8-1968

A Matrix Application to Systems of n Linear and Non-Linear Homogenous 1st Order Differential Equations

William Loy Lester
Prairie View A&M College

A MATRIX APPLICATION TO SYSTEMS OF n LINEAR AND NON-LINEAR HOMOGENEOUS 1st ORDER DIFFERENTIAL EQUATIONS

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1968

A MATRIX APPLICATION TO SYSTEMS OF n LINEAR AND

NON-LINEAR HOMOGENEOUS 1st ORDER DIFFERENTIAL EQUATIONS

A Thesis

Presented to

the Faculty of the Department of Mathematics

in the

Graduate Division

Prairie View Agricultural and Mechanical College

In Partial Fulfillment

of the Requirement for the Degree

Master of Science

by

William Loy Lester

August, 1968

This Thesis for the Degree

Master of Science

by

William Loy Lester

Has been approved for the

Department of Mathematics

by

[redacted]

(Advisor)

[redacted]

(Head of Mathematics Department)

[redacted]

(Committee Member)

[redacted]

(Committee Member)

[redacted]

(Committee Member)

7-24-68

Date

D E D I C A T I O N

This paper is dedicated to my wife, Virda and my parents, Mr. & Mrs. A. W. Lester who have been an inspiration to me throughout my graduate study.

W.L.L.

A C K N O W L E D G E M E N T

I wish to express my sincere gratitude to

Dr. A. D. Stewart

for his many helpful conversations, suggestions, criticisms and for his encouragement throughout my graduate study. Dr. Stewart is truly a great mathematician and an excellent teacher.

W.L.L.

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CHAPTER I

INTRODUCTION AND TERMINOLOGY

The theory of differential equations constitute a large and very important branch of mathematical analysis. Differential equations occur in connection with numerous problems which are encountered in various branches of science and engineering. In the study of differential equations one has been able to find solutions to some very important problems such as: the problem of determining the motion of a projecticle rocket, satellete, the change of current in an electric circuit and the reaction of chemicals.

Let us consider the system of n linear homogeneous 1st order differential equations:

$$y_1'(x) = a_{11}(x)y_1(x) + a_{12}(x)y_2(x) + \cdots + a_{1n}(x)y_n(x)$$

$$y_2'(x) = a_{21}(x)y_1(x) + a_{22}(x)y_2(x) + \cdots + a_{2n}(x)y_n(x)$$

$$\vdots$$

$$y_n'(x) = a_{n1}(x)y_1(x) + a_{n2}(x)y_2(x) + \cdots + a_{nn}(x)y_n(x)$$

this system can compactly be written in vector and matrix notation as $Y'(x) = A(x)Y(x)$, where $Y'(x)$ and $Y(x)$ are n-dimensional vector functions and $A(x)$ is an nxn matrix function.

In recent years, matrices have become very useful in the study of differential equations.

The aim of this paper is to demonstrate the application of matrices to the existence and uniqueness of solutions to systems of differential equations. This matrix theory will be applied to both the linear and non-linear homogeneous 1st order cases.

It is well known that under certain conditions the linear scalar differential equation $y'(x) = ay(x)$, $y(x_o) = y_o$ and the non-linear scalar $y'(x) = f(x,y(x))$, $y(x_o) = y_o$ have a solution and the solution is unique. The matrix and vector form of the system suggest a solution similar to the solution of a single differential equation.

Some of the basic definitions and terminology to be used in this paper are as follows:

Symbols and/or Definitions	Meaning
(1) A differential equation	An equation involving x, y(x) and the derivatives of y(x).
(2) Linear differential equation	A differential equation such that (a) y(x) is of 1st degree (b) all the derivatives of y(x) are of 1st degree (c) no product of y(x) and the derivatives of y(x) occur.
(3) A non-linear differential equation	A differential equation that is not linear.
(4) The degree of a differential equation	The degree of the highest ordered derivative appearing after all fractional exponents have been removed.
(5) A solution to a differential equation	(a) An equation free of derivatives (b) An equation that satisfies the differential equation.

