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The Lattice of Intuitionistic Fuzzy Topologies Generated by Intuitionistic Fuzzy Relations

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Abstract

We generalize the notion of fuzzy topology generated by fuzzy relation given by Mishra and Srivastava to the setting of intuitionistic fuzzy sets. Some fundamental properties and necessary examples are given. More specifically, we provide the lattice structure to a family of intuitionistic fuzzy topologies generated by intuitionistic fuzzy relations. To that end, we study necessary structural characteristics such as distributivity, modularity and complementary of this lattice.

Keywords: Fuzzy set; Atanassov's intuitionistic fuzzy set; Lattice of topologies; Binary relation

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1. Introduction

Topology generated by binary relation is one of the famous classes of general topology and play a prominent role in pure and applied mathematics. They apply in different fields especially in preference representation theorems (Bridges and Mehta (1995)) and they appear to provide the notion of nearness or proximity between two elements of an arbitrary set without using any distance function on it (Knoblauch (2009)). They are also useful for obtaining continuous representability of binary relations, which is an important optimization tool (Chateauneuf (1987), Debreu (1964)) and are used in important applications such as computing topologies (Zhao and Tsang (2008)), recombination spaces (Fotea (2008)) and information granulation which are used in biological sciences and other application fields.

The study of the topology generated by binary relation was initiated by Smithson (1969). Then after that, many researchers have been working in this topic, Knoblauch (2009) introduced topology induced by a binary relation, which are generated by the set of all upper and lower contours of this relation and he obtained a characterization about this class of topology. Salama (2008) used binary relation to generate topological structures using the lower and the upper approximations. Campión et al. (2009) have characterized topologies induced by total preorder relations by utility functions. Recently, Induráin and Knoblauch (2013) have studied the problem of characterizing which topologies on a nonempty set are generated by a binary relations by means of their lower and upper contour sets, also they extended this characterization to the context of bitopological spaces induced by binary relations. In fuzzy setting, Mishra and Srivastava (2018) have introduced the notion of fuzzy topology generated by a fuzzy relation and studied several related results.

In 1983, Atanassov (1983) introduced the concept of intuitionistic fuzzy set (IFSs) characterized by a membership function and a non-membership function, which is a generalization of Zadeh's fuzzy set. Then Çoker (1997) applied this concept to topology and introduced intuitionistic fuzzy topology on a set X as a family τ, satisfying the well-know axioms, and he referred to each member of τ as an intuitionistic fuzzy open set.

The aim of the present paper is to construct the lattice structure to a family of intuitionistic fuzzy topologies generated by intuitionistic fuzzy relations. We pay particular attention to the characteristics of this lattice such as distributivity, modularity and complementary.

The contents of the paper are organized as follows. In Section 2, we recall basic concepts and properties that will be needed throughout this paper. In Section 3, we generalize the notion of fuzzy topology generated by a fuzzy relation to the setting of intuitionistic fuzzy sets, and we study some properties of this topology in terms of its lower and upper contour sets. In Section 4, we provide a lattice structure to a family of intuitionistic fuzzy topologies generated by intuitionistic fuzzy relations, and we investigate some basic characteristics of this lattice. Finally, we present some conclusions and we discuss future research in Section 5.

2. Preliminaries

This section contains the basic definitions and properties of lattices, intuitionistic fuzzy sets, intuitionistic fuzzy relations, intuitionistic fuzzy topologies and some related notions that will be needed throughout this paper.

2.1. Lattices

In this subsection, we recall some definitions and properties of lattices that will be needed throughout this paper (for more details, see Davey and Priestley (2002), Schröder (2002)).

A partial order (order, for short) is a binary relation $\leqslant$ over a set X which is reflexive ($a \leqslant a$, for any $a \in X$), antisymmetric ($a \leqslant b$ and $b \leqslant a$ implies $a = b$, for any $a, b \in X$) and transitive ($a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$, for any $a, b, c \in X$). A set with an order relation is called an ordered set (also called a poset), denoted $(X, \leqslant)$. Let $(X, \leqslant)$ be a poset and A be a subset of X. An element $x_0 \in X$ is called a lower bound of A if $x_0 \leqslant x$, for any $x \in A$. The element x_0 is called the greatest lower bound (or the infimum) of A if x_0 is a lower bound and $m \leqslant x_0$, for any lower bound m of A. Upper bound and least upper bound (or supremum) are defined dually. Let $(X, \leqslant_X)$, $(Y, \leqslant_Y)$ be two posets. A mapping $\varphi : X \to Y$ is called an order isomorphism if it is surjective and satisfies the following condition: $x \leqslant_X y$ if and only if $\varphi(x) \leqslant_Y \varphi(y)$, for any $x, y \in X$.

2.2. Intuitionistic fuzzy sets

In this subsection, we recall some basic concepts of intuitionistic fuzzy sets.

Let X be a universe. Then a fuzzy subset $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$, of X defined by Zadeh (1965) is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of a membership of the element x in the fuzzy subset A for each $x \in X$.

Atanassov (1983) introduced another fuzzy object, called intuitionistic fuzzy set as a generalization of the concept of fuzzy set, shown as follows,

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

which is characterized by a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$, with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1\,,$$

for any $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of the element x in the intuitionistic fuzzy set A for each $x \in X$. The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

In the fuzzy set theory, the non-membership degree of an element x of the universe is defined as $\nu_A(x) = 1 - \mu_A(x)$, (using the standard negation) and thus it is fixed. In intuitionistic fuzzy setting, the non-membership degree is a more-or-less independent degree: the only condition is that $\nu_A(x) \leq 1 - \mu_A(x)$. Certainly fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$, but not conversely.

For any two IFSs A and B on a set X, several operations are defined (see, e.g., Atanassov (1983, 1986, 1989, 1999), Milles et al. (2016, 2017)). Here we will present only those which are related to the present paper.

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$, be two IFSs on a set X, then
(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x),$ for all $x \in X$,
(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x),$ for all $x \in X$,

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$,
(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$,
(v) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$,
(vi) $Supp(A) = \{x \in X \mid \mu_A(x) > 0$ or $(\mu_A(x) = 0$ and $\nu_A(x) < 1)\}$,
(vii) $Ker(A) = \{x \in X \mid \mu_A(x) = 1$ or $\nu_A(x) = 0\}$.

