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The Binomial Transform of P-Recursive Sequences And the Dilogarithm Function

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Abstract

Using a generalized binomial transform and a novel binomial coefficient identity, we will show that the set of p-recursive sequences is closed under the binomial transform. Using these results, we will derive a new series representation for the dilogarithm function that converges on its domain of analyticity. Finally, we will show that this series representation results in a scheme for numerical evaluation of the dilogarithm function that is accurate, efficient, and stable.

Keywords: Special functions; Analytic continuation; Binomial transform; Dilogarithm

MSC 2010 No.: 33B99, 33F05

1. Introduction

The binomial transform is useful numerous contexts, both in applied and pure mathematics. For an overview of the many applications of this transform, consult the comprehensive book by Boyadzhiev (2018). Specifically, the relation of the binomial transform to finite differences makes it well suited for applications to analytic continuation and series acceleration; for some of these applications, see Hirofumi (2015) and Willis (2016).

The primary result of this paper is a proof that the set of sequences that satisfy linear recursion relations with polynomial coefficients, called the set of *p-recursive sequences*, is closed under the binomial transform. Actually, we will prove this result using a generalization of the binomial transform that was first introduced by Prodinger (1994).

We will start with a derivation of generalized binomial transform that clarifies why Prodinger's generalization is particularly useful for series acceleration. After that, we will prove some novel and interesting binomial coefficient identities that we will need for our main result. Finally, we will show an application of our results, namely a new series expansion of the dilogarithm function that converges on its *entire* domain. This result is particularly noteworthy because recently Schmidt (2016) derived a series for the dilogarithm that converges only in the half-plane $\mathrm{Re}(x) < 1/2$.

2. Generalized binomial transform

For analytic continuation and series acceleration, the utility of the binomial transform stems from that fact that it can be derived from a sequence of extrapolated sequences. To show this, we build on the work of Boyadzhiev (2014). For any sequence F, we start by defining the backward shift operator S as

$$(\mathrm{S}\,F)_n = \begin{cases} 0, & n = 0, \\ F_{n-1}, & n > 0. \end{cases} \tag{1}$$

For a sequence $F^{(0)}$, we define extrapolated sequences $F^{(1)}, F^{(2)}, \ldots$, by $F^{(k)} = (\beta\,\mathrm{I} + \alpha\,\mathrm{S})^k\,F^{(0)}$, where $\alpha, \beta \in \mathbf{C}$. We call these extrapolated sequences because assuming $F^{(0)}$ converges linearly to L, there is a choice of α and β that makes the linear convergence rate of $F^{(\ell)}$ faster with larger ℓ.

Extracting the n[th] term of the n[th] extrapolated sequence yields a sequence $n \mapsto F_n^{(n)}$. We call this sequence the *generalized binomial transform* of $F^{(0)}$. Defined this way, a calculation shows that the binomial transform operator $\mathrm{B}^{(\alpha,\beta)}$ is

$$\left(\mathrm{B}^{(\alpha,\beta)} F\right)_n = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \beta^k F_k. \tag{2}$$

In a different context, Prodinger (1994) also introduced this form of the binomial transform.

The composition rule for the generalized binomial transform is $\mathrm{B}^{(\alpha,\beta)}\mathrm{B}^{(\alpha',\beta')} = \mathrm{B}^{(\alpha+\alpha'\beta,\beta\beta')}$. Since $\mathrm{B}^{(0,1)}$ is the identity operator, it follows from the composition rule that for $\beta \neq 0$, the operator $\mathrm{B}^{(\alpha,\beta)}$ is invertible and its inverse is $\mathrm{B}^{(-\alpha/\beta,1/\beta)}$. The adjoint of the binomial transform, denoted with a superscript star, is

$$\left(\mathrm{B}^{(\alpha,\beta)\star} F\right)_n = \sum_{k=n}^{\infty} \binom{k}{n} \alpha^{k-n} \beta^n F_k. \tag{3}$$

