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Generalized Smarandache curves of spacelike and equiform spacelike curves via timelike second binormal in $\Re_1^4$

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Abstract

In this paper, we investigate spacelike Smarandache curves recording to the Frenet and the equiform Frenet frame of spacelike base curve with timelike second binormal vector in four-dimensional Minkowski space. Also, we compute the formulas of Frenet and equiform Frenet apparatus recording to the base curve. Furthermore, we give the geometric properties to these curves when is general helix.

Keywords: Smarandache curves; equiform frame; Minkowski space-time

MSC 2010 No.: 53B30, 53C40, 53C50

1. Introduction

A regular spacelike curve in time-dimensional Minkowski space, whose position vector is composed by Frenet frame vectors on another regular spacelike curve, is called a Smarandache curve Ashbacher (1997). Recently some authors are studied Smarandache curves see (Bektaş and Yüce (2013); Çetin et al. (2014); Elsharkawy et al. (2018); Kalkan et al. (2018); Saad

(2016); Solouma (2017); Solouma, (2017); Solouma and Wageeda (2017); Solouma and Wageeda (2019); Taşköprü and Tosun (2014)).

During this work, Smarandache curves via the Frenet and equiform Frenet frames of ω in $\mathfrak{R}_1^4$ are obtained. In Section 2, we give the conceptions of space $\mathfrak{R}_1^4$ and give the definitions of Fremet frames. Section 3, precise to compute Frenet apparatus of tb_2-spacelike Smarandache curves via Frenet frame of the spacelike base curve ω in $\mathfrak{R}_1^4$. Furthermore, we present a special case on these curves when ω is a general helix. In Section 4, we deal with $T\xi_2$-equiform spacelike Smarandache curve and compute its Frenet apparatus according to spacelike base curve. Also, we study the case when ω laying fully in a spacelike hyperplane of $\mathfrak{R}_1^4$.

2. Preliminaries

Let $\mathfrak{R}_1^4$ be 4-dimensional Minkowski space rectangular coordinate system (x_1, x_2, x_3, x_4) and with the Lorentzian inner product

$$\mathfrak{I} = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

Definition 2.1.

Let υ be any arbitrary vector in $\mathfrak{R}_1^4$. Then, υ have one of three Lorentzian clause depicts

1. spacelike curve if $\mathfrak{I}(\upsilon, \upsilon)$ is positive or $\upsilon = 0$,
2. timelike curve if $\mathfrak{I}(\upsilon, \upsilon)$ is negative,
3. null curve if $\mathfrak{I}(\upsilon, \upsilon)$ is zero and $\upsilon \neq 0$.

Similarly, any arbitrary curve $\omega = \omega(\varsigma)$ can be spacelike, timelike, or null if all of its velocity vectors $\dot{\omega}(\varsigma)$ are spacelike , timelike, or null, respectively López (2014).

For any $u, v, w \in \mathfrak{R}_1^4$, the vector product in 4-dimensional Minkowski space $\mathfrak{R}_1^4$ is defined by Yilmaz and Turgut (2008):

$$u \wedge v \wedge w = -\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}.$$

Let $\omega = \omega(\varsigma)$ be regular spacelike curve in $\mathfrak{R}_1^4$ with timelike second binormal. Then the Frenet formulas along ω given by (Do Carmo (1976); López (2014); O'Neill (1983)).

$$\begin{pmatrix} \dot{t}(\varsigma) \\ \dot{n}(\varsigma) \\ \dot{b}_1(\varsigma) \\ \dot{b}_2(\varsigma) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(\varsigma) & 0 & 0 \\ -\kappa_1(\varsigma) & 0 & \kappa_2(\varsigma) & 0 \\ 0 & -\kappa_2(\varsigma) & 0 & \kappa_3(\varsigma) \\ 0 & 0 & \kappa_3(\varsigma) & 0 \end{pmatrix} \begin{pmatrix} t(\varsigma) \\ n(\varsigma) \\ b_1(\varsigma) \\ b_2(\varsigma) \end{pmatrix}, \tag{1}$$