2.3. Intuitionistic fuzzy relations

Burillo and Bustince (1995) introduced the concept of intuitionistic fuzzy relation as a natural generalization of fuzzy relation.

An intuitionistic fuzzy binary relation (An intuitionistic fuzzy relation, for short) from a universe X to a universe Y is an intuitionistic fuzzy subset in $X \times Y$, i.e., is an expression R given by

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in X \times Y\},$$

where,

$$\mu_R : X \times Y \to [0, 1], \text{ and } \nu_A : X \times Y \to [0, 1],$$

satisfy the condition

$$0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1,$$

for any $(x, y) \in X \times Y$. The value $\mu_R(x, y)$ is called the degree of a membership of (x, y) in R and $\nu_R(x, y)$ is called the degree of a non-membership of (x, y) in R.

The class of intuitionistic fuzzy relations on X is denoted by $IFR(X^2)$.

Example 2.1.

Let $X = \{a, b, c, d, e\}$, and R be an intuitionistic fuzzy relation defined on X by $R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid x, y \in X\}$, where μ_R and ν_R are given by the following tables.

Table 1. Intuitionistic fuzzy relation for Example 2.1

$\mu_R(.,.)$	a	b	c	d	e
a	0.35	0	0	0.35	0.30
b	0	0.40	0	0.35	0.45
c	0.20	0	0.65	0	0.70
d	0	0	0	1	0
e	0.25	0.35	0	0	0.60

$\nu_R(.,.)$	a	b	c	d	e
a	0	1	0.40	0.25	0.25
b	0.30	0.35	0.20	0.35	0.10
c	0.80	1	0	0.85	0.15
d	1	1	1	0	1
e	0.70	0.55	1	0.90	0.30

Next, the following definitions are needed to recall (see, e.g., Atanassov (1986), Burillo and Bustince (1995)). Let R and P be two intuitionistic fuzzy relations from a universe X to a universe Y.

(i) The transpose (inverse) R^t of R is the intuitionistic fuzzy relation from the universe Y to the universe X defined by

$$R^t = \{\langle (x,y), \mu_{R^t}(x,y), \nu_{R^t}(x,y)\rangle \mid (x,y) \in X \times Y\},$$

where,

$$\begin{cases} \mu_{R^t}(x,y) = \mu_R(y,x), \\ \quad \text{and} \\ \nu_{R^t}(x,y) = \nu_R(y,x)\,, \end{cases}$$

for any $(x,y) \in X \times Y$.

(ii) R is said to be contained in P or we say that P contains R, denoted by $R \subseteq P$, if for all $(x,y) \in X \times Y$ it holds that $\mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$.

(iii) The intersection (resp. *the union*) of two intuitionistic fuzzy relations R and P from a universe X to a universe Y is an intuitionistic fuzzy relation defined as

$$R \cap P = \{\langle (x,y), \min(\mu_R(x,y), \mu_P(x,y)), \max(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\},$$

and

$$R \cup P = \{\langle (x,y), \max(\mu_R(x,y), \mu_P(x,y)), \min(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}\,.$$

The following notions are crucial in this paper (see e.g., Burillo and Bustinceb (1995), Bustince and Burillo (1995), Bustince and Burillo (1996), Bustince (2003)).

Let R be an intuitionistic fuzzy relation from a universe X into itself.

(i) *Reflexivity*: $\mu_R(x,x) = 1$, for any $x \in X$. In this case we note that $\nu_R(x,x) = 0$, for any $x \in X$.

(ii) *Symmetry*: for any $x, y \in X$, then

$$\begin{cases} \mu_R(x,y) = \mu_R(y,x), \\ \nu_R(x,y) = \nu_R(y,x). \end{cases}$$

(iii) *Antisymmetry*: for any $x, y \in X$, $x \neq y$ then

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x), \\ \nu_R(x,y) \neq \nu_R(y,x), \\ \pi_R(x,y) = \pi_R(y,x), \end{cases}$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

(iv) *Perfect antisymmetry*: for any $x, y \in X$, with $x \neq y$ and

$$\begin{cases} \mu_R(x,y) > 0, \\ \quad \text{or} \\ \mu_R(x,y) = 0 \ \text{ and } \nu_R(x,y) < 1\,, \end{cases}$$

then

$$\begin{cases} \mu_R(y,x) = 0, \\ \quad \text{and} \\ \nu_R(y,x) = 1\,. \end{cases}$$

(v) *Transitivity*: $R \supseteq R \circ_{\lambda,\rho}^{\alpha,\beta} R$, where α, β, λ and ρ are t-norms or t-conorms taken under the intuitionistic fuzzy condition

$$0 \leq \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)]\} + \lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\} \leq 1\,,$$

for any $x, z \in X$.

2.4. Intuitionistic fuzzy topology

In this subsection, we recall the notion of intuitionistic fuzzy topology given by Çoker (1997).

An intuitionistic fuzzy topology (IFT, for short) on a nonempty set X is a family τ of intuitionistic fuzzy sets on X which satisfies the following axioms:
(i) $\emptyset, X \in \tau$,
(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,
(iii) $\bigcup G_i \in \tau$ for any $\{G_i :\ i \in J\} \subseteq \tau$.

In this case, the pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS, for short) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS, for short) in X. The complement of an intuitionistic fuzzy open set is called an intuitionistic fuzzy closed set (IFCS, for short) in X.

Example 2.2.

Let $X = \{x, y, z\}$ and $A, B, C \in IFS(X)$ such that

$A = \{\langle x, 0.3, 0.4\rangle, \langle y, 0.4, 0.3\rangle, \langle z, 0.1, 0.3\rangle\}$,

$B = \{\langle x, 0.4, 0.1\rangle, \langle y, 0.5, 0.2\rangle, \langle z, 0.3, 0.2\rangle\}$,

$C = \{\langle x, 0.3, 0.1\rangle, \langle y, 0.4, 0.3\rangle, \langle z, 0.3, 0.3\rangle\}$.