(6)	1st order linear homogeneous differential equation	A linear differential equation: $y'(x) = a(x)y(x) + b(x)$ is homogeneous if $b(x)$ is identically zero.
(7)	$f(x)$ is a function	$f(x)$ is a set of ordered number pairs (x,y) such that no two pairs have the same first x number.
(8)	$f(x)$ is continuous at x_o	If $e > 0$, then there exist a $d > 0$, such that if $\lvert x - x_o\rvert < d$, then $\lvert f(x) - f(x_o)\rvert < e$.
(9)	A is a matrix	A is a rectangular array of numbers.
(10)	A is a square matrix	A has the same number of rows as columns.
(11)	$A(x)$ is a matrix function	A is a matrix whose elements are functions
(12)	$V(x)$ is a k-dimensional vector function	$V(x)$ is a kx1 matrix function.
(13)	$A(x)$ is a continuous matrix function on I	A is a matrix whose elements are continuous functions on I.
(14)	I is a closed interval	$I = \{a \le x \le b\}$.
(15)	R is a rectangle	$R = I \times I$.
(16)	$\sum V_n(x)$ converges uniformly on I to $V(x)$	If $e > 0$, there exist a $N > 0$, such that $n > N$, then $\lvert \sum V_n(x) - V(x)\rvert < e$.
(17)	$\sum V_k^n(x)$ is uniformly convergent on I	$\sum V_k^n(t)$ converges uniformly for each k.
(18)	Z is the norm of the matrix A	$Z = \sum_{i,j}^{n} \lvert a_{ij}\rvert$.
(19)	The matrix function $A(x)$ is bounded	There exist a positive number K, such that $\lVert A(x)\rVert \le K$.

In Chapter II we shall state the auxillary theorems. These theorems are used at various points in this paper, however they will not be proved.

In Chapter III we shall state and prove some basic lemmas to be used in this paper.

In Chapter IV we shall state and prove the main theorem.

Chapter V gives an application of a system of two linear 1st order homogeneous differential equations.

CHAPTER II

AUXILLARY THEOREMS

Theorem 1.1

H: $f(x) \in C^0$ on I.

C: $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

Theorem 1.2

H_1: $f(x) \in C^0$ on I

H_2: $F(x) = \int_a^x f(t)\,dt$

C: $F(x) \in C^0$ on I.

Theorem 1.3

H_1: $f_n(x) \in C^0$ on I for each n = 1, 2, 3, ...

H_2: $\sum f_n(x)$ converges uniformly on I to $f(x)$

C: $f(x) \in C^0$ on I.

Theorem 1.4

H_1: $f_n(x)$ is defined for each $n = 1, 2, 3, \ldots$

H_2: $\sum M_n$ converges, $M_n > 0$ for each $n = 1, 2, 3, \ldots$

H_3: $|f_n(x)| \leq M_n$ for each n and for each x on I

C: $\sum f_n(x)$ converges uniformly on I.

Theorem 1.5

H_1: $f_n(x) \in C^0$ on I for each $n = 1, 2, 3, \ldots$

H_2: $\sum f_n(x)$ converges uniformly on I.'

C: $\int_a^b \sum f_n(x)dx = \sum \int_a^b f_n(x)dx.$

Theorem 1.6

H_1: $f_n(x)$ is a sequence of real functions which converges uniformly to f(x) on I.

H_2: $f_n(x)$ is continuous on I for each $n = 1, 2, 3, \ldots$

C: $\lim_n \int_a^b f_n(x)dx = \int_a^b \lim_n f_n(x)dx.$

CHAPTER III

BASIC LEMMAS

Lemma 2.1

H: A and B are nxn matrices

C: $\|A + B\| \leq \|A\| + \|B\|$

Proof:

1. $\|A + B\| = \sum_{ij}^{n} |a_{ij} + b_{ij}|$

2. $|a_{ij} + b_{ij}| \leq |a_{ij}| + |b_{ij}|$ for each $1 \leq i \leq n$ and
$1 \leq j \leq n$

3. $\|A + B\| = \sum |a_{ij} + b_{ij}| \leq \sum (|a_{ij}| + |b_{ij}|)$
$\leq \sum |a_{ij}| + \sum |b_{ij}|$

4. $\|A + B\| \leq \|A\| + \|B\|$ Q.E.D.