where $\left(\cdot = \frac{d}{d\varsigma}\right)$ and $\{t, n, b_1, b_2, \kappa_1, \kappa_2, \kappa_3\}$ define Frenet apparatus. Additionally, $\Im(t,t) = \Im(n,n) = \Im(b_1,b_1) = -\Im(b_2,b_2) = 1$, and $\Im(t,n) = \Im(t,b_1) = \Im(t,b_2) = \Im(n,b_1) = \Im(n,b_2) = \Im(b_1,b_2) = 0$. Also, $\{t, n, b_1, b_2, \kappa_1, \kappa_2, \kappa_3\}$ of ω can be formed as follows Yilmaz and Turgut (2008).

$$t = \frac{\dot{\omega}}{\|\dot{\omega}\|},$$

$$n = \frac{\|\dot{\omega}\|^2 \cdot \ddot{\omega} - \Im(\dot{\omega},\ddot{\omega}) \cdot \dot{\omega}}{\left\|\|\dot{\omega}\|^2 \cdot \ddot{\omega} - \Im(\dot{\omega},\ddot{\omega}) \cdot \dot{\omega}\right\|},$$

$$b_1 = \mu\, n \wedge t \wedge b_2,$$

$$b_2 = \mu\, \frac{t \wedge n \wedge \dddot{\omega}}{\|t \wedge n \wedge \dddot{\omega}\|},$$

$$\kappa_1 = \frac{\left\|\|\dot{\omega}\|^2 \cdot \ddot{\omega} - \Im(\dot{\omega},\ddot{\omega}) \cdot \dot{\omega}\right\|}{\|\dot{\omega}\|^4},$$

$$\kappa_2 = \frac{\|t \wedge n \wedge \dddot{\omega}\| \cdot \|\dot{\omega}\|}{\left\|\|\dot{\omega}\|^2 \cdot \ddot{\omega} - \Im(\dot{\omega},\ddot{\omega}) \cdot \dot{\omega}\right\|},$$

$$\kappa_3 = \frac{\Im\left(\omega^{(4)}, b_2\right)}{\|t \wedge n \wedge \dddot{\omega}\| \cdot \|\dot{\omega}\|},$$

and $\mu = \pm 1$ that make the value of matrix determinant $[t, n, b_1, b_2]$ equal $+1$.

Any arbitrary spacelike curve $\omega = \omega(\varsigma)$ in 4-dimensional Minkowski space $\mathfrak{R}_1^4$ called W-curve (or a general helix), if it Frenet curvatures are constant Miroslava and Emilija (2002).

Proposition 2.1.

Let $\omega = \omega(\varsigma)$ be spacelike curve in $\mathfrak{R}_1^4$ with timelike second binormal b_2 and with curvature $\kappa_1 > 0$, $\kappa_2 \neq 0$. Then the curve ω has $\kappa_3 \equiv 0$ if and only if ω lying fully in a spacelike hyperplane of $\mathfrak{R}_1^4$, Miroslava and Emilija (2002).

Proof:

If ω has $\kappa_3 \equiv 0$, then from equation (1) we obtain $\dot{\omega} = t$, $\ddot{\omega} = \kappa_1 n$, $\dddot{\omega} = -\kappa_1 t + \dot{\kappa_1} n + \kappa_1 \kappa_2 b_1$, $\omega^{(4)} = -3\kappa_1 \dot{\kappa_1} t + (\ddot{\kappa_1} - \kappa_1^3 - \kappa_1 \kappa_2^2) n + (2\dot{\kappa_1}\kappa_2 + \kappa_1 \dot{\kappa_2}) b_1$. Hence, all heigher derivatives of ω are linear combination of $\dot{\omega}$, $\ddot{\omega}$, $\dddot{\omega}$, so by appling Maclaurin expansion for ω we have

$$\omega(\varsigma) = \omega(0) + \dot{\omega}(0)\varsigma + \ddot{\omega}(0)\frac{\varsigma^2}{2!} + \dddot{\omega}(0)\frac{\varsigma^3}{3!} + \cdots.$$

