Applications and Applied Mathematics: An International Journal (AAM)

Volume 15 | Issue 2 Article 38

12-2020

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Recommended Citation

Al-Homidan, S.; Singh, Vivek; and Ahmad, I. (2020). On Higher-order Duality in Nondifferentiable Minimax Fractional Programming, Applications and Applied Mathematics: An International Journal (AAM), Vol. 15, Iss. 2, Article 38.
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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 15, Issue 2 (December 2020), pp. 1348 – 1368

On Higher-order Duality in Nondifferentiable Minimax Fractional Programming

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Received: January 1, 2019; Accepted: May 31, 2019

Abstract

In this paper, we consider a nondifferentiable minimax fractional programming problem with continuously differentiable functions and formulated two types of higher-order dual models for such optimization problem. Weak, strong and strict converse duality theorems are derived under higher-order generalized invexity.

Keywords: Minimax programming; Fractional programming; Nondifferentiable programming; Higher-order duality

MSC 2010 No.: 90C32, 49J35, 49N15, 26A51

1. Introduction

Fractional programming problems have become a subject of wide interest since they provide a universal apparatus for a wide class of problems in the financial analysis of a firm, educational planning, public policy decision making, corporate planning, agricultural planning, healthcare, marine transportation, and bank balance sheet management. Some results for optimality conditions and duality in multiobjective fractional programming problems have been obtained under various kinds of generalized convexities. The non-differentiable fractional programming problems play an important role in obtaining the set of most preferred solutions and a decision maker can take the good decision. In recent years, many researchers have paid attention to develop optimality conditions and duality results for a nondifferentiable minimax fractional programming problem. For more details, one can consult Ahmad and Husain (2006); Ahmad et al. (2008); Batatorescuet al. (2009); Jayswal (2011); Zalmai and Zhang (2007), and the references cited therein.

An extension of F-convexity (Hanson and Mond (1986)) and ρ-convexity (Vial (1983)) is introduced by Preda 1992, that is (F, ρ)-convexity. Later, Liang et al. 2001 presented a unified formulation of generalized convexity, called (F, α, ρ, d)-convexity and discussed optimality conditions and duality results for fractional programming problems. In Zalmai and Zhang (2013), Zalmai and Zhang obtained several parametric duality results involving generalized (α, η, ρ)-V-invex funtions for a semiinfinite multiobjective fractional programming problem. Some results for a nondifferentiable minimax fractional programming problems are established in Jayswal and Kumar (2011); Yuan et al. (2006) under (C, α, ρ, d)-convexity. Second order duality results for nondifferentiable minimax fractional programming problems are discussed in Ahmad (2013), Gupta and Dangar (2014), and Kailey and Sharma (2016).

Generalized convexity extends the validity of the results to a wider class of nonlinear programming problems. With the development of optimization problems, there has been a growing interest in the higher-order dual problems. Several researchers (Ahmad (2012); Batatorescu et al. (2007a); Batatorescu et al. (2007b); Gao (2016); Jayswal et al. (2014); Sharma and Gupta (2016); Ying (2012)) have shown their interest in higher order duality.

Motivated by the earlier work and importance of the higher order generalized convexity, we discuss the higher order duality results for the dual problems related to a minimax fractional programming problem involving generalized higher order (Φ, ρ)-V-invexity.

The structure of this paper is as follows: Basic concepts and some preliminary material from convex analysis are given in Section 2. Sections 3 and 4 deal the duality results for a minimax fractional programming problem under higher order (Φ, ρ)-V-invexity. Conclusions and future lines of research are presented in Section 5.

2. Notations and Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthant. Let X be an open subset of R^n.

Definition 2.1.

A function $\Phi : X \times X \times R^{n+1} \rightarrow R$ is convex on R^{n+1} with respect to third argument, if for any $(x, x^*) \in X \times X$, the following inequality,

$$\Phi(x, x^*; (\lambda(a_1, b_1) + (1-\lambda)(a_2, b_2))) \leq \lambda\Phi(x, x^*; (a_1, b_1)) + (1-\lambda)\Phi(x, x^*; (a_2, b_2)),$$

holds for all $a_1, a_2 \in R^n, b_1, b_2 \in R$ and for any $\lambda \in [0, 1]$.

Let $f : X \rightarrow R^k$ and $\theta : X \times R^n \rightarrow R^k$ be continuously differentiable functions at $x^* \in X$.

Definition 2.2. (Sharma and Gupta (2016))

A function f is said to be higher-order (Φ, ρ)-V-invex at $x^* \in X$ with respect to θ if there exists a function $\Phi : X \times X \times R^{n+1} \rightarrow R$, where $\Phi(x, x^*, .)$ is convex on R^{n+1}, $\Phi(x, x^*, (0, a)) \geq 0$ for all $x \in X$ and every $a \in R_+$, $\rho = (\rho_1, \rho_2, ..., \rho_k) \in R^k$ and real-valued functions $\alpha_i : X \times X \rightarrow R^+ \setminus \{0\}$, $i = 1, 2, ..., k$ such that, the following inequalities

$$\begin{aligned} &f_i(x) - f_i(x^*) - \theta_i(x^*, p) + p^T\nabla_p\theta_i(x^*, p) \\ &\qquad \geq \Phi(x, x^*, \alpha_i(x, x^*)(\nabla f_i(x^*) + \nabla_p\theta_i(x^*, p), \rho_i)), i = 1, 2, ..., k, \end{aligned} \tag{1}$$

hold for all $(x, p) \in X \times R^n$.

If each function $f_i, i = 1, 2, ..., k$, satisfies the inequality (1) at each $x \in X$, then $f_i, i = 1, 2, ..., k$ is said to be higher-order (Φ, ρ_i)-V_{α_i}-invex at x^* on X with respect to θ_i.

The function f is said to be strictly higher-order (Φ, ρ)-V-invex at $x^* \in X(x \neq x^*)$, if the above inequalities hold as strict inequalities.

In the present paper, we consider is the following nondifferentiable minimax fractional problem:

$$\textbf{(NP)} \qquad \min_{x\in R^n} \sup_{y\in Y} \frac{f(x, y) + (x^TBx)^{1/2}}{g(x, y) - (x^TCx)^{1/2}},$$

$$\text{subject to} \quad h(x) \leq 0,\ x \in X,$$

where Y is a compact subset of R^l, $f(.,.), g(.,.) : R^n \times R^l \rightarrow R$ and $h(.) : R^n \rightarrow R^m$ are continuously differentiable functions. B and C are $n \times n$ positive semi-definite matrices.

Let $\Im = \{x \in X : h(x) \leq 0\}$ denotes the set of all feasible solutions of (NP). For each $(x, y) \in R^n \times R^l$, we define

$$\phi(x, y) = \frac{f(x, y) + (x^TBx)^{1/2}}{g(x, y) - (x^TCx)^{1/2}},$$

such that for each $(x, y) \in \Im \times Y, f(x, y) + (x^T Bx)^{1/2} \geq 0$ and $g(x, y) - (x^T Cx)^{1/2} > 0$. For each $x \in \Im$, we define

$$J(x) = \{j \in J : h_j(x) = 0\},$$

where

$$J = \{1, 2, \cdots, m\},$$

$$Y(x) = \left\{ y \in Y : \frac{f(x,y) + (x^T Bx)^{1/2}}{g(x,y) - (x^T Cx)^{1/2}} = \sup_{u \in Y} \frac{f(x,u) + (x^T Bx)^{1/2}}{g(x,u) - (x^T Dx)^{1/2}} \right\},$$

$$S(x) = \Big\{ (s, t, \tilde{y}) \in N \times R_+^s \times R^{ms} : 1 \leq s \leq n+1,\ t = (t_1, t_2, \cdots, t_s) \in R_+^s$$

$$\text{with } \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_s) \text{ with } \bar{y}_i \in Y(x), i = 1, 2, \cdots, s \Big\}.$$

Since f and g are continuously differentiable and Y is compact in R^l, it follows that for each $x^* \in \Im, Y(x^*) \neq \emptyset$. Thus for any $\bar{y}_i \in Y(x^*)$, we have a positive constant

$$\lambda_\circ = \phi(x^*, \bar{y}_i) = \frac{f(x^*, \bar{y}_i) + (x^{*T} Bx^*)^{1/2}}{g(x^*, \bar{y}_i) - (x^{*T} Cx^*)^{1/2}}.$$

The following generalized Schwartz inequality and necessary conditions are required in our discussion.

