in the above case, their presence is least bothersome as the scanning region is limited to $\pm 45°$.

Beamforming is one of the key enabling technologies for 5G systems that operate in the millimeter wave frequency band [18, 19]. It is known that millimeter frequencies are more sensitive to blockages and path loss. Huge array gains and sharp directional beams obtained from antenna arrays housing hundreds of antennas can counteract the propagation effects of the millimeter channel. DOA estimation also assumes prominence in 5G as DOA-based beamforming is one of the main requirements for smart antennas [20].

1.1 Motivation for sparse arrays

Though there are many definitions of sparse arrays in the array processing literature, the focus of this book chapter would be only on grid-based sparse arrays.

In grid-based sparse arrays, the sensors are assumed to be located on the grid points defined by an integer multiple of the basic inter-element spacing. That is, the sensors are assumed to lie at $0, d, 2d,$ and so on. In general, the inter-element spacing would be half wavelength. The element positions are normalized to the half wavelength. Accordingly, a sparse array with sensor positions {0, 2, 4, 5} means that the array has an overall aperture of five units and its sensors are located at the grid points 0, $2d$, $4d$ and $5d$. There are no sensors at grid points d, $3d$. As the array is sparsely populated with sensors compared to a regular/full array, it is called as a sparse array. Sparse arrays need fewer sensors or active elements than ULAs to realize a given aperture length. Hence, they provide huge savings in the system costs associated thereof (e.g., feed, power consumption, and radio frequency chains). They are a type of green technology in array processing.

Different types of linear sparse arrays have been studied in the past. Minimum Redundancy Arrays (MRAs) have numerous useful properties and had been primarily studied in the past in relation to radio astronomy [21, 22]. MRAs have also been applied in digital communications [23–25]. Numerous properties and modern applications of MRAs can be found in [26–28].

The use of sparse or non-uniform arrays in close future communication systems such as fifth generation (5G) telecom and the Internet of Things (IoT) is appealing as sparse arrays need fewer active elements than ULAs and operate with little or no degradation in performance [29].

1.2 Contribution of this chapter

The area of sparse arrays is continuously evolving and there are at least 50 types of linear sparse arrays in the current literature. A comprehensive understanding of various sparse array configurations is essential in the current scenario as more and more sparse arrays are being rapidly introduced into the literature. To the best of our knowledge, a comprehensive review on the properties of 1D sparse arrays has not been taken up in the past (barring the works by Liu and Vaidyanathan, Cohen and Eldar). This is a sincere attempt to bring a few well-known sparse arrays under one roof so that their characteristics could be compared.

The rest of the chapter is organized as follows. Section 2 gives an introduction to linear sparse arrays. Section 3 explains some terminology related to sparse arrays and discusses the characteristics of a few well-known sparse arrays. Section 4 gives an overview of sparse arrays with special properties and also discusses the effect of sensor failures on array performance. Section 5 provides a few future directions and Section 6 concludes the chapter.

2. Introduction to sparse arrays

Sparse arrays have aperture widths equal to regular/filled/full arrays but are sparsely populated. They consist of voids that arise due to missing/inactive sensors. The vacancies are deliberately created and are woven into the design of the sparse array to achieve certain desired characteristics. The span of the array is called its aperture. It is the sum of all the inter-element spacings. A ULA with N sensors, each separated by an inter-element spacing of d has an aperture distance or aperture length of $(N-1)d$. The aperture in the units of the inter-element spacing is given by $L = N-1$. Sparse arrays need fewer than N sensors ($N_s < N$) to offer the same aperture. Failure or absence of either the first sensor or the last sensor in an array of N sensors reduces the array aperture by one unit. Failure of both sensors reduces the array aperture by two units. Hence, in the analysis of thinned/sparse arrays or when analyzing arrays with sensor failures, it is generally assumed that the first and the last sensor are always functional, intact and active so that the array aperture is preserved.

Another advantage of sparse arrays is that they are less affected by mutual coupling compared to ULAs. Sparse arrays are generally analyzed in the co-array domain. A difference co-array (DCA) is formed from the physical sparse array by considering all the spatial lags (differences) that can be generated using the available sensors. A missing spatial lag forms a hole. The DCA should be hole-free as the presence of holes introduces ambiguity in the estimation of spatial correlation and hence spatial angles.

There are several types of linear sparse arrays such as minimum hole arrays (MHAs) or Golomb Arrays, minimum redundancy arrays (MRAs), co-prime arrays, nested arrays, super-nested arrays and so on. MRAs and MHAs have been widely used for interferometry in radio astronomy [6, 21, 22, 30, 31]. While MRAs and MHAs existed for more than five decades, other sparse arrays such as co-prime arrays, nested arrays and super-nested arrays have been introduced in the past decade [32–34]. A good review on the properties of these sparse arrays can be found in the initial sections of [27, 35]. The introduction of co-prime array in the past decade can be considered as a watershed moment which has opened doors for modern applications of linear sparse arrays. Following that, nested arrays were introduced. These arrays offer hole-free co-arrays and also have closed-form expressions to determine the sensor locations. Many variants of the co-prime array [36–38] and the nested array [39, 40] have been proposed in the recent past. These arrays either improve the aperture or reduce the number of sensors needed to obtain a given aperture or make the array more immune to mutual coupling or they increase the hole-free region in the DCA.

Following are the desirable characteristics of sparse arrays:

- The chosen array should provide the largest aperture for a given number of sensors. It is well-known that arrays with larger apertures provide better resolution and DOA estimation accuracy.
- The array should be least affected by mutual coupling.
- The array should have a hole-free co-array to facilitate unambiguous parameter estimation using the entire span of the co-array.

It is desirable that the array has closed-form expressions for sensor positions. Otherwise, there should be a provision to obtain the sensor positions using a look-up table (LUT) or through tabulated entries.

3. Properties of linear sparse arrays

3.1 Sparse array terminology

3.1.1 Sparse array notations

In general, sparse arrays such as MRAs and MHAs are represented in the $\{.a.b.c.d.\}$ format or simply $\{a, b, c, d\}$ without the dots. This format has $n-1$ entries for a $n-$element array. A sparse array configuration of $\{a, b, c, d\}$ means that the array has five sensors at respective locations $\{0, a, a+b, a+b+c, a+b+c+d\}$. For example, an 8-element MRA may be denoted as {.1.3.6.6.2.3.2.} or {1, 3, 6, 6, 2, 3, 2} or {1, 3, 6^2, 2, 3, 2}. The power indicates the number of times the given spacing should be repeated. Therefore, as per the formulation, the above 8-element MRA has sensors at {0, 1, 4, 10, 16, 18, 21, 23}.

