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Globally conservative solutions for the modified Camassa–Holm (MOCH) equation

Zhaonan Luo

Zhijun Qiao
The University of Texas Rio Grande Valley, zhijun.qiao@utrgv.edu

Zhaoyang Yin

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Zhaonan Luo,[1,a)] Zhijun Qiao,[2,b)] and Zhaoyang Yin[1,3,c)]

AFFILIATIONS

[1] School of Mathematics, Sun Yat-sen University, Guangzhou 510275, China
[2] School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, Texas 78539, USA
[3] Faculty of Information Technology, Macau University of Science and Technology, Macau, China

[a)] Author to whom correspondence should be addressed: 1411919168@qq.com
[b)] Email: zhijun.qiao@utrgv.edu
[c)] Email: mcsyzy@mail.sysu.edu.cn

ABSTRACT

In this paper, we present the globally conservative solutions to the Cauchy problem for the modified Camassa–Holm (MOCH) equation. First, we transform the equation into an equivalent semi-linear system under new variables. Second, according to the standard ordinary differential equation theory with the aid of the conservation law, we give the global solutions of the semi-linear system. Finally, returning to the original variables, we obtain the globally conservative solutions to the MOCH equation.

I. INTRODUCTION

It has been known that the celebrated Camassa–Holm (CH) equation for $\lambda = 0$ has quadratic nonlinearity and is utilized to describe a shallow water wave model.[10,11] It is completely integrable[6,10] and has a bi-Hamiltonian structure.[4,17] It has peakon solutions in the form of $ce^{-|x-ct|}$ with the wave speed c, which is orbitally stable.[13] The local well-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces was presented in Refs. 1, 7, 8, 15, 16, 20, and 22. It admits not only blow-up strong solutions in a finite time but also globally strong solutions.[5,7–9] Recently, norm inflation and ill-posedness for the CH equation in the critical Sobolev Space and Besov spaces were proved in Ref. 18. The existence and uniqueness of globally weak solutions were studied in Refs. 12 and 24. The globally conservative, dissipative solutions, and algebro-geometric solutions were presented in Refs. 2, 3, and 21.

The well-known CH equation also describes pseudo-spherical surfaces, and therefore, its integrability properties can be studied by means of the geometrical approach.[14] A nonlinear differential equation is geometrically integrable if it describes a nontrivial one-parameter family of pseudo-spherical surfaces. Apparently, the CH equation is geometrically integrable.[23] Hence, it turns out from the viewpoint of differential geometry that the study of the geodesics determined by these surfaces allows one to define a Miura transform and the modified Camassa–Holm (MOCH) equation.[19] In this paper, we consider the Cauchy problem for the MOCH equation

$$\begin{cases} \gamma_t = \lambda\left(u_x - \gamma - \dfrac{1}{\lambda}u\gamma\right)_x, & t > 0,\ x \in \mathbb{R}, \\ u_{xx} - u = \gamma_x + \dfrac{\gamma^2}{2\lambda}, & t \geq 0,\ x \in \mathbb{R}, \\ \gamma(0,x) = \bar{\gamma}(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where λ is a real constant. In order to define weak solutions for Eq. (1.1), let us rewrite (1.1) as

$$\begin{cases} \gamma_t + \left(\gamma\partial_x G^{-1}\gamma + \frac{1}{2\lambda}\gamma G^{-1}\gamma^2\right)_x = \frac{\gamma^2}{2} + \lambda\partial_x G^{-1}\gamma + \frac{1}{2}G^{-1}\gamma^2, \quad t > 0,\ x \in \mathbb{R}, \\ \gamma(0,x) = \bar{\gamma}(x), \quad x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Let $G = \partial_x^2 - 1$ and $m = (\partial_x + 1)G^{-1}\gamma = (\partial_x - 1)^{-1}\gamma$. Then, we have $\gamma = (\partial_x - 1)m$ and $\|\gamma\|_{L^2} = \|m\|_{H^1}$, and therefore, Eq. (1.1) is changed to

$$\begin{aligned} m_t + um_x &= mu - G^{-1}(um_x - um) - \partial_x G^{-1}(um_x - um) + \frac{1}{2}(-m^2 + G^{-1}m_x^2 + \partial_x G^{-1}m_x^2) \\ &\quad + \lambda\partial_x G^{-1}m + \frac{1}{2}((\partial_x - 1)^{-1}G^{-1}m_x^2 - G^{-1}m^2), \end{aligned} \tag{1.3}$$

where $u = m - \partial_x G^{-1}m + G^{-1}m + \frac{1}{2\lambda}(G^{-1}m_x^2 + G^{-1}m^2 - \partial_x G^{-1}m^2)$. For the initial data, we have

$$m(0,x) = \bar{m}(x) = (\partial_x - 1)^{-1}\bar{\gamma}(x). \tag{1.4}$$

In Refs. 14 and 23, Eq. (1.1) was first investigated by the geometric approach. Pseudo-spherical surfaces, conservation laws, and the existence and uniqueness of weak solutions to the MOCH equation were studied in Ref. 19. Equation (1.1) is able to be regarded as a modified Camassa–Holm equation for the reason that if we solve Eq. (1.1), then u will formally satisfy the following shallow water wave form of the Camassa–Holm equation:[10]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} - \lambda u_x.$$

As far as we know, the globally conservative solutions of the MOCH model (1.1) have not been investigated yet. Therefore, in this paper, we aim to study the globally conservative solutions of (1.3) based on the idea of Bressan and Constantin[3] in proving the globally conservative solutions to the Camassa–Holm equation.

Before providing our main results in this paper, let us first introduce some definitions of the globally conservative solutions for (1.1) and (1.3).

Definition 1.1. Let $\bar{\gamma} \in L^2(\mathbb{R})$. We say $\gamma(t,x) \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}))$ is a globally conservative solution to the Cauchy problem (1.1) if $\gamma(t,x)$ satisfies the following equality:

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(\gamma\psi_t + \left(\gamma\partial_x G^{-1}\gamma + \frac{1}{2\lambda}\gamma G^{-1}\gamma^2\right)\psi_x\right)(t,x)\mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(\left(\frac{\gamma^2}{2} + \lambda\partial_x G^{-1}\gamma + \frac{1}{2}G^{-1}\gamma^2\right)\psi\right)(t,x)\mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}}\bar{\gamma}(x)\psi(0,x)\mathrm{d}x$$

for all $\psi \in C_0^\infty(\mathbb{R}^+; \mathcal{D}(\mathbb{R}))$. Moreover, quantities $\|\gamma\|_{L^2}$ are conserved in time.

