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Classification of Some First Order Functional Differential Equations With Constant Coefficients to Solvable Lie Algebras

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Abstract

In this paper, we shall apply symmetry analysis to some first order functional differential equations with constant coefficients. The approach used in this paper accounts for obtaining the inverse of the classification. We define the standard Lie bracket and make a complete classification of some first order linear functional differential equations with constant coefficients to solvable Lie algebras. We also classify some nonlinear functional differential equations with constant coefficients to solvable Lie algebras.

Keywords: Delay differential equations; Determining equations; Group analysis; Lie group; Neutral differential equations; Solvable Lie algebras; Symmetries

MSC 2010 No.: 34K06, 34K40, 34C14, 22E99

1. Introduction

Functional differential equations are used in modeling several physical phenomena. A particular class of functional differential equations called neutral differential equations are used in models involving flip-flop circuit studied by Schmitt and Klaus (1972), compartmental systems by Víctor et al. (2008), etc. Neutral differential equations are solved using multistep block method in Seong and Majid (2015) and by hybrid multistep block method in Ismail et al. (2020). Other methods of solution include implicit block method by Ishak and Mohd. (2015), and analyzing discontinuities of the derivatives as studied by Alfredo and Guglielmi (2009). Global exponential stability of the zero solution of a neutral differential equation with time-lags is obtained by Tunç and Altun (2017) by using the Lyapunov-Krasovskii functional approach. Further, Altun (2019) has investigated the asymptotic stability of Riemann-Liouville fractional neutral systems with variable delays by using the advantage of the Lyapunov functional. In general, these functional differential equations have a wide range of applications which include signal processing, study of epidemics and biological systems, networking problems, etc. as seen in Kyrychko and Hogan (2010). For other well known methods to solve these kinds of differential equations one can refer to Deo et al. (2013). A great detail of literature on delay differential equations can be found in Driver (1977) and Hale (1977).

Symmetries are transformations which leave an object unchanged or invariant. Symmetries make a very important tool in studying various laws governing nature. In Oliveri (2010), it is pointed that symmetry accounts for the regularities of the laws that are independent of some inessential circumstances. A very important implication of symmetry in Physics and Mathematics is the existence of conservation laws. Emmy Nöether in 1918 observed this connection in proving a relation between continuous symmetries and conservation laws. The concept of one-parameter groups which leave the differential equation invariant is the only unified understanding as to why a differential equation can be solved. Sophus Lie, over a century back, initiated this study. Lie group analysis is an excellent tool in studying the properties of solutions of functional differential equations. Applications of symmetry methods are seen for non-Newtonian fluid flow differential equations in Rehman and Malik (2019). Symmetry reduction methods have also been used to solve nonlinear heat transfer problems (Latif et al. (2015)). A lot of literature on symmetries, Lie groups, Lie algebras and related concepts can be found in Dressner (1999) and Ibragimov (1996).

The presence of delay terms makes especially the higher order nonlinear differential equations difficult to solve. As there is no analytic method to solve them directly, group analysis is the best way to study the properties of delay and neutral differential equations. Most of the existing research on symmetry analysis are done by changing the space variables. However, differential equations with deviating arguments do not possess any equivalent transformations related with the change of the variables – both dependent and independent. We consider the absence of such equivalent transformations to obtain a basis for the solvable Lie algebras of such functional differential equations. Such a classification is of immense help in problems modeled by functional differential equations which are not always easy to solve. We provide a basis for the Lie algebra given by the first order linear and nonlinear functional differential equations, for which there is no existing literature.

Tanthanuch and Meleshko (2004), obtain symmetries of delay differential equations by defining a certain operator, equivalent to the canonical Lie-Bäcklund operator. Linchuk (2001) suggests a group method based on a search of symmetries of underdetermined systems of differential equations. The method therein, encompasses the use of a basis of invariants consisting of universal and differential invariants. Recently, in Lobo and Valaulikar (2018) an admitted Lie group for first order delay differential equations with constant coefficients is defined, and the corresponding generators of the Lie group for this equation are obtained. Further, first order neutral differential equations with most general time delay have been studied by Lobo and Valaulikar (2019). Nass (2019) has employed Lie symmetry method to solve fractional neutral ordinary differential equations and to obtain the infinite dimensional symmetry algebras. The definition of the classical prolongation formulas of point transformations are extended to conformable derivatives (ordinary and partial) (Chatibi et al. (2019)). Irshad et al. (2019) used complex Lie-symmetry methods to calculate Noether-like operators and first integrals of a scalar second-order ordinary differential equation. Applications of Lie symmetries and numerical methods to solve the resulting ordinary differential equation are seen in Kumar et al. (2018) in studying the variable-coefficient modified Burgers-KdV equation by furnishing the infinitesimals of the group of transformations leaving the equation invariant.