Lemma 2.2

H: A and B are nxn matrices

C: $\|A - B\| \leq \|A\| + \|B\|$

Proof:

1. $\|-B\| = \|B\|$

2. Replace B by -B in Lemma 2.1

3. $\|A - B\| \leq \|A\| + \|B\|$ Q.E.D.

Lemma 2.3

H: A and B are nxn matrices

C: $\|AB\| \leq \|A\| \|B\|$

Proof:

1. $\|AB\| = \sum_{i,j,k} |a_{ik}b_{kj}|$

2. $\|AB\| = \sum_{i,k,j}^{n} |a_{ik}| |b_{kj}| \leq \sum_{i,k,j,l}^{n} |a_{ik}| |b_{lj}|$

3. $\|AB\| \leq (\sum_{i,k}^{n} |a_{ik}| \sum_{l,j}^{n} |b_{lj}|)$

4. $\|AB\| \leq \|A\| \|B\|$ Q.E.D.

Lemma 2.4

H: A(x) is a continuous matrix function on I

C: $\|\int_a^b A(x)dx\| \leq \int_a^b \|A(x)\| \ dx$

Proof:

1. $\|\int_a^b A(x)dx\| = \sum |\int_a^b a_{ij}(x)dx|$

2. $|\int_a^b a_{ij}(x)dx| \leq \int_a^b |a_{ij}(x)| \ dx$ By Theorem 1.1

3. $\|\int_a^b A(x)dx\| = \sum |\int_a^b a_{ij}(x)dx| \leq \sum \int_a^b |a_{ij}(x)| \ dx$

$$\leq \int_a^b \sum |a_{ij}(x)| \ dx$$

4. $\|\int_a^b A(x)dx\| \leq \int_a^b \|A(x)\| \ dx.$ Q.E.D.

Lemma 2.5

The following two statements are equivalent:

A: $Y'(x) = A(x)Y(x), \; Y(x_0) = Y_0$

B: $Y(x) = Y_0 + \int_{x_0}^{x} A(t)Y(t)dt$

Proof: A: is equivalent to B:

1. $Y'(x) = A(x)Y(x), \; Y(x_0) = Y_0$

2. $Y(x)\Big|_{x_0}^{x} = \int_{x_0}^{x} A(t)Y(t)dt$ Integrating both sides of 1.

3. $Y(x) - Y(x_0) = \int_{x_0}^{x} A(t)Y(t)dt$

4. $Y(x) = Y(x_0) + \int_{x_0}^{x} A(t)Y(t)dt$

5. $Y(x) = Y_0 + \int_{x_0}^{x} A(t)Y(t)dt$ Q.E.D.

Proof: B: is equivalent to A:

1. $Y(x) = Y_0 + \int_{x_0}^{x} A(t)Y(t)dt$

2. $Y'(x) = 0 + A(x)Y(x)$ By the Fundamental Theorem of Integral Calculus

3. $Y'(x) = A(x)Y(x), \; Y(x_0) = Y_0$ Q.E.D.

Lemma 2.6

H_1: $\sum M_n$ converges, $M_n > 0$

H_2: $V_k^n(x)$ is a k-dimensional vector of functions

H_3: $\| V_k^n(x) \| \leq M_n$ for each $n = 1, 2, 3, \ldots$

C: $\sum V_k^n(x)$ converges uniformly for each k.

Proof:

1. $\| V_k^n(x) \| = |V_1^n(x)| + |V_2^n(x)| + \ldots + |V_k^n(x)| \leq M_n$

2. $|V_k^n(x)| \leq M_n$ for each k^{th} component

3. Hence: $\sum V_k^n(x)$ converges uniformly for each k by Theorem 3.4

Q.E.D.

CHAPTER IV

Theorem I.

H_1: $A(x) \in C^0$ on I

H_2: $Y(x) \in C'$ on I

H_3: $Y(x_0) = Y_0$

C_1: There exist a solution $Y(x)$ which satisfies $Y'(x) = A(x)Y(x)$ such that $Y(x_0) = Y_0$

C_2: The solution is unique.

Proof:

1. $Y'(x) = A(x)\,Y(x),\ Y(x_0) = Y_0$

2. $Y(x) = Y_0 + \int_{x_0}^{x} A(t)Y(t)dt$, Integrating both sides of 1.