Let B be a positive semi-definite matrix of order n. Then for all $x, w \in R^n$,

$$x^T Bw \leq (x^T Bx)^{1/2} (w^T Bw)^{1/2}. \tag{2}$$

It is observe that equality holds if $Bx = \xi Bw$ for some $\xi \geq 0$. Evidently, if $(w^T Bw)^{1/2} \leq 1$, then

$$x^T Bw \leq (x^T Bx)^{1/2}.$$

Theorem 2.1. (Lai and Lee (2002))

Let x^* be an optimal solution for (NP) satisfying $x^{*T} Bx^* > 0,\ x^{*T} Cx^* > 0$ and let $\nabla h_j(x^*), j \in J(x^*)$ be linearly independent. Then there exist $(s, t^*, \bar{y}) \in S(x^*), \lambda_\circ \in R_+, w, v \in R^n$ and $\mu^* \in R_+^p$ such that

$$\sum_{i=1}^{s} t_i^* \{ \bigtriangledown f(x^*, \bar{y}_i) + Bw - \lambda_\circ (\bigtriangledown g(x^*, \bar{y}_i) - Cv) \} + \bigtriangledown \sum_{j=1}^{m} \mu_j^* h_j(x^*) = 0,$$

$$f(x^*, \bar{y}_i) + (x^{*T} Bx^*)^{1/2} - \lambda_\circ (h(x^*, \bar{y}_i) - (x^{*T} Cx^*)^{1/2}) = 0,\ i = 1, 2, \cdots, s,$$

$$\sum_{j=1}^{m} \mu_j^* h_j(x^*) = 0,$$

$$t_i^* \geq 0,\ i = 1, 2, \cdots, s, \sum_{i=1}^{s} t_i^* = 1,$$

$$w^T Bw \leq 1,\ v^T Cv \leq 1, (x^{*T} Bx^*)^{1/2} = x^{*T} Bw, (x^{*T} Cx^*)^{1/2} = x^{*T} Cv.$$

It may be noted that both the matrices B and C are positive definite in the above theorem. If one of $(x^{*T}Bx^*)$ and $(x^{*T}Cx^*)$ is zero, or both B and C are singular, then for $(s, t^*, \bar{y}) \in S(x^*)$, we can take a set $U_{\bar{y}}(x^*)$ defined in Lai et al. (1999) by
$U_{\bar{y}}(x^*) = \{u \in R^n : u^T \nabla h_j(x^*) \leq 0, j \in J(x^*)$ with any one of the following ***(i)-(iii)*** holds $\}$:

$$(i)\ \ x^{*T}Bx^* > 0, x^{*T}Cx^* = 0$$
$$\Rightarrow u^T(\sum_{i=1}^{s} t_i^*\{\nabla f(x^*, \bar{y}_i) + \tfrac{Bx^*}{(x^{*T}Bx^*)^{1/2}} - \lambda_\circ \nabla g(x^*, \bar{y}_i)\}) + (u^T(\lambda_\circ^2 C)u)^{1/2} < 0,$$

$$(ii)\ \ x^{*T}Bx^* = 0, x^{*T}Cx^* > 0$$
$$\Rightarrow u^T(\sum_{i=1}^{s} t_i^*\{\nabla f(x^*, \bar{y}_i) - \lambda_\circ(\nabla g(x^*, \bar{y}_i) - \tfrac{Cx^*}{(x^{*T}Cx^*)^{1/2}})\}) + (u^T Bu)^{1/2} < 0,$$

$$(iii)\ \ x^{*T}Bx^* = 0, x^{*T}Cx^* = 0$$
$$\Rightarrow u^T(\sum_{i=1}^{s} t_i^*\{\nabla f(x^*, \bar{y}_i) - \lambda_\circ \nabla h(x^*, \bar{y}_i)\}) + (u^T(\lambda_\circ^2 C)u)^{1/2} + (u^T Bu)^{1/2} < 0.$$

If $U_{\bar{y}}(x^*) = \emptyset$ in Theorem 2.3, then the Theorem 2.3 still holds.

3. First duality model

In this section, we formulate the following higher-order dual for (NP) and establish duality theorems:

(DMI) $$\max_{(s,t,\bar{y}) \in S(u)} \ \sup_{(u,\mu,\lambda,v,w,p) \in H_1(s,t,\bar{y})} \lambda,$$

where $H_1(s, t, \bar{y})$ denotes the set of all $(u, \mu, k, v, w, p) \in R^n \times R_+^m \times R_+ \times R^n \times R^n \times R^n$ satisfying

$$\sum_{i=1}^{s} t_i\{\bigtriangledown f(u, \bar{y}_i) + Bw - \lambda(\bigtriangledown g(u, \bar{y}_i) - Cv)\} + \bigtriangledown \sum_{j=1}^{m} \mu_j h_j(u) + \sum_{i=1}^{s} t_i[\nabla_p(F(u, \bar{y}_i, p) - \lambda G(u, \bar{y}_i, p))] + \sum_{j=1}^{m} \mu_j \nabla_p H_j(u, p) = 0, \quad (3)$$

$$\sum_{i=1}^{s} t_i\{f(u, \bar{y}_i) + u^T Bw - \lambda(g(u, \bar{y}_i) - u^T Cv) + [F(u, \bar{y}_i, p) - \lambda G(u, \bar{y}_i, p)] - p^T \nabla_p[F(u, \bar{y}_i, p) - \lambda G(u, \bar{y}_i, p)]\} + \sum_{j \in J_0} \mu_j h_j(u) + \sum_{j \in J_0} \mu_j H_j(u, p) - p^T \sum_{j \in J_0} \mu_j \nabla_p H_j(u, p) \geq 0, \quad (4)$$

$$\sum_{j \in J_\beta} \mu_j h_j(z) + \sum_{j \in J_\beta} \mu_j H_j(u, p) - p^T \sum_{j \in J_\beta} \mu_j \nabla_p H_j(u, p) \geq 0, \quad \beta = 1, 2, ..., r, \quad (5)$$

$$w^T Bw \leq 1, v^T Cv \leq 1, \tag{6}$$

where $F : R^n \times R^l \times R^n \to R, G : R^n \times R^l \times R^n \to R$ and $H : R^n \times R^n \to R^m$ are differentiable functions. $J_\beta \subseteq M = \{1, 2, ..., m\}$, $\beta = 0, 1, 2, ..., r$ with $\bigcup_{\beta=0}^{r} J_\beta = M$ and $J_\beta \cap J_\gamma = \emptyset$ if $\beta \neq \gamma$. If for a triplet $(s, t, \bar{y}) \in S(u)$, the set $H_1(s, t, \bar{y}) = \emptyset$, then we define the supremum over it to be $-\infty$.

Remark 3.1.

If $F(u, \bar{y}_i, p) = \frac{1}{2}p^T \nabla^2 f(u, \bar{y}_i)p,\ G(u, \bar{y}_i, p) = \frac{1}{2}p^T \nabla^2 g(u, \bar{y}_i)p, i = 1, 2, ..., s,\ H_j(u, p) = \frac{1}{2}p^T \nabla^2 h_j(u)p, j = 1, 2, ..., m$, then (DMI) becomes the second order dual (DM3) in (Dangar and Gupta (2013)). If, in addition, $J_0 = \emptyset$, and $p = 0$, then we get the dual (DMI) (Jayswal and Kumar (2011)).