The first novel idea in this paper is that the Lie type invariance condition for functional differential equations is obtained by using Taylor's theorem for a function of several variables. The existing method in literature uses the Lie-Bäcklund operator and an invariant manifold theorem to obtain the invariance condition for delay differential equations. This approach results in terms with magnification of the delay. No such magnification is seen in our approach which makes computation easier. We then make a classification of first order linear and some nonlinear functional differential equations to solvable Lie algebras for which there is no literature. The drawback of the analysis in all the existing papers is that the inverse of the obtained classification cannot be found. With our new classification scheme, the inverse of the classification can be obtained.

The rest of this paper is organised as follows. The next section extends the results for ordinary differential equations (Arrigo (2015); Bluman and Kumei (1989)) to functional differential equations, by obtaining a Lie type invariance condition using Taylor's theorem for a function of several variables. In the sections to follow, each section will consist of two subsections: one for linear and the other for nonlinear functional differential equations with constant coefficients. Each section will independently be concerned with (i) First order delay differential equations, and (ii) First order neutral differential equations. We conclude with a summary of our results.

2. Lie Type Invariance Condition for First Order Functional Differential Equations

In this section, we shall extend the results for ordinary differential equations to functional differential equations and obtain a Lie type invariance condition for the functional differential equation,

$$\Phi(t, x(t), x(t-r), x'(t), x'(t-r)) = 0, \tag{1}$$

where Φ is a real valued function defined on $I \times D^4$, where D is an open set in $\mathbb{R}$, I is an open interval in $\mathbb{R}$ and $r > 0$ is the delay. We use the notations $x'(t-r)$ to mean $\frac{dx}{dt}(t-r)$. We shall find a Lie group under which these functional differential equations are invariant. We call this the admitted Lie group by which we mean that one solution curve is carried to another solution curve of the same equation. We also illustrate the classification of some nonlinear delay and neutral differential equations to solvable Lie algebras. For the one-parameter group of transformations, $\bar{t} = f_1(t, x; \delta), \bar{x} = f_2(t, x; \delta)$, where f_1 and f_2 are smooth functions in t and x having a convergent Taylor series in δ. We define $\omega(t, x) = \dfrac{\partial f_1(t, x; 0)}{\partial \delta}$ and $\Upsilon(t, x) = \dfrac{\partial f_2(t, x; 0)}{\partial \delta}$, and ω and Υ are called coefficients of the infinitesimal transformations or simply infinitesimals.

The simplest examples of Lie groups include the *stretching group* given by $\bar{t} = a^{\delta} t, t \in \mathbb{R}\backslash\{0\}$, the *translational group* given by $\bar{t} = t + \delta$, and the *Rotational group* given by $\bar{t}_1 = t_1 \cos\delta - t_2 \sin\delta,\ \bar{t}_2 = t_1 \sin\delta + t_2 \cos\delta$.

We now prove a Lie type invariance condition using Taylor's theorem for a function of several variables, which is a novel approach.

Theorem 2.1.

Let a function F be defined on $I \times D^3$, where D is an open set in $\mathbb{R}$, and I is an open interval in $\mathbb{R}$. Then, with the notations, $\omega^r = \omega(t-r, x(t-r)),$ and $\Upsilon^r = \Upsilon(t-r, x(t-r))$, the Lie type invariance condition for

$$\frac{dx}{dt} = F(t, x(t), x(t-r), x'(t-r)), \tag{2}$$

is given by

$$\omega F_t + \Upsilon F_x + \Upsilon^r F_{x(t-r)} + \Upsilon^r_{[t]} F_{x'(t-r)} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2,$$

where

$$\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega) = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2,$$

$$\Upsilon^r_{[t]} = (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) - (x'(t-r))^2(\omega_x)^r,$$

where $D_t = \frac{\partial}{\partial t} + x' \frac{\partial}{\partial x}$.

Proof:

Let the neutral differential equation be invariant under the Lie group

$$\bar{t} = t + \delta\omega(t, x) + O(\delta^2), \quad \bar{x} = x + \delta\Upsilon(t, x) + O(\delta^2).$$

We then naturally define $\overline{t-r} = t - r + \delta\omega(t-r, x(t-r)) + O(\delta^2)$ and $\overline{x(t-r)} = x(t-r) + \delta\Upsilon(t-r, x(t-r)) + O(\delta^2)$.

With the notations $\omega^r = \omega(t-r, x(t-r))$ and $\Upsilon^r = \Upsilon(t-r, x(t-r)))$, it follows that,

$$\begin{aligned} \overline{x'(t-r)} &= \tfrac{d\bar{x}}{d\bar{t}}(\overline{t-r}) \\ &= x'(t-r) + (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) \\ &- (x'(t-r))^2(\omega_x)^r)\delta + O(\delta^2). \end{aligned} \tag{3}$$

For invariance, $\frac{d\bar{x}}{d\bar{t}} = F(\bar{t}, \bar{x}, \overline{(x(t-r))}, \overline{x'(t-r)})$.