Then, $\tau = \{\emptyset, X, A, B, C\}$ is an intuitionistic fuzzy topology on X.

The notion of interior (resp. closure) of an intuitionistic fuzzy set is introduced by Atanassov (1999). Later, Çoker (1997) introduced the notion of intuitionistic fuzzy interior (resp. intuitionistic fuzzy closure) of an intuitionistic fuzzy set on an intuitionistic fuzzy topological space.

Let (X, τ) be an intuitionistic fuzzy topological space, for every intuitionistic fuzzy subset A of X the intuitionistic fuzzy interior (resp. the intuitionistic fuzzy closure) of A is defined by:
$int(A) = \{\langle x, \underset{x\in X}{max}\ \mu_G(x), \underset{x\in X}{min}\ \nu_G(x)\rangle \mid G$ is an IFOS in X and $G \subseteq A\}$, and
$cl(A) = \{\langle x, \underset{x\in X}{min}\ \mu_K(x), \underset{x\in X}{max}\ \nu_K(x)\rangle \mid K$ is an IFOS in X and $A \subseteq K\}$,

Remark 2.3.

The intuitionistic fuzzy interior (resp. the intuitionistic fuzzy closure) of an intuitionistic fuzzy set

A is an intuitionistic fuzzy open set (resp. an intuitionistic fuzzy closed set), and we have
(i) $int(\overline{A}) = \overline{cl(A)}$,
(ii) $cl(\overline{A}) = \overline{int(A)}$.

3. Intuitionistic fuzzy topology generated by intuitionistic fuzzy relation

In this section, we first generalize the notion of fuzzy topology generated by fuzzy relation given by Mishra and Srivastava (2018) to the setting of intuitionistic fuzzy sets. Based on this generalization, we investigate some properties of this topology in terms of its lower and upper contour sets.

Definition 3.1.

Let X be a nonempty crisp set and $R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid x, y \in X\}$, be an intuitionistic fuzzy relation on X. Then for any $x \in X$, the intuitionistic fuzzy sets $\mathcal{L}_x$ et $\mathcal{R}_x$ are defined by
$\mu_{\mathcal{L}_x}(y) = \mu_R(y, x)$, and $\nu_{\mathcal{L}_x}(y) = \nu_R(y, x)$, for any $y \in X$,
$\mu_{\mathcal{R}_x}(y) = \mu_R(x, y)$, and $\nu_{\mathcal{R}_x}(y) = \nu_R(x, y)$, for any $y \in X$.
They are called the lower and the upper contour, respectively, of x.

We denote by τ_1, the intuitionistic fuzzy topology generated by the set of all lower contours and τ_2, the intuitionistic fuzzy topology generated by the set of all upper contours. Consequently, we denote by τ_R, the intuitionistic fuzzy topology generated by S the set of all lower and upper contours and it's called the intuitionistic fuzzy topology generated by R.

Remark 3.2.

Since the intuitionistic fuzzy set $\mathcal{L}_x$ (resp. $\mathcal{R}_x$) is defined from the intuitionistic fuzzy relation R, then it trivially holds that $0 \leq \mu_{\mathcal{L}_x} + \nu_{\mathcal{L}_x} \leq 1$, (resp. $0 \leq \mu_{\mathcal{R}_x} + \nu_{\mathcal{R}_x} \leq 1$), for any $x \in X$.

Example 3.3.

Let $X = \{x, y\}$ and R be an intuitionistic fuzzy relation on X given by the following tables.

Table 2. Intuitionistic fuzzy relation for Example 3.3

$\mu_R(.,.)$	x	y
x	0.6	0.8
y	0.3	0.7

$\nu_R(.,.)$	x	y
x	0.3	0.1
y	0.6	0.2

Then, $\mathcal{L}_x,\ \mathcal{L}_y,\ \mathcal{R}_x$ and $\mathcal{R}_y$ are the intuitionistic fuzzy sets on X given by:

$$\mathcal{L}_x = \{\langle x, 0.6, 0.3\rangle; \langle y, 0.3, 0.6\rangle\},$$
$$\mathcal{L}_y = \{\langle x, 0.8, 0.1\rangle; \langle y, 0.7, 0.2\rangle\},$$
$$\mathcal{R}_x = \{\langle x, 0.6, 0.3\rangle; \langle y, 0.8, 0.1\rangle\},$$
$$\mathcal{R}_y = \{\langle x, 0.3, 0.6\rangle; \langle y, 0.7, 0.2\rangle\}.$$

We note that, $\mathcal{L}_x \subset \mathcal{L}_y,\ \mathcal{L}_x \subset \mathcal{R}_y,\ \mathcal{R}_y \subset \mathcal{R}_x$ and $\mathcal{R}_y \subset \mathcal{L}_y$. Then the intuitionistic fuzzy topology $\tau_{\mathcal{R}}$ is generated by $S = \{\mathcal{L}_x, \mathcal{L}_y\} \cup \{\mathcal{R}_x, \mathcal{R}_y\}$. Thus, $\tau_{\mathcal{R}} = \{\emptyset, X, \mathcal{L}_x, \mathcal{L}_y, \mathcal{R}_x, \mathcal{R}_y, \mathcal{L}_x \cap \mathcal{R}_y, \mathcal{L}_y \cap \mathcal{R}_x, \mathcal{L}_x \cup \mathcal{R}_y, \mathcal{L}_y \cup R_x\}$, where
$\mathcal{L}_x \cap \mathcal{R}_y = \{\langle x, 0.6, 0.3\rangle; \langle y, 0.7, 0.2\rangle\}$, $\mathcal{L}_y \cap \mathcal{R}_x = \{\langle x, 0.8, 0.1\rangle; \langle y, 0.8, 0.1\rangle\}$,
$\mathcal{L}_x \cup \mathcal{R}_x = \{\langle x, 0.7, 0.1\rangle; \langle y, 0.7, 0.1\rangle\}$, and $\mathcal{L}_x \cup \mathcal{R}_x = \{\langle x, 0.6, 0.2\rangle; \langle y, 0.4, 0.4\rangle\}$.

Example 3.4.

Let $X = \{x, y\}$ and R be an intuitionistic fuzzy relation on X given by the following tables.