3. $Y_1(x) = Y_0 + \int_{x_0}^{x} A(t)Y_0(t)dt$

4. $Y_2(x) = Y_0 + \int_{x_0}^{x} A(t)Y_1(t)dt$

Substituting $Y_0(t)$ for $Y(t)$ in 2. and in general $Y_n(t)$ for $Y_{n-1}(t)$.

5. $Y_3(x) = Y_0 + \int_{x_0}^{x} A(t)Y_2(t)dt$

6. $Y_n(x) = Y_0 + \int_{x_0}^{x} A(t)Y_{n-1}(t)dt$

7. $Y_{n+1}(x) = Y_0 + \int_{x_0}^{x} A(t)Y_n(t)dt$

8. $\|Y_{n+1}(x) - Y_n(x)\| = \|\int_{x_0}^{x} A(t)[Y_n(t) - Y_{n-1}(t)]dt\|$

9. $\|Y_{n+1}(x) - Y_n(x)\| \leq \int_{x_0}^{x} \|A(t)[Y_n(t) - Y_{n-1}(t)]\| dt$ By Lemma 2.4

10. $\|Y_{n+1}(x) - Y_n(x)\| \leq \int_{x_0}^{x} \|A(t)\| \, \|Y_n(t) - Y_{n-1}(t)\| dt$ By Lemma 2.3

11. $\|Y_{n+1}(x) - Y_n(x)\| \leq \int_{x_0}^{x} K \, \|Y_n(t) - Y_{n-1}(t)\| dt$

By H_1 $K > 0$, $\|A(t)\| \leq K$.

12. $\|Y_2(x) - Y_1(x)\| \leq \int_{x_0}^{x} K \|Y_1(t) - Y_o(t)\| dt$

$n = 1$ in 11.

13. $\|Y_2(x) - Y_1(x)\| \leq \int_{x_0}^{x} K(\|Y_1(t)\| + \|Y_o(t)\|) dt$ By Lemma 2.2

14. $\|Y_2(x) - Y_1(x)\| \leq \int_{x_0}^{x} KMdt$

By H_1 $M_1 > 0$ $\|Y_1(t)\| \leq M_1$ and $\|Y_o(t)\|$ is a number

Let $M_1 + \|Y_o(t)\| \leq M$

15. $\|Y_2(x) - Y_1(x)\| \leq KM|x - x_o|$

16. $\|Y_3(x) - Y_2(x)\| \leq \int_{x_0}^{x} K \, \|Y_2(t) - Y_1(t)\| dt$ $n = 2$ in 11.

17. $\|Y_3(x) - Y_2(x)\| \leq \int_{x_0}^{x} K^2 M |x - x_o| dt$ Using 15.

18. $\|Y_3(x) - Y_2(x)\| \leq \dfrac{K^2 M |x - x_o|^2}{2}$

19. $\|Y_4(x) - Y_3(x)\| \leq \int_{x_0}^{x} K \, \|Y_3(t) - Y_2(t)\| dt$ $n = 3$ in 11.

20. $\|Y_4(x) - Y_3(x)\| \leq \int_{x_0}^{x} \frac{K^3M|x - x_0|^2}{2} dt$

21. $\|Y_4(x) - Y_3(x)\| \leq \frac{K^3M|x - x_0|^3}{3!}$

22. $\|Y_{n+1}(x) - Y_n(x)\| \leq \frac{K^nM|x - x_0|^n}{n!}$

23. $Y_n(x) = Y_o(x) + Y_1(x) - Y_o(x) + Y_2(x) - Y_1(x) + \ldots + Y_n(x) - Y_{n-1}(x)$

24. $\|Y_n(x)\| \leq \|Y_o(x)\| + \|Y_1(x) - Y_o(x)\| + \|Y_2(x) - Y_1(x)\| + \ldots$

$+ \|Y_n(x) - Y_{n-1}(x)\|$

Let R be on I such $|x - x_o| \leq R$

Therefore the series in 23. converges uniformly on I and each $Y_n(x)$ is continuous for each $n = 1, 2, 3, \ldots$

It follows that:

25. $\lim_n Y_n(x) = Y_o + \lim_n \int_{x_0}^{x} A(t)Y_{n-1}(t)dt$

26. $\lim_n Y_n(x) = Y_o + \int_{x_0}^{x} A(t)\lim_n Y_{n-1}(t)dt$ By Theorem 1.6

27. $Y(x) = Y_o + \int_{x_0}^{x} A(t)Y(t)dt$

Therefore $Y(x)$ satisfies the integral equation and by Lemma 2.5 $Y(x)$ satisfies the differential equation. Q.E.D.