Assuming convergence of all sums, the adjoint gives the identity

$$\sum_{k=0}^{\infty} F_k G_k = \sum_{k=0}^{\infty} \big(\mathrm{B}^{(-\alpha/\beta,1/\beta)\star} G\big)_k \big(\mathrm{B}^{(\alpha,\beta)} F\big)_k. \tag{4}$$

Specializing to $G_k = 1$ and $\beta = 1$, gives

$$\sum_{k=0}^{\infty} F_k = \sum_{k=0}^{\infty} \frac{1}{(\alpha+1)^{k+1}} \big(\mathrm{B}^{(\alpha,1)} F\big)_k. \tag{5}$$

This identity is an extension of the Euler transform. For a description of the Euler transform, see Olver et al. (2012).

We will use this summation identity to derive a new series representation for the dilogarithm function Li_2. The key to deriving this result is a new binomial coefficient identity.

3. Binomial coefficient identities

Recall that a sequence that satisfies a linear homogeneous recursion relation with polynomial coefficients is said to be *p-recursive*; for details, see Schneider et al. (2013). The set of p-recursive sequences is known to be closed under addition and multiplication; for proofs, see Kauers (2011) and Zeilberger (1990). We will show that the set of p-recursive sequences is closed under the generalized binomial transform.

A possible starting place for a proof is to extend a result from Boyadzhiev (2017) for the binomial transform of $k \mapsto k^p F_k$, where p is a positive integer, to the generalized binomial transform. Instead, our proof is based on the novel binomial coefficient identity

$$k\binom{n}{k} = n\binom{n}{k} - n\binom{n-1}{k}. \tag{6}$$

The identity is straightforward to prove and we think that using it provides more insight to the proof of closure under the binomial transform. Extending this identity by multiplying it by k and iterating, allows us to express $k^p\binom{n}{k}$, where $k \in \mathbf{Z}_{\geq 0}$, as a linear combination of the set $\left\{\binom{n}{k}, \binom{n-1}{k}, \binom{n-2}{k}, \ldots, \binom{n-p}{k}\right\}$ with coefficients that involve only n. Table 1 displays these results for p up to three. The third row of this table, for example, corresponds to the identity $k^2\binom{n}{k} = n^2\binom{n}{k} - n\,(2n-1)\binom{n-1}{k} + (n-1)\,n\binom{n-2}{k}$.

Introducing a multiplication operator M on the set of sequences defined by $(\mathrm{M}F)_n = nF_n$ and using the identity $k\binom{n}{k} = n\binom{n}{k} - n\binom{n-1}{k}$, we can show that $\mathrm{B}^{(\alpha,\beta)}\mathrm{M} = \mathrm{M}\big(\mathrm{I} - \alpha\,\mathrm{S}\big)\mathrm{B}^{(\alpha,\beta)}$, where S is the backward shift operator. Consequently, for all $p \in \mathbf{Z}_{\geq 0}$, we have

$$\mathrm{B}^{(\alpha,\beta)}\mathrm{M}^p = \big(\mathrm{M}(\mathrm{I} - \alpha S)\big)^p \mathrm{B}^{(\alpha,\beta)}. \tag{7}$$

Further, using the Pascal identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, we can show that $\beta\mathrm{B}^{(\alpha,\beta)}\,\mathrm{S}^\star = (\mathrm{S}^\star - \alpha\,\mathrm{I})\,\mathrm{B}^{(\alpha,\beta)}$. Extending this result to any positive integer power p of $\mathrm{S}^\star$ yields

$$\beta^p\mathrm{B}^{(\alpha,\beta)}\,\mathrm{S}^{\star\,p} = (\mathrm{S}^\star - \alpha\,\mathrm{I})^p\,\mathrm{B}^{(\alpha,\beta)}. \tag{8}$$