Table 3. Intuitionistic fuzzy relation for Example 3.4

$\mu_R(.,.)$	x	y
x	0.6	0.3
y	0.7	0.4

$\nu_R(.,.)$	x	y
x	0.2	0.5
y	0.1	0.4

Then, $\mathcal{L}_x,\ \mathcal{L}_y,\ \mathcal{R}_x$ and $\mathcal{R}_y$ are the intuitionistic fuzzy sets on X given by:

$$\mathcal{L}_x = \{\langle x, 0.6, 0.2\rangle; \langle y, 0.7, 0.1\rangle\},$$
$$\mathcal{L}_y = \{\langle x, 0.3, 0.5\rangle; \langle y, 0.4, 0.4\rangle\},$$
$$\mathcal{R}_x = \{\langle x, 0.6, 0.2\rangle; \langle y, 0.3, 0.5\rangle\},$$
$$\mathcal{R}_y = \{\langle x, 0.7, 0.1\rangle; \langle y, 0.4, 0.4\rangle\}.$$

We note that, $\mathcal{L}_y \subset \mathcal{L}_x,\ \mathcal{L}_y \subset \mathcal{R}_y,\ \mathcal{R}_x \subset \mathcal{R}_y$ and $\mathcal{R}_x \subset \mathcal{L}_x$. Then, the intuitionistic fuzzy topology $\tau_{\mathcal{R}}$ is generated by $S = \{\mathcal{L}_x, \mathcal{L}_y\} \cup \{\mathcal{R}_x, \mathcal{R}_y\}$. Thus, $\tau_{\mathcal{R}} = \{\emptyset, X, \mathcal{L}_x, \mathcal{L}_y, \mathcal{R}_x, \mathcal{R}_y, \mathcal{L}_x \cap \mathcal{R}_y, \mathcal{L}_y \cap \mathcal{R}_x, \mathcal{L}_x \cup \mathcal{R}_y, \mathcal{R}_y \cup \mathcal{R}_x\}$, where
$\mathcal{L}_x \cap \mathcal{R}_y = \{\langle x, 0.6, 0.2\rangle; \langle y, 0.4, 0.4\rangle\}$, $\mathcal{L}_y \cap \mathcal{R}_x = \{\langle x, 0.3, 0.5\rangle; \langle y, 0.3, 0.5\rangle\}$,
$\mathcal{L}_x \cup \mathcal{R}_y = \{\langle x, 0.7, 0.1\rangle; \langle y, 0.7, 0.1\rangle\}$, and $\mathcal{R}_y \cup \mathcal{R}_x = \{\langle x, 0.6, 0.2\rangle; \langle y, 0.4, 0.4\rangle\}$.

Proposition 3.5.

Let X be a nonempty crisp set and R be an intuitionistic fuzzy symmetric relation on X. Then it holds that $\tau_1 = \tau_2$.

Proof:

Let R be an intuitionistic fuzzy symmetric relation on X, then for any $x, y \in X$ it follows that $\mu_R(x,y) = \mu_R(y,x)$ and $\nu_R(x,y) = \nu_R(y,x)$. Then, it holds that, $\mu_{\mathcal{L}_x}(y) = \mu_{\mathcal{R}_x}(y)$ and $\nu_{\mathcal{L}_x}(y) = \nu_{\mathcal{R}_x}(y)$. Therefore, $\mathcal{L}_x = \mathcal{R}_x$, for any $x \in X$. We conclude that $\tau_1 = \tau_2$. ∎

Remark 3.6.

If R is an intuitionistic fuzzy preorder relation, then the intuitionistic fuzzy topology generated by R is a generalization of Alexandrov topology introduced by Kim (2014).

4. The lattice of intuitionistic fuzzy topologies

In this section, we mainly investigate the lattice of all intuitionistic fuzzy topologies generated by intuitionistic fuzzy relations. First, we introduce the notion of intuitionistic fuzzy inclusion between topologies.

Definition 4.1.

Let R_1, R_2 are two intuitionistic fuzzy relations on the set X and τ_{R_1}, τ_{R_2} are the intuitionistic fuzzy topologies generated by R_1 and R_2 respectively. Then, τ_{R_1} is said to be contained in τ_{R_2} (in symbols, $\tau_{R_1} \sqsubseteq \tau_{R_2}$) if $G \in \tau_{R_2}$ for any $G \in \tau_{R_1}$.

In this case, we also say that τ_{R_1} is smaller than τ_{R_2}.

Proposition 4.2.

Let τ_{R_1} and τ_{R_2} are the intuitionistic fuzzy topologies on the set X generated by R_1 and R_2 respectively. Then, it holds that, $\tau_{R_1} \sqsubseteq \tau_{R_2}$ if and only if $R_1 \subseteq R_2$ (i.e., $\mu_{R_1}(x,y) \leq \mu_{R_2}(x,y)$, and $\nu_{R_1}(x,y) \geq \nu_{R_2}(x,y),$ for any $(x,y) \in X \times X$).

Proof:

Suppose that $\tau_{R_1} \sqsubseteq \tau_{R_2}$, then for any $y \in X$ it holds that $\mu_{R_1}(x,y) = \mu_{\mathcal{R}_{1_x}}(y) \leq \mu_{\mathcal{R}_{2_x}}(y) = \mu_{R_2}(x,y)$, and $\nu_{R_1}(x,y) = \nu_{\mathcal{R}_{1_x}}(y) \geq \nu_{\mathcal{R}_{2_x}}(y) = \nu_{R_2}(x,y)$ such that $\mathcal{R}_1$ and $\mathcal{R}_2$ are the upper contours of x over R_1 and R_2 respectively . On the other hand, for any $x \in X$ it holds that $\mu_{R_1}(x,y) = \mu_{\mathcal{L}_{1_y}}(x) \leq \mu_{\mathcal{L}_{2_y}}(x) = \mu_{R_2}(x,y)$, and $\nu_{R_1}(x,y) = \nu_{\mathcal{L}_{1_y}}(x) \geq \nu_{\mathcal{L}_{2_y}}(x) = \nu_{R_2}(x,y)$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are the lower contours of y over R_1 and R_2 respectively. Hence, $\mu_{R_1}(x,y) \leq \mu_{R_2}(x,y)$ and $\nu_{R_1}(x,y) \geq \nu_{R_2}(x,y),$ for any $(x,y) \in X \times X$. Thus, $R_1 \subseteq R_2$. Conversely, suppose that $R_1 \subseteq R_2$. By using the same method as above we get that $\tau_{R_1} \sqsubseteq \tau_{R_2}$. ∎

Now, we introduce the intersection of IF-topologies generated by IF-relations.

