Chapter

# The Paradigm of Complex Probability and Isaac Newton's Classical Mechanics: On the Foundation of Statistical Physics

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> *"Imagination is more important than knowledge. Knowledge is limited. Imagination encircles the world."*
>
> *Albert Einstein*.

> *"Our minds are finite, and yet even in these circumstances of finitude we are surrounded by possibilities that are infinite, and the purpose of life is to grasp as much as we can out of that infinitude."*
>
> *Alfred North Whitehead*.

> *"The important thing is not to stop questioning. Curiosity has its own reason for existence."*
>
> *Albert Einstein*.

> *"A theory with mathematical beauty is more likely to be correct than an ugly one that fits some experimental data. God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe."*
>
> *Paul Adrien Maurice Dirac*.

## Abstract

The concept of mathematical probability was established in 1933 by *Andrey Nikolaevich Kolmogorov* by defining a system of five axioms. This system can be enhanced to encompass the imaginary numbers set after the addition of three novel axioms. As a result, any random experiment can be executed in the complex probabilities set $\mathcal{C}$ which is the sum of the real probabilities set $\mathcal{R}$ and the imaginary probabilities set $\mathcal{M}$. We aim here to incorporate supplementary imaginary dimensions to the random experiment occurring in the "real" laboratory in $\mathcal{R}$ and therefore to compute all the probabilities in the sets $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$. Accordingly, the probability in the whole set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is constantly equivalent to one independently of the distribution of the input random variable in $\mathcal{R}$, and subsequently the output of the stochastic experiment in $\mathcal{R}$ can be determined absolutely in $\mathcal{C}$. This is the consequence of the fact that the probability in $\mathcal{C}$ is computed after the subtraction of the chaotic factor from the degree of our knowledge of the nondeterministic experiment. We will apply this innovative paradigm to Isaac Newton's classical

mechanics and to prove as well in an original way an important property at the foundation of statistical physics.

**Keywords:** Chaotic factor, degree of our knowledge, complex random vector, probability norm, complex probability set, random forces, complex force, resultant force

## 1. Introduction

Firstly, classical mechanics is a theory in physics studying the macroscopic objects motion whether they are parts of machinery or projectiles or objects in astronomy like for example planets or spacecrafts or galaxies or stars. As it was established, classical mechanics is deterministic that means that we can predict the motion of objects in the future when we know their present state. It is also reversible and that means we can know the motion of objects in the past when we know their present state also. [1]

Since classical mechanics was developed at the beginning by Sir Isaac Newton therefore it is usually referred to as Newtonian mechanics. It comprises the mathematical methods and the employed physical concepts developed, as we have mentioned, by Newton, Gottfried Wilhelm Leibniz and others in the seventeenth century to study the bodies motion under the effect of a set of forces. The theory was more developed later on to embody more abstract methods which have led to the reformulations of classical mechanics and hence to the establishment of Hamiltonian mechanics and Lagrangian mechanics. These developments which were done in the eighteenth and nineteenth centuries are substantial extensions beyond the work of Newton because they used more particularly analytical mechanics. After doing some modifications, modern physics makes use of them in all its areas. [2]

Moreover, exceptionally precise results are provided by classical mechanics when considering objects with velocities far from the speed of light and when they do not possess extreme masses. It is mandatory to make use of quantum mechanics which is a sub-field of mechanics when studying objects which have an atom diameter size. Additionally, we need Albert Einstein's special relativity when considering speeds near the velocity of light. Furthermore, Einstein's general relativity is applied when objects have huge masses. It is important to note that many modern sources include in classical physics the relativistic mechanics which represents according to them the most precise, developed, and complete form of classical mechanics. [3]

Furthermore, we now present classical mechanics fundamental concepts. The theory assumes that the objects of the real world are of negligible size that means that they are point particles. And it also characterizes the point particle motion by few parameters which are: its mass, its position, and the applied forces to it. We will discuss each of these parameters in turn. [4]

In fact, and in reality, classical mechanics can describe always the kind of objects that have a non-zero size. Whereas, very small particles like electrons are described more accurately by the physics of quantum mechanics. Additionally, hypothetical point particles have more simplified behavior than non-zero size objects like for example a baseball that can spin when it is in motion. Moreover, such non-zero objects are considered as composite objects constituted of a large number of point particles acting collectively; hence, the point particles results can be used in such large objects study. [5]

Common sense notions are used by classical mechanics of how matter and forces interact and exist. Its basic assumption is that energy and matter have knowable and definite attributes such as speed and location in space. Additionally, it is assumed by non-relativistic mechanics the instantaneous action of forces or instantaneous action at a distance. [6]

The bodies motion study is very ancient, this makes classical mechanics one of the largest and oldest subjects in engineering, technology, and science. [7]

Aristotle, one among antiquity Greek philosophers and who is the founder of Aristotelian physics, may have been the first to postulate that theoretical principles can assist nature understanding and to assume that "everything happens for a reason". Many of these ideas preserved are considered as eminently reasonable by a modern reader but there is an obvious lack of controlled experiment and mathematical theory as we know it. In fact, modern science was formed by these later decisive factors and classical mechanics came to be known as their early application. [8]

The medieval mathematician Jordanus de Nemore introduced in his Elementa demonstrationem ponderum the "positional gravity" concept and the component forces use. [9]

Johannes Kepler published in 1609 Astronomia nova which was the first published causal explanation of the planets motion. Based on the observations made by Tycho Brahe on Mars orbit, he concluded that the orbits of the planet were ellipses. This epistemological revolution occurred at the same time when Galileo was proposing for objects motion abstract mathematical laws. Perhaps he may have performed the historical experiment of the two cannonballs of different weights dropping from Pisa tower. Hence, he showed that these two cannonballs hit the ground simultaneously. We doubt in fact the reality of that particular experiment, but Galileo conducted quantitative experiments which were to roll balls on an inclined plane. From such experiments results he derived his accelerated motion theory. [10]

Sir Isaac Newton laid down classical mechanics foundations by founding his natural philosophy principles on three laws of motion proposed by him: the inertia first law, the acceleration second law, and the action and reaction third law. A proper mathematical and scientific treatment in Philosophiae Naturalis Principia Mathematica of Newton was given to his second and third laws. They are in fact different from the attempts laid earlier to explain similar phenomena and which were either incorrect, incomplete, or they lack a precise mathematical expression. Moreover, the principles of conservation of angular momentum and momentum were postulated by Newton. Additionally, the universal gravitational law of Newton was also provided by him to give the first accurate mathematical and scientific formulation of gravity. The most accurate and fullest description of classical mechanics was provided by the combination of the laws of motion and gravitation of Newton. Newton showed that his three laws can be applied to the objects of everyday as well to heavenly objects. Particularly, Newton derived a theoretical explanation of the planets' laws of motion of Kepler. [11]

Newton performed the mathematical calculation by inventing previously the mathematical calculus. In fact, calculus eclipsed his book, the Principia, which was formulated totally in terms of geometric methods which were long established and to gain hence acceptability. Moreover, the notation of the integral and of the derivative which are preferred today were developed by Leibniz however. [12]

All phenomena, including light in the form of geometric optic, can be explained by classical mechanics as it was assumed by Newton and most of his contemporaries, with the notable exception of Christiaan Huygens. Newton maintained his own corpuscular light theory even when they discovered the wave interference phenomenon or the so-called Newton's rings. [13]

Classical mechanics became a major field of study in physics as well in mathematics and this after Newton. A far greater number to problems solutions were allowed by several progressive reformulations of his mechanics. Joseph Louis Lagrange was the first to reformulate in 1788 Newtons' mechanics. William Rowan Hamilton in his turn reformulated Lagrangian mechanics in 1833. [14]

More modern physics resolved some difficulties that were discovered in the late nineteenth century. Compatibility with the theory of electromagnetism and the famous Michelson-Morley experiment were some of these difficulties. Often still considered as a part of classical mechanics, the special relativity theory was led by the resolution of these problems. [15]

Explaining all thermodynamics, raised another set of difficulties and problems with classical mechanics. Gibbs paradox of classical statistical mechanics was the result of the combination of classical mechanics with thermodynamics. In this paradox, entropy is not a quantity which was well defined. We introduced quanta to explain the black-body radiation otherwise this was not possible. Classical mechanics was unable to explain, not even approximately, such basic things as the sizes of the atoms, the photo-electric effect, and the energy levels and this when experiments delved into the atomic world. Quantum mechanics was the result of the efforts to resolve these problems. [16]

Classical mechanics has no longer been considered as an independent theory since the end of the twentieth century. We consider classical mechanics now as an approximate theory to quantum mechanics which is a more general theory. The desire to understand the fundamental forces of nature has shifted our emphasis in our research and investigation and has led to the Standard Model and also has directed the studies to a unified theory of everything. For the study of the motion of low-energy, of non-quantum mechanical particles in weak gravitational fields, it is useful to make use of classical mechanics. Additionally, we were successful to extend classical mechanics to the complex domain. In fact, this extended complex classical mechanics behaves very similarly to quantum mechanics. [17]

At the end, and to conclude, this research work is organized as follows: After the introduction in section 1, Newton's laws of classical mechanics are stated in section 2, then the purpose and the advantages of the present work are presented in section 3. Afterward, in section 4, the extended Kolmogorov's axioms and hence the complex probability paradigm with their original parameters and interpretation will be explained and summarized. Moreover, in section 5, the complex probability paradigm axioms are applied to classical mechanics which will be hence extended to the imaginary and complex sets. Additionally, in section 6, the resultant complex random vector $Z$ of $CPP$ will be applied to statistical physics to prove an important property at its foundation. Also, in section 7, the flowchart of the new paradigm will be shown. Furthermore, the simulations of the novel model for various discrete and continuous stochastic distributions are illustrated in section 8. Finally, we conclude the work by doing a comprehensive summary in section 9, and then present the list of references cited in the current research work.

## 2. Isaac Newton's laws of motion

The classical mechanics foundation was laid down by Isaac Newton's three physical laws of motion. These laws define and describe the forces acting upon a body as well as the response of the body to those forces. Moreover, and more precisely, the first law defines the force qualitatively, the second law measures the force quantitively. The third law states that an isolated single force does not exist [18–21]. Throughout nearly three centuries, these three laws have been stated in many different ways and we will summarize them as follows:

First law

In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

Second law

In an inertial frame of reference, the vector sum of the forces $\vec{F}$ on an object is equal to the mass $m$ of that object multiplied by the acceleration $\vec{a}$ of the object: $\vec{F} = m\vec{a}$. (It is assumed here that the mass $m$ is constant).

Third law

When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

Isaac Newton was the first to state in his Mathematical Principles of Natural Philosophy (Philosophiae Naturalis Principia Mathematica), first published in 1687, the three laws of motion. Many systems and physical objects were investigated and explained by the three laws of motion of Newton. As an example, the planetary motion laws of Johannes Kepler were proved and demonstrated by Newton's laws when combined with the universal gravitational law, in the third volume of the text. [22–25]

Fourth law

Some also describe a Fourth law which states that forces add up like vectors, that is, that forces obey the principle of superposition.

A single point masses idealize the objects to which we apply the laws of Newton, that means that the object body shape and size are to be ignored in order to concentrate on the body's motion more easily. This is achieved when the rotation and the deformation of the body are negligible and when the object is too small compared to the distances that the analysis involves. Hence, in the planet orbital motion around a star analysis, even a planet can be idealized as a particle. [26–29]

Moreover, deformable bodies and the rigid bodies motion are not characterized by the original form of the laws of motion of Newton which reveal to be inadequate. Additionally, a generalization of the laws of motion of Newton for rigid bodies was introduced and achieved by Leonhard Euler in 1750 and they were called accordingly Euler's laws of motion. They were applied later on to deformable bodies which were postulated to be a continuum. Euler's laws can be derived from the laws of Newton if we represent a body as an assemblage of discrete particles where every particle is governed by the motion laws of Newton. Independently of the structure of any particle, the laws of Euler can be considered, however, as axioms that describe the motion laws of extended bodies. [30–33]

Newtonian inertial reference frames are a certain set of frames that verify and confirm Newton's laws. The first law defines what an inertial frame of reference is and this according to some authors interpretation. Therefore, the first law cannot be demonstrated as special case of the second law since the second law is only valid when an inertial frame of reference is used in the observation. The second law is considered as a corollary of the first law by other authors. It was long after Newton's death that we have developed the inertial frame of reference explicit concept. [34–37]

Furthermore, we assume that, momentum, acceleration, and most importantly force to be quantities defined externally in the given interpretation. This is not the only interpretation, but the most common way one can consider the definition of these quantities by Newton's laws. [38–41]

Additionally, when the speeds considered are much closer to the speed of light, then Albert Einstein's special relativity replaces Newtonian mechanics which is still useful as an approximation of the studied phenomenon. [42–44]

## 3. The purpose and the advantages of the current publication

The crucial job of the theory of classical probability is to compute and to assess probabilities. A deterministic expression of probability theory can be attained by adding supplementary dimensions to nondeterministic and stochastic experiments.

This original and novel idea is at the foundations of my new paradigm of complex probability. In its core, probability theory is a nondeterministic system of axioms that means that the phenomena and experiments outputs are the products of chance and randomness. In fact, a deterministic expression of the stochastic experiment will be realized and achieved by the addition of imaginary new dimensions to the stochastic phenomenon taking place in the real probability set $\mathcal{R}$ and hence this will lead to a certain output in the set $\mathcal{C}$ of complex probabilities. Accordingly, we will be totally capable to foretell the random events outputs that occur in all probabilistic processes in the real world. This is possible because the chaotic phenomenon becomes completely predictable. Thus, the job that has been successfully completed here was to extend the set of real and random probabilities which is the set $\mathcal{R}$ to the complex and deterministic set of probabilities which is $\mathcal{C} = \mathcal{R} + \mathcal{M}$. This is achieved by taking into account the contributions of the imaginary and complementary set of probabilities to the set $\mathcal{R}$ and that we have called accordingly the set $\mathcal{M}$. This extension proved that it was effective and consequently we were successful to create an original paradigm dealing with prognostic and stochastic sciences in which we were able to express deterministically in $\mathcal{C}$ all the nondeterministic processes happening in the 'real' world $\mathcal{R}$. This innovative paradigm was coined by the term "The Complex Probability Paradigm" and was started and established in my seventeen earlier publications and research works [45–61].

The advantages and the purpose of this current work are to:

1. Extend the theory of classical probability to encompass the complex numbers set, hence to bond the theory of probability to the field of complex variables and analysis in mathematics. This mission was elaborated and initiated in my earlier seventeen papers.

2. Apply the novel probability axioms and paradigm to Newton's classical mechanics.

3. Show that all nondeterministic phenomena can be expressed deterministically in the complex probabilities set which is $\mathcal{C}$.

4. Compute and quantify both the degree of our knowledge and the chaotic factor of all the forces acting on a body in classical mechanics and *CPP* in the sets $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$.

5. Represent and show the graphs of the functions and parameters of the innovative paradigm related to Newton's mechanics.

6. Demonstrate that the classical concept of probability is permanently equal to one in the set of complex probabilities; hence, no randomness, no chaos, no ignorance, no uncertainty, no nondeterminism, no unpredictability, and no disorder exist in:

$$\mathcal{C}\ (\text{complex set}) = \mathcal{R}\ (\text{real set}) + \mathcal{M}\ (\text{imaginary set}).$$

7. Prove an important property at the foundation of statistical physics after applying *CPP* to classical mechanics.

8. Prepare to implement this creative model to other topics in prognostics and to the field of stochastic processes. These will be the job to be accomplished in my future research publications.

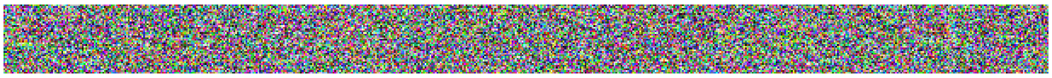

**Figure 1.**
*The diagram of the complex probability paradigm major goals.*

Concerning some applications of the novel founded paradigm and as a future work, it can be applied to any nondeterministic phenomenon using classical mechanics whether in the continuous or in the discrete cases. Moreover, compared with existing literature, the major contribution of the current research work is to apply the innovative paradigm of complex probability to Newton's classical mechanics and to statistical physics as well.

The next figure displays the major purposes and goals of the Complex Probability Paradigm ($CPP$) (**Figure 1**).

## 4. The complex probability paradigm

### 4.1 The original Andrey Nikolaevich Kolmogorov system of axioms

The simplicity of Kolmogorov's system of axioms may be surprising. Let $E$ be a collection of elements $\{E_1, E_2, \ldots\}$ called elementary events and let $F$ be a set of subsets of $E$ called random events [62–66]. The five axioms for a finite set $E$ are:

**Axiom 1:** $F$ is a field of sets.

**Axiom 2:** $F$ contains the set $E$.

**Axiom 3:** A non-negative real number $P_{rob}(A)$, called the probability of $A$, is assigned to each set $A$ in $F$. We have always $0 \le P_{rob}(A) \le 1$.

**Axiom 4:** $P_{rob}(E)$ equals 1.

**Axiom 5:** If $A$ and $B$ have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that $A$ and $B$ are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B/A) = P_{rob}(B) \times P_{rob}(A/B)$ which is the conditional probability. If both $A$ and $B$ are independent then: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B)$.

Moreover, we can generalize and say that for $N$ disjoint (mutually exclusive) events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following additivity rule:

$$P_{rob}\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} P_{rob}(A_j)$$

And we say also that for $N$ independent events $A_1, A_2, \dots, A_j, \dots, A_N$ (for $1 \leq j \leq N$), we have the following product rule:

$$P_{rob}\left(\bigcap_{j=1}^{N} A_j\right) = \prod_{j=1}^{N} P_{rob}(A_j)$$

### 4.2 Adding the imaginary part $\mathcal{M}$

Now, we can add to this system of axioms an imaginary part such that:

**Axiom 6:** Let $P_m = i \times (1 - P_r)$ be the probability of an associated complementary event in $\mathcal{M}$ (the imaginary part) to the event $A$ in $\mathcal{R}$ (the real part). It follows that $P_r + P_m/i = 1$ where $i$ is the imaginary number with $i = \sqrt{-1}$ or $i^2 = -1$.

**Axiom 7:** We construct the complex number or vector $z = P_r + P_m = P_r + i(1 - P_r)$ having a norm $|z|$ such that:

$$|z|^2 = P_r^2 + (P_m/i)^2.$$

**Axiom 8:** Let $P_c$ denote the probability of an event in the complex probability universe $\mathcal{C}$ where $\mathcal{C} = \mathcal{R} + \mathcal{M}$. We say that $P_c$ is the probability of an event $A$ in $\mathcal{R}$ with its associated event in $\mathcal{M}$ such that:

$$P_c{}^2 = (P_r + P_m/i)^2 = |z|^2 - 2iP_rP_m \text{ and is always equal to 1.}$$

We can see that by taking into consideration the set of imaginary probabilities we added three new and original axioms and consequently the system of axioms defined by Kolmogorov was hence expanded to encompass the set of imaginary numbers. [45–61]

### 4.3 A Concise Interpretation of the Original Paradigm

As a summary of the new paradigm, we declare that in the universe $\mathcal{R}$ of real probabilities we have the degree of our certain knowledge is unfortunately incomplete and therefore insufficient and unsatisfactory, hence we encompass in our analysis the set $\mathcal{C}$ of complex numbers which integrates the contributions of both the real set $\mathcal{R}$ of probabilities and its complementary imaginary probabilities set that we have called accordingly $\mathcal{M}$. Subsequently, a perfect and an absolute degree of our knowledge is obtained and achieved in the universe of probabilities $\mathcal{C} = \mathcal{R} + \mathcal{M}$ because we have constantly $P_c = 1$. In fact, a sure and certain prediction of any random phenomenon is reached in the universe $\mathcal{C}$ because in this set, we eliminate and subtract from the measured degree of our knowledge the computed chaotic factor. Consequently, this will lead to in the universe $\mathcal{C}$ a probability permanently equal to one as it is shown in the following equation: $P_c{}^2 = DOK - Chf = DOK + MChf = 1 = P_c$ deduced from the complex probability paradigm. Moreover, various discrete and continuous stochastic distributions illustrate in my seventeen previous research works this hypothesis and innovative and original model. The figure that follows shows and summarizes the Extended Kolmogorov Axioms (*EKA*) or the Complex Probability Paradigm (*CPP*) (**Figure 2**) [67–92]:

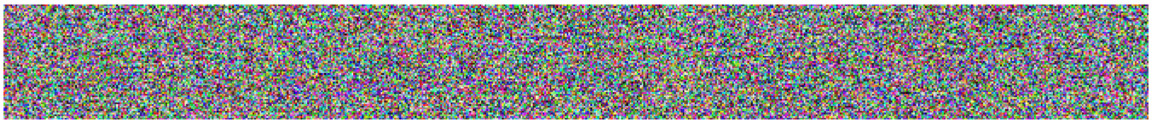

**Figure 2.**
*The* EKA *or the* CPP *diagram.*

## 5. The Newton's mechanics and the complex probability paradigm parameters

In this section we will relate and link Newton's mechanics to the complex probability paradigm with all its parameters by using four novel concepts which are: the real stochastic force $\overrightarrow{F}_r$ in the real probability set $\mathcal{R}$, the imaginary stochastic force $\overrightarrow{F}_m$ in the imaginary probability set $\mathcal{M}$, the complex resultant stochastic force $\overrightarrow{F}$ in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and the deterministic real force $\overrightarrow{F}_c$ also in the probability set $\mathcal{C}$ [45–61, 93–104].