Another common representation is in the form of a binary string of 1 s and 0 s which represent the presence or absence of a sensor element on the respective grid point. For example, in the representation $\{.a.b.c.d.\}$, the dots indicate the presence of sensors and hence have to be written as 1. Accordingly, the binary string for the above sparse array would be {1, (a-1) zeroes, 1, (b-1) zeroes, 1, (c-1) zeroes, 1, (d-1) zeroes, 1}. Considering the 8-element MRA {.1.3.6.6.2.3.2.} given above, the binary string would be {1, (1-1) zeroes, 1, (3-1) zeroes and so on}. This gives the binary string {1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1}. Zero indicates the absence of a sensor at the respective grid point. The grid positions of all sensors for which the binary entry is one gives the sensor locations of the sparse array i.e., {0, 1, 4, 10, 16, 18, 21, 23} in the case of the 8-element MRA given above. The reader has to be comfortable in changing from one form to another as different papers and textbooks follow different notations [4, 41, 42]. The auto-correlation of the above binary string gives the weight function (explained shortly) in the coarray domain.

3.1.2 Difference set and the difference co-array

For a sparse array with sensors at $\mathbb{Z} = \{z_1, z_2, \dots, z_{N_s}\}$, the difference set is defined as

$$\mathbb{H} = \left\{z_i - z_j; i, j = 1, 2, \dots, N_s\right\} \tag{1}$$

The distinct entries $(\mathbb{H}_d)$ of the difference set form the difference co-array (DCA) or simply co-array of the physical sparse array. The DCA is symmetric, that is, every $p \in \mathbb{H}_d \exists -p \in \mathbb{H}_d$.

In simple terms, the difference set is obtained by subtracting all possible sensor positions in the given sparse array. This gives rise to spatial lags. The DCA is then formed by considering only the non-repeating (distinct) spatial lags. For example, a sparse array with sensors at {0, 1, 4, 6} can generate all spatial lags (differences) between 0 and 6, resulting in a difference set of {0, −1, −4, −6, 1, 0, −3, −5, 4, 3, 0, −2, 6, 5, 2, 0}. The DCA {−6, −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, 6} is obtained by sorting the numbers and retaining only the distinct elements. Note that the repeating lag of zero is considered only once.

The number of unique lags in the DCA of a sparse array gives the number of source angles that can be detected during DOA estimation (often known as the degrees of freedom (DOFs) offered by the sparse array). Unique lags are often used in the analysis of sparse arrays with holes in the DCA. The usefulness of arrays that have holes in the DCA is limited by the span of the central continuous portion of the DCA. Though there are methods [43, 44] that can extend the continuous portion of the DCA, they are computationally intense. A more useful parameter, namely, the uniform degrees of freedom (UDOF), which indicates the number of continuous entries in the DCA; is often used to avoid ambiguity (as it clearly specifies the hole-free span of the coarray) [43].

3.1.3 The weight function

Defining w_n equal to 1 if the sensor is present at grid location nd, and 0 for the case of a missing sensor, the weight function is defined as the auto-correlation of w_n. In simple terms, the weight function gives the number of times a given spatial lag is generated or the number of sensor pairs that can generate a given spatial lag. This weight function should not be confused with the weight vector that defines element currents and phases during beamforming.

A unit spacing corresponds to a spatial lag of one and can be generated by any pair of sensors that are adjacent to each other. In other words, any two sensors that are half-wavelength apart from each other are said to have unit spacing. The concept of unit spacing is important in determining the role of mutual coupling on the array's performance. Empirically, arrays that have a large number of adjacent sensors are more susceptible to the effects of mutual coupling than those with fewer unit spacings.

3.2 Types of sparse arrays

It is necessary to understand the structure of a ULA and the normalized sensor positions in order to comprehend the properties of grid-based sparse arrays.

3.2.1 Uniform linear array (ULA)

As per definition, there would be no missing sensors in the ULA. Hence, the sensor positions are given by the set $\mathbb{Z}_{ULA} = \{0, 1, 2, \ldots, N-1\}$. All sensor spacings are normalized to half wavelength.

3.2.2 Minimum redundancy arrays (MRAs)

MRAs are synthesized from a ULA by eliminating selected sensors such that the sensors thus retained are capable of generating all possible spacings between zero and a specified number [41]. A ULA has many sensor combinations that provide a given spatial lag (e.g., considering the sensor positions from 0 to 9, a spatial lag of 4 can be obtained using any of the redundant sensor pairs {9, 5}, {8, 4}, {7, 3}, {6, 2}, {5, 1} and {4, 0}). MRAs minimize this redundancy by carefully removing select sensors. An optimum MRA has sensors at positions that are just enough to provide all the spatial lags from 0 up to a maximum number L. A zero redundancy MRA is one which generates each spatial lag exactly once. The 4-element MRA {0, 1, 4, 6} generates each spatial lag from 1 to 6 exactly once and is a zero redundancy MRA.

For arrays with more than four elements, it is not possible to get rid of the redundancy completely if one were to ensure all possible spatial lags. For example, in a 5-element MRA whose sensors are located at {0, 2, 5, 8, 9}, it can be seen that the available sensor pairs can generate all the spacings between 0 to 9. But in particular, a spacing of 3 can be obtained using the sensor pairs {5, 2} and {8, 5}. This leads to redundancy but is inevitable for large arrays.

Consider an N-element MRA with an aperture of $L\left(\leq \frac{N(N-1)}{2}\right)$. The redundancy of the MRA is given by

$$R = \frac{N(N-1)}{2L} \tag{2}$$

A value of $R = 1$ indicates zero redundancy i.e., there are no redundant sensor pairs and that each spatial lag is generated exactly once. However, in practice, $R > 1$, indicating that the array contains certain redundant sensor pairs that can generate a given spatial lag more than once. Since zero redundancy cannot be achieved in arrays with more than four sensors, MRAs are optimized to achieve the configuration that provides the minimum redundancy possible for a given number of sensors.