Then, ω lie fully in a spacelike hyperplane of $\mathfrak{R}_1^4$ spanned by $\{\dot{\omega}, \ddot{\omega}, \dddot{\omega}\}$. Conversly, let ω satsfies the assumption of the propostion and lies fully in a spacelike hyperplane Ω of $\mathfrak{R}_1^4$. Then, there exist points $p, q \in \mathfrak{R}_1^4$ such that ω satisfies $g(x(\varsigma) - p, q) = 0$ where $q \in \Omega^{\perp}$is timelike vector. Then, we have

$$g(\dot{\omega}, q) = g(\ddot{\omega}, q) = g(\dddot{\omega}, q) = 0,$$

so, $\dot{\omega}, \ddot{\omega}, \dddot{\omega} \in \Omega$. Since, $t = \dot{\omega}$, $n = \frac{\ddot{\omega}}{\|\ddot{\omega}\|}$, this yields to $g(\dot{n}, q) = 0$. From equation (1) we obtain $b_1 = \frac{1}{\kappa_2}(\dot{n} + \kappa_1 t)$, then $g(b_1, q) = 0$. Since $b_2(\varsigma)$ is perpendicular to $\{t, n, b_1\}$ then $b_2(\varsigma) = \frac{q}{\|q\|}$.

Thus, $\dot{b_2}(\varsigma) = \kappa_3 b_1 = 0$, then $\kappa_3 \equiv 0$.

For a spacelike curve $\omega: I \to \mathfrak{R}_1^4$, let ϑ be equiform parameter to $\omega(\varsigma)$ defined as $\vartheta = \int \kappa_1 \, d\varsigma$, where $\rho = \frac{1}{\kappa_1}$ is the radius of curvature of ω, then $\rho = \frac{d\varsigma}{d\vartheta}$. We recall $\{T, \eta, \xi_1, \xi_2\}$ the equiform Frenet frame of ω where $T(\vartheta) = \rho\, t(\varsigma)$, $\eta(\vartheta) = \rho\, n(\varsigma)$, $\xi_1(\vartheta) = \rho\, b_1(\varsigma)$ and $\xi_2(\vartheta) = \rho\, b_2(\varsigma)$ are the equiform tangent vector, equiform principal normal vector, equiform first binormal vector and equiform second binormal vector respectively Evren Aydin and Ergut (2013). The equiform curvatures of $\omega = \omega(\vartheta)$ defined by $k_1(\vartheta) = \dot{\rho}$, $k_2(\vartheta) = \left(\frac{\kappa_2}{\kappa_1}\right)$ and $k_3(\vartheta) = \left(\frac{\kappa_3}{\kappa_1}\right)$. Thus, the equiform Frenet formulas in $\mathfrak{R}_1^4$ have the following frame Abdel-Aziz, Khalifa and Abdel-Salam (2015):

$$\begin{pmatrix} T'(\vartheta) \\ \eta'(\vartheta) \\ \xi'_1(\vartheta) \\ \xi'_2(\vartheta) \end{pmatrix} = \begin{pmatrix} k_1(\vartheta) & 1 & 0 & 0 \\ -1 & k_1(\vartheta) & k_2(\vartheta) & 0 \\ 0 & -k_2(\vartheta) & k_1(\vartheta) & k_3(\vartheta) \\ 0 & 0 & k_3(\vartheta) & k_1(\vartheta) \end{pmatrix} \begin{pmatrix} T(\vartheta) \\ \eta(\vartheta) \\ \xi_1(\vartheta) \\ \xi_2(\vartheta) \end{pmatrix}, \tag{2}$$

where $\left(' = \frac{d}{d\vartheta}\right)$ and $\mathfrak{I}(T, T) = \mathfrak{I}(\eta, \eta) = \mathfrak{I}(\xi_1, \xi_1) = -\mathfrak{I}(\xi_2, \xi_2) = \rho^2$, and $\mathfrak{I}(T, \eta) = \mathfrak{I}(T, \xi_1) = \mathfrak{I}(T, \xi_2) = \mathfrak{I}(\eta, \xi_1) = \mathfrak{I}(\eta, \xi_2) = \mathfrak{I}(\xi_1, \xi_2) = 0$.