We denote

$$\psi(.) = \sum_{i=1}^{s} t_i \{f(., \bar{y}_i) + (.)^T Bw - \lambda(g(., \bar{y}_i) - (.)^T Cv)\},$$

and

$$\psi_1(u, \bar{y}_i, p) = \sum_{i=1}^{s} t_i \{F(u, \bar{y}_i, p) - \lambda G(u, \bar{y}_i, p)\}.$$

Theorem 3.1. (Weak duality)

Let x and $(u, \mu, \lambda, v, w, s, t, \bar{y}, p)$ be feasible solutions to (NP) and (DMI) respectively. If

(i) $\psi(.) + \sum_{j \in J_0} \mu_j h_j(.)$ is higher-order $(\Phi, \rho_i^1) - V_{\alpha_i^1}$-invex at u with respect to function $\psi_1(u, \bar{y}_i, p) + \sum_{j \in J_0} \mu_j H_j(u, p)$,

(ii) $h_j(.)$, $j \in J_\beta, \beta = 1, 2, ..., r$ is higher-order $(\Phi, \rho_j^2) - V_{\alpha_j^2}$-invex at u with respect to function $H_j, j \in J_\beta$, and

(iii) $\sum_{i=1}^{s} \rho_i^1 + \sum_{\beta=1}^{r} \sum_{j \in J_\beta} \mu_j \rho_j^2 \geq 0$,

then,

$$\sup_{\bar{y} \in Y} \frac{f(x, \bar{y}) + (x^T Bx)^{1/2}}{g(x, \bar{y}) - (x^T Cx)^{1/2}} \geq \lambda.$$

Proof:

Suppose to the contrary that

$$\sup_{\bar{y} \in Y} \frac{f(x, \bar{y}) + (x^T Bx)^{1/2}}{g(x, \bar{y}) - (x^T Cx)^{1/2}} < \lambda.$$

Then we get

$$f(x,\bar{y}_i)+(x^TBx)^{1/2}-\lambda(g(x,\bar{y}_i)-(x^TCx)^{1/2})<0,\ \text{for all } \bar{y}_i\in Y.$$

It follows from $t_i\geq 0,\ i=1,2,\cdots,s$, with $\sum_{i=1}^{s} t_i=1, t=(t_1,t_2,\cdots,t_s)\neq 0$, and by (2) and (6) that

$$\sum_{i=1}^{s} t_i[f(x,\bar{y}_i)+x^TBw-\lambda(h(x,\bar{y}_i)-x^TCv)]<0.$$

On utilizing the feasibility of x for (NP) along with dual constraint (4), we get

$$\psi(x)+\sum_{j\in J_0}\mu_j h_j(x)-\psi(u)-\psi_1(u,\bar{y}_i,p)+p^T\nabla_p\psi_1(u,\bar{y}_i,p)-\sum_{j\in J_0}\mu_j h_j(u)$$
$$-\sum_{j\in J_0}\mu_j H_j(u,p)+p^T\sum_{j\in J_0}\mu_j\nabla_p H_j(u,p)<0,$$

which by using hypothesis (i), we have

$$\Phi\big(x,u,\alpha_i^1(x,u)\big(\nabla\psi(u)+\sum_{j\in J_0}\mu_j\nabla h_j(u)+\nabla_p\psi_1(u,\bar{y}_i,p)+\sum_{j\in J_0}\mu_j\nabla_p H_j(u,p),\rho_i^1\big)\big)<0. \quad (7)$$

On one hand, by using hypothesis (ii), we get

$$h_j(x)-h_j(u)-H_j(u,p)+p^T\nabla_p H_j(u,p)$$
$$\geq\Phi\big(x,u,\alpha_j^2(x,u)\big(\nabla h_j(u)+\nabla_p H_j(u,p),\rho_j^2\big)\big),\ \forall j\in J_\beta,\ \beta=1,2,\cdots,r.$$

Multiplying the above inequalities by $\frac{\mu_j}{\alpha_j^2(x,u)}, j\in J_\beta,\ \beta=1,2,\cdots,r$, then summing up these inequalities, we get

$$\sum_{\beta=1}^{r}\sum_{j\in J_\beta}\frac{\mu_j}{\alpha_j^2(x,u)}[h_j(x)-h_j(u)-H_j(u,p)+p^T\nabla_p H_j(u,p)]$$
$$\geq\sum_{\beta=1}^{r}\sum_{j\in J_\beta}\frac{\mu_j}{\alpha_j^2(x,u)}\Phi\big(x,u,\alpha_j^2(x,u)\big(\nabla h_j(u)+\nabla_p H_j(u,p),\rho_j^2\big)\big),\ \forall j\in J_\beta,\ \beta=1,2,\cdots,r.$$

By using the feasibility of x for (NP) and dual constraint (5), the above inequality yields

$$\sum_{\beta=1}^{r}\sum_{j\in J_\beta}\frac{\mu_j}{\alpha_j^2(x,u)}\Phi\big(x,u,\alpha_j^2(x,u)\big(\nabla h_j(u)+\nabla_p H_j(u,p),\rho_j^2\big)\big)\leq 0. \quad (8)$$

Now, multiplying each inequality (7) by $\frac{1}{\alpha_i^1(x,u)},\ i=1,2,\cdots,s$ and then summing up these inequalities, we get

$$\sum_{i=1}^{s}\frac{1}{\alpha_i^1(x,u)}\Phi\big(x,u,\alpha_i^1(x,u)\big(\nabla\psi(u)+\sum_{j\in J_0}\mu_j\nabla h_j(u)+\nabla_p\psi_1(u,\bar{y}_i,p)$$
$$+\sum_{j\in J_0}\mu_j\nabla_p H_j(u,p),\rho_i^1\big)\big)<0. \quad (9)$$

By adding (8) and (10), we obtain

$$\sum_{i=1}^{s}\frac{1}{\alpha_i^1(x,u)}\Phi\big(x,u,\alpha_i^1(x,u)\big(\nabla\psi(u)+\sum_{j\in J_0}\mu_j\nabla h_j(u)+\nabla_p\psi_1(u,\bar{y}_i,p)$$
$$+\sum_{j\in J_0}\mu_j\nabla_p H_j(u,p),\rho_i^1)\big)+\sum_{\beta=1}^{r}\sum_{j\in J_\beta}\frac{\mu_j}{\alpha_j^2(x,u)}\Phi\big(x,u,\alpha_j^2(x,u)\big(\nabla h_j(u) \tag{10}$$
$$+\nabla_p H_j(u,p),\rho_j^2)\big)<0.$$

Let us introduce the following:

$$\tilde{t}_i=\frac{\frac{1}{\alpha_i^1(x,u)}}{A}, i=1,2,\cdots,s, \tag{11}$$
$$\tilde{\mu}_j=\frac{\frac{\mu_j}{\alpha_j^2(x,u)}}{A}, j\in J_\beta,\ \beta=1,2,\cdots,r \tag{12}$$

where $A=\sum\limits_{i=1}^{s}\frac{1}{\alpha_i^1(x,u)}+\sum\limits_{\beta=1}^{r}\sum\limits_{j\in J_\beta}\frac{\mu_j}{\alpha_j^2(x,u)}$.

Note that $0<\tilde{t}_i<1, i=1,2,\cdots,s$, $0<\tilde{\mu}_j<1, j\in J_\beta,\ \beta=1,2,\cdots,r$, and also $\sum\limits_{i=1}^{s}\tilde{t}_i+\sum\limits_{\beta=1}^{r}\sum\limits_{j\in J_\beta}\tilde{\mu}_j=1$.

Thus, in view of (11)-(12), inequality (10) we have

$$\sum_{i=1}^{s}\tilde{t}_i\Phi\big(x,u,\alpha_i^1(x,u)\big(\nabla\psi(u)+\sum_{j\in J_0}\mu_j\nabla h_j(u)+\nabla_p\psi_1(u,\bar{y}_i,p)$$
$$+\sum_{j\in J_0}\mu_j\nabla_p H_j(u,p),\rho_i^1)\big)+\sum_{\beta=1}^{r}\sum_{j\in J_\beta}\tilde{\mu}_j\Phi\big(x,u,\alpha_j^2(x,u)\big(\nabla h_j(u)+\nabla_p H_j(u,p),\rho_j^2)\big)<0.$$

Using the convexity of $\Phi\big(x,u,(.,.)\big)$ on R^{n+1}, we conclude that

$$\Phi\big(x,u,\sum_{i=1}^{s}\tilde{t}_i\alpha_i^1(x,u)\big(\nabla\psi(u)+\sum_{j\in J_0}\mu_j\nabla h_j(u)+\nabla_p\psi_1(u,\bar{y}_i,p)+\sum_{j\in J_0}\mu_j\nabla_p H_j(u,p),\rho_i^1\big)$$
$$+\sum_{\beta=1}^{r}\sum_{j\in J_\beta}\tilde{\mu}_j\alpha_j^2(x,u)\big(\nabla h_j(u)+\nabla_p H_j(u,p),\rho_j^2)\big)<0,$$

which yields,

$$\Phi\big(x,u,\frac{1}{A}\big(\nabla\psi(u)+\nabla\sum_{j=1}^{m}\mu_j h_j(u)+\nabla_p\psi_1(u,\bar{y}_i,p)$$
$$+\sum_{j=1}^{m}\mu_j\nabla_p H_j(u,p),\sum_{i=1}^{s}\rho_i^1+\sum_{\beta=1}^{r}\sum_{j\in J_\beta}\rho_j^2)\big)<0.$$