Proof of C_2:

Assume that $Z(x)$ is a solution to the integral equation in 27, that is, $Z(x) = Y_o + \int_{x_o}^{x} A(t)Z(t)dt$ and consider $Y_n(x) = Y_o + \int_{x_o}^{x} A(t)Y_{n-1}(t)dt$. Here we are using the same method of successive substitution as in the proof of C_1 in Theorem I.

28. $\| Z(x) - Y_n(x) \| \leq \int_{x_o}^{x} \|A(t)\| \cdot \| Z(t) - Y_{n-1}(t)\| \, dt$ By lemmas 2.3 & 2.4

29. $\| Z(x) - Y_n(x) \| \leq \int_{x_o}^{x} K \| Z(t) - Y_{n-1}(t) \| \, dt$ By H_1

30. $\| Z(x) - Y_1(x) \| \leq \int_{x_o}^{x} K \| Z(t) - Y_o(t) \| \, dt$ n = 1 in 29.

31. $\| Z(x) - Y_1(x) \| \leq \int_{x_o}^{x} KMdt$ By the same argument as in 13. & 14.

32. $\| Z(x) - Y_1(x) \| \leq KM|x - x_o|$

33. $\| Z(x) - Y_2(x) \| \leq \int_{x_o}^{x} K \| Z(t) - Y_1(t) \| \, dt$ n = 2 in 29.

34. $\| Z(x) - Y_2(x) \| \leq \int_{x_o}^{x} K^2 M |x - x_o| dt$

35. $\| Z(x) - Y_2(x) \| \leq \dfrac{K^2 M |x - x_o|^2}{2}$

36. $\| Z(x) - Y_n(x) \| \leq \dfrac{K^n M |x - x_o|^n}{n!}$

Let $R > 0$ be on I such that $|x - x_o| \leq R$.

Since the series $\dfrac{M(KR)^n}{n!}$ converges, it follows that $\lim_n \dfrac{M(KR)^n}{n!} = 0$

and $\lim_n Y_n(x) = Y(x)$, therefore $\| Z(x) - Y_n(x) \| = 0$.

This means $Z(x) = Y(x)$, therefore the solution $Y(x)$ is unique. Q.E.D.

Now let us consider the system of n non-linear homogeneous 1st order differential equations.

Theorem II

H_1: $F(x,Y(x)) \in C^o$ on R

H_2: $Y(x) \in C^o$ on I

H_3: $Y(x) \in C'$ on I

H_4: $\| F(x,Y_2(x)) - F(x,Y_1(x)) \| \leq K \| Y_2(x) - Y_1(x) \|$

$K > 0$, $Y_1(x)$ and $Y_2(x) \in R$.

H_5: $Y(x_o) = Y_0$

C_1: There exist a solution which satisfies $Y'(x) = F(x,Y(x))$ such that $Y(x_o) = Y_o$

C_2: The solution is unique.

Proof of C_1:

1. $Y'(x) = F(x,Y(x))$, $Y(x_o) = Y_o$

2. $Y(x) = Y_o + \int_{x_o}^{x} F(t,Y(t))dt$

3. $Y_1(x) = Y_o + \int_{x_o}^{x} F(t,Y_o(t))dt$ Substituting $Y_o(t)$ for $Y(t)$ in 2. and in general $Y_n(t)$ for $Y_{n-1}(t)$.