3. Spacelike Smarandache curves in $\mathfrak{R}_1^4$

Definition 3.1.

Let $\omega = \omega(\varsigma)$ be spacelike unit speed curve with $\{t, n, b_1, b_2\}$ moving frame in $\mathfrak{R}_1^4$. Then, spacelike Smarandache tb_2 curves are given by Turgut and Yilmaz (2008).

$$\Psi = \Psi(\varsigma^*) = \frac{1}{\sqrt{2}}\big(t(\varsigma) + b_2(\varsigma)\big). \tag{3}$$

Theorem 3.1.

Let $\omega: I \to \mathfrak{R}_1^4$ be spacelike curve with timelike second binormal $b_2(\varsigma)$ such that $|\dot{\omega}| = 1$ and $\Psi(\varsigma^*)$ is tb_2-spacelike Smarandache curve in $\mathfrak{R}_1^4$ reference to the base curve $\omega(\varsigma)$. Then, the

Frenet apparatus of $\Psi(\varsigma^*)$ can be formed by the Frenet apparatus of $\omega(\varsigma)$ and if $\omega(\varsigma)$ is a spacelike W-curve, then $\Psi(\varsigma^*)$ is also a spacelike W-curve.

Proof:

Let $\Psi = \Psi(\varsigma^*)$ be tb_2-spacelike Smarandache curve in $\mathfrak{R}_1^4$ according to the base curve ω. From equations (1) and (3), we get

$$\dot{\Psi}(\varsigma^*) = \frac{d\Psi}{d\varsigma^*}\frac{d\varsigma^*}{d\varsigma} = \frac{1}{\sqrt{2}}(\kappa_1 n(\varsigma) + \kappa_3 b_1(\varsigma)).$$

Hence,

$$\dot{\Psi} = \frac{1}{\sqrt{\kappa_1^2 + \kappa_3^2}}\left(\kappa_1 n(\varsigma) + \kappa_3 b_1(\varsigma)\right), \tag{4}$$

where

$$\frac{d\varsigma^*}{d\varsigma} = \frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{\sqrt{2}}.$$

The inner product $\mathfrak{J}(\dot{\Psi}, \dot{\Psi}) = 1$ implies that $\Psi = \Psi(\varsigma^*)$ is spacelike curve. The tangent of Ψ is given by

$$T_\Psi(\varsigma^*) = v_1 n(\varsigma) + v_2 b_1(\varsigma), \tag{5}$$

where

$$v_1 = \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_3^2}},$$

and

$$v_2 = \frac{\kappa_3}{\sqrt{\kappa_1^2 + \kappa_3^2}}.$$

Now,

$$\ddot{\Psi} = \frac{\sqrt{2}}{\kappa_1^2 + \kappa_3^2}\left(-\kappa_1^2 t(\varsigma) - \kappa_2\kappa_3 n(\varsigma) + \kappa_1\kappa_2 b_1(\varsigma) + \kappa_3^2 b_2(\varsigma)\right), \tag{6}$$

$$\dddot{\Psi} = \frac{1}{\sqrt{\kappa_1^2 + \kappa_3^2}}\left(\kappa_1\kappa_2\kappa_3 t(\varsigma) - \kappa_1(\kappa_1^2 + \kappa_2^2)n(\varsigma) + \kappa_3(\kappa_3^2 - \kappa_2^2)b_1(\varsigma) + \kappa_1\kappa_2\kappa_3 b_2(\varsigma)\right), \tag{7}$$

and

$$\Psi^{(4)} = \frac{\sqrt{2}}{\kappa_1^2+\kappa_3^2}\left(\kappa_1(\kappa_1^2 + \kappa_2^2)t(\varsigma) + \kappa_2\kappa_3(\kappa_1^2 + \kappa_2^2 - \kappa_3^2)n(\varsigma) + \kappa_1\kappa_2(\kappa_1^2 + \kappa_2^2 - \kappa_3^2)b_1(\varsigma)\right) \\ -\kappa_3^2(\kappa_2^2 - \kappa_3^2)b_2(\varsigma). \tag{8}$$