Table 1. Binomial coefficient identities

	$\binom{n-3}{k}$	$\binom{n-2}{k}$	$\binom{n-1}{k}$	$\binom{n}{k}$
$\binom{n}{k}$	0	0	0	1
$k\binom{n}{k}$	0	0	$-n$	n
$k^2\binom{n}{k}$	0	$(n-1)\,n$	$-n\,(2n-1)$	n^2
$k^3\binom{n}{k}$	$-(n-2)\,(n-1)\,n$	$3(n-1)^2n$	$-n\,(3n^2-3n+1)$	n^3

Using these two results, we can express the binomial transform of $n \mapsto n^p F_{n+q}$ in terms of $\mathrm{B}^{(\alpha,\beta)}F$ for all positive integers p and q. Consequently, we have shown that the set of p-recursive sequences is closed under the generalized binomial transform.

4. The Dilogarithm Function

The dilogarithm function Li_2 can be defined by its Maclaurin series

$$\mathrm{Li}_2(x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)^2}. \tag{9}$$

Inside the unit circle, the series converges linearly; on the unit circle, it converges sublinearly, and outside the unit circle, it diverges. Although various functional identities, for example $\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{1}{6}\pi^2 - \frac{1}{2}(\ln(-z))^2$, analytically continue Li_2 to $\mathbf{C} \setminus [1,\infty]$, collectively these series converge only *sublinearly* at the points $(1 \pm \mathrm{i}\sqrt{3})/2$. And near these two points, the Maclaurin series converges slowly. Using the generalized binomial transform, we will find a series representation that converges linearly on $\mathbf{C}\setminus[1,\infty)$. The currently available series require a patchwork of methods.

The summand of the Maclaurin series for Li_2, call it Q, is p-recursive. Thus we consider the convergence set for the formal identity $\mathrm{Li}_2(x) = \sum_{k=0}^{\infty} \widehat{Q}_k/(\alpha+1)^{k+1}$, where $\widehat{Q} = \mathrm{B}^{(\alpha,1)}Q$. Although we will not explicitly use this fact, the sequence $\widehat{Q}$ has an unwieldy representation in terms of a ${}_3\mathrm{F}_2$ hypergeometric function; it is

$$\widehat{Q}_k = {}_3\mathrm{F}_2\left[\begin{matrix}-k,1,1\\2,2\end{matrix};-x/\alpha\right] x\alpha^n. \tag{10}$$

The sequence Q satisfies the recursion $(k+2)^2\,Q_{k+1} = (k+1)^2\,Q_k$. Using Table 1, the recursion for $\widehat{Q}$ has the form $0 = P_0(n)\widehat{Q}_n + P_1(n)\widehat{Q}_{n+1} + P_2(n)\widehat{Q}_{n+2} + P_3(n)\widehat{Q}_{n+3}$, where the polynomials P_0 through P_3 are

$$P_0(n) = -\alpha^2(\alpha+x)(n+1)(n+2), \tag{11}$$

$$P_1(n) = \alpha(n+2)(3n\alpha+2nx+8\alpha+5x), \tag{12}$$

$$P_2(n) = -(3\,\alpha+x)\,n^2 - (19\,\alpha+6\,x)\,n - 26\,\alpha - 9\,x, \tag{13}$$

$$P_3(n) = (n+2)(n+6). \tag{14}$$

Assuming $\alpha \neq -x$, a fundamental solution set for this recursion is

$$\left\{ n \mapsto \frac{\alpha^n}{n+1}, \quad n \mapsto \frac{\alpha^n}{n+1} \sum_{k=0}^{n} \frac{1}{k+1}, \quad n \mapsto \frac{(\alpha+x)^n}{n^2} \left(1 + \mathrm{O}\left(1/n\right)\right) \right\}. \tag{15}$$

The first two members of this set are exact, but the third is an asymptotic solution that is valid toward infinity. The fundamental solution set shows that the formal series converges linearly, provided that

$$\max\left(\left| \frac{\alpha}{\alpha+1} \right|, \left| \frac{\alpha+x}{\alpha+1} \right| \right) < 1 \text{ and } \alpha \in \mathbf{C}_{\neq -x, \neq -1}. \tag{16}$$