This gives,

$$
\begin{gathered}
F(t+\delta\omega+O(\delta^2), x+\delta\Upsilon+O(\delta^2), x(t-r)+\delta\Upsilon^r+O(\delta^2), x'(t-r)+((\Upsilon_t)^r \\
+((\Upsilon_x)^r-(\omega_t)^r)x'(t-r)-(x'(t-r))^2(\omega_x)^r)\delta+O(\delta^2))
\end{gathered}
$$

$$
\begin{aligned}
&= F(t,x,x(t-r),x'(t-r))+(\omega F_t+\Upsilon F_x+\Upsilon^r F_{x(t-r)}+\Upsilon^r_{[t]}F_{x'(t-r)})\delta+O(\delta^2) \\
&= \frac{dx}{dt}+[\Upsilon_t+(\Upsilon_x-\omega_t)x'-\omega_x x'^2]\delta+O(\delta^2),
\end{aligned} \tag{4}
$$

where $\Upsilon^r_{[t]}=(\Upsilon_t)^r+((\Upsilon_x)^r-(\omega_t)^r)x'(t-r)-(x'(t-r))^2(\omega_x)^r$.

Comparing the coefficient of δ, we get

$$
\omega F_t+\Upsilon F_x+\Upsilon^r F_{x(t-r)}+\Upsilon^r_{[t]}F_{x'(t-r)}=\Upsilon_t+(\Upsilon_x-\omega_t)x'-\omega_x x'^2. \tag{5}
$$

Equation (5) is the required Lie type invariance condition. ■

Similar to the case of ordinary differential equations, we can define a prolonged operator (the general infinitesimal generator associated with the Lie algebra) for neutral differential equation as

$$
\zeta=\omega\frac{\partial}{\partial t}+\Upsilon\frac{\partial}{\partial x}+\Upsilon^r\frac{\partial}{\partial x(t-r)}.
$$

With the notation $D_t=\frac{\partial}{\partial t}+x'\frac{\partial}{\partial x}$, we can write

$$
\begin{aligned}
\tfrac{d\bar{x}}{d\bar{t}} &= \tfrac{dx}{dt}+(D_t(\Upsilon)-x'D_t(\omega))\delta+O(\delta^2) \\
&= \tfrac{dx}{dt}+\Upsilon_{[t]}\delta+O(\delta^2),
\end{aligned} \tag{6}
$$

where $\Upsilon_{[t]}=D_t(\Upsilon)-x'D_t(\omega)$. We then define the extended operator

$$
\zeta^{(1)}=\omega\frac{\partial}{\partial t}+\Upsilon\frac{\partial}{\partial x}+\Upsilon^r\frac{\partial}{\partial x(t-r)}+\Upsilon_{[t]}\frac{\partial}{\partial x'}+\Upsilon^r_{[t]}\frac{\partial}{\partial x'(t-r)}. \tag{7}
$$

Defining $\Delta=x'(t)-F(t,x(t),x(t-r),x'(t-r))=0$, we get

$$
\zeta^{(1)}\Delta=\Upsilon_{[t]}-\omega F_t-\Upsilon F_x-\Upsilon^r F_{x(t-r)}-\Upsilon^r_{[t]}F_{x'(t-r)}. \tag{8}
$$

Comparing Equations (5) and (8), we get

$$
\Upsilon_{[t]}=\Upsilon_t+(\Upsilon_x-\omega_t)x'-\omega_x x'^2.
$$

On substituting $x'=F$ into $\zeta^{(1)}\Delta=0$, we get an invariance condition for the neutral differential equation which is $\zeta^{(1)}\Delta\mid_{\Delta=0}=0$, from which we will obtain the determining equations.
We point out here that Equations (6) through (8) is another way of working with higher order differential equations as compared to Equations (3) through (5) which is simpler to use for lower order differential equations.

Remark 2.1.

If the term $x'(t-r)$ is absent, then the corresponding first order neutral differential equation reduces to a first order delay differential equation.

We conclude this section by proving a very elementary result we have established which is used in our subsequent sections. This proposition is an extension of the result known for ordinary differential equations.

Proposition 2.1.

If the linear functional differential equation is given by

$$x'(t) + ax'(t-r) + bx(t) + cx(t-r) = d(t), \tag{9}$$

then, by employing a change of variables namely $\bar{t} = t, \bar{x} = x - \tilde{x}$, where $\tilde{x}$ is a solution of the functional differential equation, we can convert the given nonhomogeneous linear functional differential equation to a homogeneous one, namely $x'(t) + ax'(t-r) + bx(t) + cx(t-r) = 0$.

Proof:

The proposition follows by substituting $t = \bar{t}$ and $x(t) = \bar{x} + \tilde{x}(\bar{t})$ in (9), by noting that
$\tilde{x}'(t) + a\tilde{x}'(t-r) + b\tilde{x}(t) + c\tilde{x}(t-r) = d(t)$. ■

In the subsequent sections, we make a complete group classification of the first order linear delay and neutral differential equations, to solvable Lie algebras — a more accurate classification scheme as compared to those present in literature.