Definition 1.2. Let $\bar{m} \in H^1(\mathbb{R})$. We say that $m(t,x) \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}))$ is a globally conservative solution to the Cauchy problem (1.3) if the map $t \to m(t,\cdot)$ is Lipschitz continuous from $[t_1,t_2]$ to $L^2(\mathbb{R})$ for any $[t_1,t_2] \subset \mathbb{R}^+$ and $m(t,x)$ satisfies the following equality:

$$\begin{aligned} m_t &= -um_x + mu - G^{-1}(um_x - um) - \partial_x G^{-1}(um_x - um) + \frac{1}{2}(-m^2 + G^{-1}m_x^2 + \partial_x G^{-1}m_x^2) \\ &\quad + \lambda\partial_x G^{-1}m + \frac{1}{2}((\partial_x - 1)^{-1}G^{-1}m_x^2 - G^{-1}m^2) \end{aligned} \tag{1.5}$$

in $L^2(\mathbb{R})$ for almost every $t \in \mathbb{R}^+$ together with the initial data (1.4). Moreover, quantities $\|m\|_{H^1}$ are conserved in time.

The main theorem of this paper is as follows.

Theorem 1.3. *Let $\bar{m}(x) \in H^1(\mathbb{R})$. Then, equation (1.3) has a globally conservative solution in the sense of Definition 1.2.*

Corollary 1.4. Let $\bar{\gamma}(x) \in L^2(\mathbb{R})$. Then, problem (1.1) has a globally conservative solution in the sense of Definition 1.1.

The rest of our paper is organized as follows. In Sec. II, we present conservation laws for the original equation, which is derived from the MOCH equation. In Sec. III, we introduce some new variables and deduce an equivalent semi-linear system under new variables. Then, we establish the global solutions to the semi-linear system. Finally, back to the original variables, we construct globally conservative solutions m to the original equation (1.3). We can easily deduce that $\gamma = (\partial_x - 1)m$ is a globally conservative solution to Eq. (1.1).

II. THE ORIGINAL EQUATION

Let us rewrite Eq. (1.3) as follows:

$$m_t + um_x = F, \tag{2.1}$$

$$F = mu - P_4 - P_{4x} + \frac{1}{2}(-m^2 + P_3 + P_{3x}) + \lambda P_{1x} + \frac{1}{2}(P_5 + P_{5x} - P_2), \tag{2.2}$$

where P_i $(i = 1, \ldots, 5)$ are defined by the following convolutions:

$$P_1 = G^{-1}m = p * m, \quad P_2 = p * m^2, \quad P_3 = p * m_x^2,$$
$$u = m - P_{1x} + P_1 + \frac{1}{2\lambda}(P_2 + P_3 - P_{2x}),$$
$$P_4 = p * (um_x - um), \quad P_5 = p * P_3,$$

with $p(x) = -\frac{1}{2}e^{-|x|}$. For smooth solutions $\gamma \in C_0^\infty(\mathbb{R})$, we first present conservation law for Eq. (1.1). Due to $u_{xx} - u = \gamma_x + \frac{\gamma^2}{2\lambda}$, multiplying (1.1) by γ, and integrating by parts, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \gamma^2 dx &= \int_{\mathbb{R}} \lambda\gamma(u_{xx} - \gamma_x) - \gamma\gamma_x u - \gamma^2 u_x dx \\ &= \int_{\mathbb{R}} -\lambda\gamma_x u_x - \gamma^2 u_x - \gamma\gamma_x u dx \\ &= \int_{\mathbb{R}} uu_x - u_x u_{xx} - \frac{1}{2}\gamma^2 u_x - \gamma\gamma_x u dx \\ &= 0.\end{aligned}$$

If $m \in C_0^\infty(\mathbb{R})$, then we have $\gamma \in C_0^\infty(\mathbb{R})$. Using the fact that $\|\gamma\|_{L^2} = \|m\|_{H^1}$ yields the following quantity:

$$E(t) := \int_{\mathbb{R}} m^2(t,x) + m_x^2(t,x)\mathrm{d}x, \tag{2.3}$$

which are not dependent on time.

Since P_i and P_{ix} $(1 \le i \le 5)$ are given by convolutions and $p(x) \in L^1 \cap L^\infty$, the conservation of the total energy (2.3) leads to

$$\|P_i\|_{L^\infty}, \quad \|P_{ix}\|_{L^\infty} \le C(E(0)).$$

III. AN EQUIVALENT SEMI-LINEAR SYSTEM AND GLOBAL SOLUTIONS OF THE SYSTEM

A. An equivalent semi-linear system

Let $\bar{m} \in H^1(\mathbb{R})$. We introduce a new variable $\xi \in \mathbb{R}$ and define the nondecreasing map $\xi \mapsto \bar{y}(\xi)$ via the following equation:

$$\int_0^{\bar{y}(\xi)} (1 + \bar{m}_x^2)\mathrm{d}x = \xi. \tag{3.1}$$

For $t \in [0, T]$, we assume that the solution m to (2.1) is Lipschitz continuous. Under the new variable (t, ξ), let us now derive a system equivalent to Eq. (2.1). We define the characteristic $y(t, \cdot)$ as

$$y_t(t,\xi) = u(t, y(t,\xi)), \qquad y(0,\xi) = \bar{y}(\xi). \tag{3.2}$$

Introduce the new variables

$$\begin{aligned}&m(t,\xi) = m(t,y(t,\xi)), \quad v(t,\xi) = 2\arctan m_x, \quad q(t,\xi) = (1 + m_x^2)y_\xi, \\ &P_i(t,\xi) = P_i(t,y(t,\xi)), \quad P_{ix}(t,\xi) = P_{ix}(t,y(t,\xi)), \quad 1 \le i \le 5, \\ &u(t,\xi) = u(t,y(t,\xi)), \quad F(t,\xi) = F(t,y(t,\xi)),\end{aligned}$$

with $m_x = m_x(t, y(t,\xi))$. From (3.1), we have

$$q(0,\xi) \equiv 1.$$

We can prove the following equalities:

$$\frac{1}{1 + m_x^2} = \cos^2\frac{v}{2}, \quad \frac{m_x}{1 + m_x^2} = \frac{1}{2}\sin v, \quad \frac{m_x^2}{1 + m_x^2} = \sin^2\frac{v}{2}, \quad y_\xi = q\cos^2\frac{v}{2}. \tag{3.3}$$