4. $Y_2(x) = Y_o + \int_{x_0}^{x} F(t, Y_1(t))dt$

5. $Y_3(x) = Y_o + \int_{x_0}^{x} F(t, Y_2(t))dt$

6. $Y_n(x) = Y_o + \int_{x_0}^{x} F(t, Y_{n-1}(t))dt$

7. $Y_{n+1}(x) = Y_o + \int_{x_0}^{x} F(t, Y_n(t))dt$

8. $\| Y_{n+1}(x) - Y_n(x) \| = \| \int_{x_0}^{x} [F(t, Y_n(t)) - F(t, Y_{n-1}(t))]dt \|$

9. $\| Y_{n+1}(x) - Y_n(x) \| \leq \int_{x_0}^{x} \| F(t, Y_n(t)) - F(t, Y_{n-1}(t)) \| \, dt$ By Lemma 2.4

10. $\| Y_2(x) - Y_1(x) \| \leq \int_{x_0}^{x} \| F(t, Y_1(t)) - F(t, Y_o(t)) \| \, dt$

$\leq \int_{x_0}^{x} K \| Y_1(t) - Y_o(t) \| \, dt$ By H_4

11. $\| Y_2(x) - Y_1(x) \| \leq \int_{x_0}^{x} K \| Y_1(t) - Y_o(t) \| \, dt$

By H_2 $M_1 > 0$, $\| Y_1(t) \| \leq M_1$ and $\| Y_o(t) \|$ is a number

Let $M_1 + \| Y_o(t) \| = M$

12. $\| Y_2(x) - Y_1(x) \| \leq \int_{x_0}^{x} KMdt$

13. $\| Y_2(x) - Y_1(x) \| \leq KM|x - x_o|$

14. $\| Y_3(x) - Y_2(x) \| \leq \int_{x_0}^{x} \| F(t, Y_2(t)) - F(t, Y_1(t)) \| \, dt$

$\leq \int_{x_0}^{x} K \| Y_2(t) - Y_1(t) \| \, dt$ $n = 2$ in 9. and H_4

15. $\| Y_3(x) - Y_2(x) \| \leq \int_{x_0}^{x} K^2 M |x - x_o| \, dt$ By using 13.

16. $\|Y_3(x) - Y_2(x)\| \leq \frac{K^2M|x - x_o|^2}{2}$

17. $\|Y_4(x) - Y_3(x)\| \leq \int_{x_o}^{x} \|F(t,Y_3(t)) - F(t,Y_2(t))\| \, dt$

$\leq \int_{x_o}^{x} K \|Y_3(t) - Y_2(t)\| \, dt$ n=3 in 9 and by H_4

18. $\|Y_4(x) - Y_3(x)\| \leq \int_{x_o}^{x} K \|Y_3(t) - Y_2(t)\| \, dt.$

19. $\|Y_4(x) - Y_3(x)\| \leq \int_{x_o}^{x} \frac{K^3M|x - x_o|^2}{2} \, dt$

20. $\|Y_4(x) - Y_3(x)\| \leq \frac{K^3M|x - x_o|^3}{3!}$

21. $\|Y_{n+1}(x) - Y_n(x)\| \leq \frac{K^nM|x - x_o|^n}{n!}$

22. $Y_n(x) = Y_o(x) + Y_1(x) - Y_o(x) + Y_2(x) - Y_1(x) + \cdots + Y_n(x) - Y_{n-1}(x)$

Identity

23. $\|Y_n(x)\| \leq \|Y_o(x)\| + \|Y_1(x) - Y_o(x)\| + \|Y_2(x) - Y_1(x)\| + \cdots$

$+ \|Y_n(x) - Y_{n-1}(x)\|$

Let R be on I such that $|x - x_o| \leq R$

Therefore the series in 22 converges uniformly on I

and each $Y_n(x)$ is continuous for each n = 1, 2, 3, ...

It follows that:

24. $\lim_n Y_n(x) = Y_o + \lim_n \int_{x_o}^{x} F(t, Y_{n-1}(t))dt$

25. $\lim_n Y_n(x) = Y_o + \int_{x_o}^{x} F(t, \lim_n Y_{n-1}(t))dt$

26. $Y(x) = Y_o + \int_{x_o}^{x} F(t, Y(t))dt$

Therefore Y(x) satisfies the integral equation and by Lemma 2.5, Y(x) satisfies the differential equation.

Q.E.D.