3. Classification of First Order Delay Differential Equations to Solvable Lie Algebras

3.1. Linear Case

We will make a classification of the first order delay differential equation with constant coefficients,

$$x'(t) + \alpha x(t) + \beta x(t-r) = 0. \tag{10}$$

The extension and prolongation operator for Equation (10) is given by

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'}. \tag{11}$$

Applying the operator defined by Equation (11) to the delay equation $g(t) = t - r$, we get

$$\omega(t,x) = \omega(t-r, x(t-r)). \tag{12}$$

Applying the operator defined by Equation (11) to Equation (10), we get

$$\Upsilon_t + (\Upsilon_x - \omega_t)(-\alpha x - \beta x^r) - \omega_x(\alpha^2 x^2 - 2\alpha\beta x x^r + \beta^2 x^{r2}) + \alpha\Upsilon + \beta\Upsilon^r = 0. \tag{13}$$

Splitting Equation (13) with respect to the constant term, we get

$$\Upsilon_t + \alpha\Upsilon + \beta\Upsilon^r = 0. \tag{14}$$

Splitting Equation (13) with respect to x, we get

$$-\alpha(\Upsilon_x - \omega_t) = 0. \tag{15}$$

Splitting Equation (13) with respect to x^2, x^{r2} or xx^r, we get

$$\omega_x = 0. \tag{16}$$

Splitting Equation (13) with respect to x^r, we get

$$-\beta(\Upsilon_x - \omega_t) = 0. \tag{17}$$

We solve the above equations by studying all possible cases and make a complete classification of Equation (10) to solvable Lie algebras by proving the following theorems, with the notation $u = x^r$.

Theorem 3.1.

The first order delay differential equation (10) for which

(1) $\alpha \neq -\beta$, admits the two dimensional Lie algebra generated by

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),$$

with the infinite dimensional Lie sub-algebra given by

$$S_3^i = -\left(\frac{\omega_t}{\alpha+\beta}\right)\frac{\partial}{\partial t} + \Big[\theta - (\alpha+\beta)\omega x\Big]\frac{\partial}{\partial x} - \left[\frac{\alpha}{\beta}\theta + \frac{1}{\beta}\theta_t + (\alpha+\beta)\omega x\right]\frac{\partial}{\partial u}.$$

(2) $\alpha = -\beta$, admits the two dimensional Lie algebra generated by

$$S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \quad S_2 = \frac{\partial}{\partial t},$$

with the infinite dimensional Lie sub-algebra given by

$$S_3^i = \theta\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right) - \frac{1}{\beta}\theta_t\frac{\partial}{\partial u}.$$

Proof:

(1) Let α, β be arbitrary non-zero constants, $\alpha \neq -\beta$. Then, from Equation (16), we get $\omega = \omega(t)$. From Equation (15), we get $\Upsilon = \omega_t x + \theta(t), \quad \Upsilon^r = \omega_t x + \psi(t-r)$.

From Equation (14), we get $\omega_t = c_1 - (\alpha+\beta)\omega, \quad \psi = -\dfrac{\alpha}{\beta}\theta - \dfrac{1}{\beta}\theta_t$, and

$$\omega = c_2 - \frac{\omega_t}{\alpha+\beta}, \tag{18}$$

where c_1 is an arbitrary constant and $c_2 = \dfrac{c_1}{\alpha+\beta}$. Hence,

$$\Upsilon = [c_1 - (\alpha+\beta)\omega]x + \theta, \tag{19}$$

and

$$\Upsilon^r = [c_1 - (\alpha+\beta)\omega]x + \psi. \tag{20}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_2 - \frac{\omega_t}{\alpha+\beta}\right)\frac{\partial}{\partial t} + \left([c_1 - (\alpha+\beta)\omega]x + \theta\right)\frac{\partial}{\partial x}\\ &\quad + \left([c_1 - (\alpha+\beta)\omega]x - \left(\frac{\alpha}{\beta}\theta + \frac{1}{\beta}\theta_t\right)\right)\frac{\partial}{\partial x^r}.\end{aligned}$$

We see that the Lie algebra is spanned by $S_1 = \dfrac{\partial}{\partial t}, \quad S_2 = x\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right)$ with

$$S_3 = -\left(\frac{\omega_t}{\alpha+\beta}\right)\frac{\partial}{\partial t} + [\theta - (\alpha+\beta)\omega x]\frac{\partial}{\partial x} - \left[(\alpha+\beta)\omega x + \frac{\alpha}{\beta}\theta + \frac{1}{\beta}\theta_t\right]\frac{\partial}{\partial u},$$

as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2
S_1	0	0
S_2	0	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra.

(2) Let α, β be arbitrary non-zero constants, $\alpha = -\beta$. Then Equation (14) becomes $\Upsilon_t + \alpha(\Upsilon - \Upsilon^r) = 0$, which can be solved to give

$$\omega = c_3 t + c_4, \tag{21}$$

$$\Upsilon = c_3 x + \theta, \tag{22}$$

$$\Upsilon^r = c_3 x + \psi, \tag{23}$$

where c_3, c_4 are arbitrary constants and $\psi = \dfrac{\theta_t}{\alpha} + \theta$.