The above inequality and equation (3) imply

$$\Phi\big(x, u, \frac{1}{A}\big(0, \sum_{i=1}^{s} \rho_i^1 + \sum_{\beta=1}^{r} \sum_{j \in J_\beta} \rho_j^2\big)\big) < 0. \tag{13}$$

On the other hand, by using the hypothesis (iii) together with the fact that $\Phi\big(x, u, (0, b)\big)$ for each $b \in R_+$, we get

$$\Phi\big(x, u, \frac{1}{A}\big(0, \sum_{i=1}^{s} \rho_i^1 + \sum_{\beta=1}^{r} \sum_{j \in J_\beta} \rho_j^2\big)\big) \geq 0,$$

which contradicts (13). This completes the proof. ■

Theorem 3.2. (Strong duality)

Let x^* be an optimal solution for (NP) and let $\nabla h_j(x^*),\ j \in J(x^*)$, be linearly independent. Assume that

$$\begin{cases} F(x^*, \bar{y}_i^*, 0) = 0;\ \nabla_p F(x^*, \bar{y}_i^*, 0) = 0, i = 1, 2, ..., s, \\ G(x^*, \bar{y}_i^*, 0) = 0;\ \nabla_p G(x^*, \bar{y}_i^*, 0) = 0, i = 1, 2, ..., s, \\ H_j(x^*, 0) = 0;\ \ \nabla_p H_j(x^*, 0) = 0, j \in J. \end{cases} \tag{14}$$

Then there exist $(\bar{s}, \bar{t}, \tilde{y}^*) \in S(x^*)$ and $(x^*, \bar{\mu}, \bar{\lambda}, \bar{v}, \bar{w}, \bar{p}) \in H_1(\bar{s}, \bar{t}, \tilde{y}^*)$ such that $z^* = (x^*, \bar{\mu}, \bar{\lambda}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \tilde{y}^*, \bar{p} = 0)$ is feasible for (DMI) and the corresponding objective values of (NP) and (DMI) are equal. Furthermore, if the hypotheses of the weak duality theorem (Theorem 3.1) hold for all feasible solutions of (DMI), then $(x^*, \bar{\mu}, \bar{\lambda}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \tilde{y}^*, \bar{p} = 0)$ is an optimal solution for (DMI).

Proof:

By assumption x^* is an optimal solution of (NP) and $\nabla h_j(x^*), j \in J(x^*)$ are linearly independent. Then, by Theorem 2.3, there exist $(\bar{s}, \bar{t}, \tilde{y}^*) \in S(x^*)$ and $(x^*, \bar{\mu}, \bar{\lambda}, \bar{v}, \bar{w}, \bar{p} = 0) \in H_1(\bar{s}, \bar{t}, \tilde{y}^*)$ such that

$$\sum_{i=1}^{\bar{s}} \bar{t}_i \{\bigtriangledown f(x^*, \bar{y}_i) + B\bar{w} - \bar{\lambda}(\bigtriangledown g(x^*, \bar{y}_i) - C\bar{v})\} + \bigtriangledown \sum_{j=1}^{m} \bar{\mu}_j h_j(x^*) = 0,$$

$$f(x^*, \bar{y}_i) + (x^{*T} B x^*)^{1/2} - \bar{\lambda}(g(x^*, \bar{y}_i) - (x^{*T} C x^*)^{1/2}) = 0,\ i = 1, 2, \cdots, \bar{s},$$

$$\sum_{j=1}^{m} \bar{\mu}_j h_j(x^*) = 0,$$

$$\bar{t}_i \geq 0,\ i = 1, 2, \cdots, \bar{s}, \sum_{i=1}^{\bar{s}} \bar{t}_i = 1,$$

$$\bar{w}^T B \bar{w} \leq 1,\ \bar{v}^T C \bar{v} \leq 1, (x^{*T} B x^*)^{1/2} = x^{*T} B \bar{w}, (x^{*T} C x^*)^{1/2} = x^{*T} C \bar{v},$$

which along with equation (14) imply that of $z^* = (x^*, \bar{\mu}, \bar{\lambda}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \tilde{y}^*, \bar{p} = 0)$ is feasible for (DMI) and the problems (NP) and (DMI) have the same objective values. Optimality of z^* for (DMI), thus follows from the weak duality theorem (Theorem 3.2). ■

Theorem 3.3. (Strict converse duality)

Let x^* and $(\bar{u}, \bar{\mu}, \bar{\lambda}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \tilde{y}^*, \bar{p})$ be the optimal solutions for (NP) and (DMI), respectively. If

(i) $\psi(.) + \sum_{j\in J_0} \mu_j h_j(.)$ is strictly higher-order $(\Phi, \rho_i^1) - V_{\alpha_i^1}$-invex at $\bar{u}$ with respect to function $\psi_1(\bar{u}, \bar{y}_i^*, \bar{p}) + \sum_{j\in J_0} \bar{\mu}_j H_j(\bar{u}, \bar{p})$,

(ii) $h_j(.)$, $j \in J_\beta, \beta = 1, 2, ..., r$ is higher-order $(\Phi, \rho_j^2) - V_{\alpha_j^2}$-invex at $\bar{u}$ with respect to function $H_j, j \in J_\beta$,

(iii) $\nabla h_j(x^*), j \in J(x^*)$ be linear independent, and

(iv) $\sum_{i=1}^{\bar{s}} \rho_i^1 + \sum_{\beta=1}^{r} \sum_{j\in J_\beta} \bar{\mu}_j \rho_j^2 \geq 0$,

then, $x^* = \bar{u}$; that is, $\bar{u}$ is an optimal solution for (NP).

Proof:

Suppose to contrary that $x^* \neq \bar{u}$. As x^* and $(\bar{u}, \bar{\mu}, \bar{\lambda}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \tilde{y}^*, \bar{p})$ be the optimal solutions for (NP) and (DMI), respectively, and $\nabla h_j(x^*), j \in J(x^*)$, be linear independent, from Theorem 3.3, we know that

$$\sup_{y\in Y} \frac{f(x^*, \bar{y}^*) + (x^{*T} B x^*)^{1/2}}{g(x^*, \bar{y}^*) - (x^{*T} C x^*)^{1/2}} = \bar{\lambda}.$$

By hypotheses (i) and (ii), we have

$$\begin{aligned}
&\psi(x^*) + \sum_{j\in J_0} \mu_j h_j(x^*) - \psi(\bar{u}) - \psi_1(\bar{u}, \bar{y}_i^*, \bar{p}) + \bar{p}^T \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p}) - \sum_{j\in J_0} \bar{\mu}_j h_j(\bar{u}) \\
&\quad - \sum_{j\in J_0} \bar{\mu}_j H_j(\bar{u}, \bar{p}) + \bar{p}^T \sum_{j\in J_0} \mu_j \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}) \\
&> \Phi\big(x^*, \bar{u}, \alpha_i^1(x^*, \bar{u})\big(\nabla\psi(\bar{u}) + \sum_{j\in J_0} \bar{\mu}_j \nabla h_j(\bar{u}) + \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p}) \\
&\quad + \sum_{j\in J_0} \bar{\mu}_j \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}), \rho_i^1\big)\big).
\end{aligned} \tag{15}$$

$$\begin{aligned}
&h_j(x^*) - h_j(\bar{u}) - H_j(\bar{u}, \bar{p}) + \bar{p}^T \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}) \\
&\quad \geq \Phi\big(x^*, \bar{u}, \alpha_j^2(x^*, \bar{u})\big(\nabla h_j(\bar{u}) + \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}), \rho_j^2\big)\big), \ \forall j \in J_\beta, \ \beta = 1, 2, \cdots, r.
\end{aligned} \tag{16}$$

Multiplying each inequality (15) by $\frac{1}{\alpha_i^1(x^*, \bar{u})}$, $i = 1, 2, \cdots, s$, and each inequality (16) by