### 5.1 The stochastic forces $\overrightarrow{F}_r$ in $\mathcal{R}$ and $\overrightarrow{F}_m$ in $\mathcal{M}$

The real stochastic force is defined by: $\overrightarrow{F}_r = P_r m\overrightarrow{a} \Leftrightarrow P_r = \frac{\overrightarrow{F}_r}{m\overrightarrow{a}}$.

Here $P_r$ measures the probability that the real stochastic force $\overrightarrow{F}_r$ acting on a body in $\mathcal{R}$ will occur.

Since $0 \leq P_r \leq 1 \Leftrightarrow 0 \leq \frac{\overrightarrow{F}_r}{m\overrightarrow{a}} \leq 1 \Leftrightarrow \overrightarrow{0} \leq \overrightarrow{F}_r \leq m\overrightarrow{a}$

If $P_r = 0$ then $\overrightarrow{F}_r = \overrightarrow{0}$ that means that the real stochastic force in $\mathcal{R}$ is totally known and is equal to $\overrightarrow{0}$ or null in this case.

If $P_r = 1$ then $\overrightarrow{F}_r = m\overrightarrow{a}$ that means that the real stochastic force in $\mathcal{R}$ is totally known and totally deterministic and is equal to $m\overrightarrow{a}$ in this case.

The imaginary stochastic force is defined by:
$\overrightarrow{F}_m = P_m m\overrightarrow{a} = i(1 - P_r)m\overrightarrow{a} \Leftrightarrow P_m = \frac{\overrightarrow{F}_m}{m\overrightarrow{a}} = i(1 - P_r)$.

Here $P_m$ measures the probability that the imaginary stochastic force $\overrightarrow{F}_m$ acting on a body in $\mathcal{M}$ will occur.

Since $0 \le P_r \le 1 \Leftrightarrow 0 \le P_m \le i \Leftrightarrow 0 \le \frac{\vec{F}_m}{m\vec{a}} \le i \Leftrightarrow \vec{0} \le \vec{F}_m \le im\vec{a}$

If $P_m = 0$ then $\vec{F}_m = \vec{0}$ that means that the imaginary stochastic force in $\mathcal{M}$ is totally known and is equal to $\vec{0}$ or null.

If $P_m = i$ then $\vec{F}_m = im\vec{a}$ that means that the imaginary stochastic force in $\mathcal{M}$ is totally known and totally deterministic and is equal to $im\vec{a}$.

#### *5.1.1 The relation between the real and the imaginary stochastic forces*

We have: $\vec{F}_m = P_m m\vec{a} = i(1-P_r)m\vec{a} \Leftrightarrow P_m = \frac{\vec{F}_m}{m\vec{a}} = i(1-P_r)$.

And since $P_r = \frac{\vec{F}_r}{m\vec{a}} \Leftrightarrow P_m = \frac{\vec{F}_m}{m\vec{a}} = i\left(1 - \frac{\vec{F}_r}{m\vec{a}}\right)$.

And we can deduce that: $P_r = 1 - P_m/i = 1 - \frac{\vec{F}_m}{im\vec{a}} \Leftrightarrow P_r = 1 + \frac{i\vec{F}_m}{m\vec{a}}$ since $i = -\frac{1}{i}$

Therefore, $\vec{F}_m = i\left(1 - \frac{\vec{F}_r}{m\vec{a}}\right)m\vec{a} = im\vec{a} - i\vec{F}_r$

$\Leftrightarrow \vec{F}_r = m\vec{a} - \frac{\vec{F}_m}{i} = m\vec{a} + i\vec{F}_m$ since $i = -\frac{1}{i}$ also.

### 5.2 The resultant complex stochastic force $\vec{F}$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$

We define the resultant complex stochastic force by: $\vec{F} = \vec{F}_r + \vec{F}_m = P_r m\vec{a} + P_m m\vec{a} = (P_r + P_m)m\vec{a} = zm\vec{a}$.

Here $z$ measures here the complex probability that the resultant stochastic force $\vec{F} = \vec{F}_r + \vec{F}_m$ acting on a body in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ will occur.

Since $z = P_r + P_m$ then:

If $P_r = 0 \Leftrightarrow P_m = i(1-P_r) = i(1-0) = i \Leftrightarrow z = 0 + i = i \Leftrightarrow \vec{F} = zm\vec{a} = im\vec{a}$.

If $P_r = 1 \Leftrightarrow P_m = i(1-P_r) = i(1-1) = 0 \Leftrightarrow z = 1 + 0 = 1 \Leftrightarrow \vec{F} = zm\vec{a} = m\vec{a}$.

#### *5.2.1 The relations between the forces $\vec{F}_r$, $\vec{F}_m$, and $\vec{F}$*

Since $\vec{F}_r = m\vec{a} + i\vec{F}_m \Leftrightarrow \vec{F} = \vec{F}_r + \vec{F}_m = m\vec{a} + i\vec{F}_m + \vec{F}_m = m\vec{a} + (1+i)\vec{F}_m$.

where $\mathrm{Re}\left(\vec{F}\right) = m\vec{a} + i\vec{F}_m$ and $\mathrm{Im}\left(\vec{F}\right) = \vec{F}_m$.

Additionally, since $\vec{F}_m = im\vec{a} - i\vec{F}_r \Leftrightarrow \vec{F} = \vec{F}_r + \vec{F}_m = \vec{F}_r + im\vec{a} - i\vec{F}_r = im\vec{a} + (1-i)\vec{F}_r$.

where $\mathrm{Re}\left(\vec{F}\right) = \vec{F}_r$ and $\mathrm{Im}\left(\vec{F}\right) = im\vec{a} - i\vec{F}_r = i\left(m\vec{a} - \vec{F}_r\right)$.

### 5.3 The deterministic real force $\vec{F}_c$ in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$

We define the deterministic real force by: $\vec{F}_c = P_c m\vec{a}$.

Since from *CPP* we have: $P_c = P_r + P_m/i = P_r + (1-P_r) = 1 \Leftrightarrow \vec{F}_c = m\vec{a}$.

Here $P_c$ measures the probability that the force $\vec{F}_c$ acting on a body in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ will occur. This means that the force acting on the body in the probability set $\mathcal{C}$ is totally known and is totally deterministic always $\forall P_r : 0 \le P_r \le 1$ and $\forall P_m : 0 \le P_m \le i$.

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#### *5.3.1 The relations between the forces $\vec{F}_r$, $\vec{F}_m$, and $\vec{F}_c$*

Furthermore,

Since $\vec{F}_r = P_r m\vec{a} \Leftrightarrow \vec{F}_r = P_r\vec{F}_c$ and $P_r = \frac{\vec{F}_r}{\vec{F}_c}$.

Since $\vec{F}_m = P_m m\vec{a} \Leftrightarrow \vec{F}_m = P_m\vec{F}_c$ and $P_m = \frac{\vec{F}_m}{\vec{F}_c}$.

Since $P_m = i(1-P_r) \Leftrightarrow P_r = 1 - \frac{P_m}{i} = 1 + iP_m$ because $i = -\frac{1}{i} \Leftrightarrow P_r = 1 + i\frac{\vec{F}_m}{\vec{F}_c}$.

Since $\vec{F} = zm\vec{a} \Leftrightarrow \vec{F} = z\vec{F}_c$, therefore:

If $P_r = 0 \Leftrightarrow P_m = i \Leftrightarrow z = 0 + i = i \Leftrightarrow \vec{F} = zm\vec{a} = im\vec{a} = i\vec{F}_c$.

If $P_r = 1 \Leftrightarrow P_m = 0 \Leftrightarrow z = 1 + 0 = 1 \Leftrightarrow \vec{F} = zm\vec{a} = m\vec{a} = \vec{F}_c$.

The second case shows and proves that if $P_r = 1$ then the complex resultant stochastic force will become equal to the real deterministic force that means that we will return directly to the classical deterministic Newtonian mechanics theory which is a special deterministic case of the stochastic complex probability paradigm general case.

Additionally, since $\vec{F}_m = im\vec{a} - i\vec{F}_r \Leftrightarrow i\vec{F}_r + \vec{F}_m = im\vec{a} = i\vec{F}_c$.

And $\vec{F}_r - i\vec{F}_m = m\vec{a} = \vec{F}_c$ since $i = -\frac{1}{i}$.

Since $\vec{F} = m\vec{a} + (1+i)\vec{F}_m \Leftrightarrow \vec{F} = \vec{F}_c + (1+i)\vec{F}_m$

And since $\vec{F} = im\vec{a} + (1-i)\vec{F}_r \Leftrightarrow \vec{F} = i\vec{F}_c + (1-i)\vec{F}_r$.

### 5.4 The relationships between the forces in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$ and all the *CPP* parameters

#### *5.4.1 The relationships between the real force in $\mathcal{R}$ and all the* CPP *parameters*

Furthermore, according to *CPP*:

$$\begin{aligned} DOK = |z|^2 = |P_r + P_m|^2 = P_r^2 + (P_m/i)^2 = P_r^2 + (1-P_r)^2 \\ = P_r^2 + 1 - 2P_r + P_r^2 \Leftrightarrow 2P_r^2 - 2P_r + 1 - DOK = 0 \end{aligned}$$

which is a second-degree equation in terms of $P_r$ whose discriminant is:

$$\Delta = 4 - 8(1 - DOK) = 8DOK - 4.$$

Since $0.5 \le DOK \le 1 \Leftrightarrow 0 \le 8DOK - 4 \le 4 \Leftrightarrow 0 \le \Delta \le 4 \Leftrightarrow \Delta \ge 0, \forall DOK$. Therefore, the equation admits two real roots which are:

$$P_{r1} = \frac{2-\sqrt{\Delta}}{4} = \frac{2-\sqrt{8DOK-4}}{4} = \frac{2-2\sqrt{2DOK-1}}{4} = \frac{1-\sqrt{2DOK-1}}{2}$$

$$\text{and } P_{r2} = \frac{2+\sqrt{\Delta}}{4} = \frac{2+\sqrt{8DOK-4}}{4} = \frac{2+2\sqrt{2DOK-1}}{4} = \frac{1+\sqrt{2DOK-1}}{2}.$$

But according to *CPP*: $\forall P_r : 0 \le P_r \le 1 \Leftrightarrow 0.5 \le DOK \le 1$ and $-0.5 \le Chf \le 0$ and $0 \le MChf \le 0.5$.

And if $P_r = 0$ or $P_r = 1$ then $DOK = 1$ and $Chf = 0$ and $MChf = 0$.

And if $P_r = 0.5$ then $DOK = 0.5$ and $Chf = -0.5$ and $MChf = 0.5$.

Consequently,

$$P_r = \begin{cases} \dfrac{1-\sqrt{2DOK-1}}{2} & \text{if } 0 \le P_r \le 0.5 \\ \dfrac{1+\sqrt{2DOK-1}}{2} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

But $\vec{F}_r = P_r m\vec{a}$ hence (**Figure 3**):

$$\vec{F}_r = \begin{cases} \left(\dfrac{1-\sqrt{2DOK-1}}{2}\right) m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left(\dfrac{1+\sqrt{2DOK-1}}{2}\right) m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

We have $DOK = 1 + Chf \Leftrightarrow 2DOK - 1 = 1 + 2Chf$ thus (**Figure 4**):

$$\vec{F}_r = \begin{cases} \left(\dfrac{1-\sqrt{1+2Chf}}{2}\right) m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left(\dfrac{1+\sqrt{1+2Chf}}{2}\right) m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

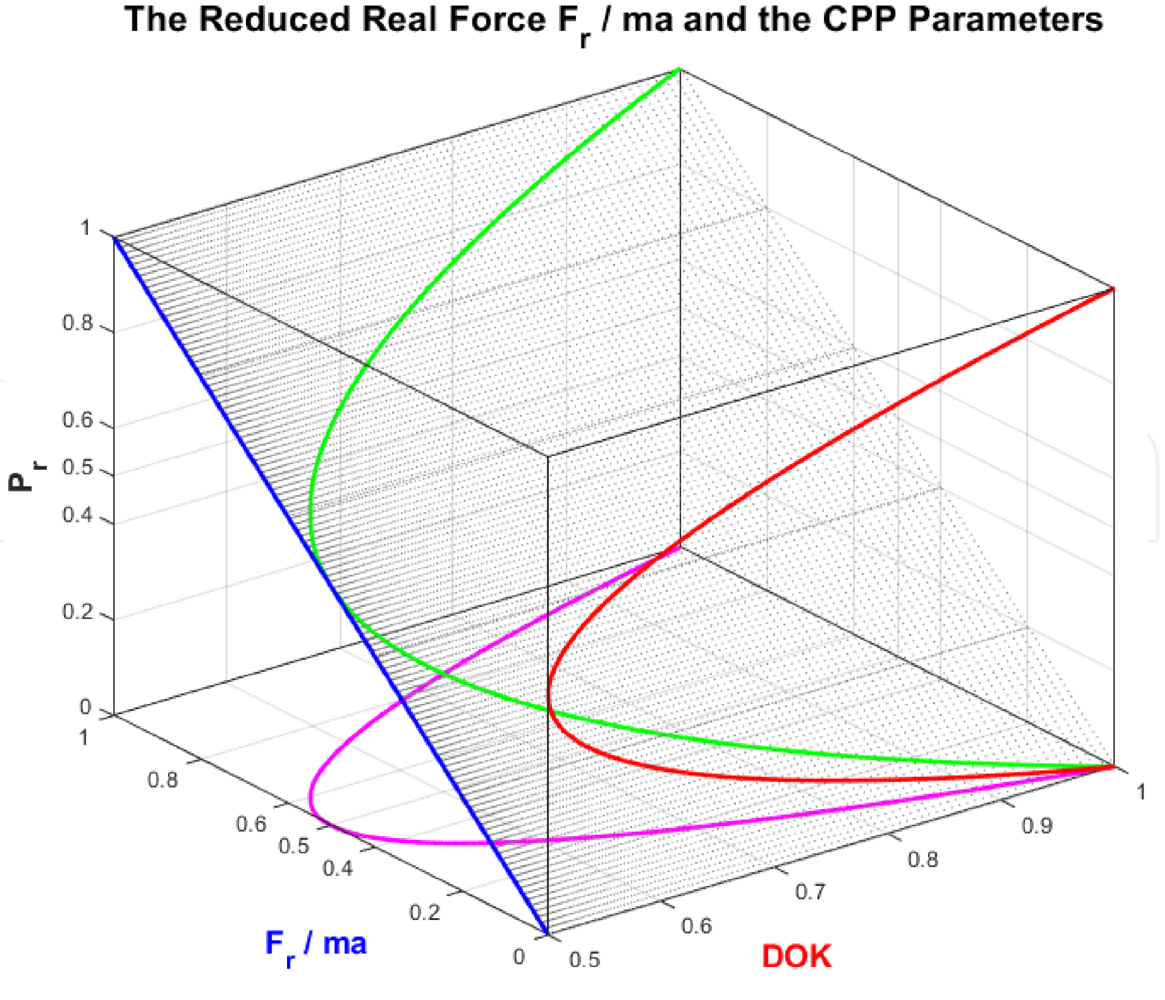


**Figure 3.**
*The graphs of the reduced real force* $F_r$ ($P_r$) / ma *in blue and of* $F_r$ (DOK) / ma *in pink and* DOK ($P_r$) *in red and of* $F_r$ (DOK) / ma *in green in the* $F_r$ ($P_r$) / ma *plane in light gray.*

We have $DOK = 1 - MChf \Leftrightarrow 2DOK - 1 = 1 - 2MChf$ thus (**Figure 5**):

$$\vec{F}_r = \begin{cases} \left(\dfrac{1 - \sqrt{1 - 2MChf}}{2}\right) m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left(\dfrac{1 + \sqrt{1 - 2MChf}}{2}\right) m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

We can deduce also from *CPP* that (**Figure 6**):



$$\vec{F}_r = \begin{cases} \left(\dfrac{1 - \sqrt{DOK + Chf}}{2}\right) m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left(\dfrac{1 + \sqrt{DOK + Chf}}{2}\right) m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

And we can infer using the fact that $MChf = -Chf$ that (**Figure 7**):

$$\vec{F}_r = \begin{cases} \left(\dfrac{1 - \sqrt{DOK - MChf}}{2}\right) m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left(\dfrac{1 + \sqrt{DOK - MChf}}{2}\right) m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

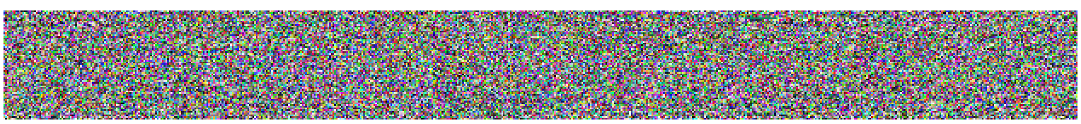



**Figure 4.**
*The graphs of the reduced real force* $F_r$ $(P_r)$ / ma *in blue and of* $F_r$ (Chf) / ma *in pink and* Chf $(P_r)$ *in red and of* $F_r$ (Chf) / ma *in green in the* $F_r$ $(P_r)$ / ma *plane in light gray.*

**Figure 5.**
*The graphs of the reduced real force* $F_r$ *(*$P_r$*)* / ma *in blue and of* $F_r$ *(*MChf*)* / ma *in pink and* MChf *(*$P_r$*) in red and of* $F_r$ *(*MChf*)* / ma *in green in the* $F_r$ *(*$P_r$*)* / ma *plane in light gray.*

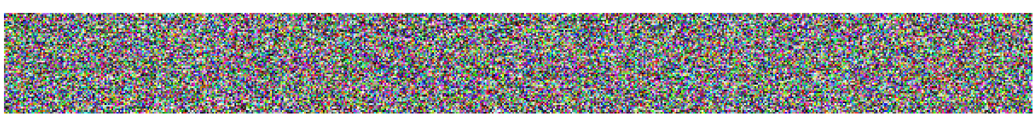

**Figure 6.**
*The graphs of the reduced real force* $F_r$ *(*Chf*)* / ma *in pink and of* $F_r$ *(*DOK*)* / ma *in red and of* $P_c^2$ = DOK – Chf = *1* = $P_c$ *(*Chf, DOK*) in cyan and of* $F_r$ *(*Chf, DOK*)* / ma *in green in the* $P_c$ *plane in light gray.*

**Figure 7.**
*The graphs of the reduced real force* $F_r$ *(MChf) / ma in pink and of* $F_r$ *(DOK) / ma in red and of* $P_c^2$ = DOK + MChf = 1 = $P_c$ (MChf, DOK) *in cyan and of* $F_r$ (MChf, DOK) / ma *in green in the* $P_c$ *plane in light gray.*

Also, we can calculate (**Figure 8**):

$$\vec{F}_r = \begin{cases} \left(\dfrac{1-\sqrt{1+Chf-MChf}}{2}\right) m\vec{a} & \text{if } 0 \leq P_r \leq 0.5 \\ \left(\dfrac{1+\sqrt{1+Chf-MChf}}{2}\right) m\vec{a} & \text{if } 0.5 \leq P_r \leq 1 \end{cases}$$

But according to *CPP*: $P_c{}^2 = DOK - Chf = DOK + MChf = 1 = P_c$ hence the real force $\vec{F}_r$ in $\mathcal{R}$ as a function of all the *CPP* parameters is the following:

$$\vec{F}_r = \begin{cases} \left(\dfrac{P_c-\sqrt{DOK-Chf-2MChf}}{2}\right) m\vec{a} & \text{if } 0 \leq P_r \leq 0.5 \\ \left(\dfrac{P_c+\sqrt{DOK-Chf-2MChf}}{2}\right) m\vec{a} & \text{if } 0.5 \leq P_r \leq 1 \end{cases}$$

### *5.4.2 The relationships between the imaginary force in* $\mathcal{M}$ *and all the* CPP *parameters*

As we have computed:

$$P_r = \begin{cases} \dfrac{1-\sqrt{2DOK-1}}{2} & \text{if } 0 \leq P_r \leq 0.5 \\ \dfrac{1+\sqrt{2DOK-1}}{2} & \text{if } 0.5 \leq P_r \leq 1 \end{cases}$$