MRAs do not have closed-form expressions to determine the optimum sensor positions and have to be synthesized using exhaustive search mechanisms. The optimization problem to find sensor positions in MRAs is given by

$$max: L(x_1, x_2, \dots, x_n)\ s.t.h(x_1, x_2, \dots, x_n) = 0, x_i \in [0, L] \tag{3}$$

where x_i indicate the positions of the array elements in the ascending order; the constraint $h(x_1, x_2, \dots, x_n) = 0$ ensures that there are no missing spatial lags within the segment $[0, L]$. The optimization wishes to maximize the segment $[0, L]$ using N sensors without any missing lags [45].

3.2.3 Minimum hole arrays (MHAs)

MHAs are obtained by optimizing the sensor positions such that a given spatial lag is obtained at most once. In other words, the sensors should be placed such that the spatial lags generated by them are unique. No two sensor pairs shall generate the same lag. Additionally, it is not bothersome in MHAs even if the available sensor pairs cannot generate all the spacings between 0 and L. For example, a 5-element MHA with sensors at {0, 1, 4, 9, 11}, can generate almost all the spatial lags between 0 and 11 but fails to generate a spacing of 6. It should also be noted that there should be no more than one sensor pair to generate a given spatial lag. A perfect MHA is one which can generate all the spacings between 0 and L exactly once. A 4-element MHA with sensor positions {0, 1, 4, 6} is perfect. MHAs are also referred to as Golomb arrays and the sensor positions represent the marks on the Golomb ruler. Perfect Golomb rulers or MHAs do not exist for arrays with more than four sensors and hence only optimum rulers can be designed for such cases. Optimal Golomb rulers up to 19 marks have been presented in the past [46]. Like MRAs, there are no closed-form expressions to determine the optimum placement of sensors in MHAs. MHAs are also known as non-redundant arrays.

Coincidentally, the definitions of a zero redundancy MRA and a perfect Golomb array (MHA) bear the same physical meaning. Arrays with fewer than four elements qualify both as zero redundancy MRAs as well as perfect MHAs (Eg: A 4-element array with sensors at {0, 1, 4, 6}). However, for arrays with more than four sensors, neither zero redundancy MRAs exist nor do perfect MHAs. MRAs and MHAs mean different things for arrays with more than four sensors.

The objective function to synthesize MHAs is described below. A Golomb ruler consists of a set of integers $A = \{a_1, a_2, \dots, a_n\}$, in ascending order, such that for each non-zero integer x, there is at most one solution to the equation $x = a_j - a_i |\, a_i, a_j \in A$. The set of integers, A represents the positions of n marks on a ruler [47]. For MHAs, $a_1 = 0; a_n = L$. The largest known optimum ruler till date has 27 sensors and can provide an aperture of 553 [31]. Obtaining optimal Golomb rulers is a computationally hard problem [48].

3.2.4 Co-prime arrays

The coprime array consists of two ULAs. It was one of the first sparse arrays introduced with closed-form expressions (CFE) for sensor positions. That is, the sensor locations or the sparse array configuration can be immediately obtained once the number of sensors is given without the need for any exhaustive search mechanisms. One ULA has $2P$ sensors with a spacing of Q units and another ULA of Q sensors spaced P units apart. P and Q are co-prime integers such that $P < Q$ [27]. The first ULA is given by $S_1 = \{qP|q = 0, 1, 2, \dots, Q - 1\}$ and the second ULA is given by $S_2 = \{pQ|p = 0, 1, 2, \dots, 2P - 1\}$.

The overall co-prime array is $S_{cp} = S_1 \cup S_2$. For example, considering $P = 2, Q = 3$; we have $S_1 = \{0, 2, 4\}, S_2 = \{0, 3, 6, 9\}$ and the resulting co-prime array has sensors at $\{0, 2, 3, 4, 6, 9\}$. The drawback of co-prime arrays is that their DCAs are not hole-free.

3.2.5 Nested array or the two-level nested array

Nested arrays (NA) can provide hole-free DCAs and were introduced as an alternative to MRAs and as an improvement over co-prime arrays. Nested arrays are better than co-prime arrays as they provide hole-free co-arrays. They too provide CFEs for element positions when the number of sensors is known. Two ULAs are needed to obtain a nested array. The first ULA has N_1 sensors with unit spacing and the second ULA has N_2 sensors with a spacing of $N_1 + 1$. The overall nested array is given by the union of these two ULAs [33]. The optimal values of N_1 and N_2 for a given number of sensors are

$$
\begin{aligned}
N_1 = N_2 = \frac{N}{2}; N\ even \\
N_1 = \frac{N-1}{2}, N_2 = \frac{N+1}{2}; N\ odd
\end{aligned} \tag{4}
$$

For example, in a 10-element NA, $N_1 = N_2 = 5$. The level 1 ULA has sensors at {0, 1, 2, 3, 4} and the level 2 ULA has sensors at {5, 11, 17, 23, 29}. The overall nested array is given by {0, 1, 2, 3, 4, 5, 11, 17, 23, 29}. These arrays are well-known by the name two-level nested arrays (TLNA). Unless otherwise specified, all instances of nested arrays mentioned in this chapter refer to the two-level nested array.

3.2.6 Super-nested arrays

Though nested arrays are better than co-prime arrays in terms of the ability to provide hole-free co-arrays, they are severely affected by mutual coupling. This is due to the dense ULA portion at the beginning (level 1). Super-nested arrays were introduced to overcome this drawback of nested arrays [34]. In super-nested arrays, the level 1 elements of the NA are re-arranged (interleaved) to different positions within the span of the array so that the number of sensors with unit spacing gets reduced, thereby making the array less susceptible to mutual coupling. A 10-element super-nested array has sensors at {0, 2, 4, 7, 9, 11, 17, 23, 28, 29}. It can be observed that the level 1 elements of the NA are interleaved to different positions. Super-nested arrays too provide the same aperture as nested arrays for a given number of sensors and have CFEs for element positions. The formulation of super-nested arrays is slightly complicated and is, therefore, not explained here.

As a continuation, augmented nested arrays (ANAs) [49] were formulated. In ANAs, the level 1 dense sub-array of NA is split into several parts and is re-arranged to the left and right of the level 2 sparse array. The design of ANAs is elegant as they provide larger apertures, higher DOFs, and are less susceptible to mutual coupling than nested and super-nested arrays.