Using (3.3), we obtain expressions for P_i and P_{ix} in terms of the new variable ξ, namely,

$$\begin{cases} P_1 = -\frac{1}{2}\int_{-\infty}^{+\infty} e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|}\, mq\cos^2\frac{v}{2}\mathrm{d}\xi', \\ P_{1x} = -\frac{1}{2}\left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi}\right)e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|}\, mq\cos^2\frac{v}{2}\mathrm{d}\xi', \\ P_2 = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|} m^2\, q\cos^2\frac{v}{2}\mathrm{d}\xi', \\ P_{2x} = -\frac{1}{2}\left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi}\right)e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|} m^2\, q\cos^2\frac{v}{2}\mathrm{d}\xi', \\ P_3 = -\frac{1}{2}\int_{-\infty}^{+\infty} e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|}\, q\sin^2\frac{v}{2}\mathrm{d}\xi', \\ P_{3x} = -\frac{1}{2}\left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi}\right)e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|}\, q\sin^2\frac{v}{2}\mathrm{d}\xi', \\ P_4 = -\frac{1}{2}\int_{-\infty}^{+\infty} e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|}\, \frac{q}{2}u\sin v - muq\cos^2\frac{v}{2}\mathrm{d}\xi', \\ P_{4x} = -\frac{1}{2}\left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi}\right)e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|}\, \frac{q}{2}u\sin v - muq\cos^2\frac{v}{2}\mathrm{d}\xi', \\ P_5 = -\frac{1}{2}\int_{-\infty}^{+\infty} e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|} P_3 q\cos^2\frac{v}{2}\mathrm{d}\xi', \\ P_{5x} = -\frac{1}{2}\left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi}\right)e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\mathrm{d}s|} P_3 q\cos^2\frac{v}{2}\mathrm{d}\xi', \end{cases} \tag{3.4}$$

where $u(t,\xi) = \left(m - P_{1x} + P_1 + \frac{1}{2\lambda}(P_2 + P_3 - P_{2x})\right)(t,\xi)$. Owing to the characteristic (3.2) and the first equation of (2.1), we can deduce that

$$m_t(t,\xi) = F(t,\xi),$$

with

$$F(t,\xi) = \left(mu - P_4 - P_{4x} + \frac{1}{2}(-m^2 + P_3 + P_{3x}) + \lambda P_{1x} + \frac{1}{2}(P_5 + P_{5x} - P_2)\right)(t,\xi). \tag{3.5}$$

Similarly, using (3.3) and (3.2) and the definitions of v and q, we obtain

$$v_t(t,\xi) = 2\cos^2\frac{v}{2}H - N\sin v - \sin^2\frac{v}{2},$$
$$q_t(t,\xi) = \frac{1}{2}q\sin v(2H+1) + qN\cos v,$$

where $N(t,\xi) = \left(-m - P_1 - P_{1x} + \frac{1}{2\lambda}(-P_2 + P_{3x} + P_{2x} - m^2)\right)(t,\xi)$ and $H = F + mN + \lambda u + \frac{m^2}{2}$.

B. Global solutions of the semi-linear system

In this section, we turn our attention to the problem of finding a global solution to system (3.6). Let $\bar{m} \in H^1(\mathbb{R})$. We rewrite the corresponding Cauchy problem for the variables (m, v, q) in the form

$$\begin{cases} m_t = F, \\ v_t = 2\cos^2\frac{v}{2}H - N\sin v - \sin^2\frac{v}{2}, \\ q_t = \frac{1}{2}q\sin v(2H+1) + qN\cos v \end{cases} \tag{3.6}$$

and

$$\begin{cases} m(0,\xi) = \bar{m}(\bar{y}(\xi)), \\ v(0,\xi) = 2\arctan\bar{m}_x(\bar{y}(\xi)), \\ q(0,\xi) = 1. \end{cases} \tag{3.7}$$

Here, F, u, N, H are given in Sec. II. We consider (3.6) as an O.D.E. in

$$X = H^1(\mathbb{R}) \times L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \times L^\infty(\mathbb{R}),$$

with $\|(m,v,q)\|_X = \|m\|_{H^1} + \|v\|_{L^\infty} + \|v\|_{L^2} + \|q\|_{L^\infty}$.

Theorem 3.1. *Let $\bar{m} \in H^1(\mathbb{R})$. For any $t \in \mathbb{R}$, the semi-linear system* [(3.6) and (3.7)] *has a unique solution.*

Proof. **Step 1** (Local existence)

It is obvious that the initial data belong to X. According to the theory of O.D.E. in Banach spaces, in order to prove the local well-posedness, we need to prove that the right-hand side of (3.6) is Lipschitz continuous on Ω, which is defined as

$$\Omega = \left\{(m,v,q) \in X : \quad \|m\|_{H^1} \le \alpha, \quad \|v\|_{L^\infty} \le \frac{3\pi}{2}, \quad \|v\|_{L^2} \le \beta, \quad q \in [q^-, q^+], \quad a.e. x \in \mathbb{R}\right\}$$

for $\alpha, \beta, q^-, q^+ > 0$. According to Ref. 3, we easily get that maps

$$m^2, \quad P_2, \quad P_{2x}, \quad P_3, \quad P_{3x}$$

are Lipschitz continuous from Ω to H^1. It is easy to check that maps

$$\cos^2\frac{v}{2}, \quad \sin v, \quad \sin^2\frac{v}{2}, \quad q\sin v, \quad q\cos v$$

are Lipschitz continuous from Ω to $L^2 \cap L^\infty$. Then, we are left to prove that the maps

$$(m,v,q) \mapsto P_1, \quad P_{1x}, \quad P_4, \quad P_{4x}, \quad P_5, \quad P_{5x}$$

are Lipschitz continous from Ω to H^1. According to Ref. 3, we obtain

$$meas\left\{\xi \in \mathbb{R} : \left|\frac{v(\xi)}{2}\right| \ge \frac{\pi}{4}\right\} \le 18\int_{\left\{\xi\in\mathbb{R}:18\sin^2\frac{v(\xi)}{2}\ge 1\right\}} \sin^2\frac{v(\xi)}{2}\mathrm{d}\xi \le \frac{9}{2}\beta^2.$$

If $\xi_1 < \xi_2$, then we have

$$\int_{\xi_1}^{\xi_2} q\cos^2\frac{v}{2}\mathrm{d}\xi \ge \int_{\left\{\xi\in[\xi_1,\xi_2]:\left|\frac{v(\xi)}{2}\right|\le\frac{\pi}{4}\right\}} \frac{q^-}{2}\mathrm{d}\xi \ge q^-\left[\frac{\xi_2-\xi_1}{2} - \frac{9}{2}\beta^2\right].$$