$\frac{\bar{\mu}_j}{\alpha_j^2(x^*,\bar{u})}, j \in J_\beta,\ \beta = 1, 2, \cdots, r$, respectively, then summing up these inequalities, we get

$$\sum_{i=1}^{\bar{s}} \frac{1}{\alpha_i^1(x^*,\bar{u})}[\psi(x^*) + \sum_{j\in J_0} \mu_j h_j(x^*) - \psi(\bar{u}) - \psi_1(\bar{u}, \bar{y}_i^*, \bar{p}) + \bar{p}^T \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p})$$
$$- \sum_{j\in J_0} \bar{\mu}_j h_j(\bar{u}) - \sum_{j\in J_0} \bar{\mu}_j H_j(\bar{u}, \bar{p}) + \bar{p}^T \sum_{j\in J_0} \mu_j \nabla_{\bar{p}} H_j(\bar{u}, \bar{p})]$$
$$> \sum_{i=1}^{\bar{s}} \frac{1}{\alpha_i^1(x^*,\bar{u})} \Phi\big(x^*, \bar{u}, \alpha_i^1(x^*,\bar{u})\big(\nabla\psi(\bar{u}) + \sum_{j\in J_0} \bar{\mu}_j \nabla h_j(\bar{u}) + \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p})$$
$$+ \sum_{j\in J_0} \bar{\mu}_j \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}), \rho_i^1\big)\big), \tag{17}$$

and

$$\sum_{\beta=1}^{r} \sum_{j\in J_\beta} \frac{\bar{\mu}_j}{\alpha_j^2(x^*,\bar{u})}[h_j(x^*) - h_j(\bar{u}) - H_j(\bar{u}, \bar{p}) + \bar{p}^T \nabla_{\bar{p}} H_j(\bar{u}, \bar{p})]$$
$$\geq \sum_{\beta=1}^{r} \sum_{j\in J_\beta} \frac{\bar{\mu}_j}{\alpha_j^2(x^*,\bar{u})} \Phi\big(x^*, \bar{u}, \alpha_j^2(x^*,\bar{u})\big(\nabla h_j(\bar{u}) + \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}), \rho_j^2\big)\big). \tag{18}$$

On the other hand, from the feasibility of x^* for (NP) and dual constraint (5), we have

$$\sum_{\beta=1}^{r} \sum_{j\in J_\beta} \frac{\bar{\mu}_j}{\alpha_j^2(x^*,\bar{u})}[h_j(x^*) - h_j(\bar{u}) - H_j(\bar{u}, \bar{p}) + \bar{p}^T \nabla_{\bar{p}} H_j(\bar{u}, \bar{p})] \leq 0.$$

Thus from (18), we obtain

$$\sum_{\beta=1}^{r} \sum_{j\in J_\beta} \frac{\bar{\mu}_j}{\alpha_j^2(x^*,\bar{u})} \Phi\big(x^*, \bar{u}, \alpha_j^2(x^*,\bar{u})\big(\nabla h_j(\bar{u}) + \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}), \rho_j^2\big)\big) \leq 0. \tag{19}$$

On combing (17) and (19), we have

$$\sum_{i=1}^{\bar{s}} \frac{1}{\alpha_i^1(x^*,\bar{u})}[\psi(x^*) + \sum_{j\in J_0} \mu_j h_j(x^*) - \psi(\bar{u}) - \psi_1(\bar{u}, \bar{y}_i^*, \bar{p}) + \bar{p}^T \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p})$$
$$- \sum_{j\in J_0} \bar{\mu}_j h_j(\bar{u}) - \sum_{j\in J_0} \bar{\mu}_j H_j(\bar{u}, \bar{p}) + \bar{p}^T \sum_{j\in J_0} \mu_j \nabla_{\bar{p}} H_j(\bar{u}, \bar{p})]$$
$$> \sum_{i=1}^{\bar{s}} \frac{1}{\alpha_i^1(x^*,\bar{u})} \Phi\big(x^*, \bar{u}, \alpha_i^1(x^*,\bar{u})\big(\nabla\psi(\bar{u}) + \sum_{j\in J_0} \bar{\mu}_j \nabla h_j(\bar{u}) + \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p})$$
$$+ \sum_{j\in J_0} \bar{\mu}_j \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}), \rho_i^1\big)\big)$$
$$+ \sum_{\beta=1}^{r} \sum_{j\in J_\beta} \frac{\bar{\mu}_j}{\alpha_j^2(x^*,\bar{u})} \Phi\big(x^*, \bar{u}, \alpha_j^2(x^*,\bar{u})\big(\nabla h_j(\bar{u}) + \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}), \rho_j^2\big)\big) \tag{20}$$

Let us introduce the following notations:

$$\tilde{t}_i = \frac{\dfrac{1}{\alpha_i^1(x^*,\bar{u})}}{\bar{A}}, i = 1, 2, \cdots, \bar{s}, \tag{21}$$

$$\tilde{\mu}_j = \frac{\dfrac{\bar{\mu}_j}{\alpha_j^2(x^*,\bar{u})}}{\bar{A}}, j \in J_\beta, ,\ \beta = 1, 2, \cdots, r \tag{22}$$

where $\bar{A} = \sum\limits_{i=1}^{\bar{s}} \dfrac{1}{\alpha_i^1(x^*,\bar{u})} + \sum\limits_{\beta=1}^{r} \sum\limits_{j \in J_\beta} \dfrac{\bar{\mu}_j}{\alpha_j^2(x^*,\bar{u})}$.

Note that $0 < \tilde{t}_i < 1, i = 1, 2, \cdots, s$, $0 < \tilde{\mu}_j < 1, j \in J_\beta,\ \beta = 1, 2, \cdots, r$, and also $\sum\limits_{i=1}^{\bar{s}} \tilde{t}_i + \sum\limits_{\beta=1}^{r} \sum\limits_{j \in J_\beta} \tilde{\mu}_j = 1$.

Thus, (20) together with (21)-(22) yield

$$\sum_{i=1}^{\bar{s}} \tilde{t}_i[\psi(x^*) + \sum_{j \in J_0} \mu_j h_j(x^*) - \psi(\bar{u}) - \psi_1(\bar{u}, \bar{y}_i^*, \bar{p}) + \bar{p}^T \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p})$$

$$- \sum_{j \in J_0} \bar{\mu}_j h_j(\bar{u}) - \sum_{j \in J_0} \bar{\mu}_j H_j(\bar{u}, \bar{p}) + \bar{p}^T \sum_{j \in J_0} \bar{\mu}_j \nabla_{\bar{p}} H_j(\bar{u}, \bar{p})]$$

$$> \sum_{i=1}^{s} \tilde{t}_i \Phi\big(x^*, \bar{u}, \alpha_i^1(x^*,\bar{u})\big(\nabla\psi(\bar{u}) + \sum_{j \in J_0} \bar{\mu}_j \nabla h_j(\bar{u}) + \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p})$$

$$+ \sum_{j \in J_0} \bar{\mu}_j \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}), \rho_i^1)\big) + \sum_{\beta=1}^{r} \sum_{j \in J_\beta} \tilde{\mu}_j \Phi\big(x^*, \bar{u}, \alpha_j^2(x^*,\bar{u})\big(\nabla h_j(\bar{u})$$

$$+ \nabla_{\bar{p}} H_j(\bar{u}, \bar{p}), \rho_j^2)\big).$$

Using the convexity of $\Phi\big(x^*, \bar{u}, (.,.)\big)$ on R^{n+1}, we conclude that

$$\sum_{i=1}^{\bar{s}} \tilde{t}_i \Bigg(\psi(x^*) + \sum_{j \in J_0} \mu_j h_j(x^*) - \psi(\bar{u}) - \psi_1(\bar{u}, \bar{y}_i^*, \bar{p}) + \bar{p}^T \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p})$$

$$- \sum_{j \in J_0} \bar{\mu}_j h_j(\bar{u}) - \sum_{j \in J_0} \bar{\mu}_j H_j(\bar{u}, \bar{p}) + \bar{p}^T \sum_{j \in J_0} \bar{\mu}_j \nabla_{\bar{p}} H_j(\bar{u}, \bar{p})\Bigg)$$

$$> \Phi\big(x^*, \bar{u}, \sum_{i=1}^{s} \tilde{t}_i \alpha_i^1(x^*,\bar{u})\big(\nabla\psi(\bar{u}) + \sum_{j \in J_0} \bar{\mu}_j \nabla h_j(\bar{u}) + \nabla_{\bar{p}} \psi_1(\bar{u}, \bar{y}_i^*, \bar{p})$$