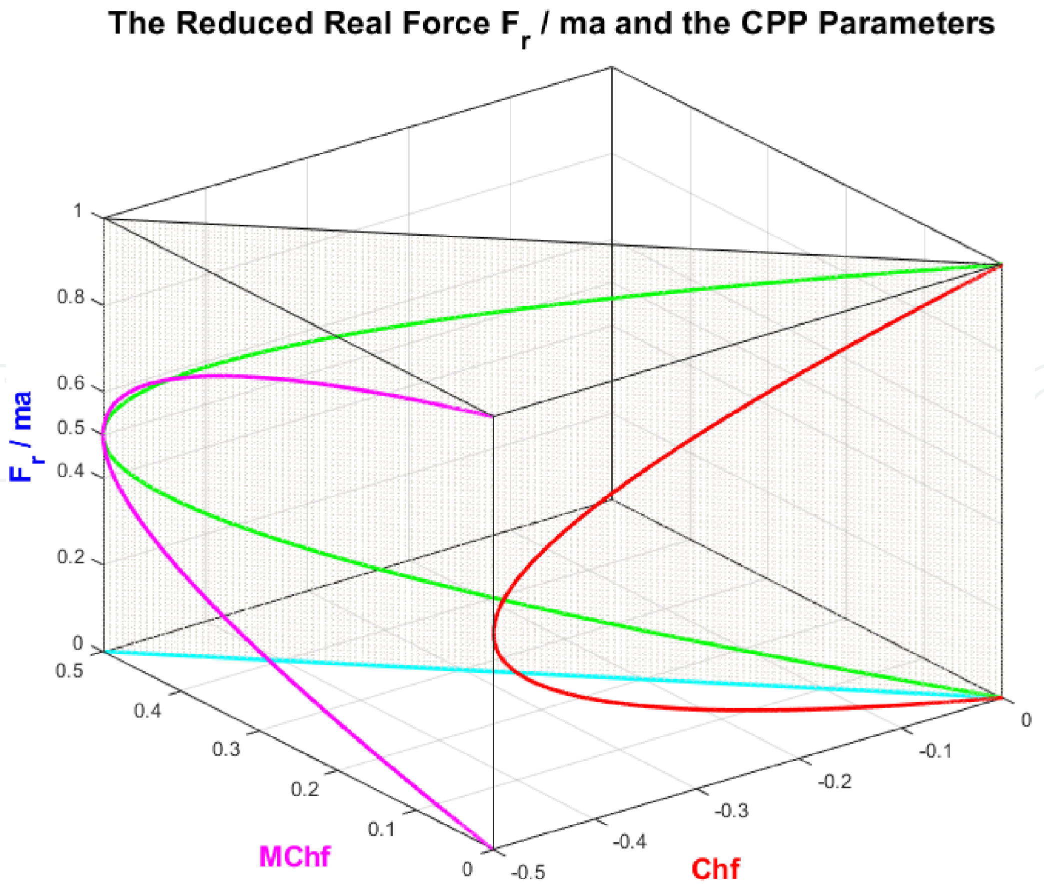


**Figure 8.**
*The graphs of the reduced real force* $F_r$ (MChf) / ma *in pink and of* $F_r$ (Chf) / ma *in red and of* Chf + MChf = 0 *in cyan and of* $F_r$ (Chf, MChf) / ma *in green in the* Chf + MChf = 0 *plane in light gray.*

And since $P_m = i(1 - P_r)$ then:

$$P_m = \begin{cases} i\left(\dfrac{1+\sqrt{2DOK-1}}{2}\right) & \text{if } 0 \le P_r \le 0.5 \Leftrightarrow \text{if } 0.5i \le P_m \le i \\ i\left(\dfrac{1-\sqrt{2DOK-1}}{2}\right) & \text{if } 0.5 \le P_r \le 1 \Leftrightarrow \text{if } 0 \le P_m \le 0.5i \end{cases}$$

We have $\vec{F}_m = P_m m\vec{a}$, so similarly to the previous section we get (**Figure 9**):

$$\vec{F}_m = \begin{cases} i\left(\dfrac{1+\sqrt{2DOK-1}}{2}\right)m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ i\left(\dfrac{1-\sqrt{2DOK-1}}{2}\right)m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

And we can deduce that (**Figure 10**):

$$\vec{F}_m = \begin{cases} i\left(\dfrac{1+\sqrt{1+2Chf}}{2}\right)m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ i\left(\dfrac{1-\sqrt{1+2Chf}}{2}\right)m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

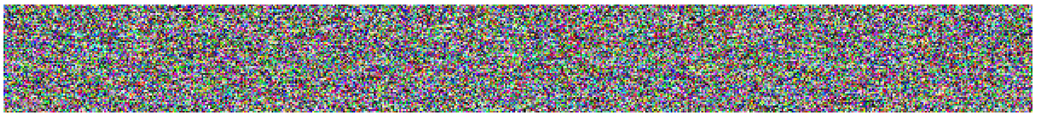

**Figure 9.**
*The graphs of the reduced imaginary force* $F_m(P_r)/ma$ *in blue and of* $F_m(DOK)/ma$ *in pink and* $DOK(P_r)$ *in red and of* $F_m(DOK)/ma$ *in green in the* $F_m(P_r)/ma$ *plane in light gray.*

**Figure 10.**
*The graphs of the reduced imaginary force* $F_m(P_r)/ma$ *in blue and of* $F_m(Chf)/ma$ *in pink and* $Chf(P_r)$ *in red and of* $F_m(Chf)/ma$ *in green in the* $F_m(P_r)/ma$ *plane in light gray.*

And we can infer that (**Figure 11**):

$$\vec{F}_m = \begin{cases} i\left(\dfrac{1+\sqrt{1-2MChf}}{2}\right)m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ i\left(\dfrac{1-\sqrt{1-2MChf}}{2}\right)m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$



We can deduce also that (**Figure 12**):

$$\vec{F}_m = \begin{cases} i\left(\dfrac{1+\sqrt{DOK+Chf}}{2}\right)m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ i\left(\dfrac{1-\sqrt{DOK+Chf}}{2}\right)m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

And we can compute (**Figure 13**):

$$\vec{F}_m = \begin{cases} i\left(\dfrac{1+\sqrt{DOK-MChf}}{2}\right)m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ i\left(\dfrac{1-\sqrt{DOK-MChf}}{2}\right)m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

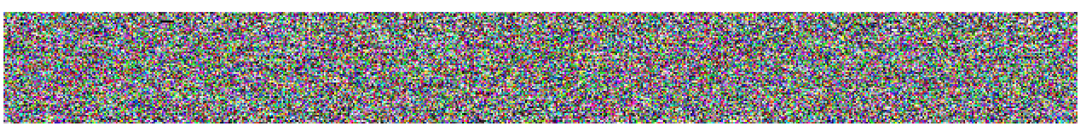



**Figure 11.**
*The graphs of the reduced imaginary force* $F_m$ $(P_r)$ / ma *in blue and of* $F_m$ (MChf) / ma *in pink and* MChf $(P_r)$ *in red and of* $F_m$ (MChf) / ma *in green in the* $F_m$ $(P_r)$ / ma *plane in light gray.*

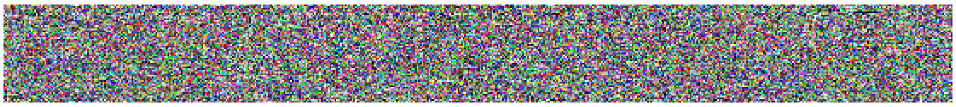

**Figure 12.**
*The graphs of the reduced imaginary force* $F_m$ *(*Chf*)* / ma *in pink and of* $F_m$ *(*DOK*)* / ma *in red and of* $P_c^2$ = DOK – Chf = *1* = $P_c$ *(*Chf, DOK*) in cyan and of* $F_m$ *(*Chf, DOK*)* / ma *in green in the* $P_c$ *plane in light gray.*

**Figure 13.**
*The graphs of the reduced imaginary force* $F_m$ *(*MChf*)* / ma *in pink and of* $F_m$ *(*DOK*)* / ma *in red and of* $P_c^2$ = DOK + MChf = *1* = $P_c$ *(*MChf, DOK*) in cyan and of* $F_m$ *(*MChf, DOK*)* / ma *in green in the* $P_c$ *plane in light gray.*

And we can calculate (**Figure 14**):

$$\vec{F}_m = \begin{cases} i\left(\dfrac{1+\sqrt{1+Chf - MChf}}{2}\right)m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ i\left(\dfrac{1-\sqrt{1+Chf - MChf}}{2}\right)m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

But according to *CPP*: $P_c^{\,2} = DOK - Chf = DOK + MChf = 1 = P_c$ hence the imaginary force $\vec{F}_m$ in $\mathcal{M}$ as a function of all the *CPP* parameters is the following:

$$\vec{F}_m = \begin{cases} i\left(\dfrac{P_c+\sqrt{DOK - Chf - 2MChf}}{2}\right)m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ i\left(\dfrac{P_c-\sqrt{DOK - Chf - 2MChf}}{2}\right)m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

### *5.4.3 The relationships between the resultant complex force in $\mathcal{C}$ and all the* CPP *parameters*

Analogously, and since $\vec{F} = \vec{F}_r + \vec{F}_m$ then:

$$\vec{F} = \begin{cases} \left[\left(\dfrac{1-\sqrt{2DOK-1}}{2}\right)+i\left(\dfrac{1+\sqrt{2DOK-1}}{2}\right)\right]m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left[\left(\dfrac{1+\sqrt{2DOK-1}}{2}\right)+i\left(\dfrac{1-\sqrt{2DOK-1}}{2}\right)\right]m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

**Figure 14.**
*The graphs of the reduced imaginary force* $F_m$ (MChf) / ma *in pink and of* $F_m$ (Chf) / ma *in red and of* Chf + MChf = 0 *in cyan and of* $F_m$ (Chf, MChf) / ma *in green in the* Chf + MChf = 0 *plane in light gray.*

And

$$\vec{F} = \begin{cases} \left[\left(\frac{1-\sqrt{1+2Chf}}{2}\right) + i\left(\frac{1+\sqrt{1+2Chf}}{2}\right)\right] m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left[\left(\frac{1+\sqrt{1+2Chf}}{2}\right) + i\left(\frac{1-\sqrt{1+2Chf}}{2}\right)\right] m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

And

$$\vec{F} = \begin{cases} \left[\left(\frac{1-\sqrt{1-2MChf}}{2}\right) + i\left(\frac{1+\sqrt{1-2MChf}}{2}\right)\right] m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left[\left(\frac{1+\sqrt{1-2MChf}}{2}\right) + i\left(\frac{1-\sqrt{1-2MChf}}{2}\right)\right] m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

We can deduce also that:

$$\vec{F} = \begin{cases} \left[\left(\frac{1-\sqrt{DOK+Chf}}{2}\right) + i\left(\frac{1+\sqrt{DOK+Chf}}{2}\right)\right] m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left[\left(\frac{1+\sqrt{DOK+Chf}}{2}\right) + i\left(\frac{1-\sqrt{DOK+Chf}}{2}\right)\right] m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

And

$$\vec{F} = \begin{cases} \left[\left(\frac{1-\sqrt{DOK-MChf}}{2}\right) + i\left(\frac{1+\sqrt{DOK-MChf}}{2}\right)\right] m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left[\left(\frac{1+\sqrt{DOK-MChf}}{2}\right) + i\left(\frac{1-\sqrt{DOK-MChf}}{2}\right)\right] m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

And

$$\vec{F} = \begin{cases} \left[\left(\frac{1-\sqrt{1+Chf-MChf}}{2}\right) + i\left(\frac{1+\sqrt{1+Chf-MChf}}{2}\right)\right] m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left[\left(\frac{1+\sqrt{1+Chf-MChf}}{2}\right) + i\left(\frac{1-\sqrt{1+Chf-MChf}}{2}\right)\right] m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

But according to *CPP*: $P_c^{\,2} = DOK - Chf = DOK + MChf = 1 = P_c$ hence the complex resultant force $\vec{F} = \vec{F}_r + \vec{F}_m$ in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of all the *CPP* parameters is the following (**Figure 15**):

$$\vec{F} = \begin{cases} \left[\left(\frac{P_c-\sqrt{DOK-Chf-2MChf}}{2}\right) + i\left(\frac{P_c+\sqrt{DOK-Chf-2MChf}}{2}\right)\right] m\vec{a} & \text{if } 0 \le P_r \le 0.5 \\ \left[\left(\frac{P_c+\sqrt{DOK-Chf-2MChf}}{2}\right) + i\left(\frac{P_c-\sqrt{DOK-Chf-2MChf}}{2}\right)\right] m\vec{a} & \text{if } 0.5 \le P_r \le 1 \end{cases}$$

And since the deterministic force in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is $\vec{F}_c = m\vec{a}$ then:

$$\vec{F} = \begin{cases} \left[\left(\dfrac{P_c - \sqrt{DOK - Chf - 2MChf}}{2}\right) + i\left(\dfrac{P_c + \sqrt{DOK - Chf - 2MChf}}{2}\right)\right]\vec{F}_c & \text{if } 0 \le P_r \le 0.5 \\ \left[\left(\dfrac{P_c + \sqrt{DOK - Chf - 2MChf}}{2}\right) + i\left(\dfrac{P_c - \sqrt{DOK - Chf - 2MChf}}{2}\right)\right]\vec{F}_c & \text{if } 0.5 \le P_r \le 1 \end{cases}$$
$$= zm\vec{a}$$

In this cube (**Figure 15**), we can notice the simulation of the complex resultant reduced force $F / ma = z(X)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real reduced force $F_r / ma = P_r(X) = \text{Re}(z)$ in $\mathcal{R}$ and of its complementary imaginary reduced force $F_m / ma = P_m(X) = i \times \text{Im}(z)$ in $\mathcal{M}$, and this in terms of the random variable $X$ for any probability and stochastic distribution. The red curve represents $F_r / ma$ in the plane $P_m(X) = 0$ and the blue curve represents $F_m / ma$ in the plane $P_r(X) = 0$. The green curve represents the complex resultant reduced force $F / ma = F_r / ma + F_m / ma = z(X) = P_r(X) + P_m(X) = \text{Re}(z) + i \times \text{Im}(z)$ in the plane $z(X) = P_r(X) + P_m(X)$ or $z(X)$ plane in cyan. The curve of $F / ma$ starts at the point J ($P_r = 0, P_m = i, X = L_b$ = lower bound of $X$) at $z = i$ and ends at the point L ($P_r = 1, P_m = 0, X = U_b$ = upper bound of $X$) at $z = 1$. The thick line in cyan is $P_r(X = L_b) + P_m(X = L_b) = z(X = L_b)$ and it is the projection of the $F / ma$ curve on the complex probability plane whose equation is



**Figure 15.**
*The graphs of the reduced real force* $F_r / ma = P_r = Re(z)$ *in red and of the reduced imaginary force* $F_m / ma = P_m = i \times Im(z)$ *in blue and of* $z = P_r + P_m$ *in cyan and of the reduced complex resultant force* $F / ma = F_r / ma + F_m / ma$ *in green in the* z *plane in light cyan.*

$z = i$. This projected thick line starts at the point J ($P_r = 0, P_m = i, X = L_b$) and ends at the point ($P_r = 1, P_m = 0, X = L_b$). Notice the importance of the point K corresponding to $z = 0.5 + 0.5i$ when $P_r = 0.5$ and $P_m = 0.5i$.

#### 5.4.3.1 The relationships between the norm of the resultant complex force and all the CPP parameters

We have: $\vec{F} = \vec{F}_r + \vec{F}_m = (P_r + P_m)m\vec{a} = zm\vec{a}$ then the norm of the complex force $\vec{F}$ can be computed as follows: $\left|\vec{F}\right|^2 = |z|^2 \times m^2\left|\vec{a}\right|^2$.

But from *CPP* we have: $|z|^2 = DOK \Leftrightarrow \left|\vec{F}\right|^2 = DOK \times m^2\left|\vec{a}\right|^2 \Leftrightarrow \left|\vec{F}\right| = \sqrt{DOK} \times m\left|\vec{a}\right|$.

According to *CPP*: $P_c^2 = DOK - Chf = 1 \Leftrightarrow DOK = 1 + Chf \Leftrightarrow \left|\vec{F}\right|^2 = (1 + Chf) \times m^2\left|\vec{a}\right|^2 \Leftrightarrow \left|\vec{F}\right| = \sqrt{1 + Chf} \times m\left|\vec{a}\right|$.

Since also $P_c^2 = DOK + MChf = 1 \Leftrightarrow DOK = 1 - MChf \Leftrightarrow \left|\vec{F}\right|^2 = (1 - MChf) \times m^2\left|\vec{a}\right|^2 \Leftrightarrow \left|\vec{F}\right| = \sqrt{1 - MChf} \times m\left|\vec{a}\right|$.

Since we have: $P_c^2 = DOK - Chf = DOK + MChf = 1 = P_c \Leftrightarrow \left|\vec{F}\right|^2 = (P_c + Chf) \times m^2\left|\vec{a}\right|^2 \Leftrightarrow \left|\vec{F}\right| = \sqrt{P_c + Chf} \times m\left|\vec{a}\right|$

And $\Leftrightarrow \left|\vec{F}\right|^2 = (P_c - MChf) \times m^2\left|\vec{a}\right|^2 \Leftrightarrow \left|\vec{F}\right| = \sqrt{P_c - MChf} \times m\left|\vec{a}\right|$

### 5.4.4 The relationships between the real deterministic force in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and all the CPP parameters

Furthermore, since $\vec{F}_c = P_c m\vec{a}$ and since $P_c^2 = DOK - Chf = DOK + MChf = 1 = P_c$ therefore:

$$\begin{aligned}
\vec{F}_c = P_c m\vec{a} &= \sqrt{DOK - Chf}.m\vec{a} \\
&= \sqrt{DOK + MChf}.m\vec{a} \\
&= \sqrt{1 + Chf + MChf}.m\vec{a} \\
&= P_c^2 m\vec{a} \\
&= 1 \times m\vec{a} = m\vec{a}
\end{aligned}$$

Hence, we can conclude that no chaos, no ignorance, no disorder, no unpredictability, no chance, and no randomness exist in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$, but complete and perfect and deterministic knowledge and experiment.

## 6. The resultant complex random vector *Z* of *CPP* and statistical physics

A powerful tool will be described in the current section which was developed in my personal previous research papers and which is founded on the concept of a complex random vector that is a vector combining the real and the imaginary probabilities of a random particle, defined in the three added axioms of *CPP* by the term $z_j = P_{rj} + P_{mj}$. Accordingly, we will define the vector $Z$ as the resultant complex random vector which is the sum of all the complex random vectors $z_j$ in the complex probability plane $\mathcal{C}$. This procedure is illustrated by considering first a general Bernoulli distribution, then we will discuss a discrete probability distribution with $N$ equiprobable random vectors as a general case. In fact, if $z$ represents one particle in a macrosystem from the uniform distribution $U$, then $Z_U$ represents all the particles in the whole macrosystem from the uniform distribution $U$ that means that $Z_U$ represents the whole random distribution in the complex probability plane $\mathcal{C}$. So, in this context, it follows directly that a Bernoulli distribution can be understood as a simplified system with two random particles (section 6-1), whereas the general case is a random system with $N$ random particles (section 6-2). Afterward, I will prove an important property at the foundation of statistical mechanics and physics using this new powerful concept (section 6-3) [45–61].

### 6.1 The resultant complex random vector $Z$ of a general Bernoulli distribution (a distribution with two random particles)

First, let us consider the following general Bernoulli distribution and let us define its complex random vectors and their resultant (**Table 1**):

Where,

$x_1$ and $x_2$ are the outcomes of the first and second random vectors respectively.

$P_{r1}$ and $P_{r2}$ are the real probabilities of $x_1$ and $x_2$ respectively.

$P_{m1}$ and $P_{m2}$ are the imaginary probabilities of $x_1$ and $x_2$ respectively.

We have:

$$\sum_{j=1}^{2} P_{rj} = P_{r1} + P_{r2} = p + q = 1$$

and

$$\begin{aligned}\sum_{j=1}^{2} P_{mj} &= P_{m1} + P_{m2} = iq + ip = i(1-p) + ip \\ &= i - ip + ip = i = i(2-1) = i(N-1)\end{aligned}$$

Where $N$ is the number of random vectors or outcomes which is equal to 2 for a Bernoulli distribution.

The complex random vector corresponding to the random outcome $x_1$ is:

$$z_1 = P_{r1} + P_{m1} = p + i(1-p) = p + iq$$

| Outcome | $x_j$ | $x_1$ | $x_2$ |
|---|---|---|---|
| In $\mathcal{R}$ | $P_{rj}$ | $P_{r1} = p$ | $P_{r2} = q$ |
| In $\mathcal{M}$ | $P_{mj}$ | $P_{m1} = i(1-p) = iq$ | $P_{m2} = i(1-q) = ip$ |
| In $\mathcal{C} = \mathcal{R} + \mathcal{M}$ | $z_j$ | $z_1 = P_{r1} + P_{m1}$ | $z_2 = P_{r2} + P_{m2}$ |

**Table 1.**
*A general Bernoulli distribution in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$.*

The complex random vector corresponding to the random outcome $x_2$ is:

$$z_2 = P_{r2} + P_{m2} = q + i(1-q) = q + ip$$

The resultant complex random vector is defined as follows:

$$\begin{aligned} Z &= \sum_{j=1}^{2} z_j = z_1 + z_2 = \sum_{j=1}^{2} P_{rj} + \sum_{j=1}^{2} P_{mj} \\ &= (p + iq) + (q + ip) = (p+q) + i(p+q) \\ &= 1 + i = 1 + i(2-1) \\ \Rightarrow Z &= 1 + i(N-1) \end{aligned}$$

The probability $P_{c1}$ in the complex plane $\mathcal{C} = \mathcal{R} + \mathcal{M}$ which corresponds to the complex random vector $z_1$ is computed as follows:

$$\begin{aligned} &|z_1|^2 = P_{r1}^2 + (P_{m1}/i)^2 = p^2 + q^2 \\ &Chf_1 = -2P_{r1}P_{m1}/i = -2pq \\ &\Rightarrow P_{c1}^2 = |z_1|^2 - Chf_1 \\ &\qquad = p^2 + q^2 + 2pq = (p+q)^2 = 1^2 = 1 \\ &\Rightarrow P_{c1} = 1 \end{aligned}$$

This is coherent with the three novel complementary axioms defined for the *CPP*.