Thereafter, from equations (4), (6), (7) and (8), we have

$$N_{\Psi}(\varsigma^*) = \varepsilon_1 t(\varsigma) + \varepsilon_2 n(\varsigma) + \varepsilon_3 b_1(\varsigma) + \varepsilon_4 b_2(\varsigma), \tag{9}$$

where

$$\varepsilon_1 = \frac{-\kappa_1^2}{\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}},$$

$$\varepsilon_2 = \frac{-\kappa_2\kappa_3}{\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}},$$

$$\varepsilon_3 = \frac{-\kappa_1\kappa_2}{\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}},$$

$$\varepsilon_4 = \frac{\kappa_3^2}{\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}}.$$

Also,

$$B_{2\Psi}(\varsigma^*) = \frac{m_1 t(\varsigma)+m_2 n(\varsigma)+m_3 b_1(\varsigma)+m_4 b_2(\varsigma)}{\sqrt{m_1^2+m_2^2+m_3^2-m_4^2}}, \tag{10}$$

where

$$m_1 = \frac{\kappa_3^2(\kappa_1^2-\kappa_2^2+\kappa_3^2)-\kappa_1^2\kappa_2^2\kappa_3}{(\kappa_1^2+\kappa_3^2)\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}},$$

$$m_2 = \frac{\kappa_1\kappa_2\kappa_3^2}{\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}},$$

$$m_3 = \frac{\kappa_1^2\kappa_2\kappa_3}{\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}},$$

$$m_4 = \frac{-\kappa_1^3\kappa_3(\kappa_1^2+\kappa_2^2)-\kappa_1\kappa_2^2\kappa_3^2}{(\kappa_1^2+\kappa_3^2)\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}}.$$

Then,

$$B_{1\Psi}(\varsigma^*) = \frac{l_1 t(\varsigma)+l_2 n(\varsigma)+l_3 b_1(\varsigma)+l_4 b_2(\varsigma)}{\sqrt{l_1^2+l_2^2+l_3^2-l_4^2}}, \tag{11}$$

where

$$l_1 = m_4(v_1\varepsilon_4+v_2\varepsilon_2)+\varepsilon_4(m_2v_2-m_3v_1),$$

$$l_2 = v_2(m_1\varepsilon_4-m_4\varepsilon_1),$$

$$l_3 = v_1(m_4\varepsilon_1 - m_1\varepsilon_4)\,,$$

$$l_4 = m_1(v_1\varepsilon_3 - v_2\varepsilon_3) + \varepsilon_1(m_2v_2 - m_3v_1).$$

So, the curvature functions of Ψ are given by

$$\kappa_{1\Psi}(\varsigma^*) = \frac{\sqrt{2}\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}}{\kappa_1^2+\kappa_3^2}\,, \tag{12}$$

$$\kappa_{2\Psi}(\varsigma^*) = \frac{(\kappa_1^2+\kappa_3^2)\sqrt{m_1^2+m_2^2+m_3^2-m_4^2}}{\sqrt{2}\sqrt{\kappa_1^2(\kappa_1^2+\kappa_2^2)+\kappa_3^2(\kappa_2^2-\kappa_3^2)}}\,, \tag{13}$$

$$\kappa_{3\Psi}(\varsigma^*) = \frac{\sqrt{2}\kappa_3^2(\kappa_2^2-\kappa_3^2)}{(\kappa_1^2+\kappa_3^2)\sqrt{m_1^2+m_2^2+m_3^2-m_4^2}}\,, \tag{14}$$

Then from equations (12), (13) and (14), the tb_2-spacelike Smarandache curve is W-curve and this complete our proof.

■

4. Equiform Smarandache curves in $\mathfrak{R}_1^4$

Definition 4.1.

Let $\phi = \phi(\vartheta)$ be equiform spacelike unit speed curve in $\mathfrak{R}_1^4$ with $\{T, \eta, \xi_1, \xi_2\}$ moving frame. Then equiform spacelike Smarandache $T\xi_2$ curves are given by

$$\mathcal{H} = \mathcal{H}(\vartheta^*) = \tfrac{1}{\sqrt{2}\rho}\big(T(\vartheta) + \xi_2(\vartheta)\big). \tag{15}$$

Theorem 4.1.