Let us introduce a function $\Gamma \in L^1 \cap L^\infty$,

$$\Gamma(\xi) = \min\left\{1, e^{-\frac{9}{2}\beta^2 q^- - \frac{|\xi|}{2}q^-}\right\},$$

with $\|\Gamma\|_{L^1} = 18\beta^2 + \frac{4}{q^-}$. First, we need to prove that $P_1, P_{1x}, P_4, P_{4x}, P_5, P_{5x} \in H^1$. For simplicity, we will concentrate on the estimates for P_{1x}, P_{4x}, P_{5x}. We can estimate the other terms in the same way. According to (3.4), we have

$$|P_{1x}(\xi)| \le \frac{q^+}{2}\Gamma * \left|m\cos^2\frac{v(\xi)}{2}\right|(\xi),$$
$$|P_{4x}(\xi)| \le \frac{q^+}{2}\Gamma * \left|\frac{1}{2}\sin vu - mu\cos^2\frac{v(\xi)}{2}\right|(\xi),$$
$$|P_{5x}(\xi)| \le \frac{q^+}{2}\Gamma * \left|P_3\cos^2\frac{v(\xi)}{2}\right|(\xi).$$

Using Young's inequality, we get

$$\|P_{1x}(\xi)\|_{L^2} \le \frac{q^+}{2}\|\Gamma\|_{L^1}\|m\|_{L^2} < \infty,$$
$$\|P_{4x}(\xi)\|_{L^2} \le \frac{q^+}{2}\|\Gamma\|_{L^1}(\|v\|_{L^2} + \|m\|_{L^2})\|u\|_{L^\infty},$$
$$\|P_{5x}(\xi)\|_{L^2} \le \frac{q^+}{2}\|\Gamma\|_{L^1}\|P_3\|_{L^2} < \infty.$$

We need to prove $\partial_\xi P_1, \partial_\xi P_{1x} \in L^2$; then, we have $u \in H^1$, which implies $u \in L^\infty$. For simplicity, we only concentrate on the estimates for $\partial_\xi P_{ix}$. From (3.4), we deduce that

$$\begin{cases} \partial_\xi P_{1x} = qm\cos^2\frac{v}{2} + qP_1\cos^2\frac{v}{2}, \\ \partial_\xi P_{4x} = qu\left(\frac{1}{2}\sin v - m\cos^2\frac{v}{2}\right) + qP_4\cos^2\frac{v}{2}, \\ \partial_\xi P_{5x} = qP_3\cos^2\frac{v}{2} + qP_5\cos^2\frac{v}{2}. \end{cases} \tag{3.8}$$

We thus get

$$\|\partial_\xi P_{1x}(\xi)\|_{L^2} \le q^+(\|P_1\|_{L^2} + \|m\|_{L^2}) < \infty,$$
$$\|\partial_\xi P_{4x}(\xi)\|_{L^2} \le q^+(\|P_4\|_{L^2} + \|u\|_{L^2} + \|u\|_{L^2}\|m\|_{L^\infty}),$$
$$\|\partial_\xi P_{5x}(\xi)\|_{L^2} \le q^+(\|P_5\|_{L^2} + \|P_3\|_{L^2}) < \infty.$$

Hence, $P_{1x}, P_{5x} \in H^1$; then, we obtain that $u \in L^\infty$. From the above equality, we have $P_{4x} \in H^1$.

Second, we need to show that all partial derivatives are uniformly bounded from Ω to H^1. In the following, we only consider $\frac{\partial P_{1x}}{\partial m}$; all the other cases can be estimated in the same way. More details can refer to Ref. 3.

For $(m, v, q) \in \Omega$, we can define the linear operator $\frac{\partial P_{1x}}{\partial m} : H^1 \mapsto L^2$ as

$$\left[\frac{\partial P_{1x}}{\partial m}\cdot\hat{m}\right] = -\frac{1}{2}\left(\int_\xi^{+\infty} - \int_{-\infty}^{\xi}\right) e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\,ds|} q\cos^2\frac{v}{2}\cdot\hat{m}\,d\xi',$$

with $\hat{m} \in H^1$. We can deduce that

$$\left\|\frac{\partial P_{1x}}{\partial m}\cdot\hat{m}\right\|_{L^2} \le \|\Gamma\|_{L^1} q^+ \|\hat{m}\|_{L^2},$$

which implies that

$$\left\|\frac{\partial P_{1x}}{\partial m}\right\| \le \|\Gamma\|_{L^1} q^+.$$

Using (3.8), we can define the linear operator $\frac{\partial(\partial_\xi P_{1x})}{\partial m} : H^1 \mapsto L^2$ as

$$\begin{aligned}\left[\frac{\partial(\partial_\xi P_{1x})}{\partial m}\cdot\hat{m}\right] &= q\cos^2\frac{v(\xi)}{2}(\xi)\cdot\hat{m} \\ &\quad + \frac{1}{2}q\cos^2\frac{v(\xi)}{2}(\xi)\left(\int_\xi^{+\infty} - \int_{-\infty}^{\xi}\right) e^{-|\int_{\xi'}^{\xi} q\cos^2\frac{v}{2}\,ds|}\,\mathrm{sign}(\xi' - \xi) q\cos^2\frac{v}{2}\cdot\hat{m}\,d\xi'.\end{aligned}$$

Therefore,

$$\left\|\frac{\partial(\partial_\xi P_{1x})}{\partial m}\cdot\hat{m}\right\|_{L^2} \le q^+\|\hat{m}\|_{L^2} + (q^+)^2\|\Gamma\|_{L^1}\|\hat{m}\|_{L^2},$$

which implies that

$$\left\|\frac{\partial(\partial_\xi P_{1x})}{\partial m}\right\| \le q^+ + (q^+)^2\|\Gamma\|_{L^1}.$$

Finally, there exists time $T > 0$ so that the semi-linear system [(3.6) and (3.7)] has a unique solution in $C([0, T]; X)$.