Definition 4.3.

Let τ_{R_1} and τ_{R_2} are the intuitionistic fuzzy topologies on the set X generated by R_1 and R_2 respectively. The intersection of τ_{R_1} and τ_{R_2} (in symbols, $\tau_{R_1} \sqcap \tau_{R_2}$) is an intuitionistic fuzzy topology τ_R such that $G \in \tau_R$ if and only if $G \in \tau_{R_1}$ and $G \in \tau_{R_2}$.

In general, if τ_{R_i} is a family of intuitionistic fuzzy topologies generated by a family of intuitionistic fuzzy relations R_i. Then, $G \in \underset{i\in I}{\sqcap} \tau_{R_i}$ if and only if $G \in \tau_{R_i}$, for any $i \in I$. Furthermore, $\underset{i\in I}{\sqcap} \tau_{R_i}$ is the smallest intuitionistic fuzzy topology on X containing all τ_{R_i}.

Example 4.4.

Let τ_{R_1} and τ_{R_2} the intuitionistic fuzzy topologies given in Examples 3.3 and 3.4. Then, it holds that $\tau_{R_1} \sqcap \tau_{R_2} = \{\phi,\ X\}$.

The following proposition shows that intuitionistic fuzzy topologies on a set are closed under the intersection and union of intuitionistic fuzzy sets.

Proposition 4.5.

Let τ_{R_1} and τ_{R_2} are the intuitionistic fuzzy topologies on the set X generated by R_1 and R_2, respectively, and $\tau_R = \tau_{R_1} \sqcap \tau_{R_2}$. Then, it holds that

$$\begin{aligned} R &= R_1 \cap R_2, \\ &= \{\langle (x,y), \min(\mu_{R_1}(x,y), \mu_{R_2}(x,y)), \max(\nu_{R_1}(x,y), \nu_{R_2}(x,y))\rangle \mid (x,y) \in X \times X\}. \end{aligned}$$

In general, if τ_{R_i} is a family of intuitionistic fuzzy topologies generated by a family of intuitionistic fuzzy relations R_i. Then, it holds that

$$R = \underset{i\in I}{\cap} R_i = \{\langle (x,y), \min(\mu_{R_i}(x,y)), \max(\nu_{R_i}(x,y))\rangle \mid (x,y) \in X \times X\}.$$

Proof:

Suppose that $\tau_R = \tau_{R_1} \sqcap \tau_{R_2}$. Then, for any $y \in X$, it holds that $\mu_R(x,y) = \mu_{\mathcal{R}_x}(y) = \mu_{\mathcal{R}_{1_x}}(y) \wedge \mu_{\mathcal{R}_{2_x}}(y) = \mu_{R_1}(x,y) \wedge \mu_{R_2}(x,y)$ and $\nu_R(x,y) = \nu_{\mathcal{R}_x}(y) = \nu_{\mathcal{R}_{1_x}}(y) \vee \nu_{\mathcal{R}_{2_x}}(y) = \nu_{R_1}(x,y) \vee \nu_{R_2}(x,y)$ such that $\mathcal{R}_1$ and $\mathcal{R}_2$ are the upper contours of x over R_1 and R_2, respectively. On the other hand, for any $x \in X$ it holds that $\mu_R(x,y) = \mu_{\mathcal{L}_y}(x) = \mu_{\mathcal{L}_{1_y}}(x) \wedge \mu_{\mathcal{L}_{2_y}}(x) = \mu_{R_1}(x,y) \wedge \mu_{R_2}(x,y)$ and $\nu_R(x,y) = \nu_{\mathcal{L}_y}(x) = \nu_{\mathcal{L}_{1_y}}(x) \vee \nu_{\mathcal{L}_{2_y}}(x) = \nu_{R_1}(x,y) \vee \nu_{R_2}(x,y)$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are the lower contours of y over R_1 and R_2 respectively. Hence, $\mu_R(x,y) = \mu_{R_1}(x,y) \wedge \mu_{R_2}(x,y)$ and $\nu_R(x,y) = \nu_{R_1}(x,y) \vee \nu_{R_2}(x,y),$ for any $(x,y) \in X \times X$. Thus, $R = R_1 \cap R_2$. ■

Next, we introduce the union of IF-topologies generated by IF-relations.

Definition 4.6.

Let τ_{R_1} and τ_{R_2} are the intuitionistic fuzzy topologies on the set X generated by R_1 and R_2 respectively. The union of τ_{R_1} and τ_{R_2} (in symbols, $\tau_{R_1} \sqcup \tau_{R_2}$) is an intuitionistic fuzzy topology τ_R such that $G \in \tau_R$ if and only if $G \in \tau_{R_1}$ or $G \in \tau_{R_2}$.

In general, if τ_{R_i} is a family of intuitionistic fuzzy topologies generated by a family of intuitionistic fuzzy relations R_i. Then, $G \in \underset{i\in I}{\sqcup}\tau_{R_i}$ if at least $G \in \tau_{R_{i_0}}$, with $i_0 \in I$.

Proposition 4.7.