3.2.7 Yang's improved nested array

An improved nested array (INA) that provides larger aperture than the nested array for the same number of sensors has been proposed [50]. The improved nested array has a hole-free co-array. However, like the original nested array, the Yang's nested array is also vulnerable to the effects of mutual coupling as it too has a dense ULA portion at the beginning. This array has a total of $N = N_1 + N_2 + 1$ sensors of which N_1 sensors at level 1 with unit spacing, N_2 sensors at level 2 with an inter-element spacing of $N_1 + 2$ and a separate sensor at $(N_1N_2 + 2N_2 + N_1 - 1)$.

The values of N_1 and N_2 are given by

$$N_1 = \frac{N}{2} - 1, N_2 = \frac{N}{2}; N\ even$$
$$N_1 = \frac{N-1}{2} - 1, N_2 = \frac{N+1}{2}; N\ odd \tag{5}$$

It follows that a 10-element Yang's nested array has $N_1 = 4$ and $N_2 = 5$. The separate sensor at the end is located at the position 33. The overall 10-element Yang's improved nested array is given by {0, 1, 2, 3, 4, 10, 16, 22, 28, 33}. It can be observed that the first few sensors are adjacent to each other and have unit spacing.

An extended nested array was also proposed in 2016 [51]. However, it does not offer apertures as large as the Yang's INA described here.

3.2.8 Huang's nested array

A nested array configuration that provides larger aperture than the two-level nested array has been recently proposed [52]. This array provides larger aperture than MRAs (obviously than nested, super-nested, improved nested, and co-prime arrays) for a given number of sensors. However, Huang's nested array suffers from holes in the co-array. The construction is similar to that of the Yang's nested array in that there is a level 1 ULA, level 2 ULA with increased spacing and a separate sensor at the end. However, the number of sensors at each level and the element spacing in level 2 determine the sensor locations and the overall behavior of the array.

3.2.9 Triply primed array

The triply primed array (TPA) is the union of three ULAs with different inter-element spacings. Three mutually prime numbers N_1, N_2 and N_3 must be selected [53]. The total sensors in the TPA are $N_1 + N_2 + N_3 - 2$. The first ULA has N_1 elements separated by an inter-element spacing of N_2N_3. The second ULA has N_2 elements separated by an inter-element spacing of N_1N_3. The third ULA has N_3 elements separated by an inter-element spacing of N_1N_2. For example, $N_1 = 3, N_2 = 4, N_3 = 5$ represents a 10-element TPA. The problem with TPAs is that their DCA has a smaller number of continuous lags. Therefore, fourth order statistics are made use of (i.e., the DCA of the DCA is used for DOA estimation).

Table 1 lists out the optimum sensor positions for different 10-element linear sparse arrays. The optimum MHA configuration for 10 sensors has been obtained through table look-up [46]. The sensor positions are shown in **Figure 2**. The continuous part of the DCAs of these sparse arrays are shown in **Table 2**.

Type of sparse array	Sensor positions	Aperture
ULA	[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]	9
MRA	[0, 1, 3, 6, 13, 20, 27, 31, 35, 36]	36
MHA	[0, 1, 6, 10, 23, 26, 34, 41, 53, 55]	55
Co-prime array	[0, 3, 5, 6, 9, 10, 12, 15, 20, 25]	25
Nested array	[0, 1, 2, 3, 4, 5, 11, 17, 23, 29]	29
Super-nested array	[0, 2, 4, 7, 9, 11, 17, 23, 28, 29]	29
Yang's Nested array	[0, 1, 2, 3, 4, 10, 16, 22, 28, 33]	33
Huang's Nested array	[0, 1, 2, 3, 7, 15, 23, 31, 39, 48]	48
Triply Primed array	[0, 12, 15, 20, 24, 30, 36, 40, 45, 48]	48

Table 1.
Sparse array configurations for 10 physical sensors.

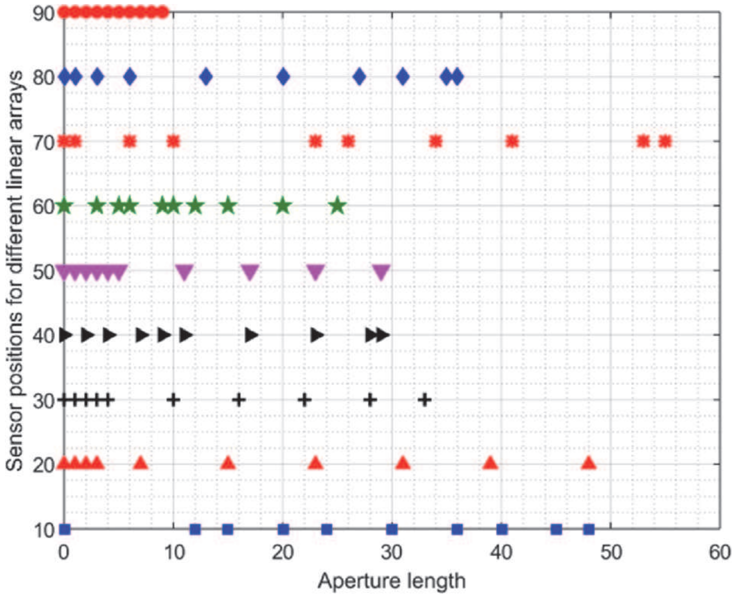

Figure 2.
*Sensor positions for various sparse arrays (arrays are considered in the same order as listed in **Table 1** from top to bottom).*

Type of sparse array	Continuous part of the DCA without holes	Uniform DOFs
ULA	$[0, \pm1, \pm2, \dots, \pm9]$	19
MRA	$[0, \pm1, \pm2, \dots, \pm36]$	73
MHA	$[0, \pm1, \pm2, \dots, \pm35]$	71
Co-prime array	$[0, \pm1, \pm2, \dots, \pm17]$	35
Nested array	$[0, \pm1, \pm2, \dots, \pm29]$	59
Super-nested array	$[0, \pm1, \pm2, \dots, \pm29]$	59
Yang's Nested array	$[0, \pm1, \pm2, \dots, \pm33]$	67
Huang's Nested array	$[0, \pm1, \pm2, \dots, \pm9]$	19
Triple Primed array	[0]	01

Table 2.
*Continuous portion of the difference co-array of the sparse arrays listed in **Table 1**.*