Similarly, $P_{c2}$ corresponding to $z_2$ is:

$$\begin{aligned} &|z_2|^2 = P_{r2}^2 + (P_{m2}/i)^2 = q^2 + p^2 \\ &Chf_2 = -2P_{r2}P_{m2}/i = -2qp \\ &\Rightarrow P_{c2}^2 = |z_2|^2 - Chf_2 \\ &\qquad = q^2 + p^2 + 2qp = (q+p)^2 = 1^2 = 1 \\ &\Rightarrow P_{c2} = 1 \end{aligned}$$

The probability $P_c$ in the complex plane $\mathcal{C}$ which corresponds to the resultant complex random vector $Z = 1 + i$ is computed as follows:

$$\begin{aligned} &|Z|^2 = \left(\sum_{j=1}^{2} P_{rj}\right)^2 + \left(\sum_{j=1}^{2} P_{mj}/i\right)^2 = 1^2 + 1^2 = 2 \\ &Chf = -2\sum_{j=1}^{2} P_{rj} \sum_{j=1}^{2} P_{mj}/i = -2(1)(1) = -2 \\ &\text{Let } s^2 = |Z|^2 - Chf = 2 + 2 = 4 \Rightarrow s = 2 \\ &\Rightarrow P_c{}^2 = \frac{s^2}{N^2} = \frac{|Z|^2 - Chf}{N^2} = \frac{|Z|^2}{N^2} - \frac{Chf}{N^2} = \frac{4}{2^2} = \frac{4}{4} = 1 \\ &\Rightarrow P_c = \frac{s}{N} = \frac{2}{2} = 1 \end{aligned}$$

Where $s$ is an intermediary quantity used in our computation of $P_c$.

$P_c$ is the probability corresponding to the resultant complex random vector $Z$ in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and is also equal to 1. Actually, $Z$ represents both

$z_1$ and $z_2$ that means the whole distribution of random vectors of the general Bernoulli distribution in the complex plane $\mathcal{C}$ and its probability $P_c$ is computed in the same way as $P_{c1}$ and $P_{c2}$ .

By analogy, for the case of one random vector $z_j$ we have:

$$P_{cj}^2 = |z_j|^2 - \ Chf_j \quad \text{with} \ \ (N = 1).$$

In general, for the vector $Z$ we have:

$$P_c{}^2 = \frac{|Z|^2}{N^2} - \ \frac{Chf}{N^2}; \quad (N \geq 1)$$

Where the degree of our knowledge of the whole distribution is equal to $DOK_Z = \frac{|Z|^2}{N^2}$, its relative chaotic factor is $Chf_Z = \frac{Chf}{N^2}$, and its relative magnitude of the chaotic factor is $MChf_Z = |Chf_Z|$.

Notice, if $N = 1$ in the previous formula, then:

$$P_c{}^2 = \frac{|Z|^2}{N^2} - \ \frac{Chf}{N^2} = \frac{|Z|^2}{1^2} - \ \frac{Chf}{1^2} = |Z|^2 - Chf = |z_j|^2 - \ Chf_j = P_{cj}^2$$

which is coherent with the calculations already done.

To illustrate the concept of the resultant complex random vector $Z$, I will use the following graph (**Figure 16**).

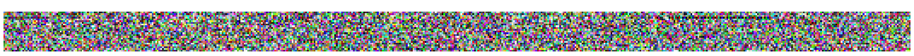

**Figure 16.**
*The resultant complex random vector $Z = z_1 + z_2$ for a general Bernoulli distribution in the complex probability plane $\mathcal{C}$.*

### 6.2 The general case: A discrete distribution with *N* Equiprobable random vectors (a uniform distribution *U* with *N* random particles)

As a general case, let us consider then this discrete probability distribution with $N$ equiprobable random vectors which is a discrete uniform probability distribution $U$ with $N$ particles (**Table 2**):

We have here in $\mathcal{C} = \mathcal{R} + \mathcal{M}$:

$$z_j = P_{rj} + P_{mj}, \quad \forall j: \ 1 \leq j \leq N,$$

$$\text{and } z_1 = z_2 = \ldots = z_N = \frac{1}{N} + \frac{i(N-1)}{N}$$

$$\Rightarrow Z_U = \sum_{j=1}^{N} z_j = z_1 + z_2 + \ldots + z_N = N z_j = N\left(\frac{1}{N} + \frac{i(N-1)}{N}\right) = 1 + i(N-1)$$

Moreover, we can notice that: $|z_1| = |z_2| = \cdots = |z_N|$, hence,

$$|Z_U| = |z_1 + z_2 + \ldots + z_N| = N|z_1| = N|z_2| = \ldots = N|z_N|$$

$$\Rightarrow |Z_U|^2 = N^2 |z_j|^2 = N^2\left(\frac{1}{N^2} + \frac{(N-1)^2}{N^2}\right) = 1 + (N-1)^2, \ \text{ where } \ 1 \leq j \leq N;$$

And

$$\begin{aligned} Chf &= N^2 \times Chf_j = -2 \times P_{rj} \times (P_{mj}/i) \times N^2 = -2N^2 \times \left(\frac{1}{N}\right)\left(\frac{N-1}{N}\right) \\ &= -2(1)(N-1) = -2(N-1) \end{aligned}$$

$$\Rightarrow s^2 = |Z_U|^2 - Chf = 1 + (N-1)^2 + 2(N-1) = [1 + (N-1)]^2 = N^2$$

$$\begin{aligned} &\Rightarrow P_c^2|_{Z_U} = \frac{s^2}{N^2} = \frac{N^2}{N^2} = 1 \\ &= \frac{|Z_U|^2}{N^2} - \frac{Chf}{N^2} = \frac{1 + (N-1)^2}{N^2} - \frac{-2(N-1)}{N^2} = \frac{1 + (N-1)^2 + 2(N-1)}{N^2} = \frac{[1 + (N-1)]^2}{N^2} = \frac{N^2}{N^2} = 1 \\ &\Rightarrow P_c|_{Z_U} = 1 \end{aligned}$$

Where $s$ is an intermediary quantity used in our computation of $P_c|_{Z_U}$.

Therefore, the degree of our knowledge corresponding to the resultant complex vector $Z_U$ representing the whole uniform distribution is:

$$DOK_{Z_U} = \frac{|Z_U|^2}{N^2} = \frac{1 + (N-1)^2}{N^2},$$

and its relative chaotic factor is:

$$Chf_{Z_U} = \frac{Chf}{N^2} = -\frac{2(N-1)}{N^2},$$

Similarly, its relative magnitude of the chaotic factor is:

| Outcome | $x_j$ | $x_1$ | $x_2$ | ... | $x_N$ |
|---|---|---|---|---|---|
| In $\mathcal{R}$ | $P_{rj}$ | $P_{r1} = \frac{1}{N}$ | $P_{r2} = \frac{1}{N}$ | ... | $P_{rN} = \frac{1}{N}$ |
| In $\mathcal{M}$ | $P_{mj}$ | $P_{m1} = i\left(1 - \frac{1}{N}\right)$ | $P_{m2} = i\left(1 - \frac{1}{N}\right)$ | ... | $P_{mN} = i\left(1 - \frac{1}{N}\right)$ |
| In $\mathcal{C} = \mathcal{R} + \mathcal{M}$ | $z_j$ | $z_1 = P_{r1} + P_{m1}$ | $z_2 = P_{r2} + P_{m2}$ | ... | $z_N = P_{rN} + P_{mN}$ |

**Table 2.**
*A discrete uniform distribution with* N *equiprobable random vectors in* $\mathcal{R}$, $\mathcal{M}$, *and* $\mathcal{C}$.

$$MChf_{Z_U} = \left|Chf_{Z_U}\right| = \left|\frac{Chf}{N^2}\right| = \left|-\frac{2(N-1)}{N^2}\right| = \frac{2(N-1)}{N^2}.$$

Thus, we can verify that we have always:

$$P_c^2|_{Z_U} = \frac{|Z_U|^2}{N^2} - \frac{Chf}{N^2} = DOK_{Z_U} - Chf_{Z_U} = DOK_{Z_U} + MChf_{Z_U} = 1 \Leftrightarrow P_c|_{Z_U} = 1$$

What is important here is that we can notice the following fact. Take for example:

$$N = 2 \Rightarrow DOK_{Z_U} = \frac{1 + (2-1)^2}{2^2} = 0.5 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(2-1)}{2^2} = -0.5$$

$$N = 4 \Rightarrow DOK_{Z_U} = \frac{1 + (4-1)^2}{4^2} = 0.625 \geq 0.5 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(4-1)}{4^2} = -0.375 \geq -0.5$$

$$N = 5 \Rightarrow DOK_{Z_U} = \frac{1 + (5-1)^2}{5^2} = 0.68 \geq 0.625 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(5-1)}{5^2} = -0.32 \geq -0.375$$

$$N = 10 \Rightarrow DOK_{Z_U} = \frac{1 + (10-1)^2}{10^2} = 0.82 \geq 0.68 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(10-1)}{10^2} = -0.18 \geq -0.32$$

$$N = 100 \Rightarrow DOK_{Z_U} = \frac{1 + (100-1)^2}{100^2} = 0.9802 \geq 0.82 \quad \text{and} \; Chf_{Z_U} = \frac{-2(100-1)}{100^2} = -0.0198 \geq -0.18$$

$$N = 1000 \Rightarrow DOK_{Z_U} = \frac{1 + (1000-1)^2}{1000^2} = 0.998002 \geq 0.9802 \quad \text{and}$$
$$Chf_{Z_U} = \frac{-2(1000-1)}{1000^2} = -0.001998 \geq -0.0198$$

$$N = 1,000,000 \Rightarrow DOK_{Z_U} = \frac{1 + \left(10^6 - 1\right)^2}{10^{12}} = 0.999998 \geq 0.998002 \quad \text{and}$$
$$Chf_{Z_U} = \frac{-2\left(10^6 - 1\right)}{10^{12}} = -0.000001999998 \geq -0.001998$$

We can deduce mathematically using calculus that:

$$\lim_{N \to +\infty} \frac{|Z_U|^2}{N^2} = \lim_{N \to +\infty} DOK_{Z_U} = \lim_{N \to +\infty} \frac{1 + (N-1)^2}{N^2} = 1,$$

$$\text{and} \lim_{N\to+\infty}\frac{Chf}{N^2} = \lim_{N\to+\infty} Chf_{Z_U} = \lim_{N\to+\infty} -\frac{2(N-1)}{N^2} = 0.$$

From the above, we can also deduce this conclusion:

As much as $N$ increases, as much as the degree of our knowledge in $\mathcal{R}$ corresponding to the resultant complex vector is perfect and absolute, that means, it is equal to one, and as much as the chaotic factor that prevents us from foretelling exactly and totally the outcome of the stochastic phenomenon in $\mathcal{R}$ approaches zero. Mathematically we state that: If $N$ tends to infinity then the degree of our knowledge in $\mathcal{R}$ tends to one and the chaotic factor always in $\mathcal{R}$ tends to zero.

### 6.3 Statistical mechanics using $Z$ and $CPP$

We have:

$P_r|_{Z_U} = \sum_{j=1}^{N} P_{rj}/N = N \times \frac{P_{rj}}{N} = P_{rj} = \frac{1}{N}$= the mean of the real probability of all the $N$ complex random vectors $z_j$ represented by $Z_U$, and.

$P_m|_{Z_U} = \sum_{j=1}^{N} P_{mj}/N = N \times \frac{P_{mj}}{N} = P_{mj} = i\left(1-\frac{1}{N}\right)$= the mean of the imaginary probability of all the $N$ complex random vectors $z_j$ represented by $Z_U$, then:

$Z_U = Nz_j = N(P_r|_{Z_U} + P_m|_{Z_U}) = N\left[\frac{1}{N} + i\left(1-\frac{1}{N}\right)\right] = 1 + i(N-1)$, as computed in section 6-2.

Where $\frac{Z_U}{N} = P_r|_{Z_U} + P_m|_{Z_U} = \frac{\sum_{j=1}^{N} z_j}{N} = \frac{Nz_j}{N} = z_j = P_{rj} + P_{mj} = \frac{1}{N} + i\left(1-\frac{1}{N}\right), \ \forall j : 1 \leq j \leq N$

= the mean of all the $N$ complex random vectors $z_j$ represented by $Z_U$.

Therefore, $P_c|_{Z_U} = P_r|_{Z_U} + \frac{P_m|_{Z_U}}{i} = \frac{1}{N} + \left(1-\frac{1}{N}\right) = 1 = P_{cj}, \ \forall j : 1 \leq j \leq N$, just as predicted by $CPP$.

Additionally, we have:

$\vec{F}_{rj} = P_{rj} m \vec{a}_j, \forall j : 1 \leq j \leq N$, that means for every particle $j$ in the macrosystem of $N$ particles, and

$$\begin{aligned}
\vec{F}_r|_{Z_U} &= \sum_{j=1}^{N} \vec{F}_{rj} = P_{r1} m\vec{a}_1 + P_{r2} m\vec{a}_2 + \ldots + P_{rj} m\vec{a}_j + \ldots + P_{rN} m\vec{a}_N \\
&= \frac{1}{N} m\vec{a}_1 + \frac{1}{N} m\vec{a}_2 + \ldots + \frac{1}{N} m\vec{a}_j + \ldots + \frac{1}{N} m\vec{a}_N \\
&= \frac{1}{N}\left(m\vec{a}_1 + m\vec{a}_2 + \ldots + m\vec{a}_j + \ldots + m\vec{a}_N\right) \\
&= P_r|_{Z_U}\left(m\vec{a}_1 + m\vec{a}_2 + \ldots + m\vec{a}_j + \ldots + m\vec{a}_N\right) \\
&= P_r|_{Z_U} m \sum_{j=1}^{N} \vec{a}_j = P_r|_{Z_U} m\vec{a} = \frac{m\vec{a}}{N}
\end{aligned}$$

= the mean real random force acting on the whole macrosystem in $\mathcal{R}$.

Moreover,

$\vec{F}_{mj} = P_{mj} m \vec{a}_j, \forall j : 1 \leq j \leq N$, that means for every particle $j$ in the macrosystem of $N$ particles, and

$$\begin{aligned}
\vec{F}_m|_{Z_U} &= \sum_{j=1}^{N} \vec{F}_{mj} = P_{m1}m\vec{a}_1 + P_{m2}m\vec{a}_2 + \ldots + P_{mj}m\vec{a}_j + \ldots + P_{mN}m\vec{a}_N \\
&= i\left(1-\frac{1}{N}\right)m\vec{a}_1 + i\left(1-\frac{1}{N}\right)m\vec{a}_2 + \ldots + i\left(1-\frac{1}{N}\right)m\vec{a}_j + \ldots + i\left(1-\frac{1}{N}\right)m\vec{a}_N \\
&= i\left(1-\frac{1}{N}\right)\left(m\vec{a}_1 + m\vec{a}_2 + \ldots + m\vec{a}_j + \ldots + m\vec{a}_N\right) \\
&= P_m|_{Z_U}\left(m\vec{a}_1 + m\vec{a}_2 + \ldots + m\vec{a}_j + \ldots + m\vec{a}_N\right) \\
&= P_m|_{Z_U} m \sum_{j=1}^{N} \vec{a}_j = P_m|_{Z_U} m\vec{a} = i\left(1-\frac{1}{N}\right)m\vec{a}
\end{aligned}$$

= the mean imaginary random force acting on the whole macrosystem in $\mathcal{M}$.

Furthermore,

$$\begin{aligned}
\vec{F}|_{Z_U} = \vec{F}_r|_{Z_U} + \vec{F}_m|_{Z_U} &= \sum_{j=1}^{N} \vec{F}_{rj} + \sum_{j=1}^{N} \vec{F}_{mj} = P_r|_{Z_U} m\vec{a} + P_m|_{Z_U} m\vec{a} \\
&= \left(P_r|_{Z_U} + P_m|_{Z_U}\right)m\vec{a} = \frac{Z_U}{N}m\vec{a} = \left[\frac{1}{N} + i\left(1-\frac{1}{N}\right)\right]m\vec{a}
\end{aligned}$$

= the mean resultant complex random force acting on the whole macrosystem in $\mathcal{C} = \mathcal{R} + \mathcal{M}$.

Also, we have:

$\vec{F}_{cj} = P_{cj}m\vec{a}_j = 1 \times m\vec{a}_j = m\vec{a}_j, \forall j : 1 \le j \le N$, that means for every particle $j$ in the macrosystem of $N$ particles, just as predicted by *CPP*.

And $\vec{F}_c|_{Z_U} = P_c\Big|_{Z_U} m\vec{a} = 1 \times m\vec{a} = m\vec{a}$ = the deterministic force acting on the whole macrosystem in $\mathcal{C} = \mathcal{R} + \mathcal{M}$, as predicted by *CPP* also.

Correspondingly, we can deduce the following result:

$$\text{If } DOK_{Z_U} = \frac{|Z_U|^2}{N^2} = \left(P_r|_{Z_U}\right)^2 + \left(\frac{P_m|_{Z_U}}{i}\right)^2 = P_r^2|_{Z_U} + \left(1 - P_r|_{Z_U}\right)^2 = 1$$

$$\Leftrightarrow \begin{cases} P_r|_{Z_U} = \dfrac{1}{N} = 0 \\ \text{or} \\ P_r|_{Z_U} = \dfrac{1}{N} = 1 \end{cases} \Leftrightarrow \begin{cases} N \to +\infty \\ \text{or} \\ N = 1 \end{cases} \Leftrightarrow \begin{cases} \vec{F}_r|_{Z_U} = P_r|_{Z_U} \times m\vec{a} = 0 \times m\vec{a} = \vec{0} \\ \text{or} \\ \vec{F}_r|_{Z_U} = P_r|_{Z_U} \times m\vec{a} = 1 \times m\vec{a} = m\vec{a} \end{cases}$$

$$\Leftrightarrow \begin{cases} P_m|_{Z_U} = i\left(1 - P_r|_{Z_U}\right) = i(1-0) = i \\ \text{or} \\ P_m|_{Z_U} = i\left(1 - P_r|_{Z_U}\right) = i(1-1) = 0 \end{cases} \Leftrightarrow \begin{cases} \vec{F}_m|_{Z_U} = P_m|_{Z_U} \times m\vec{a} = im\vec{a} \\ \text{or} \\ \vec{F}_m|_{Z_U} = P_m|_{Z_U} \times m\vec{a} = 0 \times m\vec{a} = \vec{0} \end{cases}$$

Therefore, this means that in the first case the mean real force acting on the macrosystem in the real set $\mathcal{R}$ is equal to $\vec{0}$, or that in the second case the experiment on the macrosystem is totally deterministic always in the real probability set $\mathcal{R}$.

$$\Leftrightarrow \begin{cases} \vec{F}|_{Z_U} = \vec{F}_r|_{Z_U} + \vec{F}_m|_{Z_U} = \vec{0} + im\vec{a} = im\vec{a} \\ \text{or} \\ \vec{F}|_{Z_U} = \vec{F}_r|_{Z_U} + \vec{F}_m|_{Z_U} = m\vec{a} + \vec{0} = m\vec{a} \end{cases}$$

$$\Leftrightarrow \left|\vec{F}|_{Z_U}\right| = \begin{cases} \left|im\vec{a}\right| = m\left|\vec{a}\right| \\ \text{or} \\ \left|m\vec{a}\right| = m\left|\vec{a}\right| \end{cases} \Leftrightarrow \left|\vec{F}|_{Z_U}\right| = m\left|\vec{a}\right| = \left|\vec{F}_c|_{Z_U}\right| \text{ in both cases.}$$

That means that the mean norm of the resultant force acting on the whole macrosystem is totally deterministic in both cases in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and is always equal accordingly to $m\left|\vec{a}\right|$.

Similarly, we can deduce also the following similar result:

$$\text{If } DOK_{Z_U} = 1 \Leftrightarrow Chf_{Z_U} = 2iP_r|_{Z_U} \times P_m|_{Z_U} = -2P_r|_{Z_U} \times (1 - P_r|_{Z_U}) = 0$$

$$\Leftrightarrow \begin{cases} P_r|_{Z_U} = \dfrac{1}{N} = 0 \\ \text{or} \\ P_r|_{Z_U} = \dfrac{1}{N} = 1 \end{cases} \Leftrightarrow \begin{cases} N \to +\infty \\ \text{or} \\ N = 1 \end{cases} \Leftrightarrow \begin{cases} \vec{F}_r|_{Z_U} = P_r|_{Z_U} \times m\vec{a} = 0 \times m\vec{a} = \vec{0} \\ \text{or} \\ \vec{F}_r|_{Z_U} = P_r|_{Z_U} \times m\vec{a} = 1 \times m\vec{a} = m\vec{a} \end{cases}$$

$$\Leftrightarrow \begin{cases} P_m|_{Z_U} = i(1 - P_r|_{Z_U}) = i(1-0) = i \\ \text{or} \\ P_m|_{Z_U} = i(1 - P_r|_{Z_U}) = i(1-1) = 0 \end{cases} \Leftrightarrow \begin{cases} \vec{F}_m|_{Z_U} = P_m|_{Z_U} \times m\vec{a} = im\vec{a} \\ \text{or} \\ \vec{F}_m|_{Z_U} = P_m|_{Z_U} \times m\vec{a} = 0 \times m\vec{a} = \vec{0} \end{cases}$$

Therefore, this means that in the first case the mean real force acting on the macrosystem in the real set $\mathcal{R}$ is equal to $\vec{0}$, or that in the second case the experiment on the macrosystem is totally deterministic always in the real probability set $\mathcal{R}$.

$$\Leftrightarrow \begin{cases} \vec{F}|_{Z_U} = \vec{F}_r|_{Z_U} + \vec{F}_m|_{Z_U} = \vec{0} + im\vec{a} = im\vec{a} \\ \text{or} \\ \vec{F}|_{Z_U} = \vec{F}_r|_{Z_U} + \vec{F}_m|_{Z_U} = m\vec{a} + \vec{0} = m\vec{a} \end{cases}$$

$$\Leftrightarrow \left|\vec{F}|_{Z_U}\right| = \begin{cases} \left|im\vec{a}\right| = m\left|\vec{a}\right| \\ \text{or} \\ \left|m\vec{a}\right| = m\left|\vec{a}\right| \end{cases} \Leftrightarrow \left|\vec{F}|_{Z_U}\right| = m\left|\vec{a}\right| = \left|\vec{F}_c|_{Z_U}\right| \text{ in both cases.}$$

That means that the mean norm of the resultant force acting on the whole macrosystem is totally deterministic in both cases in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and is always equal accordingly to $m\left|\vec{a}\right|$. Consequently, we reach the same conclusion if we consider $Chf_{Z_U}$ as above when we have considered $DOK_{Z_U}$.