$$+\sum_{j\in J_0}\bar{\mu}_j\nabla_{\bar{p}}H_j(\bar{u},\bar{p}),\rho_i^1)\big)+\sum_{\beta=1}^{r}\sum_{j\in J_\beta}\tilde{\mu}_j\Phi\big(x^*,\bar{u},\alpha_j^2(x^*,\bar{u})\big(\nabla h_j(\bar{u})$$

$$+\nabla_{\bar{p}}H_j(\bar{u},\bar{p}),\rho_j^2)\big).$$

Thus,

$$\sum_{i=1}^{\bar{s}}\tilde{t}_i\left(\psi(x^*)+\sum_{j\in J_0}\mu_jh_j(x^*)-\psi(\bar{u})-\psi_1(\bar{u},\bar{y}_i^*,\bar{p})+\bar{p}^T\nabla_{\bar{p}}\psi_1(\bar{u},\bar{y}_i^*,\bar{p})\right.$$

$$\left.-\sum_{j\in J_0}\bar{\mu}_jh_j(\bar{u})-\sum_{j\in J_0}\bar{\mu}_jH_j(\bar{u},\bar{p})+\bar{p}^T\sum_{j\in J_0}\bar{\mu}_j\nabla_{\bar{p}}H_j(\bar{u},\bar{p})\right)$$

$$>\Phi\big(x^*,\bar{u},\frac{1}{\bar{A}}\big(\nabla\psi(\bar{u})+\nabla\sum_{j=1}^{m}\bar{\mu}_jh_j(\bar{u})+\nabla_{\bar{p}}\psi_1(\bar{u},\bar{y}_i^*,\bar{p})$$

$$+\sum_{j=1}^{m}\bar{\mu}_j\nabla_{\bar{p}}H_j(\bar{u},\bar{p}),\sum_{i=1}^{\bar{s}}\rho_i^1+\sum_{\beta=1}^{r}\sum_{j\in J_\beta}\rho_j^2)\big)$$

The above inequality together with dual constraint (3), gives

$$\frac{1}{\bar{A}\alpha^1(x^*,\bar{u})}\left(\psi(x^*)+\sum_{j\in J_0}\mu_jh_j(x^*)-\psi(\bar{u})-\psi_1(\bar{u},\bar{y}_i^*,\bar{p})+\bar{p}^T\nabla_{\bar{p}}\psi_1(\bar{u},\bar{y}_i^*,\bar{p})\right.$$

$$\left.-\sum_{j\in J_0}\bar{\mu}_jh_j(\bar{u})-\sum_{j\in J_0}\bar{\mu}_jH_j(\bar{u},\bar{p})+\bar{p}^T\sum_{j\in J_0}\bar{\mu}_j\nabla_{\bar{p}}H_j(\bar{u},\bar{p})\right)>0.$$

Since $\frac{1}{\bar{A}}>0$, and $\alpha^1(x^*,\bar{u})>0$, the above inequality gives

$$\psi(x^*)+\sum_{j\in J_0}\mu_jh_j(x^*)-\psi(\bar{u})-\psi_1(\bar{u},\bar{y}_i^*,\bar{p})+\bar{p}^T\nabla_{\bar{p}}\psi_1(\bar{u},\bar{y}_i^*,\bar{p})$$

$$-\sum_{j\in J_0}\bar{\mu}_jh_j(\bar{u})-\sum_{j\in J_0}\bar{\mu}_jH_j(\bar{u},\bar{p})+\bar{p}^T\sum_{j\in J_0}\bar{\mu}_j\nabla_{\bar{p}}H_j(\bar{u},\bar{p})>0,$$

which together with the feasibility of x^* for (NP) and dual constraint (4) yields

$$\psi(x^*)>0,$$

$$\text{i.e., }\sum_{i=1}^{\bar{s}}\bar{t}_i\{f(x^*,\bar{y}_i^*)+(x^*)^TB\bar{w}-\bar{\lambda}[g(x^*,\bar{y}_i^*)-(x^*)^TC\bar{v}]\}>0.$$

Therefore, there exists a certain $i_\circ$, such that

$$f(x^*,\bar{y}^*_{i_\circ})+(x^{*T}Bx^*)^{1/2}-\bar{\lambda}(g(x^*,\bar{y}^*_{i_\circ})-(x^{*T}Cx^*)^{1/2})>0.$$

It follows that

$$\sup_{y\in Y}\frac{f(x^*,\bar{y}^*)+(x^{*T}Bx^*)^{1/2}}{g(x^*,\bar{y}^*)-(x^{*T}Cx^*)^{1/2}}\geq\frac{f(x^*,\bar{y}^*_{i_\circ})+(x^{*T}Bx^*)^{1/2}}{g(x^*,\bar{y}^*_{i_\circ})-(x^{*T}Cx^*)^{1/2}}>\bar{\lambda}.$$

Finally, we have a contradiction, and the proof is complete. ∎

4. Second duality model

In this section, we consider the following form of Theorem 2.3.

Theorem 4.1.

Let x^* be a solution for (NP) and let $\nabla h_j(\bar{x}), j\in J(x^*)$ be linearly independent. Then, there exist $(\bar{s},\bar{t},\bar{y})\in S(x^*)$ and $\bar{\mu}\in R^m_+$ such that

$$\sum_{i=1}^{\bar{s}}\bar{t}_i\{(g(x^*,\bar{y}_i)-(x^{*T}Cx^*)^{1/2})(\bigtriangledown f(x^*,\bar{y}_i)+Bw)-(f(x^*,\bar{y}_i)+(x^{*T}Bx^*)^{1/2})$$

$$(\bigtriangledown g(x^*,\bar{y}_i)-Cv)\}+\sum_{j=1}^{m}\bar{\mu}_j\bigtriangledown h_j(x^*)=0,$$

$$\sum_{j=1}^{m}\bar{\mu}_jh_j(x^*)\geq 0,$$

$$\bar{\mu}\in R^m_+,\bar{t}_i\geq 0,\sum_{i=1}^{\bar{s}}\bar{t}_i=1,\bar{y}_i\in Y(x^*),i=1,2,\cdots,\bar{s}.$$

Now, we consider the dual model of (NP) as follows:

(DMII) $$\max_{(s,t,\bar{y})\in S(u)}\ \sup_{(u,\mu,v,w,p)\in H_2(s,t,\bar{y})}\zeta(u),$$

where $\zeta(u)=\sup_{y\in Y}\frac{f(u,y)+(u^TBu)^{1/2}}{g(u,y)-(u^TCu)^{1/2}}$, and $H_2(s,t,\bar{y})$ denotes the set of all $(u,\mu,v,w,p)\in R^n\times R^m_+\times R^n\times R^n\times R^n$ satisfying:

$$\sum_{i=1}^{s}t_i\{(g(u,\bar{y}_i)-(u^TCu)^{1/2})(\bigtriangledown f(u,\bar{y}_i)+Bw)-(f(u,\bar{y}_i)+(u^TBu)^{1/2})$$

$$(\bigtriangledown g(u,\bar{y}_i)-Cv)\}+\sum_{j=1}^{m}\mu_j\bigtriangledown h_j(u)+\sum_{i=1}^{s}t_i\{\nabla_p\tilde{F}(u,\bar{y}_i,p)$$

$$-\nabla_p \tilde{G}(u, \bar{y}_i, p)\} + \sum_{j=1}^{m} \mu_j \nabla_p \tilde{H}_j(u, p) = 0, \tag{23}$$

$$\sum_{j=1}^{m} \mu_j h_j(u) + \sum_{i=1}^{s} t_i\{\tilde{F}(u, \bar{y}_i, p) - \tilde{G}(u, \bar{y}_i, p)\} - p^T \sum_{i=1}^{s} t_i \nabla_p \{\tilde{F}(u, \bar{y}_i, p)$$

$$-\tilde{G}(u, \bar{y}_i, p)\} + \sum_{j=1}^{m} \mu_j \tilde{H}_j(u, p) - p^T \sum_{j=1}^{m} \mu_j \nabla_p \tilde{H}_j(u, p) \geq 0, \tag{24}$$

$$(s, t, \bar{y}) \in S(u), \tag{25}$$

$$(u^T Bu)^{1/2} = u^T Bw, (u^T Cu)^{1/2} = u^T Cv, w^T Bw \leq 1, v^T Cv \leq 1, \tag{26}$$

where $\tilde{F} : R^n \times R^l \times R^n \to R$, $\tilde{G} : R^n \times R^l \times R^n \to R$ and $\tilde{H} : R^n \times R^n \to R^m$ are differentiable functions. If for a triplet $(s, t, \bar{y}) \in S(u)$, if the set $H_2(s, t, \bar{y})$ is empty, then we define the supremum over it to be $-\infty$.