Let $\phi: I \to \mathfrak{R}_1^4$ be an equiform spacelike W-curve with equiform timelike second binormal $\xi_2(\vartheta)$ and $\mathcal{H}(\vartheta^*)$ be $T\xi_2$- equiform spacelike Smarandache curves in $\mathfrak{R}_1^4$ reference to the base curve $\phi(\vartheta)$ with equiform curvatures $k_{1\mathcal{H}} > 0,\ k_{2\mathcal{H}} \neq 0$. Then, the Frenet apparatus of $\mathcal{H}(\vartheta^*)$ can be formed by the Frenet apparatus of $\phi(\vartheta)$. Furthermore, if $\phi(\vartheta)$ laying fully in a spacelike hyperplane of $\mathfrak{R}_1^4$, then $\mathcal{H}(\vartheta^*)$ laying fully also.

Proof:

Let $\mathcal{H} = \mathcal{H}(\vartheta^*)$ be $T\xi_2$- equiform spacelike Smarandache curves of $\phi(\vartheta)$. From equations (2) and (15), we have

$$\mathcal{H}'(\vartheta^*) = \frac{1}{\sqrt{2}}\big(k_1 T(\vartheta) + \eta(\vartheta) + k_3\xi_1(\vartheta) + k_1\xi_2(\vartheta)\big).$$

Then,

$$T_{\mathcal{H}}(\vartheta^*) = \frac{1}{\rho\sqrt{k_3^2+1}}\big(k_1 T(\vartheta) + \eta(\vartheta) + k_3\xi_1(\vartheta) + k_1\xi_2(\vartheta)\big), \tag{16}$$

where

$$\frac{d\vartheta^*}{d\vartheta} = \frac{\rho\sqrt{k_3^2+1}}{\sqrt{2}}.$$

Now,

$$T'_{\mathcal{H}}(\vartheta^*) = \zeta_1 T(\vartheta) + \zeta_2\eta(\vartheta) + \zeta_3\xi_1(\vartheta) + \zeta_4\xi_2(\vartheta), \tag{17}$$

where

$$\begin{cases} \zeta_1 = \dfrac{\sqrt{2}(k_1^2-1)}{\rho(k_3^2+1)}, \\ \zeta_2 = \dfrac{\sqrt{2}(2k_1 - k_2k_3)}{\rho(k_3^2+1)}, \\ \zeta_3 = \dfrac{\sqrt{2}(2k_1k_3 + k_2)}{\rho(k_3^2+1)}, \\ \zeta_4 = \dfrac{\sqrt{2}(k_1^2 + k_2k_3)}{\rho(k_3^2+1)}. \end{cases}$$

Then,

$$k_{1\mathcal{H}}(\vartheta^*) = \frac{\sqrt{2}\sqrt{2k_1^2[2(k_1^2+k_3^2) - k_2k_3 + 1] + k_2^2 + 1}}{(k_3^2+1)^{\frac{3}{2}}}, \tag{18}$$

and

$$\eta_{\mathcal{H}}(\vartheta^*) = \gamma_1 T(\vartheta) + \gamma_2\eta(\vartheta) + \gamma_3\xi_1(\vartheta) + \gamma_4\xi_2(\vartheta), \tag{19}$$

where

$$\gamma_1 = \frac{k_1^2 - 1}{\rho^2\sqrt{2k_1^2[2(k_1^2+k_3^2) - k_2k_3 + 1] + k_2^2 + 1}},$$

$$\gamma_2 = \frac{2k_1 - k_2k_3}{\rho^2\sqrt{2k_1^2[2(k_1^2+k_3^2) - k_2k_3 + 1] + k_2^2 + 1}},$$

$$\gamma_3 = \frac{2k_1k_3 + k_2}{\rho^2\sqrt{2k_1^2[2(k_1^2+k_3^2) - k_2k_3 + 1] + k_2^2 + 1}},$$

$$\gamma_4 = \frac{k_1^2 + k_2 k_3}{\rho^2\sqrt{2k_1^2[2(k_1^2 + k_3^2) - k_2 k_3 + 1] + k_2^2 + 1}}.$$