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_3 t + c_4)\frac{\partial}{\partial t} + (c_3 x + \theta)\frac{\partial}{\partial x} + \left(c_3 x + \frac{\theta_t}{\alpha} + \theta\right)\frac{\partial}{\partial x^r}.\end{aligned}$$

We see that the Lie algebra is spanned by $S_1 = t\dfrac{\partial}{\partial t} + x\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right), \quad S_2 = \dfrac{\partial}{\partial t}$ with

$S_3 = \theta\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right) - \dfrac{1}{\beta}\theta_t\dfrac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2
S_1	0	$-S_2$
S_2	S_2	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra. ∎

Corollary 3.1.

The first order delay differential equation given by Equation (10), for which $\alpha = 0$, β is an arbitrary non-zero constant, admits the same generators as in the Theorem 3.1 (part (1)) above, only that the infinite dimensional Lie sub-algebra is given by,

$$S_3 = -\left(\frac{\omega_t}{\beta}\right)\frac{\partial}{\partial t} + (\theta - \beta x\omega)\frac{\partial}{\partial x} - \left[\frac{\theta_t}{\beta} + \beta x\omega\right]\frac{\partial}{\partial u}.$$

Proof:

If $\alpha = 0$ and β is a non-zero constant, then Equation (14) becomes

$$\Upsilon_t + \beta\Upsilon^r = 0. \tag{24}$$

Equation (15) is redundant, while Equations (16) and (17) remain the same.

From Equation (16), we get $\omega = \omega(t)$. Solving Equation (17), we get $\Upsilon = \omega_t x + \theta(t), \quad \Upsilon^r = \omega_t x + \psi(t-r)$. Substituting this in Equation (24), we get $\omega_{tt} + \theta_t + \beta[\omega_t x + \psi(t-r)] = 0$, which on splitting with respect to x and the constant term and solving it gives,

$$\omega = \frac{c_1 - \omega_t}{\beta}, \quad \psi = -\frac{\theta_t}{\beta}.$$

Therefore, we can write,

$$\omega = c_2 - \frac{\omega_t}{\beta}, \tag{25}$$

where c_1 and c_2 are arbitrary constants.

For $\alpha = 0$, Equations (19) and (20) give

$$\Upsilon = [c_1 - \beta\omega]x + \theta, \tag{26}$$

$$\Upsilon^r = [c_1 - \beta\omega]x + \psi. \tag{27}$$

Using Equations (25), (26) and (27), we see that the infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_2 - \frac{\omega_t}{\beta}\right)\frac{\partial}{\partial t} + [(c_1 - \beta\omega)x + \theta]\frac{\partial}{\partial x} + [(c_1 - \beta\omega)x + \psi]\frac{\partial}{\partial x^r}.\end{aligned}$$

Let $x^r = u$, then the Lie algebra is spanned by $S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)$ with $S_3 = -\frac{\omega_t}{\beta}\frac{\partial}{\partial t} + [\theta - \beta\omega x]\frac{\partial}{\partial x} - \left[\frac{\theta_t}{\beta} + \beta\omega x\right]\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2
S_1	0	0
S_2	0	0

. ■

3.2. Nonlinear Case

We make a classification of

$$x'(t) = k\left[1 - \frac{x(t-r)}{P}\right]x(t), \tag{28}$$

a nonlinear delay differential equation extensively studied by Jones (1962) and Kakutani and Markus (1958) in modeling population growth problems.

Applying the operator defined by Equation (11), to the delay equation $g(t) = t - r$, we get Equation (12).

Applying the operator defined by Equation (11), to Equation (28), we get

$$\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2 = k\Upsilon - \frac{k}{P}[\Upsilon_x^r + x\Upsilon^r]. \tag{29}$$

Splitting Equation (29) with respect to constant term, x' and x'^2 respectively, we get

$$\Upsilon_t = k\Upsilon - \frac{k}{P}x^r\Upsilon - \frac{k}{P}x\Upsilon^r, \tag{30}$$

$$\Upsilon_x - \omega_t = 0, \tag{31}$$

$$\omega_x = 0. \tag{32}$$

These equations can be solved to give

$$\omega = c_1, \tag{33}$$

$$\Upsilon = \theta, \quad \Upsilon^r = \psi, \tag{34}$$

where c_1 is an arbitrary constant and $\theta_t = \psi\theta - \frac{k}{P}\theta x^r - \frac{k}{P}x\psi$.

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= c_1\frac{\partial}{\partial t} + \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial x^r}.\end{aligned}$$

We see that the Lie algebra is spanned by $S_1 = \dfrac{\partial}{\partial t}$, and $S_2 = \theta\dfrac{\partial}{\partial x} + \psi\dfrac{\partial}{\partial u}$ is the infinite dimensional Lie sub-algebra.