The proliferation of linear sparse arrays in the past decade has led to the development of coarray-based DOA estimation methods. Coarray methods are based on the concept of difference co-array (DCA) and are well-suited for angle estimation in sparse arrays. As the physical array has missing sensors due to the sparseness, the array correlation matrix does not represent a Toeplitz structure and is not suitable for estimation of spatial correlation. Therefore, the analysis is shifted to the co-array domain. Due to the continuity of the DCA, the co-array correlation matrix represents a complete Hermitian Toeplitz structure and can be used to estimate spatial angles. Co-array MUSIC algorithm is widely used for DOA estimation in sparse arrays [32, 33]. More recently, other algorithms such as the co-array root-MUSIC [54, 55] and the co-array ESPRIT [56] have been introduced. The Khatri Rao (KR-MUSIC) algorithm which is applicable only to quasi-stationary sources (i.e., the sources which can be assumed to be stationary for short time durations) was introduced prior to the co-array MUSIC [57]. Recently, many algorithms based on compressed sensing have been introduced for DOA estimation in sparse arrays [58, 59]. In summary, DOA estimation algorithms that operate (i) when the number of sources is unknown, (ii) in the presence of coherent arrivals, (iii) under unknown mutual coupling, (iv) under low signal-to-noise ratio (low SNR) conditions, (v) under low snapshot conditions, (vi) in the presence of non-uniform or random noise, and (vii) in a short computational time; are largely sought-after for practical applications [60, 61].

3.3 Array factors of some well-known sparse arrays

According to the principle of pattern multiplication, the overall array response is the product of the array factor and the element pattern. The element pattern for an isotropic antenna is unity. Hence, in an array of isotropic antenna elements, the far-field pattern depends only on the array factor. The array factor of sparse arrays is conveniently evaluated using the element positions x_n as given below

$$AF(\phi) = \sum_{n \in sparse} e^{jkx_n cos\phi}, \tag{6}$$

where, x_n denotes the actual grid-point location of the nth element in the sparse array. $k = \frac{2\pi}{\lambda}$ denotes the wavenumber and ϕ denotes the azimuth angle. For an MRA with sensors at $\{0, 1, 4, 6\}$, the values of x_n in the above formula would be $\{0, d, 4d, 6d\}$, respectively.

To get further idea on the characteristics of sparse arrays, the array factors of a few prominent sparse arrays given in **Table 1** were evaluated using Eq. (6) and plotted using MATLAB. **Table 3** lists the sparse arrays considered and their 3-dB beamwidths. **Figure 3** shows the plot of array factors of the considered sparse arrays. It can be observed that MRAs provide the narrowest main beam

Type of Sparse Array	Observed HPBW
ULA	10.3°
MRA	2.2°
Nested	3.1°
Co-prime	4.1°
Super-nested	2.9°

Table 3.
Beamwidths for the 10-element sparse arrays listed in ***Table 1****.*

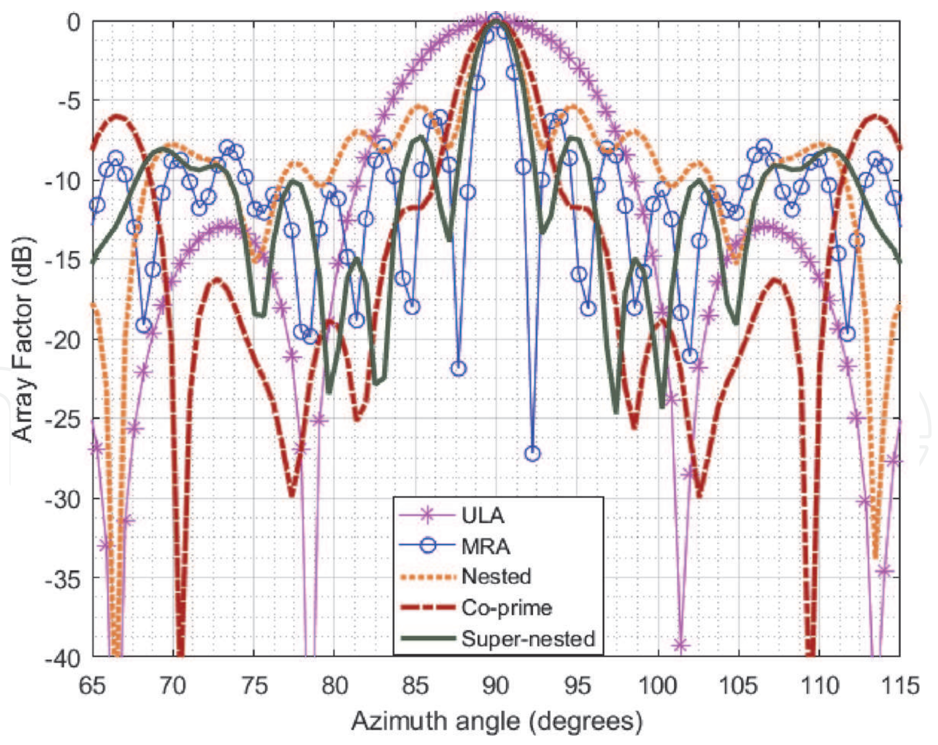

Figure 3.
Array factors of sparse arrays listed in ***Table 3****.*

characteristics for a given number of sensors as they possess the largest aperture among all the sparse arrays considered here. It has to be noted that the MHA can offer even finer beamwidth than MRAs.

It would be good to have a few more figures representing the weight functions of the above sparse arrays and their DOA estimation performance when used with co-array MUSIC. However, they are omitted from this chapter for lack of space.

3.4 MRAs with the same number of sensors and aperture might have different sensitivities

Another important aspect that a designer must be aware of, is that there can exist many sensor configurations for a given aperture in the case of MRAs and MHAs. Each of these configurations may possess different radiation characteristics or sensitivity. For example, the first column in **Table 4** shows a few MRA configurations M1 – M4, each of which has seven sensors and is capable of offering an aperture of $L = 15$. **Figure 4** shows their array patterns. It can be inferred visually from **Figure 4** and also reinforced from the remaining columns of **Table 4** that though the different MRA configurations provide the same aperture, they may not possess the same main beam and side lobe characteristics.

Hence, care must be exercised before choosing a given MRA configuration.

MRA configuration	Beamwidth (ϕ°_{HPBW})	First Side lobe level (dB)
M1 - [0, 1, 2, 3, 7, 11, 15]	5.73°	−6.24
M2 - [0, 1, 3, 6, 10, 14, 15]	5.16°	−6.50
M3 - [0, 1, 4, 8, 13, 14, 15]	4.93°	−5.26
M4 - [0, 2, 4, 5, 8, 14, 15]	5.50°	−6.58

Table 4.
Beamwidths and PSLs for different MRAs with same aperture and same number of sensors.