Let τ_{R_1} and τ_{R_2} are the intuitionistic fuzzy topologies on the set X generated by R_1 and R_2 respectively, and $\tau_R = \tau_{R_1} \sqcup \tau_{R_2}$. Then, it holds that

$$\begin{aligned} R &= R_1 \cup R_2, \\ &= \{\langle (x,y), \max(\mu_{R_1}(x,y), \mu_{R_2}(x,y)), \min(\nu_{R_1}(x,y), \nu_{R_2}(x,y))\rangle \mid (x,y) \in X \times X\}. \end{aligned}$$

In general, if τ_{R_i} is a family of intuitionistic fuzzy topologies generated by a family of intuitionistic fuzzy relations R_i. Then, it holds that

$$R = \underset{i\in I}{\cup} R_i = \{\langle (x,y), \max(\mu_{R_i}(x,y)), \min(\nu_{R_i}(x,y))\rangle \mid (x,y) \in X \times X\}.$$

Proof:

Suppose that $\tau_R = \tau_{R_1} \sqcup \tau_{R_2}$. Then, for any $y \in X$, it holds that $\mu_R(x,y) = \mu_{\mathcal{R}_x}(y) = \mu_{\mathcal{R}_{1_x}}(y) \vee \mu_{\mathcal{R}_{2_x}}(y) = \mu_{R_1}(x,y) \vee \mu_{R_2}(x,y)$ and $\nu_R(x,y) = \nu_{\mathcal{R}_x}(y) = \nu_{\mathcal{R}_{1_x}}(y) \wedge \nu_{\mathcal{R}_{2_x}}(y) = \nu_{R_1}(x,y) \wedge \nu_{R_2}(x,y)$ such that $\mathcal{R}_1$ and $\mathcal{R}_2$ are the upper contours of x over R_1 and R_2, respectively. On the other hand, for any $x \in X$ it holds that $\mu_R(x,y) = \mu_{\mathcal{L}_y}(x) = \mu_{\mathcal{L}_{1_y}}(x) \vee \mu_{\mathcal{L}_{2_y}}(x) = \mu_{R_1}(x,y) \vee \mu_{R_2}(x,y)$ and $\nu_R(x,y) = \nu_{\mathcal{L}_y}(x) = \nu_{\mathcal{L}_{1_y}}(x) \wedge \nu_{\mathcal{L}_{2_y}}(x) = \nu_{R_1}(x,y) \wedge \nu_{R_2}(x,y)$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are the lower contours of y over R_1 and R_2, respectively. Hence, $\mu_R(x,y) = \mu_{R_1}(x,y) \vee \mu_{R_2}(x,y)$ and $\nu_R(x,y) = \nu_{R_1}(x,y) \wedge \nu_{R_2}(x,y),$ for any $(x,y) \in X \times X$. Thus, $R = R_1 \cup R_2$. ■

The following theorem provides the lattice structure to a family of IF-topologies generated by IF-relations.

Theorem 4.8.

Let X be a finite set and $\mathfrak{L} = \{\tau_{R_i} \mid R_i \in IFR(X^2)\}$, is a family of all intuitionistic fuzzy topologies on X generated by the intuitionistic fuzzy relations R_i. Then, $\mathfrak{L}$ is an intuitionistic fuzzy lattice on X.

Proof:

Suppose that $\{\tau_{R_i}\}$ is a set of intuitionistic fuzzy topologies generated by the intuitionistic fuzzy relations R_i. Definition of IF-topology guaranties that $\{\tau_{R_i}\}$ is a nonempty set. Now, let τ_{R_1} and τ_{R_2} be two intuitionistic fuzzy topologies generated by the intuitionistic fuzzy relation R_1 and R_2 respectively. It is easy to cheek that $\tau_{R_1} \sqsubseteq \tau_{R_1}$, i.e., the IF-reflexivity, and if we suppose that $\tau_{R_1} \sqsubseteq \tau_{R_2}$ and $\tau_{R_2} \sqsubseteq \tau_{R_1}$, it follows that $\tau_{R_1} = \tau_{R_2}$ i.e., the IF-antisymmety. In order to verify the transitivity, we suppose that $\tau_{R_1} \sqsubseteq \tau_{R_2}$ and $\tau_{R_2} \sqsubseteq \tau_{R_3}$, it follows that $\tau_{R_1} \sqsubseteq \tau_{R_3}$, i.e., the IF-transitivity. Hence, $(\mathfrak{L}, \sqsubseteq)$ is an intuitionistic fuzzy poset on X. Moreover, the least upper bound (resp. the greatest lower bound) of τ_{R_1} and τ_{R_2} is coincides with the intersection of intuitionistic fuzzy topologies (resp. the union of intuitionistic fuzzy topologies), i.e., $\tau_{R_1} \wedge \tau_{R_2} = \tau_{R_1} \sqcap \tau_{R_2}$ (resp. $\tau_{R_1} \vee \tau_{R_2} = \tau_{R_1} \sqcup \tau_{R_2}$). Thus, we can conclude that $(\mathfrak{L}, \sqsubseteq)$ is an intuitionistic fuzzy lattice on X. ■

Proposition 4.9.

Let X be a finite set and $\mathfrak{L} = \{\tau_{R_i} \mid R_i \in IFR(X^2)\}$ is the lattice of all intuitionistic fuzzy topologies on X generated by the intuitionistic fuzzy relations R_i. Then, $\mathfrak{L}$ is complete.

Proof:

Let $\mathfrak{L} = \{\tau_{R_i} \mid R_i \in IFR(X^2)\}$ be the lattice of intuitionistic fuzzy topologies on X generated by the intuitionistic fuzzy relations R_i. Suppose that $A = \{\tau_{R_j}\}$ is a subset of $\mathfrak{L}$ under the intuitionistic fuzzy inclusion between topologies defined above. The fact that $\mathfrak{L}$ is a finite intuitionistic fuzzy lattices, this implies that $\sqcap\tau_{R_j} \in \mathfrak{L}$ which implies that A has an infimum. Hence, $\mathfrak{L}$ is complete. ■

Corollary 4.10.

Let $\mathfrak{L}$ be the complete lattice of all intuitionistic fuzzy topologies generated by intuitionistic fuzzy relations, then $\mathfrak{L}$ is bounded. Indeed, the least element of $\mathfrak{L}$ is $0_{\mathfrak{L}} = \sqcap\tau_{R_i}$ and the greatest element of $\mathfrak{L}$ is $1_{\mathfrak{L}} = \sqcup\tau_{R_i}$.