4. Classification of First Order Neutral Differential Equations to Solvable Lie Algebras

4.1. Linear Case

We will make a classification of the first order neutral differential equation with constant coefficients,

$$x'(t) + \alpha x(t) + \beta x(t-r) + \gamma x'(t-r) = 0. \tag{35}$$

The extension and prolongation operator for Equation (35) is given by

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x^{r\prime}}. \tag{36}$$

Applying the operator defined by Equation (36) to the equation $g(t) = t - r$, we get Equation (12).

Applying the operator defined by Equation (36) to Equation (35), we get

$$\begin{aligned}\Upsilon_t + (\Upsilon_x - \omega_t)(-\alpha x - \beta x^r - \gamma x^{r\prime}) - \omega_x(\alpha^2 x^2 + 2\alpha\beta x x^r + 2\beta\gamma x^r x^{r\prime} + 2\alpha\gamma x x^{r\prime}\\ + \beta^2 x^{r2} + \gamma^2 x^{r'2}) + \alpha\Upsilon + \beta\Upsilon^r + \gamma[\Upsilon^r_t + (\Upsilon^r_x - \omega^r_t)x^{r\prime} - \omega^r_x x^{r'2}] = 0.\end{aligned} \tag{37}$$

Splitting Equation (37) with respect to the constant term, we get

$$\Upsilon_t + \alpha\Upsilon + \beta\Upsilon^r + \gamma\Upsilon^r_t = 0. \tag{38}$$

Splitting Equation (37) with respect to x, we get

$$-\alpha(\Upsilon_x - \omega_t) = 0. \tag{39}$$

Splitting Equation (37) with respect to $x^2, x^{r2}, x^r x^{r\prime}, x x^{r\prime}$ and $x x^r$, we get

$$\omega_x = 0. \tag{40}$$

Splitting Equation (37) with respect to x^r, we get

$$-\beta(\Upsilon_x - \omega_t) = 0. \tag{41}$$

Splitting Equation (37) with respect to $x^{r'2}$, we get

$$\gamma\omega_x - \omega^r_x = 0. \tag{42}$$

Splitting Equation (37) with respect to $x^{r'}$, we get

$$\Upsilon_x - \omega_t = \Upsilon^r_x - \omega^r_t. \tag{43}$$

We solve the above equations by studying all possible cases and make a complete classification of Equation (35) to solvable Lie algebras by proving the following theorems, with the notation $u = x^r$.

Theorem 4.1.

The first order neutral differential equation (35) for which

(1) $\alpha \neq -\beta$, admits the two dimensional Lie algebra generated by

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),$$

with the infinite dimensional Lie sub-algebra given by

$$S_3^i = -\left(\frac{1+\gamma}{\alpha+\beta}\right)\omega_t\frac{\partial}{\partial t} + \left[\theta - \left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial x} + \left[\psi - \left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial u}.$$

(2) $\alpha = -\beta$, admits the two dimensional Lie algebra generated by

$$S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \quad S_2 = \frac{\partial}{\partial t},$$

with the infinite dimensional Lie sub-algebra given by $S_3^i = \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}$.

(3) $\alpha = -\beta,\ \gamma = -1$, admits the one dimensional Lie algebra generated by $S_1 = \frac{\partial}{\partial t}$ with the infinite dimensional Lie sub-algebra given by $S_2^i = \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}$.

Proof:

(1) Let α, β, γ be arbitrary non-zero constants, $\alpha \neq -\beta, \gamma \neq -1$. Then, from Equation (40), we get $\omega = \omega(t)$. From Equations (39), (41) and (43), we get

$$\Upsilon = \omega_t x + \theta(t), \quad \Upsilon^r = \omega_t x + \psi(t-r).$$

From Equation (38) we get $\omega_t = c_3 - \frac{(\alpha+\beta)}{1+\gamma}\omega, \quad \theta_t + \alpha\theta + \beta\psi + \gamma\psi_t = 0$, and

$$\omega = c_2 - \frac{1+\gamma}{\alpha+\beta}\omega_t, \tag{44}$$

where c_1 is an arbitrary constant, $c_2 = \frac{c_1}{\alpha+\beta}$ and $c_3 = \frac{c_1}{1+\gamma}$. Hence,

$$\Upsilon = \left[c_3 - \frac{(\alpha+\beta)}{1+\gamma}\omega\right]x + \theta, \tag{45}$$

and

$$\Upsilon^r = \left[c_3 - \frac{(\alpha+\beta)}{1+\gamma}\omega\right] x + \psi. \tag{46}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\ &= \left(c_2 - \frac{1+\gamma}{\alpha+\beta}\omega_t\right)\frac{\partial}{\partial t} + \left(\left[c_3 - \frac{(\alpha+\beta)}{1+\gamma}\omega\right] x + \theta\right)\frac{\partial}{\partial x} \\ &\quad + \left(\left[c_3 - \frac{(\alpha+\beta)}{1+\gamma}\omega\right] x + \psi\right)\frac{\partial}{\partial x^r}.\end{aligned}$$

We see that the Lie algebra is spanned by $S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)$ with

$$S_3 = -\left(\frac{1+\gamma}{\alpha+\beta}\right)\omega_t\frac{\partial}{\partial t} + \left[\theta - \left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial x} + \left[\psi - \left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial u},$$

as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2
S_1	0	0
S_2	0	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra.