Proof of C_2:

Assume that Z(x) is a solution to the integral equation in 26., that is,

$Z(x) = Y_o + \int_{x_o}^{x} F(t, Z(t))dt$ and consider $Y_n(x) = Y_o + \int_{x_o}^{x} F(t, Y_{n-1}(t))dt$. Here we are using the same method of successive substitution as in the proof of C_1 in Theorem II.

27. $\|Z(x) - Y_n(x)\| \leq \int_{x_o}^{x} \| Z(t) - Y_{n-1}(t)\| \, dt$ By Lemma 2.4

28. $\| Z(x) - Y_1(x)\| \leq \int_{x_o}^{x} K\| Z(t) - Y_o(t)\| \, dt$ By H_4

29. $\| Z(x) - Y_1(x)\| \leq \int_{x_o}^{x} KMdt$ By the same argument as in 11.

30. $\| Z(x) - Y_1(x)\| \leq KM|x - x_o|$

31. $\| Z(x) - Y_2(x)\| \leq \int_{x_o}^{x} K \| Z(t) - Y_1(t)\| dt$

32. $\| Z(x) - Y_2(x)\| \leq \int_{x_o}^{x} K^2M|x - x_o| dt$

33. $\| Z(x) - Y_2(x)\| \leq \frac{K^2M|x - x_o|^2}{2!}$

34. $$\|Z(x) - Y_n(x)\| \leq \frac{K^n M |x - x_o|^n}{n!}$$

Let $R > 0$ on I such that $|x - x_o| \leq R$

By the same argument as in 36. of the Proof of C_2 of Theorem I, the solution Y(x) is unique. Q.E.D.

CHAPTER V

APPLICATION

Let us consider a system of two 1st order linear homogeneous differential equations with constant coefficients.

$$y_1'(x) = 2y_1(x) + y_2(x) \quad , \qquad y_1(0) = 2$$

$$y_2'(x) = 3y_1(x) + 4y_2(x) \quad , \qquad y_2(0) = 5$$

In matrix and vector notation we have

$$Y'(x) = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} \quad , \qquad Y(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} .$$

In order to find a solution $Y(x)$ to this system we use the well known fact that the solution is $Y(x) = P\cdot\text{diag}[e^{rx}]C$, where P is a non-singular matrix such that $P^{-1}AP$ is a diagonal matrix (diag), r is given by $(A - rI) = 0$ and C is a column matrix of arbitrary constants.

In order to find the matrix P we use the following method:

$$|A - rI| = \begin{vmatrix} 2-r & 1 \\ 3 & 4-r \end{vmatrix} = 0 \qquad r^2 - 6r + 5 = 0$$

$$r_1 = 1, \quad r_2 = 5$$

$$(A - rI)V = 0$$

for $r_1 = 1$, we get

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} v_1 + v_2 &= 0 \\ 3v_1 + 3v_2 &= 0 \end{aligned} \qquad v_1 = -v_2 \quad \text{Let } v_1 = 1$$

Then $v_2 = -1$

for $r_2 = 5$, we get

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -3v_1 + v_2 &= 0 \\ 3v_1 - v_2 &= 0 \end{aligned} \qquad v_2 = 3v_1 \quad \text{Let } v_1 = 1$$

Then $v_2 = 3$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$Y(x) = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} e^x & 0 \\ 0 & e^{5x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$y_1(x) = c_1e^x + c_2e^{5x}$$

$$y_2(x) = -c_1e^x + 3c_2e^{5x}$$

Using $Y(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ we find c_1 and c_2

$$2 = c_1 + c_2$$
$$5 = -c_1 + 3c_2$$

$$c_1 = 1/4$$
$$c_2 = 7/4$$

therefore

$$y_1(x) = 1/4e^{x} + 7/4e^{5x}$$

$$y_2(x) = -1/4e^{x} + 21/4e^{5x} \quad .$$

BIBLIOGRAPHY

Hochstadt, Harry. Differential Equations. New York: Holt, Rinehart and Winston, 1964.

Ross, Shepley L. Differential Equations. New York: Blaisdell Publishing Company, 1965.

Taylor, Angus E. Advanced Calculus. Massachusetts: Blaisdell Publishing Company, 1955.

Taylor, Angus E. General Theory of Functions and Integration. Massachusetts: Blaisdell Publishing Company, 1965.