In addition, for $N = 1 \Rightarrow \frac{|Z_U|^2}{N^2} = DOK_{Z_U} = \frac{1+(N-1)^2}{N^2} = \frac{1+(1-1)^2}{1^2} = 1$

$$\text{and } \frac{Chf}{N^2} = Chf_{Z_U} = -\frac{2(N-1)}{N^2} = -\frac{2(1-1)}{1^2} = 0$$

This means that we have a random experiment with only one outcome or vector, hence, $P_r|_{Z_U} = \frac{1}{N} = \frac{1}{1} = 1$, that means we have a sure event in $\mathcal{R}$. Consequently, we have accordingly the degree of our knowledge is equal to one (perfect macrosystem knowledge) and the chaotic factor is equal to zero (no chaos) since the experiment is certain and totally deterministic in $\mathcal{R}$, which is absolutely logical.

### 6.4 Analysis and interpretation of all the results

The law of large numbers states that:

"As $N$ increases, then the probability that the value of sample mean to be close to population mean approaches 1".

We can deduce now the following conclusion related to the law of large numbers:

We can see, as we have proved, that as much as $N$ increases, as much as the degree of knowledge of the resultant complex vector $DOK_{Z_U} = \frac{|Z_U|^2}{N^2}$ tends to 1 and its relative chaotic factor $Chf_{Z_U} = \frac{Chf}{N^2}$ tends to 0. Assume now that the random variables $x_j$'s correspond to the atoms or particles or molecules moving randomly in a gas or a liquid. So, if we study a gas or a liquid with billions of such particles, then $N$ is big enough (e.g. Avogadro's number $\approx 6.02214 \times 10^{23}$ / mole in the International System of Units) to allow that its corresponding temperature, pressure, energy etc. ... tend to the mean of these quantities corresponding to the whole system. This because the chaotic factor of the whole macrosystem (gas, liquid, etc.), that is, of the resultant complex random vector $Z_U$ representing all the random particles or vectors, tends to 0; thus, the behavior and characteristics of the whole system in $\mathcal{R}$ is predictable with great precision since the degree of our knowledge of

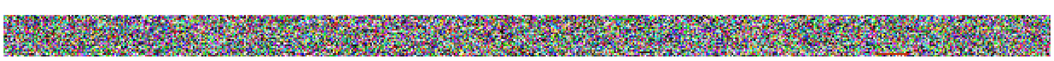

**Figure 17.**
*$Chf_{Z_U}$, $DOK_{Z_U}$, and $P_c|_{Z_U}$, as functions of the particles number* N *in 2D.*

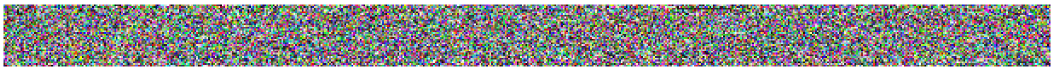

**Figure 18.**
*$Chf_{Z_U}$,$DOK_{Z_U}$, and $P_c|_{Z_U}$, as functions of the particles number* N *in 3D.*

the whole macrosystem tends to 1. Subsequently, we can deduce from the above that since for $DOK_{Z_U} = 1$ or for $Chf_{Z_U} = 0$ the mean norm of the resultant force acting on the macrosystem that consists of $N > > 1$ individual particles is totally known and deterministic in $\mathcal{R}$ then all the properties of the macrosystem are totally and completely known and determined like the macrosystem energy which should be equal to the mean of the individual particles energies, or the macrosystem pressure which should be equal to the mean of the individual particles pressures or the macrosystem temperature which should be equal to the mean of the individual particles temperatures, etc.

Hence, what we have done here is that we have proved the law of large numbers (already discussed in the published papers [46, 50, 57, 61]) as well as an important property of statistical mechanics using *CPP*. In fact, as it is very well known in the classical probability theory and statistics, the law of large numbers is tightly related and linked to statistical mechanics. Here *CPP* comes and proves both of them in a novel and original way. This looks very interesting and fruitful and shows the validity and the benefits of extending Kolmogorov's axioms to the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$. The following figures (**Figures 17** and **18**) show the convergence of $Chf_{Z_U}$ to 0 and of $DOK_{Z_U}$ to 1 as functions of the particles or atoms or molecules number $N$.

## 7. Flowchart of the complex probability and Newton's mechanics prognostic model

The following flowchart summarizes all the procedures of the proposed complex probability prognostic model where $X$ is between the lower bound $L_b$ and the upper bound $U_b$:

## 8. The new paradigm applied to various discrete and continuous stochastic distributions

In this section, the simulation of the novel *CPP* model for various discrete and continuous random distributions will be done. Note that all the numerical values found in the paradigm functions analysis for all the simulations were computed using the 64-Bit MATLAB version 2020 software. It is important to mention here that a few important and well-known probability distributions were considered although the original *CPP* model can be applied to any stochastic distribution beside the studied random cases below. This will lead to similar results and conclusions. Hence, the new paradigm is successful with any discrete or continuous random case.

### 8.1 Simulation of discrete probability distributions

#### *8.1.1 The discrete uniform probability distribution*

The probability density function (*PDF*) of this discrete stochastic distribution is:

$$f(X = x_k; N) = \begin{cases} 0 & \text{for } X = x_0 = L_b, k = 0 \\ \dfrac{1}{N} & \text{for } X = x_1, x_2, \dots, x_k, \dots, (x_N = U_b), \forall k : 1 \le k \le N \end{cases}$$

Note that in the simulation we have considered: $L_b = -21$ and $U_b = 21$ and $N = 60$ and $\forall k : 1 \le k \le (N = 60)$ we have : $\Delta x_k = x_k - x_{k-1} = 0.7$ .

The cumulative distribution function ($CDF$) is:

$$CDF(x) = P_{rob}(X \le x) = \sum_{j=0}^{k} f(x_j; N) = f(x_0; N) + \sum_{j=1}^{k} f(x_j; N) = 0 + \sum_{j=1}^{k} \frac{1}{N} = \frac{k}{N}$$
$$= \frac{k}{60}, \forall k : 0 \le k \le (N = 60)$$

Note that:

If $k = 0 \Leftrightarrow CDF(x) = P_{rob}(X \le x) = f(X = x_0 = L_b; N) = 0$.

If $k = N \Leftrightarrow X = x_N = U_b$

$$\Leftrightarrow CDF(x) = P_{rob}(X \le x) = f(x_0; N) + \sum_{j=1}^{k=N} f(x_j; N) = 0 + \sum_{j=1}^{k=N} \frac{1}{N} = \frac{N}{N} = \frac{60}{60} = 1$$

The mean or average or expectation is:

$$\mu = \frac{\sum_{j=0}^{N} x_j}{N+1} = 0$$

The variance is:

$$\sigma^2 = \frac{\sum_{j=0}^{N} (x_j - \mu)^2}{N+1} = 151.9000$$

The standard deviation is:

$$\sigma = \sqrt{\frac{\sum_{j=0}^{N} (x_j - \mu)^2}{N+1}} = \sqrt{151.9000} = 12.3247718$$

The median $Md = 0 = \mu$ since it is a symmetric distribution.

Since the distribution is uniform then it has no mode.

The real probability $P_r(x)$ and force are:

$$P_r(x) = CDF(x) = \sum_{j=0}^{k} f(x_j; N) = \frac{k}{N} = \frac{k}{60}, \forall k : 0 \le k \le (N = 60)$$
$$\Leftrightarrow \vec{F}_r(x) = P_r(x) m\vec{a} = \left(\frac{k}{N}\right) m\vec{a} = \left(\frac{k}{60}\right) m\vec{a}$$

The imaginary complementary probability $P_m(x)$ and force are:

$$P_m(x) = i[1 - P_r(x)] = i[1 - CDF(x)] = i\left[1 - \sum_{j=0}^{k} f(x_j; N)\right]$$
$$= i \sum_{j=k+1}^{N} f(x_j; N) = i\left(1 - \frac{k}{N}\right) = i\left(1 - \frac{k}{60}\right), \quad \forall k : 0 \le k \le (N = 60)$$

$$\Leftrightarrow \vec{F}_m(x) = P_m(x)m\vec{a} = i\left(1-\frac{k}{N}\right)m\vec{a} = i\left(1-\frac{k}{60}\right)m\vec{a}$$

The real complementary probability $P_m(x)/i$ and force are:

$$P_m(x)/i = 1 - P_r(x) = 1 - CDF(x) = 1 - \sum_{j=0}^{k} f(x_j;N) = \sum_{j=k+1}^{N} f(x_j;N) = 1 - \frac{k}{N}$$
$$= 1 - \frac{k}{60}, \forall k: 0 \le k \le (N = 60)$$
$$\Leftrightarrow \vec{F}_m(x)/i = \frac{P_m(x)}{i}m\vec{a} = \left(1-\frac{k}{N}\right)m\vec{a} = \left(1-\frac{k}{60}\right)m\vec{a}$$

The complex probability or random vector and force are:

$$z(x) = P_r(x) + P_m(x) = \frac{k}{N} + i\left(1-\frac{k}{N}\right) = \frac{k}{60} + i\left(1-\frac{k}{60}\right), \forall k: 0 \le k \le (N = 60)$$
$$\Leftrightarrow \vec{F}(x) = \vec{F}_r(x) + \vec{F}_m(x) = P_r(x)m\vec{a} + P_m(x)m\vec{a} = [P_r(x) + P_m(x)]m\vec{a} = zm\vec{a}$$
$$= \left(\frac{k}{N}\right)m\vec{a} + i\left(1-\frac{k}{N}\right)m\vec{a} = \left[\left(\frac{k}{N}\right) + i\left(1-\frac{k}{N}\right)\right]m\vec{a}$$
$$= \left[\left(\frac{k}{60}\right) + i\left(1-\frac{k}{60}\right)\right]m\vec{a}$$

The Degree of Our Knowledge:

$$DOK(x) = |z(x)|^2 = P_r^2(x) + [P_m(x)/i]^2 = \left(\frac{k}{N}\right)^2 + \left(1-\frac{k}{N}\right)^2$$
$$= 1 + 2iP_r(x)P_m(x) = 1 - 2P_r(x)[1 - P_r(x)] = 1 - 2P_r(x) + 2P_r^2(x)$$
$$= 1 - 2\left(\frac{k}{N}\right) + 2\left(\frac{k}{N}\right)^2$$
$$= 1 - 2\left(\frac{k}{60}\right) + 2\left(\frac{k}{60}\right)^2, \quad \forall k: 0 \le k \le (N = 60)$$

$DOK(x)$ is equal to 1 when $P_r(x) = P_r(L_b = -21) = 0$ and when $P_r(x) = P_r(U_b = 21) = 1$.

The Chaotic Factor:

$$Chf(x) = 2iP_r(x)P_m(x) = -2P_r(x)[1 - P_r(x)] = -2P_r(x) + 2P_r^2(x)$$
$$= -2\left(\frac{k}{N}\right) + 2\left(\frac{k}{N}\right)^2$$
$$= -2\left(\frac{k}{60}\right) + 2\left(\frac{k}{60}\right)^2, \quad \forall k: 0 \le k \le (N = 60)$$

$Chf(x)$ is null when $P_r(x) = P_r(L_b = -21) = 0$ and when $P_r(x) = P_r(U_b = 21) = 1$.

The Magnitude of the Chaotic Factor $MChf$:

$$MChf(x) = |Chf(x)| = -2iP_r(x)P_m(x) = 2P_r(x)[1 - P_r(x)] = 2P_r(x) - 2P_r^2(x)$$

*DOI: http://dx.doi.org/10.5772/intechopen.98341*

$$= 2\left(\frac{k}{N}\right) - 2\left(\frac{k}{N}\right)^2$$
$$= 2\left(\frac{k}{60}\right) - 2\left(\frac{k}{60}\right)^2, \quad \forall k : 0 \le k \le (N = 60)$$

$MChf(x)$ is null when $P_r(x) = P_r(L_b = -21) = 0$ and when $P_r(x) = P_r(U_b = 21) = 1$ .

At any value of $x$: $\forall x : (L_b = -21) \le x \le (U_b = 21)$ and $\forall k : 0 \le k \le (N = 60)$, the probability expressed in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is the following:

$$\begin{aligned} P_c{}^2(x) &= [P_r(x) + P_m(x)/i]^2 = |z(x)|^2 - 2iP_r(x)P_m(x) \\ &= DOK(x) - Chf(x) \\ &= DOK(x) + MChf(x) \\ &= 1 \end{aligned}$$

then,

$$P_c{}^2(x) = [P_r(x) + P_m(x)/i]^2 = \{P_r(x) + [1 - P_r(x)]\}^2 = 1^2 = 1 \Leftrightarrow P_c(x) = 1 \text{ always}$$

$\Leftrightarrow \vec{F}_c(x) = P_c(x)m\vec{a} = 1 \times m\vec{a} = m\vec{a}$ always also.

Hence, the prediction of all the probabilities and forces of the stochastic experiment in the universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is permanently certain and perfectly deterministic (**Figure 19**).

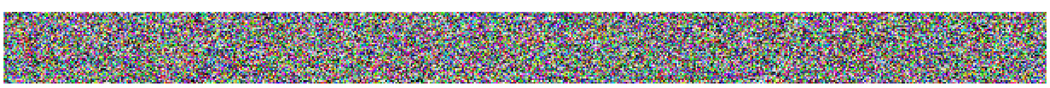

**Figure 19.**
*The graphs of* $F_r$ / ma, $F_m$ / ima, *and* $F_c$ / ma *and of all the* CPP *parameters as functions of the random variable* X *for this discrete uniform probability distribution.*

*8.1.1.1 The complex probability cubes*

In the first cube (**Figure 20**), the simulation of $DOK$ and $Chf$ as functions of each other and of the random variable $X$ for the discrete uniform probability distribution can be seen. The dotted line in cyan is the projection of the plane $P_c^2(X) = DOK(X) - Chf(X) = 1 = P_c(X) = F_c / ma$ on the plane $X = L_b$ = lower bound of $X = -21$. This dotted line starts at the point J ($DOK = 1$, $Chf = 0$) when $X = L_b = -21$, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $X = 0$, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $X = U_b$ = upper bound of $X = 21$. The other curves are the graphs of $DOK(X)$ (red) and $Chf(X)$ (green, blue, pink) in different simulation planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $X = 0$). The point L corresponds to ($DOK = 1$, $Chf = 0$, $X = U_b = 21$). The three points J, K, L are the same as in **Figure 19**.

In the second cube (**Figure 21**), we can notice the simulation of the real reduced force $F_r / ma = P_r(X)$ in $\mathcal{R}$ and its complementary real reduced force $F_m / ima = P_m(X)/i$ in $\mathcal{R}$ also in terms of the random variable $X$ for the discrete uniform probability distribution. The dotted line in cyan is the projection of the plane $P_c^2(X) = P_r(X) + P_m(X)/i = 1 = P_c(X) = F_c / ma$ on the plane $X = L_b$ = lower

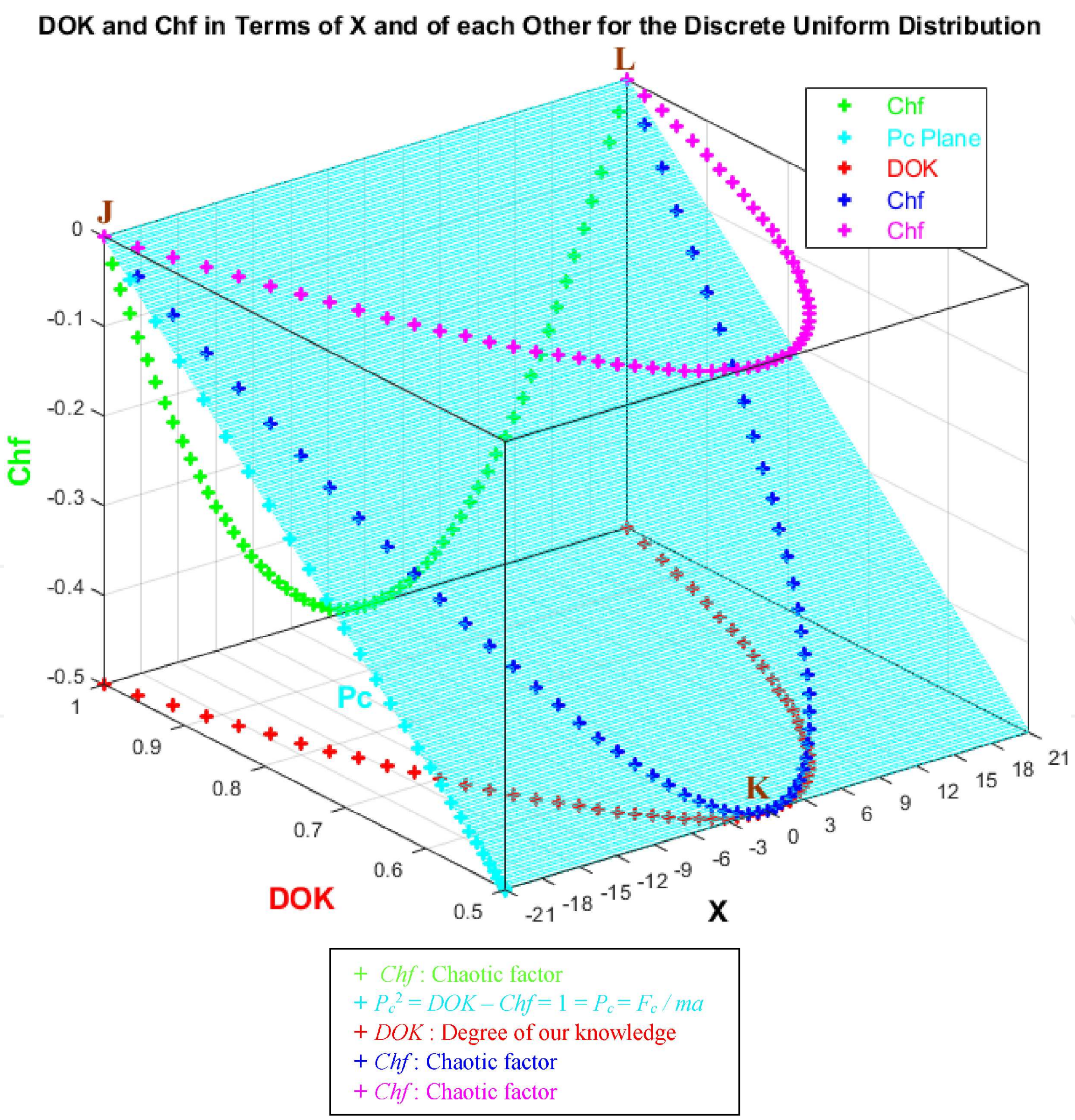


**Figure 20.**
*The graphs of* DOK *and* Chf *and the deterministic reduced force* $F_c / ma = P_c$ *in terms of* X *and of each other for this discrete uniform probability distribution.*

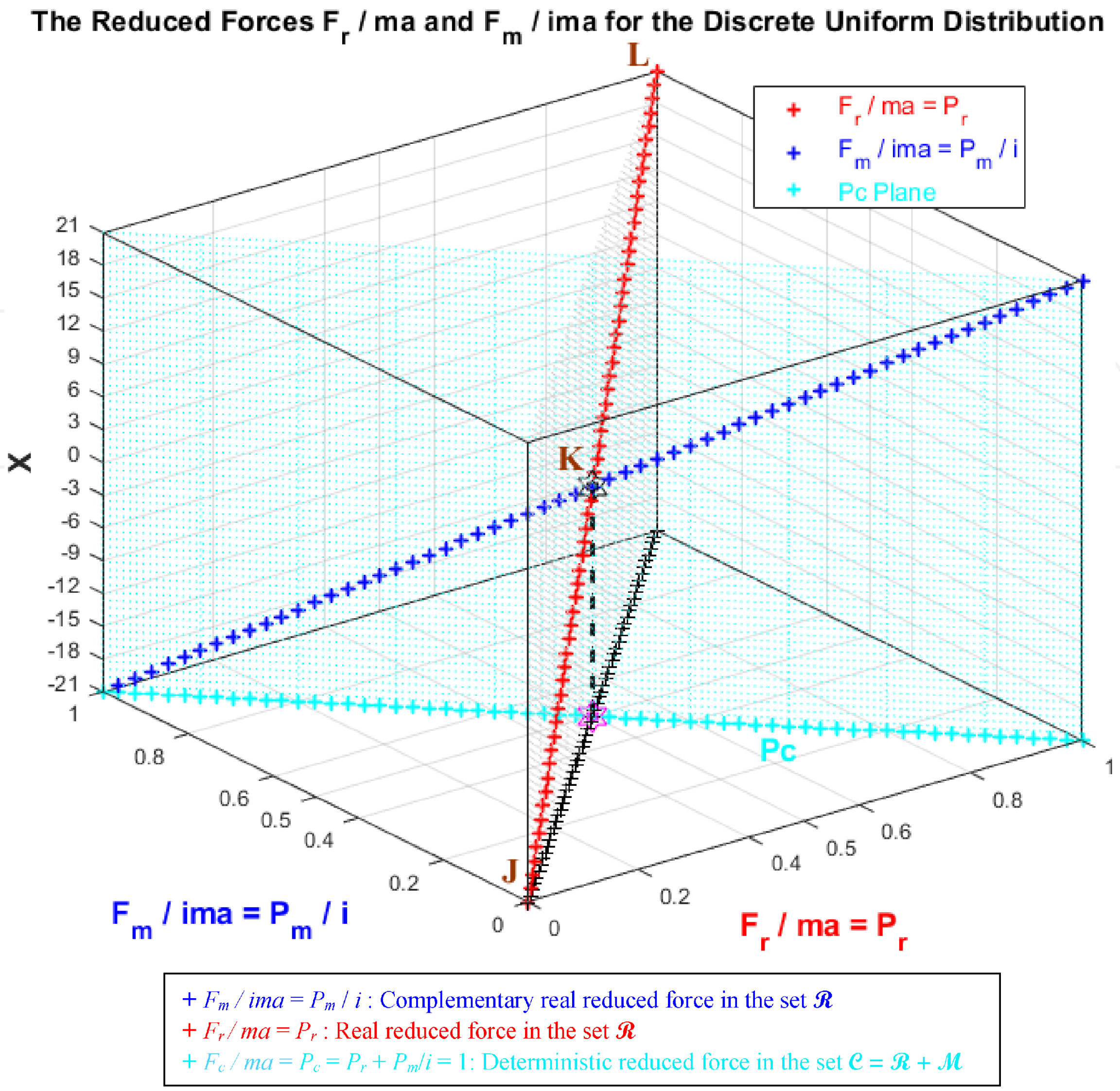


**Figure 21.**
*The graphs of* $F_r / ma = P_r$ *and* $F_m / ima = P_m / i$ *and* $F_c / ma = P_c$ *in terms of* X *and of each other for this discrete uniform probability distribution.*

bound of $X = -21$. This dotted line starts at the point ($P_r = 0, P_m/i = 1$) and ends at the point ($P_r = 1, P_m/i = 0$). The red curve represents $F_r / ma = P_r(X)$ in the plane $P_r(X) = P_m(X)/i$ in light gray. This curve starts at the point J ($P_r = 0, P_m/i = 1$, $X = L_b$ = lower bound of $X = -21$), reaches the point K ($P_r = 0.5, P_m/i = 0.5, X = 0$), and gets at the end to L ($P_r = 1, P_m/i = 0, X = U_b$ = upper bound of $X = 21$). The blue curve represents $F_m / ima = P_m(X)/i$ in the plane in cyan $P_r(X) + P_m(X)/i = 1 = P_c(X) = F_c / ma$. Notice the importance of the point K which is the intersection of the red and blue curves at $X = 0$ and when $P_r(X) = P_m(X)/i = 0.5$. The three points J, K, L are the same as in **Figure 19**.