(2) Let α, β be arbitrary non-zero constants, $\alpha = -\beta$, $\gamma \neq -1$. Then Equation (38) becomes $\Upsilon_t + \alpha(\Upsilon - \Upsilon^r) + \gamma\Upsilon^r_t = 0$, which can be solved to give

$$\omega = c_4 t + c_5, \tag{47}$$

$$\Upsilon = c_4 x + \theta, \tag{48}$$

$$\Upsilon^r = c_4 x + \psi, \tag{49}$$

where c_4, c_5 are arbitrary constants and $\theta_t + \alpha(\theta - \psi) + \gamma\psi_t = 0$.

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\ &= (c_4 t + c_5)\frac{\partial}{\partial t} + (c_4 x + \theta)\frac{\partial}{\partial x} + (c_4 x + \psi)\frac{\partial}{\partial x^r}.\end{aligned}$$

We see that the Lie algebra is spanned by $S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \quad S_2 = \frac{\partial}{\partial t}$ with

$S_3 = \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2
S_1	0	$-S_2$
S_2	S_2	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra.

(3) Let $\alpha \neq -\beta$, $\gamma = -1$. Then Equation (38) becomes $\Upsilon_t + \alpha\Upsilon + \beta\Upsilon^r - \Upsilon^r_t = 0$, which can be solved to give

$$\omega = c_7, \tag{50}$$

$$\Upsilon = \theta(t), \tag{51}$$

$$\Upsilon^r = \psi(t - r), \tag{52}$$

where c_6 is an arbitrary constant, $c_7 = \dfrac{c_6}{\alpha + \beta}$ and $\theta_t + \alpha\theta + \beta\psi - \psi_t = 0$.

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= c_7\frac{\partial}{\partial t} + \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial x^r}.\end{aligned}$$

We see that the Lie algebra is spanned by $S_1 = \dfrac{\partial}{\partial t}$ with $S_2 = \theta\dfrac{\partial}{\partial x} + \psi\dfrac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra. ∎

Corollary 4.1.

Subject to the condition $\gamma = -1$, in Equation (35), the same result of Theorem 4.1 (part (3)) is obtained if $\alpha \neq -\beta$ but either $\alpha = 0$ or $\beta = 0$.

Proof:

At the start, we see that Equation (40) gives $\omega = \omega(t)$.

If $\alpha = 0$, consider Equation (39) and if $\beta = 0$, consider Equation (41). In either case, solution of the corresponding equation gives

$$\Upsilon = \omega_t x + \theta(t), \quad \Upsilon^r = \omega_t x + \psi(t - r).$$

Without loss of generality, consider $\alpha = 0$. Then, Equation (38) becomes $\Upsilon_t + \beta\Upsilon^r - \Upsilon^r_t = 0$.

Substituting the values of Υ and Υ^r, we get $\theta_t + \beta[\omega_t x + \psi] - \psi_t = 0$, which on splitting with respect to x and the constant term give, $\omega = c_1$, a constant. Consequently, $\Upsilon = \theta, \quad \Upsilon^r = \psi$. Thus, the infinitesimal generator of the admitted Lie group is

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= c_1\frac{\partial}{\partial t} + \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial x^r}.\end{aligned}$$

We see that the Lie algebra is spanned by $S_1 = \dfrac{\partial}{\partial t}$ with $S_2 = \theta\dfrac{\partial}{\partial x} + \psi\dfrac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra. ∎

Theorem 4.2.

The first order neutral differential equation (35) for which $\alpha = 0 = \beta,\ \gamma = 1$ admits the three dimensional Lie algebra generated by

$$S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = \frac{\partial}{\partial u},$$

with the infinite dimensional Lie sub-algebra given by $S_4^i = \theta\left[\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial u}\right]$.

Proof:

Let $\alpha = 0 = \beta$, $\gamma = 1$. Then Equation (38) becomes $\Upsilon_t + \Upsilon_t^r = 0$, which can be solved to give

$$\omega = c_8 t + c_9, \tag{53}$$

$$\Upsilon = c_8 x + \theta(t), \tag{54}$$

$$\Upsilon^r = c_8 x + \psi(t-r), \tag{55}$$

where c_8, c_9, c_{10} are arbitrary constants and $\psi = c_{10} - \theta$.