Step 2 (Extension to a global solution)

We shall extend the local solution constructed above to a global solution if

$$\|m\|_{H^1} + \|v\|_{L^2} + \|v\|_{L^\infty} + \|q\|_{L^\infty} + \left\|\frac{1}{q}\right\|_{L^\infty} \tag{3.9}$$

is uniformly bounded $[0, T]$ with any $T > 0$. The proof is actually relying on the conservation law (2.3) and a contradiction argument. We will re-derive the conservation law under the new variables. We first claim that

$$m_\xi = \frac{q}{2}\sin v. \tag{3.10}$$

In fact, (3.6) yields

$$\begin{aligned}\left(\frac{q}{2}\sin v\right)_t &= \frac{q_t}{2}\sin v + \frac{q}{2}v_t\cos v\\ &= \frac{q}{4}\sin^2 v(2H+1) + \frac{q}{2}N\cos v\sin v + \frac{q}{2}\cos v\left(2H\cos^2\frac{v}{2} - N\sin v - \sin^2\frac{v}{2}\right)\\ &= 2H\left(\frac{q}{4}\sin^2 v + \frac{q}{2}\cos v\cos^2\frac{v}{2}\right) + \frac{q}{4}\sin^2 v - \frac{q}{2}\cos v\sin^2\frac{v}{2}\\ &= qH\cos^2\frac{v}{2} + \frac{q}{2}\sin^2\frac{v}{2}.\end{aligned}$$

To calculate $m_{\xi t} = F_\xi$, we will give some basic calculations result about P_i, P_{ix}, which is similar to (3.8),

$$\begin{cases}\partial_\xi P_i = qP_{ix}\cos^2\frac{v}{2},\\ \partial_\xi P_{2x} = qP_2\cos^2\frac{v}{2} + qm^2\cos^2\frac{v}{2},\\ \partial_\xi P_{3x} = qP_3\cos^2\frac{v}{2} + q\sin^2\frac{v}{2}.\end{cases} \tag{3.11}$$

Since $u_\xi = m_\xi + qN\cos^2\frac{v}{2}$, then we get

$$\begin{aligned}m_{\xi t} &= F_\xi\\ &= m_\xi u + mm_\xi + mqN\cos^2\frac{v}{2} - q\cos^2\frac{v}{2}(P_4 + P_{4x}) - \frac{q}{2}u\sin v + mqu\cos^2\frac{v}{2} - mm_\xi + \frac{q}{2}\sin^2\frac{v}{2}\\ &\quad + \frac{q}{2}\cos^2\frac{v}{2}(P_3 + P_{3x}) + \lambda\left(mq\cos^2\frac{v}{2} + qP_1\cos^2\frac{v}{2}\right) + \frac{q}{2}P_5\cos^2\frac{v}{2} + \frac{q}{2}P_3\cos^2\frac{v}{2} - \frac{q}{2}P_{2x}\cos^2\frac{v}{2}\\ &= qH\cos^2\frac{v}{2} + \frac{q}{2}\sin^2\frac{v}{2} + \left(m_\xi - \frac{q}{2}\sin v\right)u.\end{aligned}$$

Since $(m_\xi - \frac{q}{2}\sin v)_t = (m_\xi - \frac{q}{2}\sin v)u$ and $(m_\xi - \frac{q}{2}\sin v)(t,0) = 0$, then we complete the proof of (3.10). Next, we infer that

$$E(t) = \int_{\mathbb{R}}\left(m^2\cos^2\frac{v}{2} + \sin^2\frac{v}{2}\right)q\,d\xi = E(0) = E_0, \tag{3.12}$$

which is exactly the expression for the conservation law (2.3) in the new variable. We deduce that $u + N = \frac{1}{2\lambda}(P_3 + P_{3x} - m^2)$ and $N_\xi = \frac{1}{2\lambda}\left(q\sin^2\frac{v}{2} + q\cos^2\frac{v}{2}(P_3 + P_{3x})\right) - \frac{1}{\lambda}mm_\xi - m_\xi - qN\cos^2\frac{v}{2}$. Then, we can prove that

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{\mathbb{R}} q_t\left(m^2\cos^2\frac{v}{2} + \sin^2\frac{v}{2}\right) + 2mqF\cos^2\frac{v}{2} + qv_t\left(-\frac{m^2}{2}\sin v + \frac{1}{2}\sin v\right)d\xi\\ &= \int_{\mathbb{R}} -2mH_\xi + 2mqF\cos^2\frac{v}{2} + qm^2N\cos^2\frac{v}{2} - qN\sin^2\frac{v}{2} + m^2m_\xi d\xi\\ &= \int_{\mathbb{R}} -2mF_\xi - m^2N_\xi - 2\lambda mu_\xi + qm^2N\cos^2\frac{v}{2} - qN\sin^2\frac{v}{2} + 2mqF\cos^2\frac{v}{2}d\xi\\ &= \int_{\mathbb{R}} -2mqu\lambda\cos^2\frac{v}{2} - qm^3\cos^2\frac{v}{2} - 2mqN\lambda\cos^2\frac{v}{2} - qN\sin^2\frac{v}{2} - mq\sin^2\frac{v}{2} - \frac{1}{2\lambda}qm^2\left(\cos^2\frac{v}{2}(P_3 + P_{3x}) + \sin^2\frac{v}{2}\right)d\xi\\ &= \int_{\mathbb{R}} -(P_3 + P_{3x})mq\cos^2\frac{v}{2} - qN\sin^2\frac{v}{2} - mq\sin^2\frac{v}{2} - \frac{1}{2\lambda}qm^2\left(\cos^2\frac{v}{2}(P_3 + P_{3x}) + \sin^2\frac{v}{2}\right)d\xi\\ &= \int_{\mathbb{R}}(-P_1 + P_{1x})q\sin^2\frac{v}{2} - qN\sin^2\frac{v}{2} - mq\sin^2\frac{v}{2} + \frac{1}{2\lambda}q\sin^2\frac{v}{2}(-P_2 + P_{2x}) - \frac{1}{2\lambda}qm^2\sin^2\frac{v}{2}d\xi\\ &= \int_{\mathbb{R}} -\frac{1}{2\lambda}qP_{3x}\sin^2\frac{v}{2}d\xi.\end{aligned}$$

Set $I = \int_{\mathbb{R}} -\frac{1}{2\lambda}qP_{3x}\sin^2\frac{v}{2}d\xi$. Applying Fubini's theorem, we have

$$I = \int_{\mathbb{R}} -\frac{1}{2\lambda}q(\xi)\sin^2\frac{v}{2}(\xi)\int_{\mathbb{R}} e^{-\left|\int_{\xi'}^{\xi}\cos^2\frac{v}{2}q\,ds\right|}q(\xi')\sin^2\frac{v}{2}(\xi')\,\mathrm{sign}(\xi' - \xi)d\xi'd\xi = -I.$$

It means that $\frac{d}{dt}E(t) = I = 0$; we complete the proof of (3.12). As long as the solution is defined, using (3.10) and (3.12), we have

$$\sup_{\xi\in R}|m^2(t,\xi)| \le 2\int_{\mathbb{R}}|mm_\xi|d\xi \le 2\int_{\mathbb{R}}|m\|\cos\frac{v}{2}\sin\frac{v}{2}|qd\xi \le E_0, \tag{3.13}$$

which implies the uniform boundedness of m. From (3.12), we deduce that

$$\|P_2\|_{L^\infty}, \quad \|P_{2x}\|_{L^\infty}, \quad \|P_3\|_{L^\infty}, \quad \|P_{3x}\|_{L^\infty} \le \frac{1}{2}E_0. \tag{3.14}$$

However, we cannot get L^∞ boundedness of other terms from (3.12), for example, P_1. Using variable transformations and contradiction argument will overcome the difficulty. Here are the variable transformation lemma and corollary, which can be referenced to Ref. 25.