In the third cube (**Figure 22**), we can notice the simulation of the complex resultant reduced force $F / ma = z(X)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real reduced force $F_r / ma = P_r(X) = \mathrm{Re}(z)$ in $\mathcal{R}$ and of its complementary imaginary reduced force $F_m / ma = P_m(X) = i \times \mathrm{Im}(z)$ in $\mathcal{M}$, and this in terms of the random variable $X$ for the discrete uniform probability distribution. The red curve represents $F_r / ma$ in the plane $P_m(X) = 0$ and the blue curve represents $F_m / ma$ in the plane $P_r(X) = 0$. The green curve represents the complex resultant reduced force $F / ma = F_r / ma + F_m / ma = z(X) = P_r(X) + P_m(X) = \mathrm{Re}(z) + i \times \mathrm{Im}(z)$ in the plane $P_r(X) = iP_m(X) + 1$ or $z(X)$ plane in cyan. The curve of $F / ma$ starts at the point J ($P_r = 0, P_m = i$, $X = L_b$ = lower bound of $X = -21$) and ends at the point L ($P_r = 1, P_m = 0$,

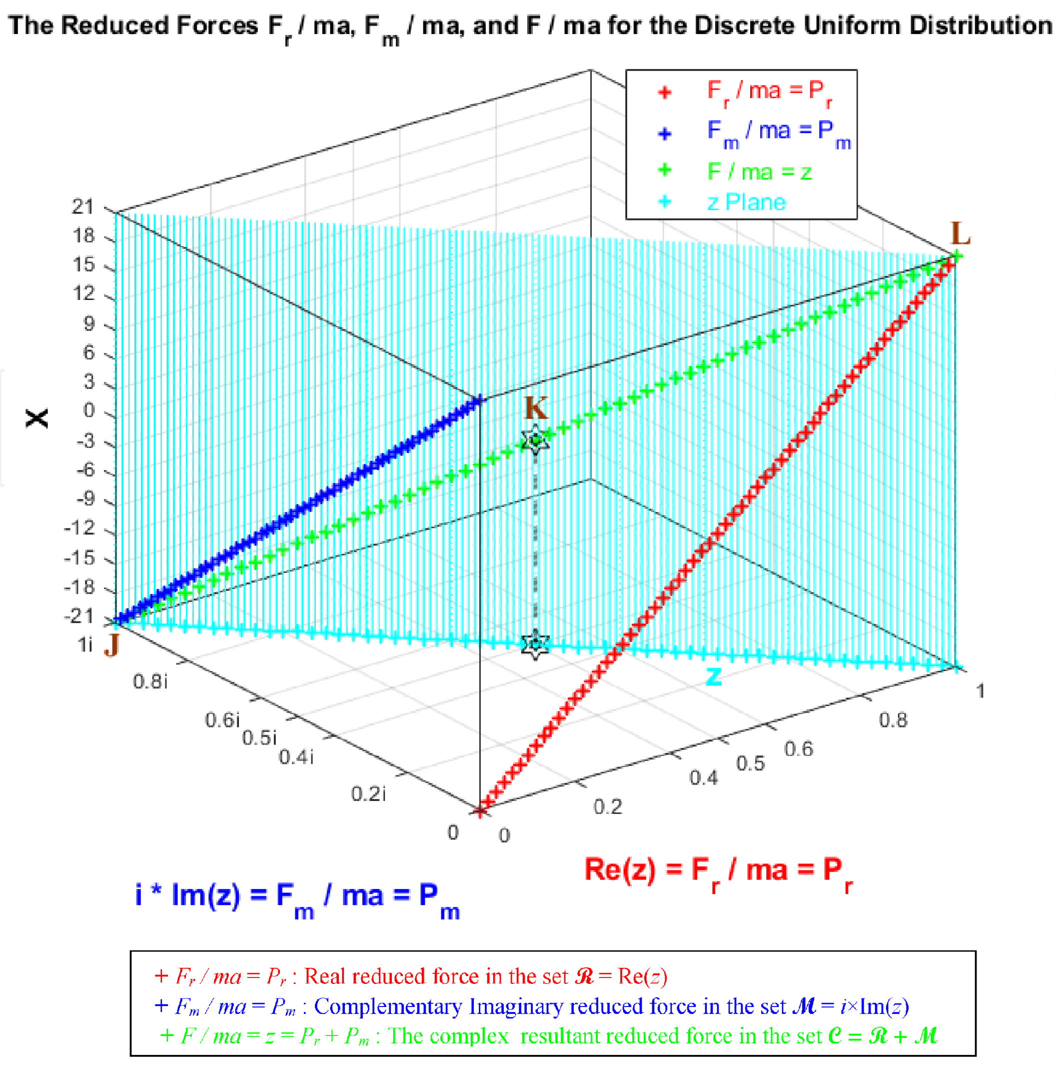


**Figure 22.**
*The graphs of the reduced forces* $F_r$ / ma = $P_r$ *and* $F_m$ / ma = $P_m$ *and* F / ma = z *in terms of* X *for this discrete uniform probability distribution.*

$X = U_b$ = upper bound of $X = 21$). The dotted line in cyan is $P_r(X = L_b = -21) = iP_m(X = L_b = -21) + 1$ and it is the projection of the $F/ma$ curve on the complex probability plane whose equation is $X = L_b = -21$. This projected dotted line starts at the point J ($P_r = 0$, $P_m = i$, $X = L_b = -21$) and ends at the point ($P_r = 1$, $P_m = 0$, $X = L_b = -21$). Notice the importance of the point K corresponding to $X = 0$ and $z = 0.5 + 0.5i$ when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in **Figure 19**.

### *8.1.2 The binomial probability distribution*

The probability density function (*PDF*) of this discrete stochastic distribution is:

$$f(x) = {}_N C_x p^x q^{N-x} = \binom{N}{x} p^x q^{N-x}, \quad \text{for } (L_b = 0) \leq x \leq (U_b = N)$$

I have taken the domain for the binomial random variable to be: $x \in [L_b = 0, U_b = N = 12]$ and $\forall k : 1 \leq k \leq 12$ we have $\Delta x_k = x_k - x_{k-1} = 1$, then: $x = 0, 1, 2, \ldots, 12$.

Taking in our simulation $N = 12$ and $p + q = 1$, $p = q = 0.5$ then:

The mean of this binomial discrete random distribution is: $\mu = Np = 12 \times 0.5 = 6$.

The standard deviation is: $\sigma = \sqrt{Npq} = \sqrt{12 \times 0.5 \times 0.5} = \sqrt{3} = 1.73205\ldots$.
The median is $Md = \mu = 6$.
The mode for this symmetric distribution is = 6 = $Md$ = $\mu$.
The cumulative distribution function ($CDF$) is:

$$CDF(x) = P_{rob}(X \leq x) = \sum_{k=0}^{x} f(k;N) = \sum_{k=0}^{x} {}_N C_k p^k q^{N-k} = \sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k},$$
$$\forall x : 0 \leq x \leq (N = 12)$$

Note that:
If $x = 0 \Leftrightarrow X = L_b \Leftrightarrow CDF(x) = P_{rob}(X \leq 0) = f(X = L_b; N) = {}_N C_0 p^0 q^{N-0} = q^N = 0.5^{12} \cong 0$.
If $x = N = 12 \Leftrightarrow X = U_b \Leftrightarrow CDF(x) = P_{rob}(X \leq x) = \sum_{k=0}^{N} {}_N C_k p^k q^{N-k} = (p+q)^N = 1^N = 1^{12} = 1$ by the binomial theorem.
The real probability $P_r(x)$ and force are:

$$P_r(x) = CDF(x) = \sum_{k=0}^{x} f(k;N) = \sum_{k=0}^{x} {}_N C_k p^k q^{N-k} = \sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k},$$
$$\forall x : 0 \leq x \leq (N = 12)$$

$$\Leftrightarrow \vec{F}_r(x) = P_r(x) m\vec{a} = \left( \sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k} \right) m\vec{a}$$

The imaginary complementary probability $P_m(x)$ and force are:

$$P_m(x) = i[1 - P_r(x)] = i[1 - CDF(x)] = i \left[ 1 - \sum_{k=0}^{x} f(k;N) \right]$$

$$= i \left( 1 - \sum_{k=0}^{x} {}_N C_k p^k q^{N-k} \right) = i \sum_{k=x+1}^{N} {}_N C_k p^k q^{N-k} = i \sum_{k=x+1}^{12} {}_{12} C_k p^k q^{12-k},$$
$$\forall x : 0 \leq x \leq (N = 12)$$

$$\Leftrightarrow \vec{F}_m(x) = P_m(x) m\vec{a} = i \left( \sum_{k=x+1}^{12} {}_{12} C_k p^k q^{12-k} \right) m\vec{a}$$

The real complementary probability $P_m(x)/i$ and force are:

$$P_m(x)/i = 1 - P_r(x) = 1 - CDF(x) = 1 - \sum_{k=0}^{x} f(k;N) = \sum_{k=x+1}^{N} {}_N C_k p^k q^{N-k}$$

$$= \sum_{k=x+1}^{12} {}_{12} C_k p^k q^{12-k}, \quad \forall x : 0 \leq x \leq (N = 12)$$

$$\Leftrightarrow \vec{F}_m(x)/i = \frac{P_m(x)}{i} m\vec{a} = \left( \sum_{k=x+1}^{12} {}_{12} C_k p^k q^{12-k} \right) m\vec{a}$$

The complex probability or random vector and force are:

$$z(x) = P_r(x) + P_m(x) = \sum_{k=0}^{x} {}_N C_k p^k q^{N-k} + i\left(\sum_{k=x+1}^{N} {}_N C_k p^k q^{N-k}\right)$$

$$= \sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k} + i\left(\sum_{k=x+1}^{12} {}_{12} C_k p^k q^{12-k}\right), \quad \forall x : 0 \le x \le (N = 12)$$

$$\Leftrightarrow \vec{F}(x) = \vec{F}_r(x) + \vec{F}_m(x) = P_r(x)m\vec{a} + P_m(x)m\vec{a} = [P_r(x) + P_m(x)]m\vec{a} = zm\vec{a}$$

$$= \left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)m\vec{a} + i\left(\sum_{k=x+1}^{N} {}_N C_k p^k q^{N-k}\right)m\vec{a}$$

$$= \left[\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right) + i\left(\sum_{k=x+1}^{N} {}_N C_k p^k q^{N-k}\right)\right]m\vec{a}$$

$$= \left[\left(\sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k}\right) + i\left(\sum_{k=x+1}^{12} {}_{12} C_k p^k q^{12-k}\right)\right]m\vec{a}, \quad \forall x : 0 \le x \le (N = 12)$$

The Degree of Our Knowledge:

$$DOK(x) = |z(x)|^2 = P_r^2(x) + [P_m(x)/i]^2 = \left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2 + \left(1 - \sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2$$

$$= \left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2 + \left(\sum_{k=x+1}^{N} {}_N C_k p^k q^{N-k}\right)^2 = \left(\sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k}\right)^2 + \left(\sum_{k=x+1}^{12} {}_{12} C_k p^k q^{12-k}\right)^2$$

$$= 1 + 2iP_r(x)P_m(x) = 1 - 2P_r(x)[1 - P_r(x)] = 1 - 2P_r(x) + 2P_r^2(x) \quad .$$

$$= 1 - 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right) + 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2$$

$$= 1 - 2\left(\sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k}\right) + 2\left(\sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k}\right)^2, \quad \forall x : 0 \le x \le (N = 12)$$

$DOK(x)$ is equal to 1 when $P_r(x) = P_r(L_b = 0) = 0$ and when $P_r(x) = P_r(U_b = 12) = 1$ .

The Chaotic Factor:

$$Chf(x) = 2iP_r(x)P_m(x) = -2P_r(x)[1 - P_r(x)] = -2P_r(x) + 2P_r^2(x)$$

$$= -2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right) + 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2$$

$$= -2\left(\sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k}\right) + 2\left(\sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k}\right)^2, \forall x : 0 \le x \le (N = 12)$$

$Chf(x)$ is null when $P_r(x) = P_r(L_b = 0) = 0$ and when $P_r(x) = P_r(U_b = 12) = 1$ .

The Magnitude of the Chaotic Factor $MChf$:

$$\begin{aligned}
MChf(x) = |Chf(x)| = -2iP_r(x)P_m(x) = 2P_r(x)[1 - P_r(x)] = 2P_r(x) - 2P_r^2(x) \\
= 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right) - 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2 \\
= 2\left(\sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k}\right) - 2\left(\sum_{k=0}^{x} {}_{12} C_k p^k q^{12-k}\right)^2, \quad \forall x : 0 \le x \le (N = 12)
\end{aligned}$$

$MChf(x)$ is null when $P_r(x) = P_r(L_b = 0) = 0$ and when $P_r(x) = P_r(U_b = 12) = 1$.

At any value of $x$: $\forall x : (L_b = 0) \le x \le (U_b = N = 12)$, the probability expressed in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is the following:

$$\begin{aligned}
P_c{}^2(x) &= [P_r(x) + P_m(x)/i]^2 = |z(x)|^2 - 2iP_r(x)P_m(x) \\
&= DOK(x) - Chf(x) \\
&= DOK(x) + MChf(x) \\
&= 1
\end{aligned}$$



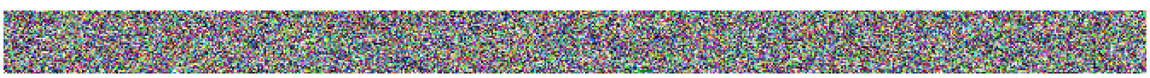



**Figure 23.**
*The graphs of* $F_r$ / ma, $F_m$ / ima, *and* $F_c$ / ma *and of all the* CPP *parameters as functions of the random variable* X *for this discrete binomial probability distribution.*

then,

$$P_c{}^2(x) = [P_r(x) + P_m(x)/i]^2 = \{P_r(x) + [1 - P_r(x)]\}^2 = 1^2 = 1 \Leftrightarrow P_c(x) = 1 \text{ always}$$

$\Leftrightarrow \vec{F}_c(x) = P_c(x)m\vec{a} = 1 \times m\vec{a} = m\vec{a}$ always also.

Hence, the prediction of all the probabilities and forces of the stochastic experiment in the universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is permanently certain and perfectly deterministic (**Figure 23**).

*8.1.2.1 The complex probability cubes*

In the first cube (**Figure 24**), the simulation of *DOK* and *Chf* as functions of each other and of the random variable $X$ for the binomial probability distribution can be seen. The thick line in cyan is the projection of the plane $P_c{}^2(X) = DOK(X) - Chf(X) = 1 = P_c(X) = F_c / ma$ on the plane $X = L_b$ = lower bound of $X = 0$. This thick line starts at the point J ($DOK = 1$, $Chf = 0$) when $X = L_b = 0$, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $X = 6$, and returns at the end to J ($DOK = 1$, $Chf = 0$)

**Figure 24.**
*The graphs of* DOK *and* Chf *and the deterministic reduced force* $F_c$ / ma = $P_c$ *in terms of* X *and of each other for this binomial probability distribution.*

when $X = U_b$ = upper bound of $X = 12$. The other curves are the graphs of $DOK(X)$ (red) and $Chf(X)$ (green, blue, pink) in different simulation planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $X = 6$). The point L corresponds to ($DOK = 1$, $Chf = 0$, $X = U_b = 12$). The three points J, K, L are the same as in **Figure 23**.

In the second cube (**Figure 25**), we can notice the simulation of the real reduced force $F_r\,/\,ma = P_r(X)$ in $\mathcal{R}$ and its complementary real reduced force $F_m\,/\,ima = P_m(X)/i$ in $\mathcal{R}$ also in terms of the random variable $X$ for the binomial probability distribution. The thick line in cyan is the projection of the plane $P_c^2(X) = P_r(X) + P_m(X)/i = 1 = P_c(X) = F_c\,/\,ma$ on the plane $X = L_b$ = lower bound of $X = 0$. This thick line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $F_r\,/\,ma = P_r(X)$ in the plane $P_r(X) = P_m(X)/i$ in light gray. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $X = L_b$ = lower bound of $X = 0$), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, $X = 6$), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $X = U_b$ = upper bound of $X = 12$). The blue curve represents $F_m\,/\,ima = P_m(X)/i$ in the plane in cyan $P_r(X) + P_m(X)/i = 1 = P_c(X) = F_c\,/\,ma$. Notice the importance of the point K which is the intersection of the red and blue curves at $X = 6$ and when $P_r(X) = P_m(X)/i = 0.5$. The three points J, K, L are the same as in **Figure 23**.

**Figure 25.**
*The graphs of* $\mathrm{F_r\,/\,ma = P_r}$ *and* $\mathrm{F_m\,/\,ima = P_m\,/\,i}$ *and* $\mathrm{F_c\,/\,ma = P_c}$ *in terms of* X *and of each other for this binomial probability distribution.*

In the third cube (**Figure 26**), we can notice the simulation of the complex resultant reduced force $F / ma = z(X)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real reduced force $F_r / ma = P_r(X) = \mathrm{Re}(z)$ in $\mathcal{R}$ and of its complementary imaginary reduced force $F_m / ma = P_m(X) = i \times \mathrm{Im}(z)$ in $\mathcal{M}$, and this in terms of the random variable $X$ for the binomial probability distribution. The red curve represents $F_r / ma$ in the plane $P_m(X) = 0$ and the blue curve represents $F_m / ma$ in the plane $P_r(X) = 0$. The green curve represents the complex resultant reduced force $F / ma = F_r / ma + F_m / ma = z(X) = P_r(X) + P_m(X) = \mathrm{Re}(z) + i \times \mathrm{Im}(z)$ in the plane $P_r(X) = iP_m(X) + 1$ or $z(X)$ plane in cyan. The curve of $F / ma$ starts at the point J ($P_r = 0, P_m = i, X = L_b =$ lower bound of $X = 0$) and ends at the point L ($P_r = 1, P_m = 0, X = U_b =$ upper bound of $X = 12$). The thick line in cyan is $P_r(X = L_b = 0) = iP_m(X = L_b = 0) + 1$ and it is the projection of the $F / ma$ curve on the complex probability plane whose equation is $X = L_b = 0$. This projected thick line starts at the point J ($P_r = 0, P_m = i, X = L_b = 0$) and ends at the point ($P_r = 1, P_m = 0, X = L_b = 0$). Notice the importance of the point K corresponding to $X = 6$ and $z = 0.5 + 0.5i$ when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in **Figure 23**.

**Figure 26.**
*The graphs of the reduced forces* $F_r / ma = P_r$ *and* $F_m / ma = P_m$ *and* $F / ma = z$ *in terms of* $X$ *for this binomial probability distribution.*

*8.1.3 The Poisson probability distribution*

The probability density function (*PDF*) of this discrete stochastic distribution is:

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \text{ where } 0 \le x < \infty.$$

For the Poisson discrete random variable: $x \in [L_b = 0, \infty)$ and $\forall k : k \ge 1$ we have $\Delta x_k = x_k - x_{k-1} = 1$, then $x = 0, 1, 2, \dots, \infty$.