Now,

$$\mathcal{H}'''(\vartheta^*) = r_1 T(\vartheta) + r_2 \eta(\vartheta) + r_3 \xi_1(\vartheta) + r_4 \xi_2(\vartheta)\,, \tag{20}$$

where

$$r_1 = \frac{\sqrt{2}(\zeta_1 k_1 - \zeta_2)}{\rho(k_3^2 + 1)},$$

$$r_2 = \frac{\sqrt{2}(\zeta_1 + \zeta_2 k_1 - \zeta_3 k_2)}{\rho(k_3^2 + 1)},$$

$$r_3 = \frac{\sqrt{2}(\zeta_3 k_3 + \zeta_4 k_1)}{\rho(k_3^2 + 1)},$$

$$r_4 = \frac{\zeta_3 k_3 + \zeta_4 k_1}{\rho(k_3^2 + 1)}.$$

Since,

$$T_{\mathcal{H}} \wedge \eta_{\mathcal{H}} \wedge \mathcal{H}''' = \delta_1 T(\vartheta) + \delta_2 \eta(\vartheta) + \delta_3 \xi_1(\vartheta) + \delta_4 \xi_2(\vartheta),$$

where

$$\begin{cases} \delta_1 = \dfrac{1}{\rho\sqrt{k_3^2+1}}(r_2(\gamma_3 k_1 - \gamma_4 k_3) + r_3(\gamma_4 - \gamma_2 k_1) + r_4(\gamma_2 k_3 - \gamma_3)), \\ \delta_2 = \dfrac{1}{\rho\sqrt{k_3^2+1}}(r_1(\gamma_3 k_1 - \gamma_4 k_3) + r_3 k_1(\gamma_4 - \gamma_1) + r_4(\gamma_1 k_3 - \gamma_3 k_1)), \\ \delta_3 = \dfrac{1}{\rho\sqrt{k_3^2+1}}\big(r_1(\gamma_3 - \gamma_2 k_1) + r_2 k_1(\gamma_1 - \gamma_3) + r_4(\gamma_2 k_1 - \gamma_1)\big), \\ \delta_4 = \dfrac{1}{\rho\sqrt{k_3^2+1}}\big(r_1(\gamma_2 k_3 - \gamma_3) + r_2(\gamma_3 k_1 - \gamma_1 k_3) + r_3(\gamma_1 - \gamma_2 k_1)\big), \end{cases}$$

then,

$$\xi_{2_{\mathcal{H}}}(\vartheta^*) = \frac{\delta_1 T(\vartheta) + \delta_2 \eta(\vartheta) + \delta_3 \xi_1(\vartheta) + \delta_4 \xi_2(\vartheta)}{\rho\sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2 - \delta_4^2}}\,, \tag{21}$$

and

$$\xi_{1_{\mathcal{H}}}(\vartheta^*) = \eta_{\mathcal{H}} \wedge T_{\mathcal{H}} \wedge \xi_{2_{\mathcal{H}}} = \alpha_1 T(\vartheta) + \alpha_2 \eta(\vartheta) + \alpha_3 \xi_1(\vartheta) + \alpha_4 \xi_2(\vartheta), \tag{22}$$

where

$$\alpha_1 = \frac{k_1(\gamma_2\delta_3 - \gamma_3\delta_2) - k_3(\gamma_2\delta_4 + \gamma_4\delta_2) + \gamma_3\delta_4 + \gamma_4\delta_3}{\rho\sqrt{k_3^2+1}\sqrt{\delta_1^2+\delta_2^2+\delta_3^2-\delta_4^2}},$$

$$\alpha_2 = \frac{k_1(\gamma_1\delta_3 + \gamma_3\delta_4 + \gamma_4\delta_3 - \gamma_3\delta_1) - k_3(\gamma_1\delta_4 + \gamma_4\delta_1)}{\rho\sqrt{k_3^2+1}\sqrt{\delta_1^2+\delta_2^2+\delta_3^2-\delta_4^2}},$$