Lemma 3.2. If $g(x)$ is differentiable on $a.e.[a,b]$, $f(x)\in L^1[c,d]$, and $g([a,b])\subset[c,d]$, then we have that $F(g(t))$ is absolutely continuous on $[a,b]$ if and only if $f(g(t))g'(t)\in L^1[c,d]$ and $\int_{g(a)}^{g(b)}f(x)dx = \int_a^b f(g(t))g'(t)dt$, with $F(x) = \int_c^x f(t)dt$.

Corollary 3.3. Let $g(x)$ be absolutely continuous on $[a,b]$, $f(x)\in L^1[c,d]$, and $g([a,b])\subset[c,d]$. If $g(x)$ is monotonous or $f(x)\in L^\infty[c,d]$, then we get $\int_{g(a)}^{g(b)}f(x)dx = \int_a^b f(g(t))g'(t)dt$.

First, we define $y_t = u(t,\xi)$, with $y(0,\xi) = \bar{y}(\xi)$. From (3.1), we obtain that $y(0,\xi)\in L^\infty_{loc}$ is strictly monotonous and

$$|\bar{y}(\xi_1) - \bar{y}(\xi_2)| = \left|\int_{\bar{y}(\xi_2)}^{\bar{y}(\xi_1)} 1dx\right| \le \left|\int_{\bar{y}(\xi_2)}^{\bar{y}(\xi_1)} 1 + \bar{m}_x^2 dx\right| \le |\xi_1 - \xi_2|.$$

Then, we can know that $y(0,\xi)$ is a local Lipschitz continuous function. From step 1, we have deduced that $u(t,\xi)$ is Lipschitz continuous as it maps from Ω to H^1. Using (3.2), there exists $T\in(0,\infty)$ such that $y(t,\xi)\in H^1_{loc}$ for $t\in[0,T)$. It is obvious that $y(t,\xi)$ is a local absolutely continuous function for $t\in[0,T)$. The key point is to prove that T is any positive number.

Second, we claim that

$$y_\xi = q\cos^2\frac{v}{2}. \tag{3.15}$$

According to (3.8) and (3.11), we get

$$\partial_t y_\xi = u_\xi = m_\xi + qN\cos^2\frac{v}{2}.$$

In fact, (3.6) and (3.10) yield

$$\begin{aligned}\left(q\cos^2\frac{v}{2}\right)_t &= q_t\cos^2\frac{v}{2} - qv_t\cos\frac{v}{2}\sin\frac{v}{2}\\ &= \left(\frac{q}{2}\sin v(2H+1) + qN\cos v\right)\cos^2\frac{v}{2} - \frac{q}{2}\sin v\left(2H\cos^2\frac{v}{2} - N\sin v - \sin^2\frac{v}{2}\right)\\ &= \frac{q}{2}\sin v\left(\cos^2\frac{v}{2} + N\sin v + \sin^2\frac{v}{2}\right) + qN\cos v\cos^2\frac{v}{2}\\ &= m_\xi + qN\cos^2\frac{v}{2}.\end{aligned}$$

Since $(y_\xi - q\cos^2\frac{v}{2})(t,0) = 0$, then (3.15) remains valid for all times $t\in[0,T)$.

Third, for $t\in[0,T)$ and $[a,b]\subset\mathbb{R}$, using corollary (3.3), we have

$$\left\|\frac{1}{2}\int_a^b e^{-|\int_{\xi'}^{\xi}\cos^2\frac{v}{2}qds|}|m|\cos^2\frac{v}{2}qd\xi'\right\|_{L^\infty} \le \frac{1}{2}\|m\|_{L^\infty}\left\|\int_a^b e^{-|y(\xi)-y(\xi')|}y_\xi d\xi'\right\|_{L^\infty} \tag{3.16}$$

$$\le \frac{1}{2}\|m\|_{L^\infty}\int_{-\infty}^{+\infty}e^{-|s|}ds. \tag{3.17}$$

When $a\to-\infty$ and $b\to+\infty$, the left-hand side of (3.16) is monotonous. According to the monotone convergence theorem, we deduce that there exists a limit of the left-hand side for (3.16). Then, we know that $\|P_1\|_{L^\infty}$ exists and $\|P_1\|_{L^\infty} \le \frac{1}{2}\|m\|_{L^\infty}\int_{-\infty}^{+\infty}e^{-|s|}ds \le CE_0^{\frac{1}{2}}$.

The estimate for $\|P_{1x}\|_{L^\infty}$ is similar. Now, we can easily get that $\|u\|_{L^\infty}, \|N\|_{L^\infty} \le C(E_0^{\frac{1}{2}} + E_0)$ and $\|P_5\|_{L^\infty}, \|P_{5x}\|_{L^\infty} \le CE_0$. Since (3.12), then we have

$$\begin{aligned}\|P_4\|_{L^\infty}, \|P_{4x}\|_{L^\infty} &\le C\|u\|_{L^\infty}\| \int_{-\infty}^{+\infty} e^{-|\int_{\xi'}^{\xi}\cos^2\frac{v}{2}q\mathrm{d}s|}\Big(\sin^2\frac{v}{2}q + (|m|+1)\cos^2\frac{v}{2}q\Big)\mathrm{d}\xi'\|_{L^\infty}\\ &\le C\Big(E_0^{\frac{1}{2}} + E_0\Big)\Big(E_0 + E_0^{\frac{1}{2}} + 1\Big)\\ &\le C\Big(E_0^{\frac{1}{2}} + E_0 + E_0^{\frac{3}{2}} + E_0^2\Big).\end{aligned}$$

Therefore, $\|F\|_{L^\infty}, \|H\|_{L^\infty} \le C(E_0^{\frac{1}{2}} + E_0 + E_0^{\frac{3}{2}} + E_0^2)$. Using (3.6), we get

$$\begin{aligned}|q_t| &\le C\Big(E_0^{\frac{1}{2}} + E_0 + E_0^{\frac{3}{2}} + E_0^2\Big)q,\\ e^{-C\left(E_0^{\frac{1}{2}}+E_0+E_0^{\frac{3}{2}}+E_0^2\right)t} &\le q \le e^{C\left(E_0^{\frac{1}{2}}+E_0+E_0^{\frac{3}{2}}+E_0^2\right)t}\end{aligned} \tag{3.18}$$

for $t \in [0, T)$. Using (3.15), then we have

$$\|y_\xi\|_{L^\infty} \le \|q\|_{L^\infty} \le e^{C\left(E_0^{\frac{1}{2}}+E_0+E_0^{\frac{3}{2}}+E_0^2\right)T}.$$