I have taken in the simulation the domain for the Poisson random variable to be equal to: $x \in [L_b = 0, U_b = 16]$, then: $x = 0, 1, 2, \dots, 16$.

The mean of this Poisson discrete random distribution is: $\mu = \lambda = 6.7$.

The standard deviation is: $\sigma = \sqrt{\lambda} = \sqrt{6.7} = 2.588435821\dots$.

The median $Md$ is $= 6$.

The mode is = $= \lfloor \lambda \rfloor = \lfloor 6.7 \rfloor = 6$.

Since $Md$ = mode $< \mu$ then this distribution is skewed to the right or positively skewed.

The cumulative distribution function (*CDF*) is:

$$CDF(x) = P_{rob}(X \le x) = \sum_{k=0}^{x} f(k;\lambda) = \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}, \quad \forall x : 0 \le x \le 16$$

Note that:

If $x = 0 \Leftrightarrow CDF(x) = P_{rob}(X \le 0) = f(X = L_b;\lambda) = e^{-\lambda} = e^{-6.7} \cong 0$.

If $x = U_b \Leftrightarrow X >> 1 \Leftrightarrow X \to +\infty \Leftrightarrow CDF(x) = P_{rob}(X \le x) \to \sum_{k=0}^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \times e^{\lambda} = 1$ by the properties of infinite series from calculus.

The real probability $P_r(x)$ and force are:

$$P_r(x) = CDF(x) = \sum_{k=0}^{x} f(k;\lambda) = \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}, \quad \forall x : 0 \le x \le 16$$

$$\Leftrightarrow \vec{F}_r(x) = P_r(x) m\vec{a} = \left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right) m\vec{a} = \left(\sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}\right) m\vec{a}$$

The imaginary complementary probability $P_m(x)$ and force are:

$$P_m(x) = i[1 - P_r(x)] = i[1 - CDF(x)] = i\left[1 - \sum_{k=0}^{x} f(k;\lambda)\right]$$

$$= i\left(1 - \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right) = i\left(\sum_{k=x+1}^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!}\right) = i\left(\sum_{k=x+1}^{16} \frac{e^{-6.7}6.7^k}{k!}\right), \forall x : 0 \le x \le 16$$

$$\Leftrightarrow \vec{F}_m(x) = P_m(x) m\vec{a} = i\left(\sum_{k=x+1}^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!}\right) m\vec{a} = i\left(\sum_{k=x+1}^{16} \frac{e^{-6.7}6.7^k}{k!}\right) m\vec{a}$$

The real complementary probability $P_m(x)/i$ and force are:

$$P_m(x)/i = 1 - P_r(x) = 1 - CDF(x) = 1 - \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=x+1}^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=x+1}^{16} \frac{e^{-6.7}6.7^k}{k!}, \forall x : 0 \leq x \leq 16$$

$$\Leftrightarrow \vec{F}_m(x)/i = \frac{P_m(x)}{i} m\vec{a} = \left(\sum_{k=x+1}^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!}\right) m\vec{a} = \left(\sum_{k=x+1}^{16} \frac{e^{-6.7}6.7^k}{k!}\right) m\vec{a}$$

The complex probability or random vector and force are:

$$z(x) = P_r(x) + P_m(x) = \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!} + i\left(\sum_{k=x+1}^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!}\right)$$
$$= \sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!} + i\left(\sum_{k=x+1}^{16} \frac{e^{-6.7}6.7^k}{k!}\right),$$

$$\Leftrightarrow \vec{F}(x) = \vec{F}_r(x) + \vec{F}_m(x) = P_r(x)m\vec{a} + P_m(x)m\vec{a} = [P_r(x) + P_m(x)]m\vec{a} = zm\vec{a}$$

$$= \left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right) m\vec{a} + i\left(\sum_{k=x+1}^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!}\right) m\vec{a}$$

$$= \left[\left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right) + i\left(\sum_{k=x+1}^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!}\right)\right] m\vec{a}$$

$$= \left[\left(\sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}\right) + i\left(\sum_{k=x+1}^{16} \frac{e^{-6.7}6.7^k}{k!}\right)\right] m\vec{a}, \quad \forall x : 0 \leq x \leq 16$$

The Degree of Our Knowledge:

$$DOK(x) = |z(x)|^2 = P_r^2(x) + [P_m(x)/i]^2 = \left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right)^2 + \left(1 - \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right)^2$$

$$= \left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right)^2 + \left(\sum_{k=x+1}^{+\infty} \frac{e^{-\lambda}\lambda^k}{k!}\right)^2 = \left(\sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}\right)^2 + \left(\sum_{k=x+1}^{16} \frac{e^{-6.7}6.7^k}{k!}\right)^2$$

$$= 1 + 2iP_r(x)P_m(x) = 1 - 2P_r(x)[1 - P_r(x)] = 1 - 2P_r(x) + 2P_r^2(x) \qquad .$$

$$= 1 - 2\left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right) + 2\left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right)^2$$

$$= 1 - 2\left(\sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}\right) + 2\left(\sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}\right)^2, \quad \forall x : 0 \leq x \leq 16$$

$DOK(x)$ is equal to 1 when $P_r(x) = P_r(L_b = 0) = 0$ and when $P_r(x) = P_r(U_b = 16) = 1$ .

The Chaotic Factor:

$$Chf(x) = 2iP_r(x)P_m(x) = -2P_r(x)[1 - P_r(x)] = -2P_r(x) + 2P_r^2(x)$$

$$= -2\left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right) + 2\left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right)^2$$

$$= -2\left(\sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}\right) + 2\left(\sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}\right)^2, \quad \forall x : 0 \leq x \leq 16$$

$Chf(x)$ is null when $P_r(x) = P_r(L_b = 0) = 0$ and when $P_r(x) = P_r(U_b = 16) = 1$.

The Magnitude of the Chaotic Factor *MChf*:

$$MChf(x) = |Chf(x)| = -2iP_r(x)P_m(x) = 2P_r(x)[1 - P_r(x)] = 2P_r(x) - 2P_r^2(x)$$

$$= 2\left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right) - 2\left(\sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}\right)^2$$

$$= 2\left(\sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}\right) - 2\left(\sum_{k=0}^{x} \frac{e^{-6.7}6.7^k}{k!}\right)^2, \quad \forall x : 0 \leq x \leq 16$$

$MChf(x)$ is null when $P_r(x) = P_r(L_b = 0) = 0$ and when $P_r(x) = P_r(U_b = 16) = 1$.

At any value of $x$: $\forall x : (L_b = 0) \leq x \leq (U_b = 16)$, the probability expressed in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is the following:

$$\begin{aligned} P_c{}^2(x) &= [P_r(x) + P_m(x)/i]^2 = |z(x)|^2 - 2iP_r(x)P_m(x) \\ &= DOK(x) - Chf(x) \\ &= DOK(x) + MChf(x) \\ &= 1 \end{aligned}$$

then,

$$P_c{}^2(x) = [P_r(x) + P_m(x)/i]^2 = \{P_r(x) + [1 - P_r(x)]\}^2 = 1^2 = 1 \Leftrightarrow P_c(x) = 1 \text{ always}$$

$\Leftrightarrow \vec{F}_c(x) = P_c(x)m\vec{a} = 1 \times m\vec{a} = m\vec{a}$ always also.

Hence, the prediction of all the probabilities and forces of the stochastic experiment in the universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is permanently certain and perfectly deterministic (**Figure 27**).

#### *8.1.3.1 The complex probability cubes*

In the first cube (**Figure 28**), the simulation of *DOK* and *Chf* as functions of each other and of the random variable $X$ for the Poisson probability distribution can be seen. The thick line in cyan is the projection of the plane $P_c{}^2(X) = DOK(X) - Chf(X) = 1 = P_c(X) = F_c / ma$ on the plane $X = L_b$ = lower bound of $X = 0$. This thick line starts at the point J ($DOK = 1$, $Chf = 0$) when $X = L_b = 0$, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $X = 6$, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $X = U_b$ = upper bound of $X = 16$. The other curves are the graphs of $DOK(X)$ (red) and $Chf(X)$ (green, blue, pink) in different simulation planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $X = 6$). The point L corresponds to ($DOK = 1$, $Chf = 0$, $X = U_b = 16$). The three points J, K, L are the same as in **Figure 27**.

In the second cube (**Figure 29**), we can notice the simulation of the real reduced force $F_r / ma = P_r(X)$ in $\mathcal{R}$ and its complementary real reduced force $F_m / ima = P_m(X)/i$ in $\mathcal{R}$ also in terms of the random variable $X$ for the Poisson probability distribution. The thick line in cyan is the projection of the plane $P_c{}^2(X) = P_r(X) + P_m(X)/i = 1 = P_c(X) = F_c / ma$ on the plane $X = L_b$ = lower bound of $X = 0$. This thick line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point

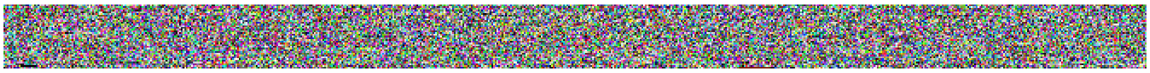



**Figure 27.**
*The graphs of* $F_r$ / ma, $F_m$ / ima, *and* $F_c$ / ma *and of all the* CPP *parameters as functions of the random variable* X *for this discrete Poisson probability distribution.*

($P_r = 1$, $P_m/i = 0$). The red curve represents $F_r\,/\,ma = P_r(X)$ in the plane $P_r(X) = P_m(X)/i$ in light gray. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $X = L_b$ = lower bound of $X = 0$), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, $X = 6$), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $X = U_b$ = upper bound of $X = 16$). The blue curve represents $F_m\,/\,ima = P_m(X)/i$ in the plane in cyan $P_r(X) + P_m(X)/i = 1 = P_c(X) = F_c\,/\,ma$. Notice the importance of the point K which is the intersection of the red and blue curves at $X = 6$ and when $P_r(X) = P_m(X)/i = 0.5$. The three points J, K, L are the same as in **Figure 27**.

In the third cube (**Figure 30**), we can notice the simulation of the complex resultant reduced force $F\,/\,ma = z(X)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real reduced force $F_r\,/\,ma = P_r(X) = \mathrm{Re}(z)$ in $\mathcal{R}$ and of its complementary imaginary reduced force $F_m\,/\,ma = P_m(X) = i \times \mathrm{Im}(z)$ in $\mathcal{M}$, and this in terms of the random variable $X$ for the Poisson probability distribution. The red curve represents $F_r\,/\,ma$ in the plane $P_m(X) = 0$ and the blue curve represents $F_m\,/\,ma$ in the plane $P_r(X) = 0$. The green curve represents the complex resultant reduced force $F\,/\,ma = F_r\,/\,ma + F_m\,/\,ma = z(X) = P_r(X) + P_m(X) = \mathrm{Re}(z) + i \times \mathrm{Im}(z)$ in the plane $P_r(X) = iP_m(X) + 1$ or $z(X)$ plane in cyan. The curve of $F\,/\,ma$ starts at the point J ($P_r = 0$, $P_m = i$, $X = L_b$ = lower bound of $X = 0$) and ends at the point L ($P_r = 1$, $P_m = 0$, $X = U_b$ = upper bound of $X = 16$). The thick line in cyan is $P_r(X = L_b = 0) = iP_m(X = L_b = 0) + 1$ and it is the projection of the $F\,/\,ma$ curve on the complex probability plane whose equation is $X = L_b = 0$. This projected thick line starts at the point J ($P_r = 0$, $P_m = i$, $X = L_b = 0$) and ends at the point ($P_r = 1$, $P_m = 0$, $X = L_b = 0$). Notice the importance of the point K corresponding to $X = 6$ and $z = 0.5 + 0.5i$ when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in **Figure 27**.

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**Figure 28.**
*The graphs of* DOK *and* Chf *and the deterministic reduced force* $F_c$ / ma = $P_c$ *in terms of* X *and of each other for this Poisson probability distribution.*

## 8.2 Simulation of continuous probability distributions

### *8.2.1 The continuous uniform probability distribution*

The probability density function (*PDF*) of this continuous stochastic distribution is:

$$f(x) = \frac{d\,[CDF(x)]}{dx} = \begin{cases} \dfrac{1}{U_b - L_b} & \text{if} \quad L_b \leq x \leq U_b \\ 0 & \text{elsewhere} \end{cases}$$

and the cumulative distribution function (*CDF*) is:

$$CDF(x) = P_{rob}(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{L_b}^{x} f(t)dt = \begin{cases} \dfrac{x - L_b}{U_b - L_b} & \text{if } L_b \leq x \leq U_b \\ 0 & \text{elsewhere} \end{cases}$$

**Figure 29.**
*The graphs of* $F_r$ / ma = $P_r$ *and* $F_m$ / ima = $P_m$ / i *and* $F_c$ / ma = $P_c$ *in terms of* X *and of each other for this Poisson probability distribution.*

I have taken the domain for the continuous uniform random variable to be equal to: $x \in [L_b = -3, U_b = 3]$ and $dx = 0.01$.

$$\text{Then } CDF(x) = \begin{cases} \dfrac{x+3}{6} & \text{if } (L_b = -3) \leq x \leq (U_b = 3) \\ 0 & \text{elsewhere} \end{cases}$$

Note that:
If $x = L_b = -3 \Leftrightarrow CDF(x) = P_{rob}(X \leq -3) = \frac{-3+3}{6} = 0$.
If $x = U_b = +3 \Leftrightarrow CDF(x) = P_{rob}(X \leq +3) = \frac{3+3}{6} = 1$.
The mean of this continuous uniform random distribution is: $\mu = \frac{L_b + U_b}{2} = \frac{-3+3}{2} = 0$.
The variance is: $\sigma^2 = \frac{(L_b - U_b)^2}{12} = \frac{(-3-3)^2}{12} = \frac{36}{12} = 3$.
The standard deviation is: $\sigma = \frac{|L_b - U_b|}{\sqrt{12}} = \frac{|-3-3|}{\sqrt{12}} = \frac{6}{\sqrt{12}} = \sqrt{3} = 1.732050808\ldots$.
The median is $Md = 0 = \mu$ since the distribution is symmetric.
Since the distribution is uniform then it has no mode.
The real probability $P_r(x)$ and force are:

**Figure 30.**
*The graphs of the reduced forces* $F_r$ / ma = $P_r$ *and* $F_m$ / ma = $P_m$ *and* F / ma = z *in terms of* X *for this Poisson probability distribution.*

$$P_r(x) = CDF(x) = \frac{x+3}{6}, \quad \forall x : -3 \le x \le 3$$

$$\Leftrightarrow \vec{F}_r(x) = P_r(x) m\vec{a} = \left(\frac{x+3}{6}\right) m\vec{a}$$

The imaginary complementary probability $P_m(x)$ and force are:

$$P_m(x) = i[1 - P_r(x)] = i[1 - CDF(x)] = i\left[1 - \int_{-\infty}^{x} f(t)dt\right] = i\left[1 - \int_{-3}^{x} f(t)dt\right]$$

$$= i\left[\int_{x}^{+\infty} f(t)dt\right] = i\left[\int_{x}^{3} f(t)dt\right] = i\left(1 - \frac{x+3}{6}\right) = i\left(\frac{3-x}{6}\right), \forall x : -3 \le x \le 3$$

$$\Leftrightarrow \vec{F}_m(x) = P_m(x) m\vec{a} = i\left(\frac{3-x}{6}\right) m\vec{a}$$

The real complementary probability $P_m(x)/i$ and force are:

$$P_m(x)/i = 1 - P_r(x) = 1 - CDF(x) = 1 - \int_{-\infty}^{x} f(t)dt = \int_{x}^{+\infty} f(t)dt = \int_{x}^{3} f(t)dt$$
$$= \frac{3-x}{6}, \forall x : -3 \leq x \leq 3$$
$$\Leftrightarrow \vec{F}_m(x)/i = \frac{P_m(x)}{i} m\vec{a} = \left(\frac{3-x}{6}\right) m\vec{a}$$

The complex probability or random vector and force are:

$$z(x) = P_r(x) + P_m(x) = \left(\frac{x+3}{6}\right) + i\left(\frac{3-x}{6}\right), \forall x : -3 \leq x \leq 3$$
$$\Leftrightarrow \vec{F}(x) = \vec{F}_r(x) + \vec{F}_m(x) = P_r(x)m\vec{a} + P_m(x)m\vec{a} = [P_r(x) + P_m(x)]m\vec{a} = zm\vec{a}$$
$$= \left(\frac{x+3}{6}\right)m\vec{a} + i\left(\frac{3-x}{6}\right)m\vec{a}$$
$$= \left[\left(\frac{x+3}{6}\right) + i\left(\frac{3-x}{6}\right)\right]m\vec{a}$$

The Degree of Our Knowledge:

$$DOK(x) = |z(x)|^2 = P_r^2(x) + [P_m(x)/i]^2 = \left(\frac{x+3}{6}\right)^2 + \left(1 - \frac{x+3}{6}\right)^2$$
$$= \left(\frac{x+3}{6}\right)^2 + \left(\frac{3-x}{6}\right)^2$$
$$= 1 + 2iP_r(x)P_m(x) = 1 - 2P_r(x)[1 - P_r(x)] = 1 - 2P_r(x) + 2P_r^2(x)$$
$$= 1 - 2\left(\frac{x+3}{6}\right) + 2\left(\frac{x+3}{6}\right)^2, \quad \forall x : -3 \leq x \leq 3$$

$DOK(x)$ is equal to 1 when $P_r(x) = P_r(L_b = -3) = 0$ and when $P_r(x) = P_r(U_b = 3) = 1$.

The Chaotic Factor:

$$Chf(x) = 2iP_r(x)P_m(x) = -2P_r(x)[1 - P_r(x)] = -2P_r(x) + 2P_r^2(x)$$
$$= -2\left(\frac{x+3}{6}\right) + 2\left(\frac{x+3}{6}\right)^2, \quad \forall x : -3 \leq x \leq 3$$

$Chf(x)$ is null when $P_r(x) = P_r(L_b = -3) = 0$ and when $P_r(x) = P_r(U_b = 3) = 1$.

The Magnitude of the Chaotic Factor $MChf$:

$$MChf(x) = |Chf(x)| = -2iP_r(x)P_m(x) = 2P_r(x)[1 - P_r(x)] = 2P_r(x) - 2P_r^2(x)$$
$$= 2\left(\frac{x+3}{6}\right) - 2\left(\frac{x+3}{6}\right)^2, \quad \forall x : -3 \leq x \leq 3$$

$MChf(x)$ is null when $P_r(x) = P_r(L_b = -3) = 0$ and when $P_r(x) = P_r(U_b = 3) = 1$.

At any value of $x$: $\forall x : (L_b = -3) \leq x \leq (U_b = 3)$, the probability expressed in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is the following:

$$
\begin{aligned}
P_c{}^2(x) &= [P_r(x) + P_m(x)/i]^2 = |z(x)|^2 - 2iP_r(x)P_m(x) \\
&= DOK(x) - Chf(x) \\
&= DOK(x) + MChf(x) \\
&= 1
\end{aligned}
$$

then,

$$P_c{}^2(x) = [P_r(x) + P_m(x)/i]^2 = \{P_r(x) + [1 - P_r(x)]\}^2 = 1^2 = 1 \Leftrightarrow P_c(x) = 1 \text{ always}$$

$\Leftrightarrow \vec{F}_c(x) = P_c(x)m\vec{a} = 1 \times m\vec{a} = m\vec{a}$ always also.

Hence, the prediction of all the probabilities and forces of the stochastic experiment in the universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is permanently certain and perfectly deterministic (**Figure 31**).

*8.2.1.1 The complex probability cubes*

In the first cube (**Figure 32**), the simulation of *DOK* and *Chf* as functions of each other and of the random variable *X* for the continuous uniform probability distribution can be seen. The thick line in cyan is the projection of the plane $P_c{}^2(X) = DOK(X) - Chf(X) = 1 = P_c(X) = F_c / ma$ on the plane $X = L_b =$ lower bound of $X = -3$. This thick line starts at the point J ($DOK = 1$, $Chf = 0$) when $X = L_b = -3$,

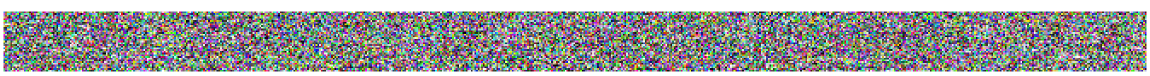

**Figure 31.**
*The graphs of* $F_r$ / ma, $F_m$ / ima, *and* $F_c$ / ma *and of all the* CPP *parameters as functions of the random variable* X *for this continuous uniform probability distribution.*

reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $X = 0$, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $X = U_b$ = upper bound of $X = 3$. The other curves are the graphs of $DOK(X)$ (red) and $Chf(X)$ (green, blue, pink) in different simulation planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $X = 0$). The point L corresponds to ($DOK = 1$, $Chf = 0$, $X = U_b = 3$). The three points J, K, L are the same as in **Figure 31**.