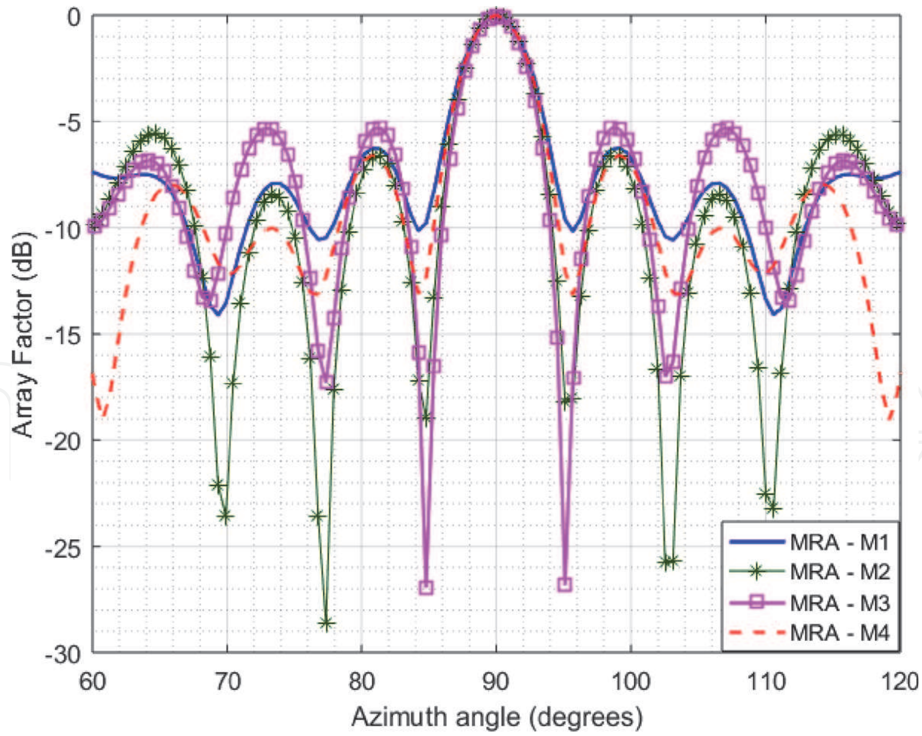

Figure 4.
Array factors of MRAs M1-M4 mentioned in ***Table 4.***

4. More about linear sparse arrays

4.1 Sparse arrays with special properties

Many modifications to coprime arrays have been proposed in the Literature. For example, the thinned coprime array, the coprime array with compressed inter-element spacing (CACIS), coprime array with displaced subarrays (CADiS), coprime array with reduced sensors (CARS) etc. [37, 38, 62]. Most recent is the generalized thinned coprime array (GTCA) [63], of which, the above ones can be considered as special cases. A multi-level prime array which extends the concept of coprime arrays has also been proposed [36]. The three-level prime array (3LPA) is a special case of the multi-level prime array and should not be confused with the triply primed array described in Section 3.2.9.

Recently, a unified array geometry in the form of a generalized nested subarray (GNSA) was proposed [55]. The geometry has a nested structure with two prototype arrays A and B. Specifically, if the prototype arrays A and B are MRAs, then the geometry leads to the nested MRA (NMRA) thereby providing the largest sparse array and associated hole-free coarray. On the other hand, when A and B are ULAs, the design provides a sparse array with the least aperture and DOFs. It can be said that the GNSA has revolutionized the development of sparse arrays in recent times. The prototype arrays could be nested arrays or super-nested arrays, or for that matter, any sparse array that has a hole-free coarray [55].

4.1.1 Sparse arrays for active sensing

Active sensing applications need hole-free sum co-arrays. Symmetric sparse arrays are also useful for certain applications. The concatenated nested array (CNA) is one such array which is obtained by appending the level 1 elements of the nested array just after the level 2 elements such that the overall array is symmetric [40]. However, this array heavily suffers from the effects of mutual coupling owing to the closely spaced elements at both the ends. The Interleaved Wichmann Array (IWA) was proposed to overcome the mutual coupling problem of the CNA by re-arranging the sensors such that the number of sensor pairs with unit spacing is reduced [39]. On similar lines, a nested structure using two CNAs, namely, the Kløve array has been introduced with hole-free sum and difference coarray and is suitable for active as well as passive sensing [64]. More recently, low redundancy arrays with nested arrays and Kløve-Mossige as basis were proposed with hole-free sum coarray [65]. The sum coarray is defined as

$$\mathbb{P} = \{z_i + z_j | i,j = 1, 2, \cdots, N_s\}, \tag{7}$$

where all the cross summations between sensor positions are considered.

4.1.2 Sparse arrays for DOA estimation of non-circular sources

While the difference coarray approach is well suited for the DOA estimation of circular sources, many non-circular source signals exist in practice. For example, binary phase shift keying (BPSK), minimum shift keying (MSK), unbalanced quadrature phase shift keying (UQPSK) etc. Non-circular sources have non-zero pseudo covariances (non-zero ellipse covariance matrix) which can be used to enhance the aperture of the virtual array to further exploit the received information for parameter estimation [66, 67]. To fully leverage the special properties of non-circular sources, the DOA estimation is performed using the sum difference co-array (SDCA) which is defined as

$$\mathbb{R} = \mathbb{H} \cup \mathbb{P} \cup \mathbb{Q} \tag{8}$$

where $\mathbb{H}$ is the difference co-array, $\mathbb{P}$ is the sum co-array and $\mathbb{Q}$ is the mirrored sum co-array and equal to - $\mathbb{P}$. In short, for a sparse array with sensors at $\mathbb{Z} = \{z_1, z_2, \ldots, z_{N_s}\}$, the locations of the virtual sensors in the SDCA are given by

$$\mathbb{R} = \{z_i - z_j, z_i + z_j, -z_i - z_j | i,j = 1, 2, \cdots, N_s\} \tag{9}$$

The use of difference co-array along with the sum co-array increases the virtual array span and leads to increased DOFs than possible by using the DCA alone. In SDCA-based designs, the vectorized conjugate augmented MUSIC (VCAM) algorithm is generally used for DOA estimation [68].