Using the uniform bound $\|u\|_{L^\infty} \le C\Big(E_0^{\frac{1}{2}} + E_0\Big)$, we obtain the estimate

$$\bar{y}(\xi) - C\Big(E_0^{\frac{1}{2}} + E_0\Big)t \le y(t,\xi) \le \bar{y}(\xi) + C\Big(E_0^{\frac{1}{2}} + E_0\Big)t. \tag{3.19}$$

Therefore, $y(t,\xi) \in L^\infty_{loc}$ for $t \in [0, T]$; then, we have $y(t,\xi) \in H^1_{loc}$ for $t \in [0, T]$.

Finally, using contradiction argument, we can see that T in the above results cannot have a upper bound, which means that the above results are valid for every $t \in \mathbb{R}$.

According to (3.6), we obtain

$$\|v\|_{L^\infty} \le e^{Bt},$$

for $B = B(E_0) > 0$. Using (3.6), we can easily deduce that

$$\frac{d}{dt}\|m\|_{L^2}^2 = \int_{\mathbb{R}} 2mF\mathrm{d}\xi \le 2\|m\|_{L^2}\|F\|_{L^2},$$

with

$$\|F\|_{L^2} \le C(\|m\|_{L^2}\|u\|_{L^\infty} + \|P_4 + P_{4x}\|_{L^2} + \|P_3 + P_{3x}\|_{L^2} + \|P_5 + P_{5x}\|_{L^2} + \|P_2\|_{L^2} + \|P_{1x}\|_{L^2} + \|m\|_{L^2}\|m\|_{L^\infty}).$$

To estimate $\|m\|_{L^2}$, we need to estimate the L^2 norms of P_i and P_{ix}. Let κ be the right-hand side of (3.18). Then, we have $\kappa^{-1} \le q(t) \le \kappa$ and

$$\begin{aligned}\|P_2\|_{L^2} + \|P_{2x}\|_{L^2} + \|P_3\|_{L^2} + \|P_{3x}\|_{L^2} &\le C\|\Gamma * \Big(q\sin^2\frac{v}{2} + m^2 q\cos^2\frac{v}{2}\Big)\|_{L^2} \le C\|\Gamma\|_{L^2}E_0,\\ \|P_1\|_{L^2} + \|P_{1x}\|_{L^2} &\le C\|\Gamma * \Big(mq\cos^2\frac{v}{2}\Big)\|_{L^2} \le C\kappa\|\Gamma\|_{L^1}\|m\|_{L^2},\end{aligned}$$

where

$$\Gamma(\xi) = \min\Big\{1, e^{18\kappa^{-1}E_0 - \frac{|\xi|}{2}\kappa^{-1}}\Big\}.$$

Similar to step 1, we can easily deduce that $\Gamma \in L^1 \cap L^\infty$. Thus, we have

$$\begin{aligned}\|P_4\|_{L^2} + \|P_{4x}\|_{L^2} &\le C\|\Gamma * \Big(\frac{q}{2}u\sin v - muq\cos^2\frac{v}{2}\Big)\|_{L^2} \le C\kappa\|\Gamma\|_{L^1}\|u\|_{L^2}(1 + \|m\|_{L^\infty}),\\ \|P_5\|_{L^2} + \|P_{5x}\|_{L^2} &\le C\|\Gamma * \Big(P_3 q\cos^2\frac{v}{2}\Big)\|_{L^2} \le C\kappa\|\Gamma\|_{L^1}\|P_3\|_{L^2},\end{aligned}$$

with

$$\|u\|_{L^2} \leq C\|m\|_{L^2}(1+\kappa\|\Gamma\|_{L^1}) + C\|\Gamma\|_{L^2}E_0.$$

In a word, we get

$$\frac{d}{dt}\|m\|_{L^2}^2 \leq C(E_0,\kappa)(\|m\|_{L^2}^2 + \|m\|_{L^2}).$$

Appling Gronwall's inequality, we finally have an estimate on $\|m\|_{L^2}$. Hence, we get an estimate on $\|P_i\|_{L^2}$, $\|P_{ix}\|_{L^2}$, $\|u\|_{L^2}$, $\|N\|_{L^2}$, $\|F\|_{L^2}$, and $\|H\|_{L^2}$. By the second equation in (3.6), it is now clear that

$$\begin{aligned}\frac{d}{dt}\|v\|_{L^2}^2 &\leq 2\|v\|_{L^2}\|2H\cos^2\frac{v}{2} - N\sin v - \sin^2\frac{v}{2}\|_{L^2}\\ &\leq C(E_0,\kappa)(\|v\|_{L^2}^2 + \|v\|_{L^2}).\end{aligned}$$

Appling Gronwall's inequality, we obtain an estimate on $\|v\|_{L^2}$. Finally, we only need to get an estimate on $\|m_\xi\|_{L^2}$,

$$\|m_\xi\|_{L^2}^2 = \|\frac{q}{2}\sin v\|_{L^2}^2 \leq \frac{\kappa^2}{4}\|v\|_{L^2}^2.$$

We thus complete the Proof of Theorem 3.1. □

For future use, we give an important property of the above solutions. Let us introduce the set

$$\Phi = \{t \geq 0 : meas\{\xi \in \mathbb{R} : v(t,\xi) = -\pi\} > 0\}. \tag{3.20}$$

We claim that

$$meas(\Phi) = 0. \tag{3.21}$$

Indeed, if $v(t,\xi) = -\pi$, then we have $v_t = -1$ by (3.6). Since $\|H\|_{L^\infty}$ and $\|N\|_{L^\infty}$ is finite for any $t \in R^+$, we get $v_t < -\frac{1}{2}$ whenever $\cos^2\frac{v}{2}$, $\sin v < \delta$ with $\delta > 0$. Since $\|v(t)\|_{L^2}$ is finite for any $t \in R^+$, then we can prove that the map $t \to v(t,\xi)$ is absolute continuous. This implies $v_t = 0$ on a.e. $\{v(t,\xi) = -\pi\}$. Therefore, we get (3.21) by contradiction argument.

IV. GLOBALLY CONSERVATIVE SOLUTIONS FOR THE ORIGINAL EQUATION

In this section, we use the global solutions of system (3.6) to construct globally conservative weak solution to the original equation (1.3) in the original variables (t,x).