In the second cube (**Figure 33**), we can notice the simulation of the real reduced force $F_r / ma = P_r(X)$ in $\mathcal{R}$ and its complementary real reduced force $F_m / ima = P_m(X)/i$ in $\mathcal{R}$ also in terms of the random variable $X$ for the continuous uniform probability distribution. The thick line in cyan is the projection of the plane $P_c^2(X) = P_r(X) + P_m(X)/i = 1 = P_c(X) = F_c / ma$ on the plane $X = L_b$ = lower bound of $X = -3$. This thick line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $F_r / ma = P_r(X)$ in the plane $P_r(X) = P_m(X)/i$ in light gray. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $X = L_b$ = lower bound of $X = -3$), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, $X = 0$), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $X = U_b$ = upper bound of $X = 3$). The blue curve represents $F_m / ima = P_m(X)/i$ in the plane in cyan $P_r(X) + P_m(X)/i = 1 = P_c(X) = F_c / ma$. Notice the importance of the point K which is the intersection of

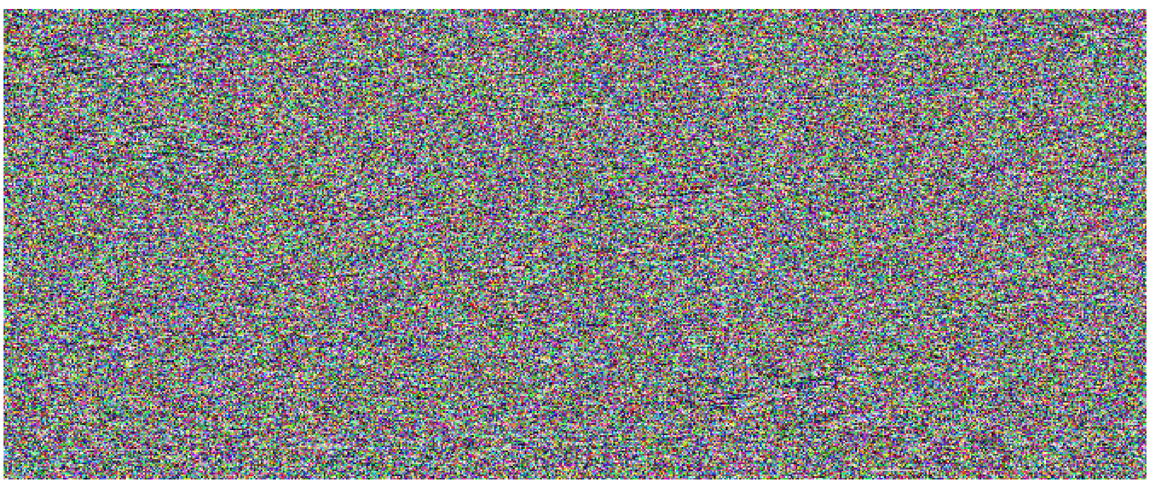

**Figure 32.**
*The graphs of* DOK *and* Chf *and the deterministic reduced force* $F_c / ma = P_c$ *in terms of* X *and of each other for this continuous uniform probability distribution.*

**Figure 33.**
*The graphs of* $F_r / ma = P_r$ *and* $F_m / ima = P_m / i$ *and* $F_c / ma = P_c$ *in terms of* X *and of each other for this continuous uniform probability distribution.*

the red and blue curves at $X = 0$ and when $P_r(X) = P_m(X)/i = 0.5$. The three points J, K, L are the same as in **Figure 31**.

In the third cube (**Figure 34**), we can notice the simulation of the complex resultant reduced force $F / ma = z(X)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real reduced force $F_r / ma = P_r(X) = \mathrm{Re}(z)$ in $\mathcal{R}$ and of its complementary imaginary reduced force $F_m / ma = P_m(X) = i \times \mathrm{Im}(z)$ in $\mathcal{M}$, and this in terms of the random variable $X$ for the continuous uniform probability distribution. The red curve represents $F_r / ma$ in the plane $P_m(X) = 0$ and the blue curve represents $F_m / ma$ in the plane $P_r(X) = 0$. The green curve represents the complex resultant reduced force $F / ma = F_r / ma + F_m / ma = z(X) = P_r(X) + P_m(X) = \mathrm{Re}(z) + i \times \mathrm{Im}(z)$ in the plane $P_r(X) = iP_m(X) + 1$ or $z(X)$ plane in cyan. The curve of $F / ma$ starts at the point J ($P_r = 0, P_m = i, X = L_b =$ lower bound of $X = -3$) and ends at the point L ($P_r = 1$, $P_m = 0, X = U_b =$ upper bound of $X = 3$). The thick line in cyan is $P_r(X = L_b = -3) = iP_m(X = L_b = -3) + 1$ and it is the projection of the $F / ma$ curve on the complex probability plane whose equation is $X = L_b = -3$. This projected thick line starts at the point J ($P_r = 0, P_m = i, X = L_b = -3$) and ends at the point ($P_r = 1$, $P_m = 0, X = L_b = -3$). Notice the importance of the point K corresponding to $X = 0$ and $z = 0.5 + 0.5i$ when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in **Figure 31**.

**Figure 34.**
*The graphs of the reduced forces* $F_r$ / ma = $P_r$ *and* $F_m$ / ma = $P_m$ *and* F / ma = z *in terms of* X *for this continuous uniform probability distribution.*

### *8.2.2 The standard Gaussian normal probability distribution*

The probability density function (*PDF*) of this continuous stochastic distribution is:

$$f(x) = \frac{d\,[CDF(x)]}{dx} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \text{for} \quad -\infty < x < \infty$$

and the cumulative distribution function (*CDF*) is:

$$CDF(x) = P_{rob}(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)\, dt$$

The domain for this standard Gaussian normal variable is considered in the simulations to be equal to: $x \in [L_b = -4, U_b = 4]$ and I have taken $dx = 0.01$.

In the simulations, the mean of this standard normal random distribution is $\mu = 0$.

The variance is $\sigma^2 = 1$.

The standard deviation is $\sigma = 1$.
The median is $Md = 0$.
The mode for this symmetric distribution is = 0 = $Md$ = $\mu$.
The real probability $P_r(x)$ and force are:

$$P_r(x) = CDF(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt = \int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt, \forall x : -4 \le x \le 4$$

$$\Leftrightarrow \vec{F}_r(x) = P_r(x) m\vec{a} = \left[\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] m\vec{a}$$
$$= \left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] m\vec{a}$$

The imaginary complementary probability $P_m(x)$ and force are:

$$P_m(x) = i[1 - P_r(x)] = i[1 - CDF(x)] = i\left[1 - \int_{-\infty}^{x} f(t) dt\right]$$

$$= i\left[\int_{x}^{+\infty} f(t) dt\right] = i\left[\int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] = i\left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right], \forall x : -4 \le x \le 4$$

$$\Leftrightarrow \vec{F}_m(x) = P_m(x) m\vec{a} = i\left[\int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] m\vec{a}$$
$$= i\left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] m\vec{a}$$

The real complementary probability $P_m(x)/i$ and force are:

$$P_m(x)/i = 1 - P_r(x) = 1 - CDF(x) = 1 - \int_{-\infty}^{x} f(t) dt = \int_{x}^{+\infty} f(t) dt$$
$$= \int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt, \forall x : -4 \le x \le 4$$

$$\Leftrightarrow \vec{F}_m(x)/i = \frac{P_m(x)}{i} m\vec{a} = \left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] m\vec{a}$$

The complex probability or random vector and force are:

$$z(x) = P_r(x) + P_m(x) = \left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + i\left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right], \forall x : -4 \le x \le 4$$

$$\Leftrightarrow \vec{F}(x) = \vec{F}_r(x) + \vec{F}_m(x) = P_r(x)m\vec{a} + P_m(x)m\vec{a} = [P_r(x) + P_m(x)]m\vec{a} = zm\vec{a}$$

$$= \left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] m\vec{a} + i\left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] m\vec{a}$$

$$= \left\{\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + i\left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]\right\} m\vec{a}$$

The Degree of Our Knowledge:

$$DOK(x) = |z(x)|^2 = P_r^2(x) + [P_m(x)/i]^2 = \left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2$$

$$+ \left(1 - \left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]\right)^2$$

$$= \left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2 + \left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2$$

$$= 1 + 2iP_r(x)P_m(x) = 1 - 2P_r(x)[1 - P_r(x)] = 1 - 2P_r(x) + 2P_r^2(x)$$

$$= 1 - 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2, \quad \forall x : -4 \le x \le 4$$

$DOK(x)$ is equal to 1 when $P_r(x) = P_r(L_b = -4) = 0$ and when $P_r(x) = P_r(U_b = 4) = 1$.

The Chaotic Factor:

$$Chf(x) = 2iP_r(x)P_m(x) = -2P_r(x)[1 - P_r(x)] = -2P_r(x) + 2P_r^2(x)$$

$$= -2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2, \quad \forall x : -4 \le x \le 4$$

$Chf(x)$ is null when $P_r(x) = P_r(L_b = -4) = 0$ and when$P_r(x) = P_r(U_b = 4) = 1$.

The Magnitude of the Chaotic Factor $MChf$:

$$MChf(x) = |Chf(x)| = -2iP_r(x)P_m(x) = 2P_r(x)[1 - P_r(x)] = 2P_r(x) - 2P_r^2(x)$$

$$= 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] - 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2, \quad \forall x : -4 \le x \le 4$$

$MChf(x)$ is null when $P_r(x) = P_r(L_b = -4) = 0$ and when $P_r(x) = P_r(U_b = 4) = 1$.

At any value of $x$: $\forall x : (L_b = -4) \le x \le (U_b = 4)$, the probability expressed in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is the following:

$$\begin{aligned} P_c{}^2(x) &= [P_r(x) + P_m(x)/i]^2 = |z(x)|^2 - 2iP_r(x)P_m(x) \\ &= DOK(x) - Chf(x) \\ &= DOK(x) + MChf(x) \\ &= 1 \end{aligned}$$

then,

$$P_c{}^2(x) = [P_r(x) + P_m(x)/i]^2 = \{P_r(x) + [1 - P_r(x)]\}^2 = 1^2 = 1 \Leftrightarrow P_c(x) = 1 \text{ always}$$

$\Leftrightarrow \vec{F}_c(x) = P_c(x) m\vec{a} = 1 \times m\vec{a} = m\vec{a}$ always also.

Hence, the prediction of all the probabilities and forces of the stochastic experiment in the universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is permanently certain and perfectly deterministic (**Figure 35**).

#### *8.2.2.1 The complex probability cubes*

In the first cube (**Figure 36**), the simulation of *DOK* and *Chf* as functions of each other and of the random variable $X$ for the standard Gaussian normal probability distribution can be seen. The thick line in cyan is the projection of the plane $P_c{}^2(X) = DOK(X) - Chf(X) = 1 = P_c(X) = F_c / ma$ on the plane $X = L_b$ = lower bound of $X = -4$. This thick line starts at the point J ($DOK = 1$, $Chf = 0$) when $X = L_b = -4$,

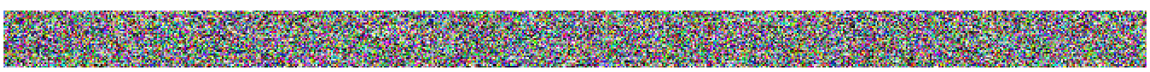

**Figure 35.**
*The graphs of* $F_r$ / ma, $F_m$ / ima, *and* $F_c$ / ma *and of all the* CPP *parameters as functions of the random variable* X *for the continuous standard Gaussian normal distribution.*

**Figure 36.**
*The graphs of* DOK *and* Chf *and the deterministic reduced force* $F_c$ / ma = $P_c$ *in terms of* X *and of each other for the standard Gaussian normal probability distribution.*

reaches the point ($DOK$ = 0.5, $Chf$ = $-0.5$) when $X$ = 0, and returns at the end to J ($DOK$ = 1, $Chf$ = 0) when $X = U_b$ = upper bound of $X$ = 4. The other curves are the graphs of $DOK(X)$ (red) and $Chf(X)$ (green, blue, pink) in different simulation planes. Notice that they all have a minimum at the point K ($DOK$ = 0.5, $Chf$ = $-0.5$, $X$ = 0). The point L corresponds to ($DOK$ = 1, $Chf$ = 0, $X = U_b$ = 4). The three points J, K, L are the same as in **Figure 35**.

In the second cube (**Figure 37**), we can notice the simulation of the real reduced force $F_r / ma = P_r(X)$ in $\mathcal{R}$ and its complementary real reduced force $F_m / ima$ = $P_m(X)/i$ in $\mathcal{R}$ also in terms of the random variable $X$ for the standard Gaussian normal probability distribution. The thick line in cyan is the projection of the plane $P_c^2(X) = P_r(X) + P_m(X)/i = 1 = P_c(X) = F_c / ma$ on the plane $X = L_b$ = lower bound of $X = -4$. This thick line starts at the point ($P_r$ = 0, $P_m/i$ = 1) and ends at the point ($P_r$ = 1, $P_m/i$ = 0). The red curve represents $F_r / ma = P_r(X)$ in the plane $P_r(X) = P_m(X)/i$ in light gray. This curve starts at the point J ($P_r$ = 0, $P_m/i$ = 1, $X = L_b$ = lower bound of $X = -4$), reaches the point K ($P_r$ = 0.5, $P_m/i$ = 0.5, $X$ = 0), and gets at the end to L ($P_r$ = 1, $P_m/i$ = 0, $X = U_b$ = upper bound of $X$ = 4). The blue curve represents $F_m / ima = P_m(X)/i$ in the plane in cyan $P_r(X) + P_m(X)/i = 1$ = $P_c(X) = F_c / ma$. Notice the importance of the point K which is the intersection of the

**Figure 37.**
*The graphs of* $\mathrm{F_r / ma = P_r}$ *and* $\mathrm{F_m / ima = P_m / i}$ *and* $\mathrm{F_c / ma = P_c}$ *in terms of* $\mathrm{X}$ *and of each other for the standard Gaussian normal probability distribution.*

red and blue curves at $X = 0$ and when $P_r(X) = P_m(X)/i = 0.5$. The three points J, K, L are the same as in **Figure 35**.

In the third cube (**Figure 38**), we can notice the simulation of the complex resultant reduced force $F / ma = z(X)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real reduced force $F_r / ma = P_r(X) = \mathrm{Re}(z)$ in $\mathcal{R}$ and of its complementary imaginary reduced force $F_m / ma = P_m(X) = i \times \mathrm{Im}(z)$ in $\mathcal{M}$, and this in terms of the random variable $X$ for the standard Gaussian normal probability distribution. The red curve represents $F_r / ma$ in the plane $P_m(X) = 0$ and the blue curve represents $F_m / ma$ in the plane $P_r(X) = 0$. The green curve represents the complex resultant reduced force $F / ma = F_r / ma + F_m / ma = z(X) = P_r(X) + P_m(X) = \mathrm{Re}(z) + i \times \mathrm{Im}(z)$ in the plane $P_r(X) = iP_m(X) + 1$ or $z(X)$ plane in cyan. The curve of $F / ma$ starts at the point J ($P_r = 0, P_m = i, X = L_b =$ lower bound of $X = -4$) and ends at the point L ($P_r = 1, P_m = 0, X = U_b =$ upper bound of $X = 4$). The thick line in cyan is $P_r(X = L_b = -4) = iP_m(X = L_b = -4) + 1$ and it is the projection of the $F / ma$ curve on the complex probability plane whose equation is $X = L_b = -4$. This projected thick line starts at the point J ($P_r = 0, P_m = i, X = L_b = -4$) and ends at the point ($P_r = 1, P_m = 0, X = L_b = -4$). Notice the importance of the point K corresponding to $X = 0$ and $z = 0.5 + 0.5i$ when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in **Figure 35**.

**Figure 38.**
*The graphs of the reduced forces* $F_r$ / ma = $P_r$ *and* $F_m$ / ma = $P_m$ *and* F / ma = z *in terms of* X *for the standard Gaussian normal probability distribution.*

## 9. Conclusion and perspectives

In the current research work, the original extended model of eight axioms (*EKA*) of A. N. Kolmogorov was connected and applied to Isaac Newton's classical mechanics theory. Thus, a tight link between classical mechanics and the novel paradigm was achieved. Consequently, the model of "Complex Probability" was more developed beyond the scope of my seventeen previous research works on this topic.

Additionally, as it was proved and verified in the novel model, before the beginning of the random phenomenon simulation and at its end we have the chaotic factor (*Chf* and *MChf*) is zero and the degree of our knowledge (*DOK*) is one since the stochastic fluctuations and effects have either not started yet or they have terminated and finished their task on the probabilistic phenomenon. During the execution of the nondeterministic phenomenon and experiment we also have: $0.5 \le DOK < 1$, $-0.5 \le Chf < 0$, and $0 < MChf \le 0.5$. We can see that during this entire process we have incessantly and continually $P_c^2 = DOK - Chf = DOK + MChf = 1 = P_c$, that means that the simulation which behaved randomly and stochastically in the set $\mathcal{R}$ is now certain and deterministic in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding to the random experiment executed in $\mathcal{R}$ the contributions of the set $\mathcal{M}$ and hence after eliminating and subtracting the chaotic

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factor from the degree of our knowledge. Furthermore, the probabilities of the real, imaginary, complex, and deterministic forces acting on a body and that correspond to each value of the random variable $X$ have been determined in the three probabilities sets which are $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$ by $P_r$, $P_m$, $z$ and $P_c$ respectively. Consequently, at each value of $X$, the novel classical mechanics and *CPP* parameters $F_r$, $F_m$, $F$, $F_c$, $P_r$, $P_m$, $P_m/i$, $DOK$, *Chf*, *MChf*, $P_c$, and $z$ are surely and perfectly predicted in the complex probabilities set $\mathcal{C}$ with $P_c$ maintained equal to one permanently and repeatedly. Also, as it was shown and proved in the equations above that if the real probability $P_r$ is equal to one then we will return directly to the classical deterministic Newtonian mechanics theory which is a special deterministic case of the stochastic complex probability paradigm general case.

In addition, referring to all these obtained graphs and executed simulations throughout the whole research work, we are able to quantify and to visualize both the system chaos and stochastic effects and influences (expressed and materialized by *Chf* and *MChf*) and the certain knowledge (expressed and materialized by *DOK* and $P_c$) of the new paradigm. This is without any doubt very fruitful, wonderful, and fascinating and proves and reveals once again the advantages of extending A. N. Kolmogorov's five axioms of probability and hence the novelty and benefits of this inventive and original model in the fields of prognostics and applied mathematics that can be called truly: "The Complex Probability Paradigm".

Moreover, it is important to mention here that one very well-known and important random distribution was considered in the current work which is the discrete and uniform random distribution that was used to prove an important and essential result at the foundation of statistical mechanics and physics, knowing that the novel *CPP* paradigm can be implemented to any probability distribution that exists in literature as it was shown in the simulation section. This will lead without any doubt to analogous and similar conclusions and results and will confirm certainly the success of my innovative and original model.

As a future and prospective research and challenges, we aim to more develop the novel prognostic paradigm conceived and to implement it to a large set of random and nondeterministic events like for other probabilistic phenomena as in stochastic processes and in the classical theory of probability. Additionally, we will apply *CPP* to the random walk problems which have huge and very interesting consequences when implemented to chemistry, to physics, to economics, to applied and pure mathematics.

## Conflicts of interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

## Data availability

The data used to support the findings of this study are available from the author upon request.

## Nomenclature

| | |
|---|---|
| $\mathcal{R}$ | real set of events |
| $\mathcal{M}$ | imaginary set of events |
| $\mathcal{C}$ | complex set of events |

| | |
|---|---|
| $i$ | the imaginary number where $i = \sqrt{-1}$ or $i^2 = -1$ |
| $EKA$ | Extended Kolmogorov's Axioms |
| $CPP$ | Complex Probability Paradigm |
| $P_{rob}$ | probability of any event |
| $P_r$ | probability in the real set $\mathcal{R}$ = probability of the real random force in $\mathcal{R}$ |
| $P_m$ | probability in the imaginary set $\mathcal{M}$ corresponding to the real probability in $\mathcal{R}$ = probability of the imaginary random force in $\mathcal{M}$ |
| $P_c$ | probability of an event in $\mathcal{R}$ with its associated complementary event in $\mathcal{M}$ = probability of the real deterministic force in the complex probability set $\mathcal{C}$ |
| $\vec{F}_r$ | the real stochastic force in $\mathcal{R}$ |
| $\vec{F}_m$ | the imaginary stochastic force in $\mathcal{M}$ |
| $\vec{F}$ | the resultant complex stochastic force in $\mathcal{C}$ |
| $\vec{F}_c$ | the real deterministic force in $\mathcal{C}$ |
| $z$ | complex probability number = sum of $P_r$ and $P_m$ = complex random vector = probability of the resultant complex stochastic force in $\mathcal{C}$ |
| $DOK = \|z\|^2$ | the degree of our knowledge of the random system or experiment, it is the square of the norm of $z$ |
| $Chf$ | the chaotic factor of $z$ |
| $MChf$ | magnitude of the chaotic factor of $z$ |
| $N$ | number of random vectors = number of random atoms or particles or molecules |
| $Z$ | the resultant complex random vector = $\sum_{j=1}^{N} z_j$ |
| $DOK_Z = \frac{\|Z\|^2}{N^2}$ | the degree of our knowledge of the whole stochastic system |
| $Chf_Z = \frac{Chf}{N^2}$ | the chaotic factor of the whole stochastic system |
| $MChf_Z$ | magnitude of the chaotic factor of the whole stochastic system |
| $Z_U$ | the resultant complex random vector corresponding to a uniform random distribution |
| $DOK_{Z_U}$ | the degree of our knowledge of the whole stochastic system corresponding to a uniform random distribution |
| $Chf_{Z_U}$ | the chaotic factor of the whole stochastic system corresponding to a uniform random distribution |
| $MChf_{Z_U}$ | the magnitude of the chaotic factor of the whole stochastic system corresponding to a uniform random distribution |
| $P_c\|_{Z_U}$ | probability in the complex probability set $\mathcal{C}$ of the whole stochastic system corresponding to a uniform random distribution |

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