Proof:

Let $\mathfrak{L} = \{\tau_{R_i} \mid R_i \in IFR(X^2)\}$, be the lattice of intuitionistic fuzzy topologies on X generated by the intuitionistic fuzzy relations R_i. On the one hand, from Definition 4.3, it follows that $\sqcap\tau_{R_i}$ is an intuitionistic fuzzy topology on X generated by $R = \underset{i\in I}{\cap} R_i$. Hence, $\sqcap\tau_{R_i}$ is the smallest intuitionistic fuzzy topology on X denoted by $0_{\mathfrak{L}}$. On the other hand, from Definition 4.6, it follows that $\sqcup\tau_{R_i}$ is an intuitionistic fuzzy topology on X generated by $R = \underset{i\in I}{\cup} R_i$. Hence, $\sqcup\tau_{R_i}$ is the greatest intuitionistic fuzzy topology on X denoted by $0_{\mathfrak{L}}$. We conclude that $\mathfrak{L}$ is bounded. ■

The following proposition discuss the distributivity property for the lattice of all IF-topologies generated by IF-relations.

Proposition 4.11.

Let $\mathfrak{L}$ be the intuitionistic fuzzy lattice of intuitionistic fuzzy topologies generated by the intuitionistic fuzzy relations R_i, then $\mathfrak{L}$ is distributive.

Proof:

Suppose that τ_{R_1}, τ_{R_2} and τ_{R_3} are an intuitionistic fuzzy topologies generated by R_1, R_2 and R_3, respectively. We will show that $\tau_{R_1} \sqcap (\tau_{R_2} \sqcup \tau_{R_3}) = (\tau_{R_1} \sqcap \tau_{R_2}) \sqcup (\tau_{R_1} \sqcap \tau_{R_3})$. We have

$$\begin{aligned}
\mu_{R_1\cap(R_2\cup R_3)}(x,y) &= \mu_{R_1}(x,y) \wedge (\mu_{R_2}(x,y) \vee \mu_{R_3(x,y)}),\\
&= (\mu_{R_1}(x,y) \wedge \mu_{R_2}(x,y)) \vee (\mu_{R_1}(x,y) \wedge \mu_{R_3}(x,y)),\\
&= \mu_{R_1\cap R_2}(x,y) \vee \mu_{R_1\cap R_3}(x,y),\\
&= \mu_{(R_1\cap R_2)\cup(R_1\cap R_3)}(x,y).
\end{aligned}$$

Similarly, for any $x, y \in X$, it holds that

$$\begin{aligned}\nu_{R_1\cap(R_2\cup R_3)}(x,y) &= \nu_{R_1}(x,y) \vee (\nu_{R_2}(x,y) \wedge \nu_{R_{3(x,y)}}),\\ &= (\nu_{R_1}(x,y) \vee \nu_{R_2}(x,y)) \wedge (\nu_{R_1}(x,y) \vee \nu_{R_3}(x,y)),\\ &= \nu_{R_1\cap R_2}(x,y) \wedge \nu_{R_1\cap R_3}(x,y),\\ &= \nu_{(R_1\cap R_2)\cup(R_1\cap R_3)}(x,y).\end{aligned}$$

Hence, $R_1\cap(R_2\cup R_3) = (R_1\cap R_2)\cup(R_1\cap R_3)$. From Proposition 4.2, it holds that $\tau_{R_1}\sqcap(\tau_{R_2}\sqcup\tau_{R_3}) = (\tau_{R_1} \sqcap \tau_{R_2}) \sqcup (\tau_{R_1} \sqcap \tau_{R_3})$. We conclude that $\mathfrak{L}$ is a distributive lattice. ■

Corollary 4.12.

Since $\mathfrak{L}$ is a distributive lattice, then it holds that $\mathfrak{L}$ is modular.

Proof:

Let τ_{R_1}, τ_{R_2} and τ_{R_3} be an intuitionistic fuzzy topologies generated by R_1, R_2 and R_3, respectively. Since $\mathfrak{L}$ is a distributive lattice, then it follows that $\tau_{R_1}\sqcup(\tau_{R_2}\sqcap\tau_{R_3}) = (\tau_{R_1}\sqcup\tau_{R_2})\sqcap(\tau_{R_1}\sqcup\tau_{R_3})$. Since $\tau_{R_1} \sqsubseteq \tau_{R_3}$, then it holds that $\tau_{R_1}\sqcup\tau_{R_3} = \tau_{R_3}$. This implies that $\tau_{R_1}\sqcup(\tau_{R_2}\sqcap\tau_{R_3}) = (\tau_{R_1}\sqcup\tau_{R_2})\sqcap\tau_{R_3}$. Hence, $\mathfrak{L}$ is modular. ■

In 1958, Hartmanis (1958) proved that the lattice of all topologies on a finite set is complemented. In the following proposition, we prove that the lattice of intuitionistic fuzzy topologies generated by intuitionistic fuzzy relations is also complemented.

Proposition 4.13.

Let $\mathfrak{L}$ be the lattice of all intuitionistic fuzzy topologies generated by the intuitionistic fuzzy relations R_i, then $\mathfrak{L}$ is complemented.

Proof:

Indeed, every element $\tau_{R_{i_0}}$ has a complement $\tau_{R_{j_0}}$ such that $\tau_{R_{i_0}} \sqcap \tau_{R_{j_0}} = 0_{\mathfrak{L}}$ and $\tau_{R_{i_0}} \sqcup \tau_{R_{j_0}} = 1_{\mathfrak{L}}$. Hence, $\mathfrak{L}$ is complemented. ■

Corollary 4.14.

Since $\mathfrak{L}$ is a distributive lattice and complemented with a least element $0_{\mathfrak{L}}$ and a greatest element $1_{\mathfrak{L}}$, then $\mathfrak{L}$ is a Boolean algebra denoted by $(\mathfrak{L}, \sqcap, \sqcup, 0_{\mathfrak{L}}, 1_{\mathfrak{L}})$.

Proof:

The proof is directly from Proposition 4.11 and Proposition 4.13. ■

In the following theorem, we will show that the lattice of IF-relations and the lattice of all IF-topologies generated by IF-relations are isomorphic.

Theorem 4.15.

Let $\varphi : IFR(X^2) \to \mathfrak{L}$ be a mapping defined as $\varphi(R) = \tau_R$, for any $R \in IFR(X^2)$, then $(IFR(X^2), \cap, \cup)$ and $\mathfrak{L}$ are isomorphic.