$$\alpha_3 = \frac{k_1(\gamma_1\delta_2 + \gamma_2\delta_1 + \gamma_4\delta_2 - \gamma_2\delta_4) + \gamma_1\delta_4 - \gamma_4\delta_1}{\rho\sqrt{k_3^2+1}\sqrt{\delta_1^2+\delta_2^2+\delta_3^2-\delta_4^2}},$$

$$\alpha_4 = \frac{k_1(\gamma_2\delta_3 - \gamma_2\delta_1 - \gamma_3\delta_2 + \gamma_2\delta_4) + \gamma_3\delta_1 - \gamma_1\delta_3}{\rho\sqrt{k_3^2+1}\sqrt{\delta_1^2+\delta_2^2+\delta_3^2-\delta_4^2}}.$$

Then,

$$k_{2\mathcal{H}}(\vartheta^*) = \sqrt{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2-\alpha_4^2}{\zeta_1^2+\zeta_2^2+\zeta_3^2}}. \tag{23}$$

Now,

$$\mathcal{H}^{(4)}(\vartheta^*) = \lambda_1 T(\vartheta) + \lambda_2\eta(\vartheta) + \lambda_3\xi_1(\vartheta) + \lambda_4\xi_2(\vartheta), \tag{24}$$

where

$$\begin{cases} \lambda_1 = \dfrac{\sqrt{2}[\zeta_1(k_3^2+1) - \zeta_2(k_1+1) + \zeta_3 k_3]}{\rho\sqrt{k_3^2+1}}, \\ \lambda_2 = \dfrac{\sqrt{2}[\zeta_1(k_1-k_2+1) - \zeta_2(k_1^2 - k_1k_2 - 1) - \zeta_3 k_1k_3 - \zeta_4 k_2k_3]}{\rho\sqrt{k_3^2+1}}, \\ \lambda_3 = \dfrac{\sqrt{2}[\zeta_1 k_2 + 2\zeta_2 k_1k_2 - \zeta_3 k_2^2 + \zeta_4 k_1k_3]}{\rho\sqrt{k_3^2+1}}, \\ \lambda_4 = \dfrac{\sqrt{2}[\zeta_2 k_2k_3 + \zeta_3(k_3^2 + 2k_1k_3) + \zeta_4(k_3^2 + k_1k_3)]}{\rho\sqrt{k_3^2+1}}. \end{cases}$$

Then,

$$k_{3\mathcal{H}}(\vartheta^*) = \frac{\sqrt{2}k_3[\zeta_2k_2 + \zeta_3(k_3+2k_2) + \zeta_4(k_3+k_3)]}{\rho\sqrt{k_3^2+1}\sqrt{\delta_1^2+\delta_2^2+\delta_3^2-\delta_4^2}}. \tag{25}$$

So, if the spacelike base curve $\phi(\vartheta)$ laying fully in a spacelike hyperplane of $\mathfrak{R}_1^4$ then from equation (25) we have $k_{3\mathcal{H}}(\vartheta^*) \equiv 0$, which mean that $\mathcal{H}(\vartheta^*)$ lie fully in a spacelike hyperplane of $\mathfrak{R}_1^4$.

5. Conclusion

As a conclusion of results, the Frenet apparatus of spacelike and equiform spacelike Smarandache curves recording to the Frenet and the equiform Frenet frame of spacelike base curve ω with timelike second binormal vector in 4-dimensional Minkowski space $\mathfrak{R}_1^4$ can be formed by the Frenet apparatus of ω. Furthermore, its curvature functions related directly with the geometric properties of curvature function of ω which mean that if the base curve $\phi(\vartheta)$ laying fully in a spacelike hyperplane of $\mathfrak{R}_1^4$, then $\mathcal{H}(\vartheta^*)$ laying fully also ($k_{3\mathcal{H}}(\vartheta^*)$ which given by equation (25) is identically vanished).

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