Remark 4.1.

Let $\tilde{F}(u, \bar{y}_i, p) = (g(u, \bar{y}_i) - (u^T Cu)^{1/2})\frac{1}{2}p^T \nabla^2 f(u, \bar{y}_i)p$, $\tilde{G}(u, \bar{y}_i, p) = (f(u, \bar{y}_i) + (u^T Bu)^{1/2})\frac{1}{2}p^T \nabla^2 g(u, \bar{y}_i)p, i = 1, 2, ..., s$, $\tilde{H}_j(u, p) = \frac{1}{2}p^T \nabla^2 h_j(u)p, j = 1, 2, ..., m$. Then (DMII) reduces to the second order dual (DM1) (Gupta et al. (2012)). If in addition, $p = 0$, then we get the dual (DII) (Jayswal and Kumar (2011)).

In this section we denote

$$\psi_2(.) = \sum_{i=1}^{s} t_i\{(g(u, \bar{y}_i) - u^T Cv)(f(., \bar{y}_i) + (.)^T Bw) - (f(u, \bar{y}_i) + u^T Bw)(g(., \bar{y}_i) - (.)^T Cv)\},$$

and

$$\psi_3(u, \bar{y}_i, p) = \sum_{i=1}^{s} t_i[\tilde{F}(u, \bar{y}_i, p) - \tilde{G}(u, \bar{y}_i, p)].$$

Theorem 4.2. (Weak Duality)

Let x and $(u, \mu, v, w, s, t, \bar{y}, p)$ be feasible solutions of (NP) and (DMII) respectively. If

(i) $\psi_2(.)$ is higher-order $(\Phi, \sigma_i^1) - V_{\beta_i^1}$-invex at u with respect to function $\psi_1(u, \bar{y}_i, p)$,
(ii) $h_j(.)$, $j = 1, 2, ..., m$ is higher-order $(\Phi, \sigma_j^2) - V_{\beta_j^2}$-invex at u with respect to function $\tilde{H}_j, j = 1, 2, ..., m$, and
(iii) $\sum_{i=1}^{s} \sigma_i^1 + \sum_{j=1}^{m} \mu_j \sigma_j^2 \geq 0$,

then,

$$\sup_{y \in Y} \frac{f(x, y) + (x^T Bx)^{1/2}}{g(x, y) - (x^T Cx)^{1/2}} \geq \zeta(u).$$

Proof:

Suppose to the contrary that

$$\sup_{y \in Y} \frac{f(x,y) + (x^T Bx)^{1/2}}{g(x,y) - (x^T Cx)^{1/2}} < \zeta(u). \tag{27}$$

For any $y_i \in Y(u), i = 1, 2, ..., s$, we have

$$\zeta(u) = \frac{f(u,y_i) + (u^T Bu)^{1/2}}{g(u,y_i) - (u^T Cu)^{1/2}}. \tag{28}$$

From (27) and (28), we get

$$\frac{f(x,y_i) + (x^T Bx)^{1/2}}{g(x,y_i) - (x^T Cx)^{1/2}} \leq \sup_{y \in Y} \frac{f(x,y) + (x^T Bx)^{1/2}}{g(x,y) - (x^T Cx)^{1/2}} < \frac{f(u,y_i) + (u^T Bu)^{1/2}}{g(u,y_i) - (u^T Cu)^{1/2}}, i = 1, 2, ..., s.$$

Since $g(., y_i) - ((.)^T C(.))^{\frac{1}{2}} > 0$, therefore, we have

$$[(g(u,y_i) - (u^T Cu)^{1/2})(f(x,y_i) + (x^T Bx)^{1/2}) - (f(u,y_i) + (u^T Bu)^{1/2})$$
$$(g(x,y_i) - (x^T Cx)^{1/2})] < 0, i = 1, 2, ..., s.$$

It follows from $t_i \geq 0, i = 1, 2, ..., s$ and $t = (t_1, t_2, \cdots, t_s) \neq 0$ that

$$\sum_{i=1}^{s} t_i[(g(u,y_i) - (u^T Cu)^{1/2})(f(x,y_i) + (x^T Bx)^{1/2}) - (f(u,y_i)$$
$$+(u^T Bu)^{1/2})(g(x,y_i) - (x^T Cx)^{1/2})] < 0.$$

From (2) and (26), it follows that

$$\begin{aligned}\psi_2(x) &= \sum_{i=1}^{s} t_i\{(g(u,\bar{y}_i) - u^T Cv)(f(x,\bar{y}_i) + x^T Bw) - (f(u,\bar{y}_i) + u^T Bw)(g(x,\bar{y}_i) - x^T Cv)\} \\ &\leq \sum_{i=1}^{s} t_i\{(g(u,\bar{y}_i) - (u^T Cu)^{1/2})(f(x,\bar{y}_i) + (x^T Bx)^{1/2}) \\ &\quad - (f(u,\bar{y}_i) + (u^T Bu)^{1/2})(g(x,\bar{y}_i) - (x^T Cx))^{1/2}\} \\ &< 0 = \psi_2(u).\end{aligned}$$

That is,

$$\psi_2(x) - \psi_2(u) < 0. \tag{29}$$

Now, by hypotheses (i) and (ii), we have

$$\psi_2(x) - \psi_2(u) - \psi_3(u,\bar{y}_i,p) + p^T \nabla_p \psi_3(u,\bar{y}_i,p)$$
$$\geq \Phi\big(x, u, \beta_i^1(x,u)\big(\nabla\psi_2(u) + \nabla_p\psi_3(u,\bar{y}_i,p), \sigma_i^1\big)\big), \tag{30}$$

and

$$h_j(x) - h_j(u) - \tilde{H}_j(u,p) + p^T \nabla_p \tilde{H}_j(u,p)$$
$$\geq \Phi\big(x, u, \beta_j^2(x,u)\big(\nabla h_j(u) + \nabla_p \tilde{H}_j(u,p), \sigma_j^2\big)\big). \tag{31}$$

Multiplying each inequality (32) by $\frac{1}{\beta_i^1(x,u)}, i = 1, 2, \cdots, s$, and each inequality (33) by $\frac{\mu_j}{\beta_j^2(x,u)}, j = 1, 2, \cdots, m$, respectively, then summing up these inequalities, we get

$$\sum_{i=1}^{s} \frac{1}{\beta_i^1(x,u)} [\psi_2(x) - \psi_2(u) - \psi_3(u, \bar{y}_i, p) + p^T \nabla_p \psi_3(u, \bar{y}_i, p)]$$
$$\geq \sum_{i=1}^{s} \frac{1}{\beta_i^1(x,u)} \Phi\big(x, u, \beta_i^1(x,u)\big(\nabla \psi_2(u) + \nabla_p \psi_3(u, \bar{y}_i, p), \sigma_i^1\big)\big), \tag{32}$$

and

$$\sum_{j=1}^{m} \frac{\mu_j}{\beta_j^2(x,u)} [h_j(x) - h_j(u) - \tilde{H}_j(u,p) + p^T \nabla_p \tilde{H}_j(u,p)]$$
$$\geq \sum_{j=1}^{m} \frac{\mu_j}{\beta_j^2(x,u)} \Phi\big(x, u, \beta_j^2(x,u)\big(\nabla h_j(u) + \nabla_p \tilde{H}_j(u,p), \sigma_j^2\big)\big). \tag{33}$$

Let us introduce the following notations:

$$\tilde{\beta}_i(x,u) = \frac{\overline{\beta_i^1(x,u)}^{\,1}}{\tilde{A}}, i = 1, 2, \cdots, \bar{s}, \tag{34}$$

$$\tilde{\beta}_j(x,u) = \frac{\frac{\mu_j}{\beta_j^2(x,u)}}{\tilde{A}}, j = 1, 2, \cdots, r, \tag{35}$$

where $\tilde{A} = \sum\limits_{i=1}^{s} \frac{1}{\beta_i^1(x,u)} + \sum\limits_{j=1}^{m} \frac{\mu_j}{\beta_j^2(x,u)}$.