Proof:

Assume that $\varphi : IFR(X^2) \to \mathfrak{L}$ is a mapping. It is obvious to verify that φ is surjective. Furthermore, Proposition 4.2 guarantees that $R_1 \subseteq R_2$ if and only if $\tau_{R_1} \sqsubseteq \tau_{R_1}$, for any $R_1, R_2 \in IFR(X^2)$. Thus, φ is an order isomorphism between $IFR(X^2)$ and $\mathfrak{L}$, which is equivalent to saying that φ is a lattice isomorphism. Therefore, $(IFR(X^2), \cap, \cup)$ and $\mathfrak{L}$ are isomorphic.■

Remark 4.16.

By the virtue of the two lattices $IFR(X^2)$ and $\mathfrak{L}$ are isomorphic, we mention that the lattice $IFR(X^2)$ inherits all the properties of $\mathfrak{L}$, like distributivity, modularity and complementary.

5. Conclusion

In this work, we have introduced the intuitionistic fuzzy topologies generated by intuitionistic fuzzy relations and we have investigated its most important properties. In particular, we have focused on the lattice structure of these topologies and we have shown that this lattice is complete, distributive, modular and complemented.

Future work is anticipated in multiple directions. We think it makes sense to study the notions of intuitionistic fuzzy ideal and intuitionistic fuzzy filter in this class of topological spaces. Moreover, we intend to extend this work to neutrosophic sets, and to characterize their ideals and filters.

REFERENCES

Atanassov, K.T. (1983). *Intuitionistic fuzzy sets*. VII ITKRs Scientific Session, Sofia.

Atanassov, K.T. (1986). Intuitionistic fuzzy sets, Fuzzy Set Syst., Vol. 20, pp. 87–96.

Atanassov, K.T. (1989). More on intuitionistic fuzzy sets, Fuzzy Set Syst., Vol. 33, pp. 37–45.

Atanassov, K.T. (1999). *Intuitionistic Fuzzy Sets*, Physica-Verlag, Heidelberg, New York.

Bridges, D.S. and Mehta, G.B. (1995). *Representations of Preference Orderings*, Springer, Berlin.

Burillo, P. and Bustince, H. (1995a). Intuitionistic fuzzy relations (Part I), Mathware Soft Comput., Vol. 2, pp. 5–38.

Burillo, P. and Bustince, H. (1995b). Intuitionistic fuzzy relations (Part II), Effect of Atanassov's operators on the properties of the intuitionistic fuzzy relations, Mathware Soft Comput., Vol. 2, pp. 117–148.

Bustince, H. (2003). Construction of intuitionistic fuzzy relations with predetermined properties, Fuzzy Set Syst., Vol. 3, pp. 79–403.

Bustince, H. and Burillo, P. (1995). Antisymmetrical intuitionistic fuzzy relation. Order on the referential set induced by an bi fuzzy relation, Fuzzy Set Syst., Vol. 62, pp. 17–22.

Bustince, H. and Burillo, P. (1996). Structures on intuitionistic fuzzy relations, Fuzzy Set Syst., Vol. 3, pp. 293–303.

Campión, M.J., Candeal, J.C. and Induráin, E. (2009). Preorderable topologies and orderrepresentability of topological spaces, Topol. Appl., Vol. 156, pp. 2971-2978.

Chateauneuf, A. (1987). Continuous representation of a preference relation on a connected topological space, J. Math. Econ., Vol. 16, pp. 139-146.

Çoker, D. (1997). An introduction to intuitionistic fuzzy topological spaces, Fuzzy Set Syst., Vol. 88, pp. 81-89.

Davey, B.A. and Priestley, H.A. (2002). *Introduction to Lattices and Order*, Cambridge University Press.

Debreu, G. (1964). Continuity properties of Paretian utility, Int. Econ. Rev., Vol. 5, pp. 285-293.

Fotea, V.L. (2008). The lower and upper approximations in a hypergroup, Inform. Sci., Vol. 178, pp. 3605-3615.

Hartmanis, J. (1958). On the lattice of topologies, Quaest Math., Vol. 10, pp. 547-553.

Induráin, E. and Knoblauch, V. (2013). On topological spaces whose topology is induced by a binary relation, Quaest Math., Vol. 36, pp. 47-65.

Kim, Y.C. (2014). Alexandrov L-topologies, Int. J. Pure Appl. Math., Vol. 93, pp. 165-179.

Knoblauch, V. (2009). *Topologies Defined by Binary Relations*, Department of Economics Working Paper Series, University of Connecticut.

Mishra, S. and Srivastava, R. (2018). Fuzzy topologies generated by fuzzy relations, Soft Comput., Vol. 22, pp. 373385.

Milles, S., Rak, E. and Zedam, L. (2016). Characterizations of intuitionistic fuzzy complete lattices, In *Novel Developments in Uncertainty Representation and Processing Advances in Intelligent Systems and Computing*, Springer International Publishing Switzerland, pp. 149–160.

Milles, S., Zedam, L. and Rak, E. (2017). Characterizations of intuitionistic fuzzy ideals and filters based on lattice operations, J. Fuzzy Set Valued Anal., Vol. 2, pp. 143–159.

Salama, A.S. (2008). Topologies induced by relations with applications, J. Comput. Sci., Vol. 4, pp. 877-887.

Schröder, B.S. (2002). *Ordered Sets*, Birkhauser, Boston.

Smithson, R.E. (1969). Topologies generated by relations, Bull Austral. Math. Soc., Vol. 1, pp. 297–306.

Saravanakumar, G., Tamilselvan, S. and Vadivel, A. (2019). Regular semiopen sets on intuitionistic fuzzy topological spaces in Ŝostak's sense, Appl. Appl. Math., Vol. 14, No. 2, pp. 820–835.

Zadeh, L.A. (2008). Fuzzy sets, Inform Comput., Vol. 8, pp. 338–353.

Zhao, S.E. and Tsang, C.C. (2008). On fuzzy approximation operators in attribute reduction with fuzzy rough sets, Inform. Sci., Vol. 178, pp. 3163-3176.