One of the prominent designs for sparse arrays that are suitable for non-circular sources is the nested array with displaced subarray (NADiS) as it provides CFEs for element positions, virtual apertures, and DOFs. The NADiS array has a large central continuous portion in the SDCA thereby providing large uniform DOFs. However, the SDCA is not completely hole-free. As an improvement, sparse array for non-circular sources (SANC) array was proposed with a hole-free SDCA [69]. A drawback of the SANC array is that it has no CFE for element positions. Therefore, the sensor positions in SANC have to be determined through exhaustive searching. Though an improved nested array with SDCA (INAwSDCA) [68] was proposed recently, there is no comparison with the NADiS and SANC arrays. An ultimate design of the nested array for non-circular signals is the translated nested array [67] which has CFEs for sensor positions and provides larger apertures than the NADis and the SANC. In addition, the SDCA of translated nested array is hole-free.

4.1.3 Array motion

In recent years, moving array platforms or array motion are being exploited to obtain higher DOFs in sparse arrays. The synthetic coprime array [70], dilated nested array [71] and the multi-level dilated nested array [72] are few examples of sparse array designs that leverage platform motion to fill the holes in the DCAs. This topic is pretty new and is widely being explored.

4.2 Sensor failures in antenna arrays

4.2.1 Sensor failures in ULAs

In an array of identical elements, the overall pattern depends on parameters such as the array geometry, inter-element spacing, amplitudes and phases of the individual elements and the inherent pattern of each element [16]. In most cases, the array geometry and the type of array elements is fixed. For example, assume a linear array with patch antennas. In such cases, only the spacing of elements, their relative amplitudes and phases can be altered to modify the array pattern.

Perturbations in any of these parameters can distort the array's response. Worst of all is the partial or complete failure of one or more sensors in the array. Element failures in sensor arrays can cause distortions in the main beam, side lobe levels and null placements, thereby disrupting the normal functioning of the array. Fault diagnosis and fault compensation are needed to ensure smooth operation of arrays. Several methods have been reported in literature that can (i) identify the location of the faulty element(s) and (ii) compensate or restore the array response through suitable weighting of the remaining healthy antennas in the array. Consider a 10-element ULA with uniform feeding. Imagine that the seventh sensor fails. In this case, the algorithm should be able to identify the position of the failed element and also determine suitable weights to be applied to the remaining nine sensors so that the compensated pattern closely resembles the response of the healthy array. As the weights are no more uniform, the use of digital beamforming is called for. Many bio-inspired algorithms or compressed sensing techniques have been used in the past either to detect sensor failures or to compensate the pattern of a faulty array or for both [73–80]. An extreme case and new perspective is presented in [81], where a sparse array is said to be formed when one or more elements of an ULA fail at random.

4.2.2 Fault tolerance in sparse arrays a.k.a. fragility

The notions of robustness, fragility, essentialness etc., in relation to sparse sensor arrays have been introduced in recent years by Liu and Vaidyanathan [82–84]. The fragility of a sparse array gives a measure of how vulnerable the array is to its' sensor failures. Fragility is defined as the number of essential sensors to the total number of sensors in the sparse array. A sensor is said to be essential if its failure/absence alters the difference coarray or introduces holes into the coarray. Arrays in which all sensors are essential are known as maximally economic sparse arrays (MESA). MRAs, nested and super nested arrays are maximally economic as all their sensors are essential. For this reason, these arrays are highly fragile with a fragility of $\frac{N}{N} = 1$.

It is well-known in MRA theory that, in an array of N elements, the failure of a single-element can cause up to $N - 1$ missing spatial lags, thereby rendering the sparse array useless for parameter estimation in the coarray domain. Robust MRAs (RMRAs) have been recently proposed with the aim of designing resilient sparse arrays which can offer reliable and smooth operation even in the presence of a single-element failure. RMRAs ensure that each spatial lag is generated at least twice [85]. That is, there are at least two separate sensor pairs that can generate a given spatial lag. Even if a single sensor fails, the remaining sensors can generate all the necessary spatial lags. The RMRA is, therefore, preferred when array reliability is the foremost design concern. ULAs and RMRAs have only two essential sensors (the first and the last to preserve the aperture). Therefore, both these arrays have a fragility of 2/N. ULAs are the most robust and least fragile as they have many redundant sensors. RMRAs have been designed with the specific aim of achieving the least fragility in a sparse array while retaining the hole-free coarray properties of MRA.

As an example, a 10-element RMRA has sensors at $\{0, 1, 2, 6, 7, 11, 15, 16, 18, 19\}$ [85]. The weight function of the RMRA is plotted in **Figure 5**. It can be seen that all the spatial lags from $\{0, \pm 1, \pm 2, \ldots, \pm 18\}$ have a weight of two or more. That means there are at least two sensor pairs that can generate the required lags from 0 to 18. RMRAs are very much similar in concept to two-fold redundancy arrays (TFRAs). TFRAs are based on double difference sets [86]. However, RMRAs have a much broader scope and their design is more elegant. It can be observed from **Figure 5** that the weight w_1 is five which indicates that there are five sensor pairs with unit spacing. As per the empirical relation between the weight function and mutual coupling, it is easy to predict that the array is heavily prone to mutual coupling. Nevertheless, the array is robust to sensor failures.

Consider a situation where a particular element in the above RMRA fails (say the element at position 11). The weight function of the RMRA with failed sensor is shown in **Figure 6**. It can be observed that the weights of a few spatial lags fall down to one but none of them becomes zero. As long as there is just a single-element failure in RMRAs, the weight of any given spatial lag never becomes zero, meaning that holes would never occur in the DCA. This justifies the robustness of RMRAs.

In recent years, sparse arrays based on fractal geometries have been proposed. Such arrays use a small sparse array as a base (called generator) to obtain larger sparse arrays through recursive formulations. Examples include the Cantor arrays proposed by Liu and Vaidyanathan and the generalized fractal sparse arrays proposed by Cohen and Eldar [35, 87].