The convergence set is maximized when $\left|\frac{\alpha}{\alpha+1}\right| = \left|\frac{\alpha+x}{\alpha+1}\right|$. Assuming $x \in \mathbf{R}$, the convergence set is maximized when $\alpha = -x/2$. For this choice, the linear convergence rate is $|x/(x-2)|$ and the series converges in the half plane $\mathrm{Re}(x) < 1$. For $x \in \mathbf{C} \setminus \mathbf{R}$, the convergence set is maximized when

$$\alpha = \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta} - 1} x \text{ , where } \mathrm{e}^{\mathrm{i}\theta} = \pm \sqrt{\frac{\overline{x} - 1}{x - 1}}. \tag{17}$$

Setting $x = 1 + R\exp(\mathrm{i}\omega)$, where $R \in \mathbf{R}_{\geq 0}$ and $\omega \in [0, 2\pi)$, the minimum of the linear convergence rate $|\alpha/(\alpha+1)|$ is

$$\min\left(\frac{2R\cos(\omega) + R^2 + 1}{(R-1)^2}, \frac{2R\cos(\omega) + R^2 + 1}{(R+1)^2} \right). \tag{18}$$

For $\omega \in (0, 2\pi)$, or equivalently for $x \in \mathbf{C} \setminus [1, \infty)$, the linear convergence rate is less than one. We have shown that there is a value of α that makes the series $\sum_{k=0}^{\infty} \widehat{Q}_k/(\alpha+1)^{k+1}$ converge linearly on $\mathbf{C} \setminus [1, \infty)$. Recently, Schmidt (2016) derived a series that only converges in the half-plane $\mathrm{Re}(x) < 1/2$.

In the next section, we will comment on some of the practical considerations of using this series to numerically evaluate Li_2.

5. Accuracy, efficiency, and stability

For our series representation to be useful for numerical evaluation, the sum must be well conditioned (accuracy), the convergence must be fast (efficiency), and every solution to the fundamental solution set to the recursion for the summand must converge to zero (stability).

Of these three conditions, we have already shown that for a particular choice of parameter α each member of the fundamental solution set to the recursion relation converges to zero; thus the recursion for the summand is stable.

We can achieve greater efficiency by leveraging various functional identities. The algorithm can automatically choose between them to minimize the linear convergence rate. Making an optional choice, our tests show that to achieve full accuracy with IEEE binary64 numbers at most 70 terms need to be summed, including all points on the unit circle, including the difficult points $(1 \pm \mathrm{i}\sqrt{3})/2$.

Additionally, our numerical experiments show that the sum is well-conditioned. Higham (2002) has many details about the condition number of a sum.

6. Conclusion

Our derivation of the generalized binomial transform highlights the reason for its effectiveness for series acceleration methods and for analytic continuation. Additionally, by introducing the operator adjoint, we have shown how our method extends the venerable Euler summation method.

Further, using a simple, yet novel binomial coefficient identity, we were able to show that the set of p-recursive sequences is closed under the generalized binomial transform. Our proof is constructive; that is, our proof gives a mechanical method for determining the recursion for the transformed sequence. Using operator notation, we are able to express these results in a form that is both brief and intelligible.

We merged these results to derive a *single* series representation for the dilogarithm function that converges linearly on its entire domain of analyticity. The series available currently involve a patchwork of series that cover various proper subsets of the domain. In particular, our series converges linearly at the points $(1 \pm \mathrm{i}\sqrt{3})/2$. In part we think this is a noteworthy result because other series, for example, the various power series for the Gauss hypergeometric function, also fail to converge linearly at these same two points.

Finally, we gave a brief discussion of the utility our series representation. In particular, the recursion for the summand is stable, the sum is well conditioned, and it converges linearly on $\mathbf{C} \setminus [1, \infty)$. This is in contrast to recent work on series representations for Li_2 that converge on a smaller portion of the complex plane.

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