Proof of Theorem 1.3. Given is a global solution (y,m,v,q) to system (3.6). Hence, the map $t \mapsto y(t,\xi)$ gives a solution to the following problem:

$$y_t(t,\xi) = u(t,\xi), \quad y(0,\xi) = \bar{y}(\xi), \tag{4.1}$$

where $u(t,\xi) = \big(m - P_{1x} + P_1 + \frac{1}{2\lambda}(P_2 + P_3 - P_{2x})\big)(t,\xi)$ and P_i, P_{ix} is defined in (3.4). Write

$$m(t,x) = m(t,\xi) \qquad \text{if} \quad x = y(t,\xi). \tag{4.2}$$

We have to explain such that the definition makes sense. Using (3.1) and (3.19), we obtain

$$\lim_{\xi\to\pm\infty} y(t,\xi) = \pm\infty.$$

Since (3.15), we deduce that $y_\xi(t,\xi) \geq 0$ for all $t \geq 0$ and a.e. ξ. Therefore, the map $\xi \mapsto y(t,\xi)$ is nondecreasing. Moreover, if $\xi_1 < \xi_2$ but $y(t,\xi_1) = y(t,\xi_2)$, we have

$$0 = \int_{\xi_1}^{\xi_2} y_\xi(t,\eta)\mathrm{d}\eta = \int_{\xi_1}^{\xi_2}\Big(q\cos^2\frac{v}{2}\Big)(t,\eta)\mathrm{d}\eta.$$

Then, we get $\cos\frac{v}{2} = 0$. By (3.10), we have

$$m(t,\xi_2) - m(t,\xi_1) = \int_{\xi_1}^{\xi_2}\Big(\frac{q}{2}\sin v\Big)(t,\eta)\mathrm{d}\eta = 0.$$

Hence, the map $(t,x) \mapsto m(t,x)$ is well-defined for any $t \geq 0$ and $x \in \mathbb{R}$.

From definition (4.2), we give

$$m_x(t,y(t,\xi)) = \frac{\sin v(t,\xi)}{1+\cos v(t,\xi)} \qquad \text{if} \quad x = y(t,\xi), \quad \cos v(t,\xi) \neq -1. \tag{4.3}$$

Changing the variables and applying (3.12) and (4.3), we find that

$$E(t) = \int_{\mathbb{R}} m^2(t,x) + m_x^2(t,x)\mathrm{d}x = \int_{\{\cos\ v\neq -1\}} \big(m^2(t,y(t,\xi)) + m_x^2(t,y(t,\xi))\big)y_\xi \mathrm{d}\xi$$
$$= \int_{\{\cos\ v\neq -1\}} \left(m^2\cos^2\frac{v}{2} + \sin^2\frac{v}{2}\right)q(t,\xi)\mathrm{d}\xi \le E_0. \tag{4.4}$$

By $H^1 \hookrightarrow C^{0,\frac{1}{2}}$, we get $m(t,\cdot) \in C^{0,\frac{1}{2}}$ for any fixed $t \in \mathbb{R}$. According to the first equation in (3.6) and the bound for F, it follows that the map $t \mapsto m(t,y(t))$ is uniformly Lipschitz continuous along every characteristic curve $t \mapsto y(t)$. Therefore, $m = m(t,x)$ is globally Hölder continuous.

Now, we claim that the map $t \mapsto m(t)$ is Lipschitz continuous with values in L^2. Considering any time interval $[\tau, \tau+h]$ and given x, we choose $\xi \in \mathbb{R}$ such that the function $t \mapsto y(t,\xi)$ passes through (τ, x). Using (3.6) and (3.13), we obtain

$$|m(\tau+h,x) - m(\tau,x)| \le |m(\tau+h,x) - m(\tau+h,y(\tau+h,\xi))| + |m(\tau+h,y(\tau+h,\xi)) - m(\tau,x)|$$
$$\le \sup_{|y-x|\le E_0^{\frac{1}{2}} h} |m(\tau+h,y) - m(\tau+h,x)| + \int_\tau^{\tau+h} |F|\mathrm{d}t.$$

Integrating over $\mathbb{R}$. By (4.4) and the bound for $\|F(t)\|_{L^2}$, we get

$$\|m(\tau+h,x) - m(\tau,x)\|_{L^2}^2 \le 2\int_{\mathbb{R}} \left(\int_{x-E_0^{\frac{1}{2}}}^{x+E_0^{\frac{1}{2}}} |m_x(\tau+h,y)|\mathrm{d}y\right)^2 \mathrm{d}x + 2h\|q(\tau)\|_{L^\infty} \int_\tau^{\tau+h} \|F(t)\|_{L^2}^2 \mathrm{d}t$$
$$\le 8E_0h^2\|m_x(\tau+h)\|_{L^2}^2 + 2h\|q(\tau)\|_{L^\infty}\int_\tau^{\tau+h}\|F(t)\|_{L^2}^2\mathrm{d}t$$
$$\le Ch^2.$$

We thus complete the proof of the claim. Using the fact that L^2 is a reflexive space, we know that the left-hand side of (1.5) is a well-defined function and the right-hand side of (1.5) also lies in L^2 for *a.e.t* $\in \mathbb{R}$. By (3.6), we have

$$\frac{d}{dt} m(t,y(t,\xi)) = F(t,\xi),$$

where F is the function defined at (3.5). On the other hand, for every $t \notin \Phi$, we deduce that the map $t \mapsto x(t,\xi)$ is one to one. Using (3.10) and (3.15), we get

$$P_{1x}(t,\xi) = -\frac{1}{2}\left(\int_\xi^{+\infty} - \int_{-\infty}^{\xi}\right) e^{-|\int_{\xi'}^{\xi}\cos^2\frac{v}{2}q\mathrm{d}s|}\, m\cos^2\frac{v}{2} q\mathrm{d}\xi'$$
$$= \frac{1}{2}\left(\int_{y(t,\xi)}^{+\infty} - \int_{-\infty}^{y(t,\xi)}\right) e^{-|y(t,\xi)-x|}\, m(t,x)\mathrm{d}x$$
$$= P_{1x}(t,y(t,\xi))$$

for every $\xi \in \mathbb{R}$. Similarly, we deduce that $F(t,\xi) = F(t,y(t,\xi))$. According to (3.21), we know that Eq. (1.5) is satisfied for almost every $t \in \mathbb{R}$. Using (3.12) and (4.4), we can present the conservation property. This completes the Proof of Theorem 1.3.

□

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DATA AVAILABILITY

The data that support the findings of this study are available on citation and from the corresponding author upon reasonable request.

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