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_8 t + c_9)\frac{\partial}{\partial t} + (c_8 x + \theta)\frac{\partial}{\partial x} + (c_8 x + c_{10} - \theta)\frac{\partial}{\partial x^r}.\end{aligned}$$

We see that the Lie algebra is spanned by $S_1 = t\dfrac{\partial}{\partial t} + x\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right), \quad S_2 = \dfrac{\partial}{\partial t}, S_3 = \dfrac{\partial}{\partial u}$, with $S_4 = \theta\left(\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial u}\right)$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3
S_1	0	$-S_2$	0
S_2	S_2	0	0
S_3	0	0	0

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. ■

Corollary 4.2.

For the first order neutral differential equation given by (35) with $\alpha = 0 = \beta$, $\gamma \neq 0, -1$, the same generators as in the previous Theorem are obtained, only that the infinite dimensional Lie sub-algebra is given by $S_4 = \theta \left(\frac{\partial}{\partial x} - \frac{1}{\gamma} \frac{\partial}{\partial u} \right)$.

Proof:

Subject to the given conditions and using Equation (43), we see that

$$\omega = \omega(t), \quad \Upsilon = \omega_t x + \theta(t), \quad \Upsilon^r = \omega_t x + \psi(t - r).$$

Further, the given conditions reduce Equation (38) to $\Upsilon_t + \gamma \Upsilon_t^r = 0$, which on substituting the values of Υ and Υ^r and splitting the resulting equation with respect to x and the constant term give

$$(1 + \gamma)\omega_{tt} = 0, \quad \theta_t + \gamma \psi_t = 0.$$

These equations can be solved to give

$$\omega = c_1 t + c_2, \quad \psi = c_4 - \frac{\theta}{\gamma},$$

where c_1, c_2, c_3 are arbitrary constants and $c_4 = \frac{c_3}{\gamma}$.

Therefore, the infinitesimal generator of the Lie group is given by

$$\begin{aligned} \zeta^* &= \omega \frac{\partial}{\partial t} + \Upsilon \frac{\partial}{\partial x} + \Upsilon^r \frac{\partial}{\partial x^r} \\ &= (c_1 t + c_2) \frac{\partial}{\partial t} + (c_1 x + \theta) \frac{\partial}{\partial x} + (c_1 x + c_4 - \frac{\theta}{\gamma}) \frac{\partial}{\partial x^r}. \end{aligned}$$

We see that the Lie algebra is spanned by

$$S_1 = t \frac{\partial}{\partial t} + x \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right), \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = \frac{\partial}{\partial u},$$

with $S_4 = \theta \left[\frac{\partial}{\partial x} - \frac{1}{\gamma} \frac{\partial}{\partial u} \right]$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3
S_1	0	$-S_2$	0
S_2	S_2	0	0
S_3	0	0	0

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. ■

4.2. Nonlinear Case

We make a classification of

$$x'(t) + x(t)x(t-r) + x'(t-r) = v(t). \tag{56}$$

This is a nonlinear and nonhomogeneous equation.

Applying the operator defined by Equation (36), to the equation $g(t) = t - r$, we get Equation (12).

Applying the operator defined by Equation (36), to Equation (56), we get

$$\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2 + x\Upsilon^r + x^r\Upsilon + \Upsilon_t^r + (\Upsilon_x^r - \omega_t^r)x^{r\prime} - \omega_x^r {x^{r\prime}}^2 = \omega v'. \tag{57}$$

Splitting Equation (57) with respect to constant term, x', x'^2, $x^{r\prime}$ and ${x^{r\prime}}^2$ respectively, we get

$$\Upsilon_t + x\Upsilon^r + x^r\Upsilon + \Upsilon_t^r = \omega v', \tag{58}$$

$$\Upsilon_x - \omega_t = 0, \tag{59}$$

$$\omega_x = 0, \tag{60}$$

$$\Upsilon_x^r - \omega_t^r = 0, \tag{61}$$

$$\omega_x^r = 0. \tag{62}$$

These equations can be solved to give

$$\omega = c_1, \tag{63}$$

$$\Upsilon = \theta, \quad \Upsilon^r = \psi, \tag{64}$$

where c_1, c_2 are arbitrary constants and $\theta = c_1 v + c_2 - \psi_t$.

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\ &= c_1\frac{\partial}{\partial t} + (c_1 v + c_2 - \psi_t)\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial x^r}.\end{aligned}$$

We see that the Lie algebra is spanned by

$$S_1 = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}, \quad S_2 = \frac{\partial}{\partial x},$$

and $S_3 = -\psi_t\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}$ is the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2
S_1	0	0
S_2	0	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra.

5. Conclusion

We have established a Lie type invariance condition based on Taylor's theorem for a function of several variables. Using this condition, we have thoroughly classified first order linear and some nonlinear functional differential equations to solvable Lie algebras. We see that the dimension of the Lie algebra ranges from 1 to 3 with the maximum dimension occurring when the differential equation is free from the unknown function at time t and the earlier instants (delay term). In total, we have obtained 10 different solvable Lie algebras admitted by these first order functional differential equations.

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