Note that $0 < \tilde{\beta}_i(x,u) < 1, i = 1, 2, \cdots, s$, $0 < \tilde{\beta}_j(x,u) < 1, j = 1, 2, \cdots, m$, and also $\sum\limits_{i=1}^{s} \tilde{\beta}_i(x,u) + \sum\limits_{j=1}^{m} \tilde{\beta}_j(x,u) = 1$.

Thus, (32)-(33) together with (34)-(35) yield, respectively

$$\sum_{i=1}^{s} \tilde{\beta}_i(x,u)[\psi_2(x) - \psi_2(u) - \psi_3(u, \bar{y}_i, p) + p^T \nabla_p \psi_3(u, \bar{y}_i, p)]$$
$$\geq \sum_{i=1}^{s} \tilde{\beta}_i(x,z) \Phi\big(x, u, \beta_i^1(x,u)\big(\nabla \psi_2(u) + \nabla_p \psi_3(u, \bar{y}_i, p), \sigma_i^1\big)\big), \tag{36}$$

and

$$\sum_{j=1}^{m}\tilde{\beta}_j(x,u)[h_j(x)-h_j(u)-\tilde{H}_j(u,p)+p^T\nabla_p\tilde{H}_j(u,p)] \geq \sum_{j=1}^{m}\tilde{\beta}_j(x,u)\Phi\big(x,u,\beta_j^2(x,u)\big(\nabla h_j(u)+\nabla_p\tilde{H}_j(u,p),\sigma_j^2\big)\big). \tag{37}$$

On adding (36), (37) and using the convexity of $\Phi(x,z,(.,.))$ on R^{n+1}, we get

$$\sum_{i=1}^{s}\tilde{\beta}_i(x,u)[\psi_2(x)-\psi_2(u)-\psi_3(u,\bar{y}_i,p)+p^T\nabla_p\psi_3(u,\bar{y}_i,p)]$$

$$+\sum_{j=1}^{m}\tilde{\beta}_j(x,u)[h_j(x)-h_j(u)-\tilde{H}_j(u,p)+p^T\nabla_p\tilde{H}_j(u,p)]$$

$$\geq \sum_{i=1}^{s}\Phi\big(x,u,\tilde{\beta}_i(x,u)\beta_i^1(x,u)\big(\nabla\psi_2(u)+\nabla_p\psi_3(u,\bar{y}_i,p),\sigma_i^1\big)\big)$$

$$+\sum_{j=1}^{m}\Phi\big(x,u,\tilde{\beta}_j(x,u)\beta_j^2(x,u)\big(\nabla h_j(u)+\nabla_p\tilde{H}_j(u,p),\sigma_j^2\big)\big)$$

$$\geq \Phi\big(x,u,\frac{1}{\tilde{A}}(\nabla\psi_2(u)+\nabla_p\psi_3(u,\bar{y}_i,p)+\sum_{j=1}^{m}\mu_j\nabla h_j(u)+\sum_{j=1}^{m}\mu_j\nabla_p\tilde{H}_j(u,p),\sum_{i=1}^{s}\sigma_i^1+\sum_{j=1}^{m}\mu_j\sigma_j^2)).$$

The above inequality together with dual constraint (23), hypothesis (iii), and the fact $\Phi(x,u,(0,a))$ for each $a\in R_+$, gives

$$\sum_{i=1}^{s}\frac{1}{\tilde{A}\beta_i^1(x,u)}\{\psi_2(x)-\psi_2(u)-\psi_3(u,\bar{y}_i,p)+p^T\nabla_p\psi_3(u,\bar{y}_i,p)\}$$

$$+\sum_{j=1}^{m}\frac{\mu_j}{\tilde{A}\beta_j^2(x,u)}\{h_j(x)-h_j(u)-\tilde{H}_j(u,p)+p^T\nabla_p\tilde{H}_j(u,p)\}\geq 0.$$

Now, assume $\beta_i^1(x,u)=\beta_j^2(x,u)=\beta(x,u)>0$, then the above inequality gives

$$\psi_2(x)-\psi_2(u)-\psi_3(u,\bar{y}_i,p)+p^T\nabla_p\psi_3(u,\bar{y}_i,p)$$

$$+\sum_{j=1}^{m}\mu_j\left\{h_j(x)-h_j(u)-\tilde{H}_j(u,p)+p^T\nabla_p\tilde{H}_j(u,p)\right\}\geq 0.$$

Utilizing the feasibility of x for (NP), dual constraint (24), we conclude from the above inequality

$$\psi_2(x)-\psi_2(z)\geq 0,$$

which contradicts (29). This completes the proof. ∎

In a similar way as discussed in section 3, we can prove the following theorems 4.3, 4.4 between (NP) and (DMII). Therefore, we simply state them here.

Theorem 4.3. (Strong duality)

Let x^* be an optimal solution for (NP), and let $\nabla h_j(x^*),\ j \in J(x^*)$, be linearly independent. Assume that

$$\tilde{F}(x^*, \bar{y}_i^*, 0) = 0; \qquad \nabla_p \tilde{F}(x^*, \bar{y}_i^*, 0) = 0, i = 1, 2, ..., s,$$

$$\tilde{G}(x^*, \bar{y}_i^*, 0) = 0; \qquad \nabla_p \tilde{G}(x^*, \bar{y}_i^*, 0) = 0, i = 1, 2, ..., s,$$

$$\tilde{H}_j(x^*, 0) = 0; \qquad \nabla_p \tilde{H}_j(x^*, 0) = 0, j \in J.$$

Then there exist $(\bar{s}, \bar{t}, \tilde{y}^*) \in S(x^*)$ and $(x^*, \bar{\mu}, \bar{v}, \bar{w}, \bar{p}) \in H_2(\bar{s}, \bar{t}, \tilde{y}^*)$ such that $(x^*, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \tilde{y}^*, \bar{p} = 0)$ is feasible for (DMII) and the two objectives have same value. If, in addition, the hypotheses of the weak duality theorem (Theorem 4.3) hold for all feasible solutions of (DMII), then $(x^*, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \tilde{y}^*, \bar{p} = 0)$ is an optimal solution for (DMII).

Theorem 4.4. (Strict converse duality)

Let x^* and $(\bar{u}, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \tilde{y}^*, \bar{p})$ be the optimal solutions for] (NP) and (DMII), respectively. If

(i) $\psi_2(.)$ is strictly higher-order $(\Phi, \sigma_i^1) - V_{\beta_i^1}$-invex at $\bar{u}$ with respect to function $\psi_3(\bar{u}, \bar{y}_i^*, \bar{p})$,
(ii) $h_j(.), j = 1, 2, ..., m$ is higher-order $(\Phi, \sigma_j^2) - V_{\beta_j^2}$-invex at $\bar{u}$ with respect to function $\tilde{H}_j$,
(iii) $\nabla h_j(x^*), j \in J(x^*)$ be linear independent, and
(iv) $\sum_{i=1}^{\bar{s}} \sigma_i^1 + \sum_{j=1}^{m} \bar{\mu}_j \sigma_j^2 \geq 0$,

then $x^* = \bar{u}$; that is, $\bar{u}$ is an optimal solution for (NP).

5. Conclusion

In this paper, we have formulated two types of higher order dual models for a nondifferentiable minimax fractional programming problem and proved an appropriate duality relations involving higher-order (Φ, ρ)-V-invex functions. This work can be further extended to study for the following nondifferentiable minimax fractional programming problem:

$$\textbf{(CNP)} \qquad \min_{x \in R^n} \sup_{\nu \in W} \frac{Re[f(\xi, \nu) + (x^H A x)^{1/2}]}{Re[g(\xi, \nu) - (x^H B x)^{1/2}]},$$

$$\text{subject to} \quad -h(\xi) \in S,\ \xi \in C^{2n},$$

where $\xi = (z, \bar{z}),\ \nu = (w, \bar{w})$ for $z \in C^n,\ w \in C^m$. $f(.,.) : C^{2n} \times C^{2m} \to C$ and $h(.,.) : C^{2n} \times C^{2m} \to C$ are analytic with respect to ξ, W is a specified compact subset in C^{2m}, S is a polyhedral cone in C^p and $g : C^{2n} \to C^p$ is analytic. Also $A, B \in C^{n\times n}$ are positive semi-definite Hermitian matrices. This will orient the future research of the authors.

Acknowledgement:

This research is supported by the King Fahd University of Petroleum and Minerals, Saudi Arabia under the Internal Research Project No. IN171012.

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