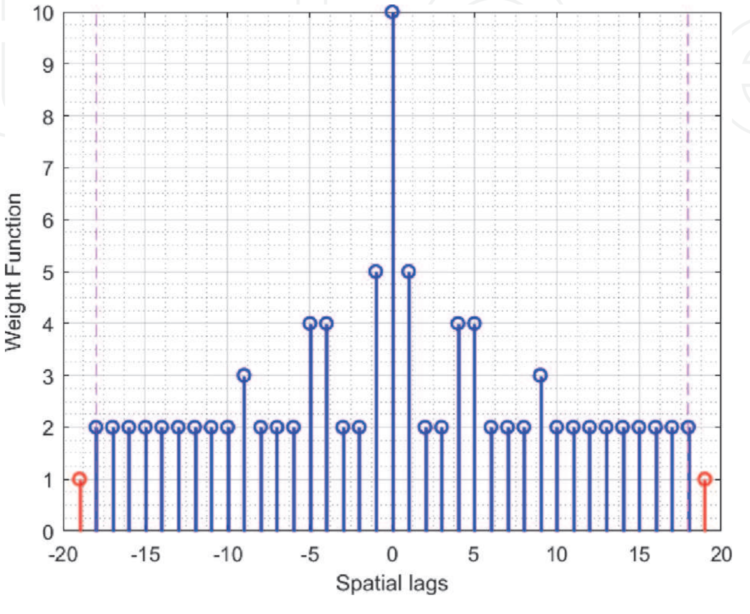

Figure 5.
Weight function of the 10-element RMRA described above.

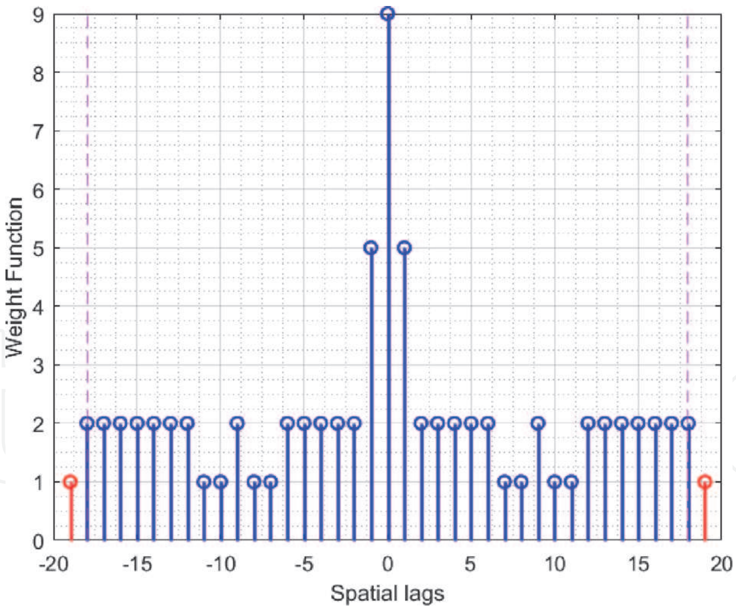

Figure 6.
Weight function of the RMRA with failed sensor at position 11.

4.3 Sparse multiple input multiple output (MIMO) radar arrays

Any review of linear sparse arrays would be incomplete without a mention of their usage in the design of sparse MIMO radar arrays. MIMO radar arrays make use of co-located transmit and receive arrays. These transmit and receive arrays work in tandem and produce the effect of a large virtual array. The transmitting array consists of M antennas and the receiving array contains N antennas. Hence, the total number of physical antennas in the MIMO array is $K = M + N$. Each transmit antenna emits an independent waveform. At each receiving antenna, there are M matched filters that are used to extract the reflected signals. As there are N receive elements, each equipped with M matched filters, the total number of extracted signals is MN. It has been shown that the matched filter output is equivalent to the signals received by an array of MN elements. This gives the effect of a large virtual array with MN elements [88], whereas physically there are only $M + N$ elements. The virtual array is also known as the sum co-array as it is obtained by adding all possible element positions in the transmit and receive arrays.

Following the introduction of the minimum redundancy MIMO radar [45] in 2008, several other sparse MIMO configurations have been proposed in the literature [89–93]. However, a thorough review of the properties of sparse MIMO arrays is beyond the scope of this article. However, it is easy to foresee that if the transmit and receive arrays are maximally sparse with hole-free co-arrays (like MRAs), the resulting MIMO radar array would also be highly sparse and, therefore, capable of providing the largest virtual array aperture for a given number of sensors and hence the highest DOFs as reported recently [94]. Several novel designs of sparse MIMO radar arrays are being proposed. Another research area which has gained traction in recent years is the co-existent MIMO radar and MIMO communications.

5. Future scope

Evolutionary and swarm-based algorithms have been extensively applied for array pattern optimization. The No-free-lunch (NFL) theorem in optimization theory says that there is no single optimization algorithm that works well against all the optimization problems (objective functions). Going by the above fact, it might so turn out that there might exist a few algorithms which could be more suitable over others for synthesizing MRAs and MHAs. As many powerful nature inspired optimization algorithms (grey wolf optimizer, moth flame optimizer, whale optimization, sparrow search algorithm and other hybrid meta-heuristic approaches etc. [95–98]) have been introduced in the recent past, it would be a worthwhile effort to try synthesizing MRAs and MHAs using such algorithms. One such attempt to determine large MRAs using parallel processing has been recently reported [99]. In recent times, deep learning methods are being employed to synthesize sparse arrays for joint requirements such as hole-free coarrays, low peak side lobes, and optimum far-field performance [100, 101].

In the future, research could be done to determine large RMRAs such that tabulated entries on the optimum RMRA configurations for a given number of sensors could be widely made available to the scientific community. Similarly, efforts could be made to find robust nested arrays with closed-form expressions for sensor positions as an alternative to RMRAs [33, 85].

In sparse arrays, beamforming is usually performed in the co-array domain [102–104]. The weights of the virtual sensors in the coarray are adjusted to obtain the desired beam pattern. It would be interesting to see how the failure of one or more sensors in the physical array affects these beamforming weights. Detection of failed sensors in the physical array and subsequent compensation of the beam pattern in the coarray are a few open research challenges for sparse array analysis.

6. Conclusion

In this chapter, the properties of various linear sparse arrays have been compared. It is found that different sparse arrays have different design criteria and trade-offs. It remains as a future scope to synthesize new MRAs and RMRAs using the latest bio-inspired and/or deep learning algorithms. Antenna array processing techniques will be in the limelight for many years to come as many future wireless communication systems (infrastructure-based and ad hoc) heavily rely on them. DOA estimation algorithms and adaptive beamforming methods using sparse/full arrays in the presence of array miscalibrations/failures are a trending research topic at the moment. It is believed that this chapter serves as a comprehensive guide to new researchers in the field of sparse array signal processing.

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Conflict of interest

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